

Globally Optimal SISO H_2 -Norm Model Reduction Using Walsh's Theorem

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Table of Contents

Problem statement

Characterizing first-order optimality

Methodology: polynomial root finding

Methodology: multiparameter eigenvalue problem (MEVP)

Model reduction

- Class \mathcal{M} : minimal, stable, SISO LTI systems.
- Given $H(s) \in \mathcal{M}$ of order n ,

$$H(s) = \frac{b(s)}{a(s)} = \frac{b_{n-1}s^{n-1} + \dots + b_1s + b_0}{s^n + a_{n-1}s^{n-1} + \dots + a_1s + a_0}, \quad (1)$$

find $\hat{H}(s) \in \mathcal{M}$ of order $m < n$,

$$\hat{H}(s) = \frac{\hat{b}(s)}{\hat{a}(s)} = \frac{\hat{b}_{m-1}s^{m-1} + \dots + \hat{b}_1s + \hat{b}_0}{s^m + \hat{a}_{m-1}s^{m-1} + \dots + \hat{a}_1s + \hat{a}_0}, \quad (2)$$

so that $\hat{H}(s)$ is a 'good approximation' of $H(s)$.

- $2n$ unknowns: $\hat{\mathbf{a}} = (\hat{a}_{m-1}, \dots, \hat{a}_0)^T \in \mathbb{R}^n$, $\hat{\mathbf{b}} = (\hat{b}_{m-1}, \dots, \hat{b}_0)^T \in \mathbb{R}^n$.

H_2 -norm model reduction

- Minimize H_2 -norm of approximation error $E(s) = H(s) - \hat{H}(s)$:

$$\hat{H}(s) \in \underset{\hat{H}(s) \in \mathcal{M}}{\operatorname{argmin}} J^2, \quad (3)$$

where,

$$\begin{aligned} J^2 = \|E(s)\|_{H_2}^2 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} |H(j\omega) - \hat{H}(j\omega)|^2 d\omega \\ &= \int_0^{\infty} (h(t) - \hat{h}(t))^2 dt, \end{aligned} \quad (4)$$

with $h(t)$ and $\hat{h}(t)$ the impulse responses of $H(s)$, $\hat{H}(s)$, respectively.

H_2 -norm model reduction

- Minimize H_2 -norm of approximation error $E(s) = H(s) - \hat{H}(s)$:

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with $h(t)$ and $\hat{h}(t)$ the impulse responses of $H(s)$, $\hat{H}(s)$, respectively.

- Opt. problem in (3) is **non-convex** \rightarrow (many) local minimizers.

H_2 -norm model reduction

- State-of-the-art solvers:
 - Interpolation-based methods (e.g., Gugercin et al., 2008; Van Dooren et al., 2010)
 - Lyapunov-based methods (e.g., Spanos et al., 1992)
 - Strategy: solve iteratively for stationary point of (3)

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- Non-convexity \longleftrightarrow iterative solvers:
 - Performance depends on heuristic choice of initial point (heuristics)
 - Impossible to guarantee global optimality
 - Suboptimal from a mathematical point of view \times

H_2 -norm model reduction

- State-of-the-art solvers:
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- Non-convexity \longleftrightarrow iterative solvers:
 - Performance depends on heuristic choice of initial point (heuristics)
 - Impossible to guarantee global optimality
 - Suboptimal from a mathematical point of view \times
- Globally optimal approaches¹:
 - Strategy: characterize first-order necessary conditions for optimality
 - Solve for all stationary points → identify global minimizer(s) \checkmark

¹E.g., Agudelo et al., 2021; Ahmad et al., 2011; Alsubaie, 2019; Hanzon et al., 2007.

Table of Contents

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Interpolatory conditions for optimality

Theorem (Meier and Luenberger, 1967)

Given a stable SISO model $H(s) \in \mathcal{M}$ of order n , let $\hat{H}(s)$ of order m ($m < n$) be a stationary point of the H_2 -norm model reduction problem in (3). Then for all poles p_i of $\hat{H}(s)$,

$$H(-p_i)^{(j)} = \hat{H}(-p_i)^{(j)}, \quad j = 0, \dots, d_i,$$

where d_i is the multiplicity of the pole p_i and the superscript j denotes the j th derivative with respect to s , i.e.,

$$F(a)^{(j)} = \left. \frac{d^j F(s)}{ds^j} \right|_{s=a},$$

for any function $F(s)$ and $a \in \mathbb{C}$.

Walsh's theorem

Theorem (Regalia, 1995)

Given a stable SISO model $H(s) \in \mathcal{M}$ of order n , let $\hat{H}(s)$ of order m with $m < n$ be a stationary point of the model reduction problem (3). Then for all $s \in \mathbb{C}$:

$$H(s) - \hat{H}(s) = \frac{b(s)}{a(s)} - \frac{\hat{b}(s)}{\hat{a}(s)} = \left[\frac{\hat{a}(-s)}{\hat{a}(s)} \right]^2 G(s),$$

with $G(s)$ the Laplace-transform of some real-valued, stable and causal signal, and where $a(s)$, $b(s)$, $\hat{a}(s)$ and $\hat{b}(s)$ are defined as in (1)–(2).

- Roots of $\hat{a}(-s)$ are $\{-p_i\}_{i=1,\dots,m}$
- Origins within rational approximation theory (Walsh, 1960)

Walsh's theorem - revisited

Corollary

For any given stable SISO model $H(s)$ of order n and m th order approximant $\hat{H}(s)$ with $m < n$ as defined in (1)–(2), define the polynomial,

$$I(s) = b(s)\hat{a}(s) - a(s)\hat{b}(s) - [\hat{a}(-s)]^2 \tilde{G}(s), \quad (3)$$

where $\tilde{G}(s)$ is a polynomial parametrized in the coefficients $\mathbf{g} = (g_0, \dots, g_{n-m-1})^T \in \mathbb{R}^{m-n}$:

$$\tilde{G}(s) = g_{n-m-1}s^{n-m-1} + \dots + g_1s + g_0.$$

Then, $\hat{H}(s)$ is a stationary point of (3) if and only if,

$$\exists \mathbf{g} \quad \text{s.t.} \quad I(s) = 0, \quad \forall s \in \mathbb{C}.$$

Walsh's theorem - revisited

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→ Elegant and compact characterization of the stationary points of (3)

Table of Contents

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System of polynomial equations

Example

- Consider the third order model ($n = 3$) used in Agudelo et al., 2021:

$$H(s) = \frac{s^2 + 9s - 10}{s^3 + 12s^2 + 49s + 78}.$$

System of polynomial equations

Example

- Consider the third order model ($n = 3$) used in Agudelo et al., 2021:

$$H(s) = \frac{s^2 + 9s - 10}{s^3 + 12s^2 + 49s + 78}.$$

- For $m = 1$, the polynomial $l(s)$ from (3) is given as,

$$l(s) = \underbrace{(1 - g_1 - \hat{b}_0)}_{f_3} s^3 + \underbrace{(\hat{a}_0 - 12\hat{b}_0 - g_0 + 2\hat{a}_0 g_1 + 9)}_{f_2} s^2 + \underbrace{(9\hat{a}_0 - 49\hat{b}_0 + 2\hat{a}_0 g_0 - \hat{a}_0^2 g_1 - 10)}_{f_1} s + \underbrace{(-g_0 \hat{a}_0^2 - 10\hat{a}_0 - 78\hat{b}_0)}_{f_0} 1.$$

System of polynomial equations

Example (continued)

Strategy: find all $(\hat{\mathbf{a}}, \hat{\mathbf{b}}, \mathbf{g})$ for which $l(s) = 0, \forall s \in \mathbb{C}$ "

$$\iff \begin{cases} 0 = f_3 = 1 - g_1 - \hat{b}_0, \\ 0 = f_2 = \hat{a}_0 - 12\hat{b}_0 - g_0 + 2\hat{a}_0g_1 + 9, \\ 0 = f_1 = 9\hat{a}_0 - 49\hat{b}_0 + 2\hat{a}_0g_0 - \hat{a}_0^2g_1 - 10, \\ 0 = f_0 = -g_0\hat{a}_0^2 - 10\hat{a}_0 - 78\hat{b}_0. \end{cases} \quad (3)$$

System of polynomial equations

Example (continued)

Strategy: find all $(\hat{\mathbf{a}}, \hat{\mathbf{b}}, \mathbf{g})$ for which $l(s) = 0, \forall s \in \mathbb{C}$

$$\Leftrightarrow \begin{cases} 0 = f_3 = 1 - g_1 - \hat{b}_0, \\ 0 = f_2 = \hat{a}_0 - 12\hat{b}_0 - g_0 + 2\hat{a}_0g_1 + 9, \\ 0 = f_1 = 9\hat{a}_0 - 49\hat{b}_0 + 2\hat{a}_0g_0 - \hat{a}_0^2g_1 - 10, \\ 0 = f_0 = -g_0\hat{a}_0^2 - 10\hat{a}_0 - 78\hat{b}_0. \end{cases}$$

- Common roots of (3) contain all stationary points $\hat{H}(s)$ of (3):

J	\hat{a}_0	\hat{b}_0	g_0	g_1	stable
8.9403	$-4.1639 + 0.9026j$	$24.930 - 6.5393j$	$-106.84 - 18.287j$	$-23.930 + 6.5393j$	X
8.9403	$-4.1639 - 0.9026j$	$24.930 + 6.5393j$	$-106.84 + 18.287j$	$-23.930 - 6.5393j$	X
0.3982	0.2671	-0.0437	10.349	1.0437	✓
0.2784	0.6914	9.6796	1.2799	-2.0986	✓
0.5232	-16.618	1.9264	0.0576	-0.9264	X

Summary

- Let f_k be the coefficient corresponding to s^k in the polynomial,

$$l(s) = b(s)\hat{a}(s) - a(s)\hat{b}(s) - [\hat{a}(-s)]^2 \tilde{G}(s),$$

and define the algebraic variety,

$$\mathcal{V}_{\mathbb{R}} = \{(\hat{\mathbf{a}}, \hat{\mathbf{b}}, \mathbf{g}) \in \mathbb{R}^{m+n} : f_k(\hat{\mathbf{a}}, \hat{\mathbf{b}}, \mathbf{g}) = 0, \quad \forall k = 0, \dots, m+n-1\}.$$

Summary

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- Consider the subvariety $\mathcal{V}'_{\mathbb{R}} \subseteq \mathcal{V}_{\mathbb{R}}$ s.t.,

$\mathcal{V}'_{\mathbb{R}}$ contains the $(\hat{\mathbf{a}}, \hat{\mathbf{b}}, \mathbf{g}) \in \mathcal{V}_{\mathbb{R}}$ for which $\hat{H}(s)$ is stable.

Summary

- Let f_k be the coefficient corresponding to s^k in the polynomial,

$$l(s) = b(s)\hat{a}(s) - a(s)\hat{b}(s) - [\hat{a}(-s)]^2 \tilde{G}(s),$$

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$$\mathcal{V}_{\mathbb{R}} = \{(\hat{\mathbf{a}}, \hat{\mathbf{b}}, \mathbf{g}) \in \mathbb{R}^{m+n} : f_k(\hat{\mathbf{a}}, \hat{\mathbf{b}}, \mathbf{g}) = 0, \quad \forall k = 0, \dots, m+n-1\}.$$

- Consider the subvariety $\mathcal{V}'_{\mathbb{R}} \subseteq \mathcal{V}_{\mathbb{R}}$ s.t.,

$\mathcal{V}'_{\mathbb{R}}$ contains the $(\hat{\mathbf{a}}, \hat{\mathbf{b}}, \mathbf{g}) \in \mathcal{V}_{\mathbb{R}}$ for which $\hat{H}(s)$ is stable.

Then, by the Corollary we know that,

→ $\mathcal{V}'_{\mathbb{R}}$ describes all m th order stationary points $\hat{H}(s)$ of (3) ✓

- Computing $\mathcal{V}_{\mathbb{R}}$ is a polynomial root finding problem:
 - square system of $m+n$ polynomial equations, solve using (e.g., Breiding and Timme, 2018)

Numerical example I

We search for the globally optimal 4th order reduced model ($m = 4$) of the state-space model² ($n = 17$) considered in Žigić et al., 1992.

- System of 21 polynomial equations with $d_{\max} = 3$
- Takes 3h24m2s to solve using Breiding and Timme, 2018:
→ $\mathcal{V}_{\mathbb{R}}$ contains 290 tuples, 69 remain in $\mathcal{V}'_{\mathbb{R}}$
- Four best-performing stationary points $\hat{H}(s)$:

	J	$p_{1,2}$	$p_{3,4}$
*	9.14×10^{-3}	$-0.032 \pm 78.54j$	$-0.111 \pm 15.43j$
l_1	9.22×10^{-3}	$-0.032 \pm 78.54j$	$-5.713 \pm 52.57j$
l_2	1.03×10^{-2}	$-0.032 \pm 78.54j$	$-0.023 \pm 3.842j$
l_3	1.09×10^{-2}	$-0.032 \pm 78.54j$	$-4.663 \pm 15.88j$

²The model describes the interaction between a torque activator and a torsional rate sensor for the ACES structure Collins et al., 1991.

Numerical example I

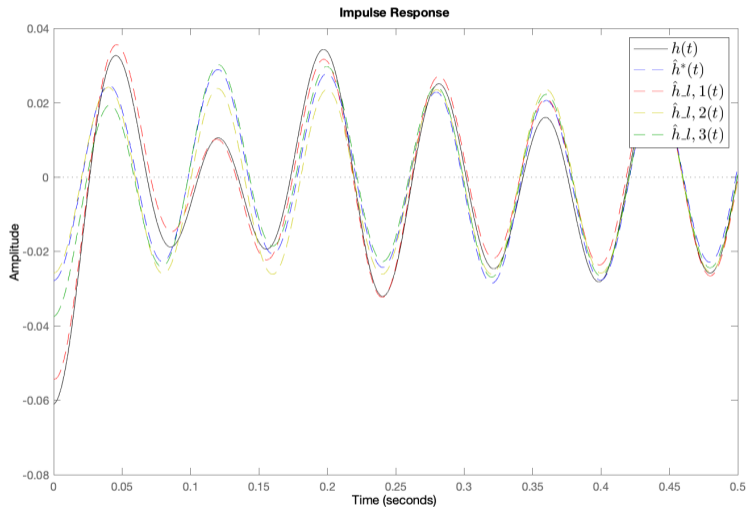


Table of Contents

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Multiparameter eigenvalue problem (MEVP)

Example (continued)

$$\begin{cases} 0 = f_3 = 1 - g_1 - \hat{b}_0, \\ 0 = f_2 = \hat{a}_0 - 12\hat{b}_0 - g_0 + 2\hat{a}_0g_1 + 9, \\ 0 = f_1 = 9\hat{a}_0 - 49\hat{b}_0 + 2\hat{a}_0g_0 - \hat{a}_0^2g_1 - 10, \\ 0 = f_0 = -g_0\hat{a}_0^2 - 10\hat{a}_0 - 78\hat{b}_0. \end{cases}$$

Multiparameter eigenvalue problem (MEVP)

Example (continued)

Exploit partial linear structure in (3) ($\{\hat{b}_i\}_{i=0,\dots,m-1}, \{g_i\}_{i=0,\dots,n-m-1}$),

$$\underbrace{\begin{bmatrix} 1 & -1 & -1 & 0 \\ 9 + \hat{a}_0 & -12 & 2\hat{a}_0 & -1 \\ -10 + 9\hat{a}_0 & -49 & -\hat{a}_0^2 & 2\hat{a}_0 \\ -10\hat{a}_0 & -78 & 0 & -\hat{a}_0^2 \end{bmatrix}}_{\mathcal{M}(\hat{a}_0)} \begin{bmatrix} 1 \\ \hat{b}_0 \\ g_1 \\ g_0 \end{bmatrix} = \mathbf{0},$$

which is a quadratic m -parameter eigenvalue problem (MEVP):

$$\left(\underbrace{\begin{bmatrix} 1 & -1 & -1 & 0 \\ 9 & -12 & 0 & -1 \\ -10 & -49 & 0 & 0 \\ 0 & -78 & 0 & 0 \end{bmatrix}}_{M_0} + \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 9 & 0 & 0 & 2 \\ -10 & 0 & 0 & 0 \end{bmatrix}}_{M_1} \hat{a}_0 + \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}}_{M_2} \hat{a}_0^2 \right) \begin{bmatrix} 1 \\ \hat{b}_0 \\ g_1 \\ g_0 \end{bmatrix} = \mathbf{0}.$$

Multiparameter eigenvalue problem (MEVP)

Proposition

Given the above-mentioned square system of $m+n$ equations $f_0 = \dots = f_{n+m-1} = 0$ and the variety $\mathcal{V}_{\mathbb{R}}$, consider the quadratic m -parameter eigenvalue problem obtained by 'extracting' $\hat{\mathbf{b}}$ and \mathbf{g} ,

$$\mathcal{M}(\hat{\mathbf{a}}) \begin{bmatrix} 1 \\ \hat{\mathbf{b}} \\ \mathbf{g} \end{bmatrix} = \mathbf{0},$$

with coefficient matrices $\mathbf{M}_{\alpha} \in \mathbb{R}^{(n+m) \times (n+1)}$. Denote the set of real-valued eigentuples of this MEVP by $\mathcal{V}_{\mathbb{R}}(\hat{\mathbf{a}})$. Then, $\mathcal{V}_{\mathbb{R}}(\hat{\mathbf{a}})$ is equal to the projection of $\mathcal{V}_{\mathbb{R}}$ onto $\hat{\mathbf{a}}$.

- Computing the set of stationary points $\mathcal{V}'_{\mathbb{R}}$ is a (multiparameter) eigenvalue problem.
- Specialized MEVP solvers (e.g., Plestenjak, 2023; Vermeersch and De Moor, 2022)

Numerical example II

- Consider the discrete-time system described in De Moor et al., 1993, and take $m = 2$:

$$H(z) = \frac{0.0448z^5 + 0.2368z^4 + 0.0013z^3 + 0.0211z^2 + 0.2250z + 0.0219}{z^6 - 1.2024z^5 + 2.3675z^4 - 2.0039z^3 + 2.2337z^2 - 1.0420z + 0.8513}$$

- Equating all coefficients of $l(z)$ to zero gives:

$$\underbrace{\left(\mathbf{M}_0 + \mathbf{M}_{10}\hat{a}_0 + \mathbf{M}_{01}\hat{a}_1 + \mathbf{M}_{20}\hat{a}_0^2 + \mathbf{M}_{11}\hat{a}_0\hat{a}_1 + \mathbf{M}_{02}\hat{a}_1^2 \right)}_{\mathcal{M}(\hat{\mathbf{a}})} \begin{bmatrix} 1 & \hat{b}_1 & \hat{b}_0 & g_3 & g_2 & g_1 & g_0 \end{bmatrix}^T = \mathbf{0},$$

where,

$$\mathcal{M}(\hat{\mathbf{a}}) = \begin{bmatrix} 0.045 & \vdots -1 & \vdots 0 & \vdots -\hat{a}_0 & \vdots 0 & \vdots 0 & \vdots 0 \\ 0.045\hat{a}_1 + 0.24 & \vdots 1.20 & \vdots -1 & \vdots -2\hat{a}_0\hat{a}_1 & \vdots -\hat{a}_0^2 & \vdots 0 & \vdots 0 \\ 0.045\hat{a}_0 + 0.24\hat{a}_1 + 0.001 & \vdots -2.37 & \vdots 1.20 & \vdots -\hat{a}_1^2 - 2\hat{a}_0 & \vdots -2\hat{a}_0\hat{a}_1 & \vdots -\hat{a}_0^2 & \vdots 0 \\ 0.24\hat{a}_0 + 0.001\hat{a}_1 + 0.02 & \vdots 2.01 & \vdots -2.37 & \vdots -2\hat{a}_1 & \vdots -\hat{a}_1^2 - 2\hat{a}_0 & \vdots -2\hat{a}_0\hat{a}_1 & \vdots -\hat{a}_0^2 \\ 0.001\hat{a}_0 + 0.02\hat{a}_1 + 0.22 & \vdots -2.23 & \vdots 2.01 & \vdots -1 & \vdots -2\hat{a}_1 & \vdots -\hat{a}_1^2 - 2\hat{a}_0 & \vdots -2\hat{a}_0\hat{a}_1 \\ 0.02\hat{a}_0 + 0.22\hat{a}_1 + 0.02 & \vdots 1.04 & \vdots -2.23 & \vdots 0 & \vdots -1 & \vdots -2\hat{a}_1 & \vdots -\hat{a}_1^2 - 2\hat{a}_0 \\ 0.22\hat{a}_0 + 0.02\hat{a}_1 & \vdots -0.85 & \vdots 1.04 & \vdots 0 & \vdots 0 & \vdots -1 & \vdots -2\hat{a}_1 \\ 0.02\hat{a}_0 & \vdots 0 & \vdots -0.85 & \vdots 0 & \vdots 0 & \vdots 0 & \vdots -1 \end{bmatrix}.$$

Numerical example II

- The block Macaulay matrix (De Cock and De Moor, 2021; Vermeersch and De Moor, 2022) of degree 10 suffices to retrieve the 49 eigentuples, 11 of which real-valued.
- The variety $\mathcal{V}_{\mathbb{R}}$ contains 5 stationary points:

J^\dagger	\hat{a}_1	\hat{a}_0	\hat{b}_1	\hat{b}_0
0.868	-0.293	0.941	0.139	0.266
1.076	0.505	0.930	-0.254	-0.120
1.124	0.267	0.820	-0.294	0.167
1.174	-1.423	0.969	0.069	0.028
1.254	-0.992	0.534	0.132	0.086

[†]the discrete-time equivalent of (4).

S. Lagauw et al. (2023). “Globally Optimal SISO H_2 -Norm Model Reduction Using Walsh’s Theorem”. In: *IEEE Control Systems Letters* 7, pp. 1670–1675.

Questions?

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