Global Identifiability of parameterized nonlinear dynamical models

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Identifiability

Given an *arbitrary* model

$$\Sigma = \begin{cases} \dot{\boldsymbol{x}}(t) = f(\boldsymbol{x}(t), u(t), \boldsymbol{\theta}_m) \\ y(t) = g(\boldsymbol{x}(t), u(t), \boldsymbol{\theta}_m) \\ \boldsymbol{x}(0) = \boldsymbol{\theta}_{ic} \end{cases}$$

does

$$\{y(u(t), \bar{\boldsymbol{\theta}})\}_{t_0}^{t_1} = \{y(u(t), \hat{\boldsymbol{\theta}})\}_{t_0}^{t_1}$$

always imply,

$$\bar{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}}$$

No !

Consider the following model

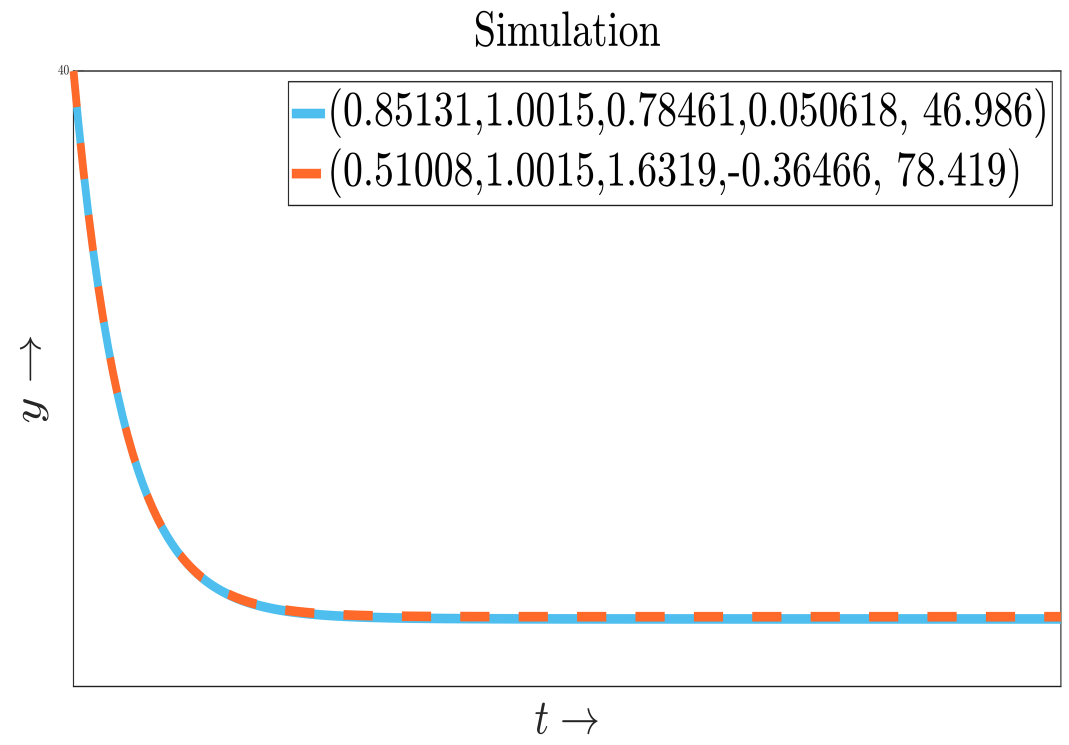
$$\begin{cases} \dot{x}(t) = -\frac{V_m x(t)}{k_m + x(t)} - k_{01} x(t) \\ x(0) = D \\ y(t) = c x(t) \end{cases}$$

where

$$\theta_m = \{c, k_{01}, k_m, V_m, D\},$$

$$\theta_{ic} = x_0$$

only k_{01} is identifiable



Can we asses identifiability by mimicking identification ?

Two approaches via Polynomial Optimization !

Consider the Goodwin Napkin example

$$\begin{cases} \dot{x}_1 = & x_2 \\ \dot{x}_2 = & -2\theta x_2 - \theta^2 x_1 \\ y = & x_1 \end{cases},$$

then obtain the IO equations via differential elimination [1]

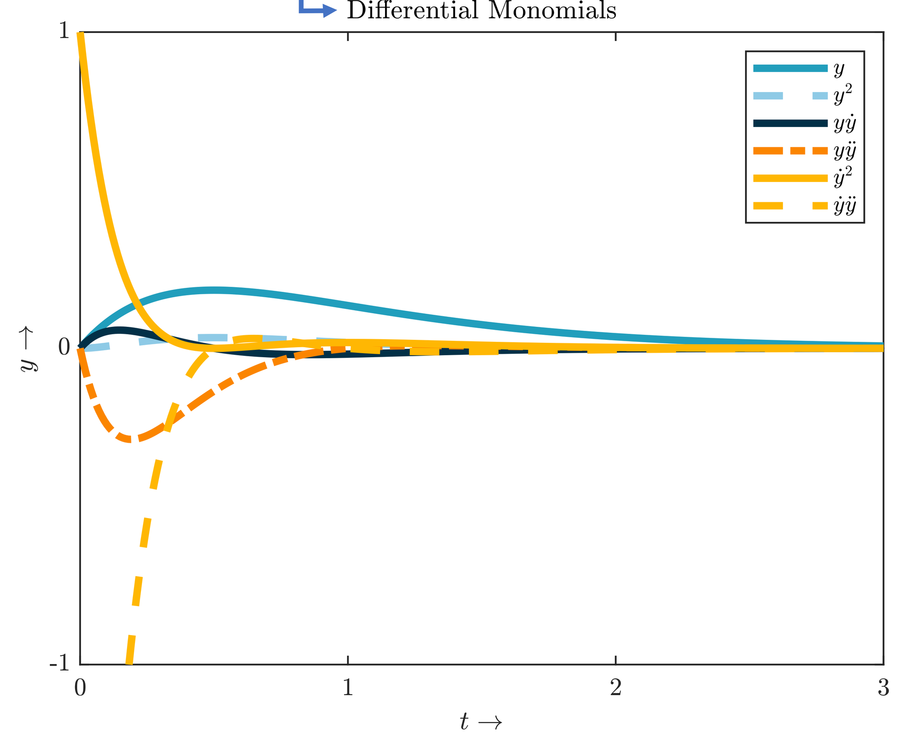
$$\ddot{y} + 2\theta\dot{y} + \theta^2 y = 0.$$

1 *Equation error method*

Minimize the IO equation error to identify parameters

$$\min_{\theta} \underbrace{\int_0^T (\ddot{y} + 2\theta\dot{y} + \theta^2 y)^2 dt}_{V(\theta)}$$

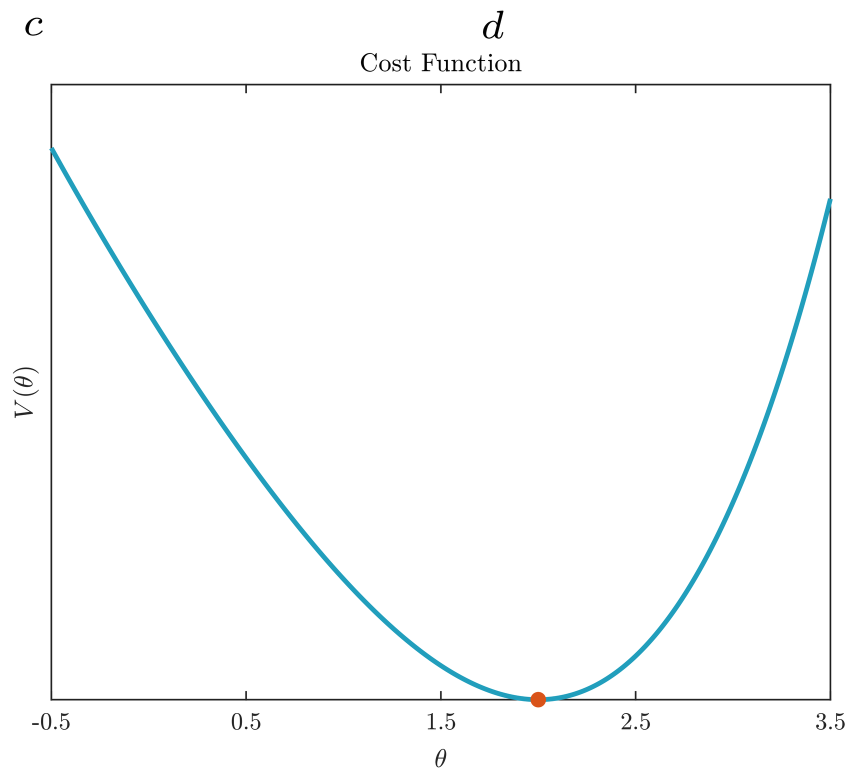
Differential Monomials



Find the global optima by solving a **system of polynomial equation(s) in the model parameters** associated with the FONCs **!**

$$\frac{dV(\theta)}{d\theta} = 4\overbrace{\left(\int_0^T y^2 dt\right)}^{a}\theta^3 + 12\overbrace{\left(\int_0^T (y\dot{y}) dt\right)}^{b}\theta^2$$

$$+4\underbrace{\left(\int_0^T (y\ddot{y}) dt\right)}_{c}\theta + 8\underbrace{\left(\int_0^T \dot{y}^2 dt\right)}_{d}\theta + 4\underbrace{\left(\int_0^T (\dot{y}\ddot{y}) dt\right)}_{e} = 0$$



2 *Misfit identification method*

Package IO equation recursively in a Toeplitz matrix

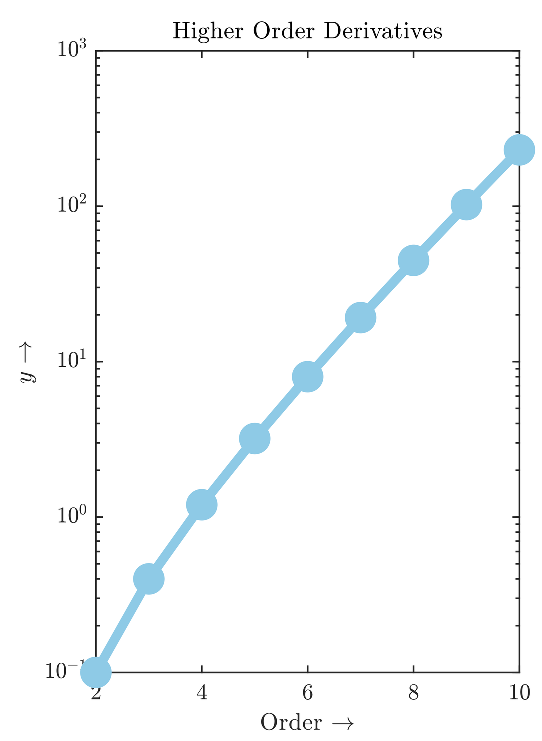
$$\begin{bmatrix} \theta^2 & 2\theta & 1 & & & \\ & \theta^2 & 2\theta & 1 & & \\ & & \ddots & & \ddots & \\ & & & \theta^2 & 2\theta & 1 \end{bmatrix} \begin{bmatrix} y \\ \dot{y} \\ \vdots \\ y^{(N+2)} \end{bmatrix} = 0.$$

The observed data can be broken down as

$$\boldsymbol{y}_{\text{obs}} = \boldsymbol{y} + \tilde{\boldsymbol{y}}$$ → misfit

The misfit in terms of the h.o.ds of the observed data written as

$$\tilde{\boldsymbol{y}} = \boldsymbol{T}_\theta^\dagger \boldsymbol{T}_\theta \boldsymbol{y}_{\text{obs}}$$

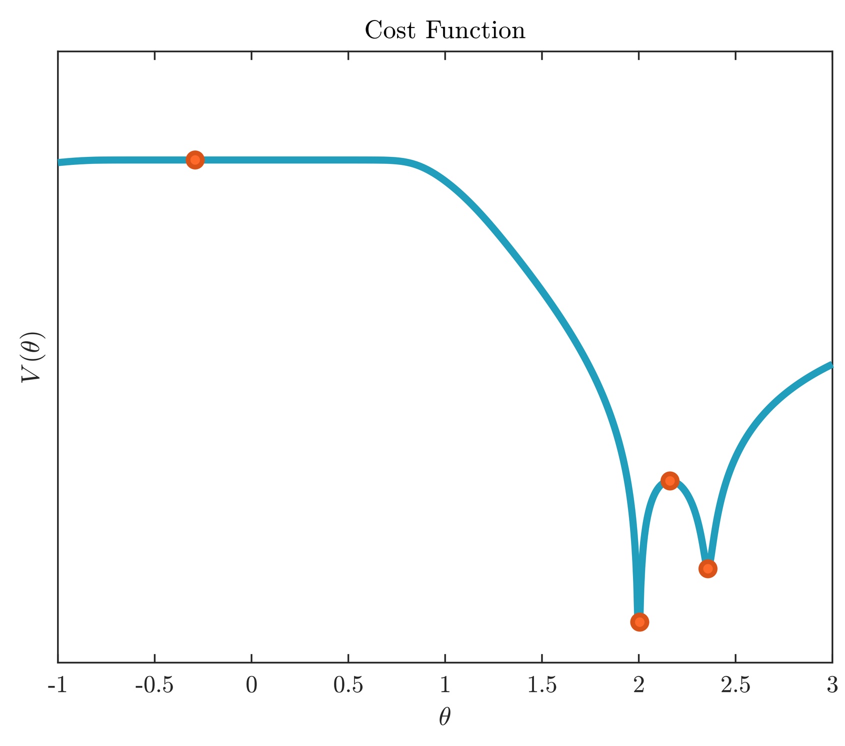


Minimize the misfit in a least squares sense to identify the parameters

$$\min_{\theta} \underbrace{||\tilde{\boldsymbol{y}}||^2}_{V(\theta)}$$

Find the global optima by solving a **multiparameter eigenvalue problem !** [2]

$$\begin{pmatrix} \boldsymbol{T}_\theta \boldsymbol{y}_{\text{obs}} & \boldsymbol{D}_\theta & \boldsymbol{0} \\ \boldsymbol{T}_\theta^\theta \boldsymbol{y}_{\text{obs}} & \boldsymbol{D}_\theta^\theta & \boldsymbol{D}_\theta \\ 0 & \boldsymbol{y}_{\text{obs}}^T \left(\boldsymbol{T}_\theta^\theta\right)^T & \boldsymbol{y}_{\text{obs}}^T \boldsymbol{T}_\theta^T \end{pmatrix} \boldsymbol{z} = 0$$



A unique global minima confirms global identifiability of the model parameter !

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