Numerical methods for rectangular multiparameter eigenvalue problems

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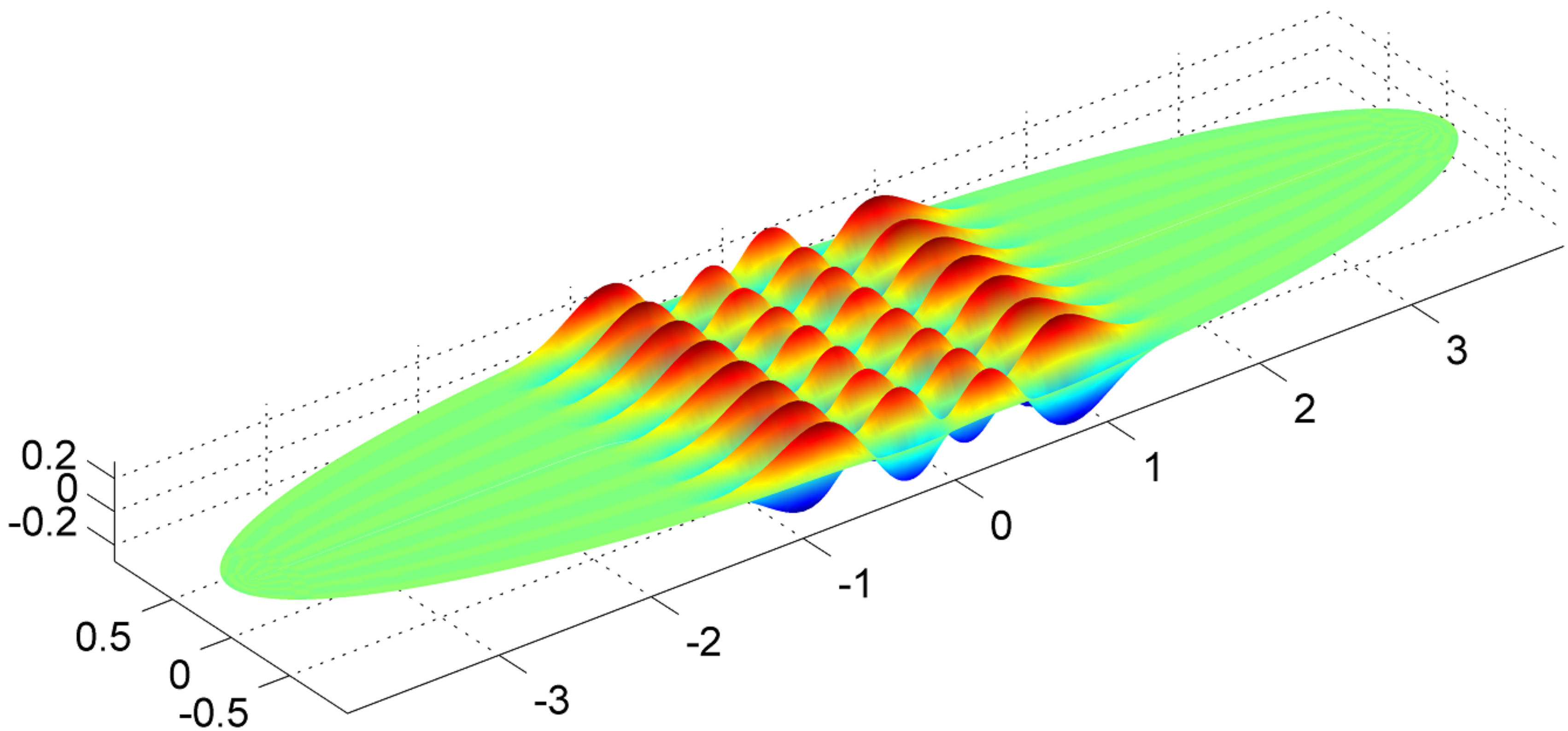
Outline

- Standard (square) multiparameter eigenvalue problems
- Rectangular multiparameter eigenvalue problems

Motivational example

We want to compute efficiently and accurately a couple of hundreds eigenmodes of an elliptic membrane Ω with a fixed boundary:

$$\left(\nabla^2 + \omega^2\right) u\,(x, y) = 0, \quad (x, y) \in \Omega = \{(x/\alpha)^2 + (y/\beta)^2 \leq 1\}, \quad u|_{\partial\Omega} = 0.$$



eigenmode $\omega_{298} = 24.45490912$ for $\alpha = 4$ and $\beta = 1$

(Gheorghiu, Hochstenbach, P., Rommes 2012)

Separation of variables: $\nabla^2 \mathbf{u} + \omega^2 \mathbf{u} = 0$ on Ω, $\mathbf{u}|_{\delta\Omega} = 0$

Rectangle: $\Omega = [0, a] \times [0, b] \Longrightarrow$ two S-L equations ($\omega^2 = \lambda + \mu$)

$$x'' + \lambda x = 0, \qquad x(0) = x(a) = 0,$$
$$y'' + \mu y = 0, \qquad y(0) = y(b) = 0.$$

Disc: $\Omega = \{x^2 + y^2 \leq a^2\}$, polar coordinates $\Longrightarrow$ a triangular situation

$$\Phi'' + \lambda \Phi = 0, \qquad \Phi(0) = \Phi(2\pi) = 0,$$
$$r^{-1}(rR')' + (\omega^2 - \lambda r^{-2})R = 0, \qquad R(0) < \infty, R(a) = 0.$$

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Ellipse: $\Omega = \{(x/\alpha)^2 + (y/\beta)^2 \leq 1\}$, elliptic coordinates ($\alpha > \beta$)

$\Longrightarrow$ modified Mathieu's and Mathieu's DE

$$F''(\xi) - (\lambda - 2\mu \cosh(2\xi))F(\xi) = 0, \quad F(0) = F(\xi_0) = 0,$$
$$G''(\eta) + (\lambda - 2\mu \cos(2\eta))G(\eta) = 0, \quad G(0) = G(\pi/2) = 0,$$

where $h = \sqrt{\alpha^2 - \beta^2}$, $\xi_0 = \operatorname{arccosh}\frac{\alpha}{h}$, and $\mu = h^2\omega^2/4$.

This is a two-parameter eigenvalue problem.

Multiparameter eigenvalue problem (MEP)

In several coordinate systems, when separation of variables is applied to a PDE (Helmholtz, Laplace, Schrödinger,...), we obtain a MEP. A general form is

$$p_j(t_j)\, y_j''(t_j) + q_j(t_j)\, y_j'(t_j) + r_j(t_j)\, y_j(t_j) + \sum_{\ell=1}^{k} \lambda_\ell\, s_{j\ell}(t_j)\, y_j(t_j) = 0, \quad j = 1, \ldots, k,$$

where $t_j \in [a_j, b_j]$, with the appropriate b.c. We are looking for $(\lambda_1, \ldots, \lambda_k)$ and nontrivial functions $y_1, \ldots, y_k$ that satisfy the above equations and b.c.

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Discretization (e.g., Chebyshev collocation) leads to an algebraic MEP

$$\begin{aligned} (A_{10} + \lambda_1 A_{11} + \cdots + \lambda_k A_{1k})x_1 &= 0 \\ &\vdots \\ (A_{k0} + \lambda_1 A_{k1} + \cdots + \lambda_k A_{kk})x_k &= 0, \end{aligned} \qquad \text{(MEP)}$$

where $A_{ij} \in \mathbb{C}^{n_i \times n_i}$

- eigenvalue: $(\lambda_1, \ldots, \lambda_k)$, that satisfies (MEP) for nonzero $x_1, \ldots, x_k$,
- eigenvector: $x_1 \otimes \cdots \otimes x_k$.

Generically, the above (MEP) has $N := n_1 n_2 \cdots n_k$ eigenvalues.

Operator determinants

$$\begin{aligned}(A_{10}+\lambda_1 A_{11}+\cdots+\lambda_k A_{1k})x_1 &= 0\\ &\vdots\\ (A_{k0}+\lambda_1 A_{k1}+\cdots+\lambda_k A_{kk})x_k &= 0\end{aligned} \qquad \text{(MEP)}$$

is related to $N \times N$ matrices, called operator determinants,

$$\Delta_0 = \begin{vmatrix} A_{11} & \cdots & A_{1k} \\ \vdots & & \vdots \\ A_{k1} & \cdots & A_{kk} \end{vmatrix}_{\otimes} := \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma)\, A_{1\sigma_1} \otimes A_{2\sigma_2} \otimes \cdots \otimes A_{k\sigma_k},$$

$$\Delta_i = (-1) \begin{vmatrix} A_{11} & \cdots & A_{1,i-1} & A_{10} & A_{1,i+1} & \cdots & A_{1k} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ A_{k1} & \cdots & A_{k,i-1} & A_{k0} & A_{k,i+1} & \cdots & A_{kk} \end{vmatrix}_{\otimes}, \qquad i = 1, \ldots, k.$$

(Atkinson 1972) If Δ_0 is nonsingular, then $\Delta_0^{-1}\Delta_1, \ldots, \Delta_0^{-1}\Delta_k$ commute and (MEP) is equivalent to a system of generalized eigenvalue problems (GEPs)

$$\begin{aligned}\Delta_1\, z &= \lambda_1\, \Delta_0\, z\\ &\vdots\\ \Delta_k\, z &= \lambda_k\, \Delta_0\, z\end{aligned} \qquad (\Delta)$$

for $z = x_1 \otimes \cdots \otimes x_k$.

Example

For a 2EP

$$(A_1 + \lambda B_1 + \mu C_1)x_1 = \left(\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix} + \lambda \begin{bmatrix}3 & 1\\1 & 1\end{bmatrix} + \mu \begin{bmatrix}2 & 1\\2 & 1\end{bmatrix}\right)x_1 \quad = \quad 0$$

$$(A_2 + \lambda B_2 + \mu C_2)x_2 = \left(\begin{bmatrix}1 & 0\\1 & 3\end{bmatrix} + \lambda \begin{bmatrix}1 & 2\\2 & 3\end{bmatrix} + \mu \begin{bmatrix}2 & 1\\1 & 1\end{bmatrix}\right)x_2 \quad = \quad 0.$$

we use $\Delta_0 = B_1 \otimes C_2 - C_1 \otimes B_2$, $\Delta_1 = C_1 \otimes A_2 - A_1 \otimes C_2$, $\Delta_2 = A_1 \otimes B_2 - B_1 \otimes A_2$,

$$\Delta_0 = \begin{bmatrix} 4 & -1 & 1 & -1\\ -1 & -3 & -1 & -2\\ 0 & -3 & 1 & -1\\ -3 & -5 & -1 & -2 \end{bmatrix}, \quad \Delta_1 = \begin{bmatrix} 0 & -1 & -3 & -2\\ 1 & 5 & -1 & 1\\ -4 & -3 & -7 & -4\\ -1 & 3 & -3 & -1 \end{bmatrix}, \quad \Delta_2 = \begin{bmatrix} -2 & 2 & 1 & 4\\ -1 & -6 & 3 & 3\\ 2 & 6 & 3 & 8\\ 5 & 6 & 7 & 9 \end{bmatrix}.$$

Example

For a 2EP

$$(A_1+\lambda B_1+\mu C_1)x_1 = \left(\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix} + \lambda \begin{bmatrix}3 & 1\\ 1 & 1\end{bmatrix} + \mu \begin{bmatrix}2 & 1\\ 2 & 1\end{bmatrix}\right)x_1 = 0$$
$$(A_2+\lambda B_2+\mu C_2)x_2 = \left(\begin{bmatrix}1 & 0\\ 1 & 3\end{bmatrix} + \lambda \begin{bmatrix}1 & 2\\ 2 & 3\end{bmatrix} + \mu \begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}\right)x_2 = 0.$$

we use $\Delta_0 = B_1\otimes C_2 - C_1\otimes B_2$, $\Delta_1 = C_1\otimes A_2 - A_1\otimes C_2$, $\Delta_2 = A_1\otimes B_2 - B_1\otimes A_2$,

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Then,

$$\texttt{eig}(\Delta_1,\Delta_0) = (-2.5737, \quad 0.4496, -0.7713 \pm 1.1518i)$$
$$\texttt{eig}(\Delta_2,\Delta_0) = (-2.9682, -0.6903, -1.8374 \mp 4.0984i)$$

and the 2EP has 4 eigenvalues

$$\begin{aligned}(\lambda_1,\mu_1) &= (-2.5737, -2.9682)\\ (\lambda_2,\mu_2) &= (\ \ 0.4496, -0.6903)\\ (\lambda_{3,4},\mu_{3,4}) &= (-0.7713 \pm 1.1518i, -1.8374 \mp 4.0984i).\end{aligned}$$

Quadratic 2EP

A Q2EP has the form

$$\begin{aligned}(A_1 + \lambda B_1 + \mu C_1 + \lambda^2 D_1 + \lambda\mu E_1 + \mu^2 F_1)x_1 &= 0,\\ (A_2 + \lambda B_2 + \mu C_2 + \lambda^2 D_2 + \lambda\mu E_2 + \mu^2 F_2)x_2 &= 0.\end{aligned}$$

and generically has $4n_1 n_2$ solutions.

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We can linearize the Q2EP into a linear 2EP

$$\begin{aligned}\left(\begin{bmatrix} A_1 & B_1 & C_1 \\ 0 & -I & 0 \\ 0 & 0 & -I \end{bmatrix} + \lambda \begin{bmatrix} 0 & D_1 & E_1 \\ I & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \mu \begin{bmatrix} 0 & 0 & F_1 \\ 0 & 0 & 0 \\ I & 0 & 0 \end{bmatrix}\right) \begin{bmatrix} x_1 \\ \lambda x_1 \\ \mu x_1 \end{bmatrix} &= 0,\\ \left(\begin{bmatrix} A_2 & B_2 & C_2 \\ 0 & -I & 0 \\ 0 & 0 & -I \end{bmatrix} + \lambda \begin{bmatrix} 0 & D_2 & E_2 \\ I & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \mu \begin{bmatrix} 0 & 0 & F_2 \\ 0 & 0 & 0 \\ I & 0 & 0 \end{bmatrix}\right) \begin{bmatrix} x_2 \\ \lambda x_2 \\ \mu x_2 \end{bmatrix} &= 0.\end{aligned}$$

The associated GEPs with operator determinants $\Delta_1 z = \lambda\Delta_0 z$ and $\Delta_2 z = \mu\Delta_0 z$, which are of size $9n_1n_2 \times 9n_1n_2$, are both singular.

However, they have $4n_1n_2$ finite regular eigenvalues that give solutions of Q2EP.

Singular GEP

A GEP $A - \lambda B$ is singular iff $\det(A - \lambda B) \equiv 0$. Then $\lambda_0 \in \mathbb{C}$ is an eigenvalue if

$$\operatorname{rank}(A - \lambda_0 B) < \operatorname{nrank}(A, B) := \max_{\zeta \in \mathbb{C}} \operatorname{rank}(A - \zeta B)$$

(or $\operatorname{rank}(B) < \operatorname{nrank}(A, B)$ for $\lambda_0 = \infty$). Example:

$$A - \lambda B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

is singular and has one eigenvalue 1.

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Singular problems are very challenging to solve numerically, both with respect to accuracy and efficiency:

- QZ usually returns eigenvalues close to the regular ones mixed with fake ones (Lotz, Noferini 2020),
- Recommended approach (Wilkinson 1979) - extract the regular part by the staircase method (Van Dooren 1979) and then apply QZ to the regular part,
- Guptri, a robust software for the extraction (Demmel, Kågström 1993),
- Rank-completing perturbations or projections to normal rank (Hochstenbach, Mehl, P. 2019, 2023).

Singular MEP

$$\begin{aligned}(A_{10}+\lambda_1 A_{11}+\cdots+\lambda_k A_{1k})x_1 &= 0\\ &\ \vdots \\ (A_{k0}+\lambda_1 A_{k1}+\cdots+\lambda_k A_{kk})x_k &= 0\end{aligned} \quad \text{(MEP)} \qquad \begin{aligned}\Delta_1 z &= \lambda_1 \Delta_0 z\\ &\ \vdots \\ \Delta_k z &= \lambda_k \Delta_0 z\end{aligned} \quad (\Delta)$$

In some applications (MEP) is singular, i.e., all GEPs in (Δ) are singular.

We can extract the regular part of (Δ) by a staircase-type algorithm and compute the eigenvalues (Muhič, P. 2010). The algorithm returns matrices P and Q with orthogonal columns that define $\widehat{\Delta}_i = P^*\Delta_i Q$ for $i = 0, \ldots, k$ such that

a) $\widehat{\Delta}_0$ is nonsingular,

b) matrices $\widehat{\Delta}_0^{-1}\widehat{\Delta}_1, \ldots, \widehat{\Delta}_0^{-1}\widehat{\Delta}_k$ commute,

c) finite regular eigenvalues of (Δ) are eigenvalues of

$$\begin{aligned}\widehat{\Delta}_1\, w &= \lambda_1\, \widehat{\Delta}_0\, w\\ &\ \vdots \\ \widehat{\Delta}_k\, w &= \lambda_k\, \widehat{\Delta}_0\, w\end{aligned}$$

Singular MEP

$$
\begin{array}{rcl}
(A_{10}+\lambda_1 A_{11}+\cdots+\lambda_k A_{1k})x_1 & = & 0\\
 & \vdots & \\
(A_{k0}+\lambda_1 A_{k1}+\cdots+\lambda_k A_{kk})x_k & = & 0
\end{array}
\quad \text{(MEP)}
\qquad
\begin{array}{rcl}
\Delta_1 z & = & \lambda_1 \Delta_0 z\\
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\Delta_k z & = & \lambda_k \Delta_0 z
\end{array}
\quad (\Delta)
$$

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$$
\begin{array}{rcl}
\widehat{\Delta}_1 w & = & \lambda_1 \widehat{\Delta}_0 w\\
 & \vdots & \\
\widehat{\Delta}_k w & = & \lambda_k \widehat{\Delta}_0 w
\end{array}
$$

For singular MEPs, the equivalence of eigenvalues of (MEP) and (Δ) is known only for $k = 2$ (Muhič, P. 2009), (Košir, P. 2022).

Numerical methods for standard MEPs

$$\begin{array}{rclc}
(A_{10}+\lambda_1 A_{11}+\cdots+\lambda_k A_{1k})x_1 & = & 0 & \\
& \vdots & & \text{(MEP)} \\
(A_{k0}+\lambda_1 A_{k1}+\cdots+\lambda_k A_{kk})x_k & = & 0 &
\end{array}
\qquad
\begin{array}{rclc}
\Delta_1 z & = & \lambda_1 \Delta_0 z & \\
& \vdots & & (\Delta) \\
\Delta_k z & = & \lambda_k \Delta_0 z &
\end{array}$$

All eigenvalues:

- simultaneous Schur decomposition of (Δ): (Hochstenbach, Košir, P. 2005)
- staircase-type algorithm for singular MEPs: (Muhič, P. 2009)
- linearization of quadratic 2EP: (Muhič, P. 2010), (Hochstenbach, Muhič, P. 2012)
- continuation methods: (Dong, Yu, Yu 2016), (Rodriguez, Du, You, Lim 2021)

Numerical methods for standard MEPs

$$\begin{array}{rcl} (A_{10} + \lambda_1 A_{11} + \cdots + \lambda_k A_{1k}) x_1 & = & 0 \\ & \vdots & \\ (A_{k0} + \lambda_1 A_{k1} + \cdots + \lambda_k A_{kk}) x_k & = & 0 \end{array} \quad \text{(MEP)} \qquad \begin{array}{rcl} \Delta_1 z & = & \lambda_1 \Delta_0 z \\ & \vdots & \\ \Delta_k z & = & \lambda_k \Delta_0 z \end{array} \quad (\Delta)$$

All eigenvalues:

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Small subset of eigenvalues for large problems:

- Jacobi–Davidson: (Hochstenbach, Košir, P. 2005), (Hochstenbach, P. 2008)
- Sylvester–Arnoldi: (Meerbergen, P. 2015), (Hochstenbach, Meerbergen, Mengi, P. 2019)
- Jacobi–Davidson for PMEP: (Hochstenbach, Muhič, P. 2015)
- Subspace method using tensor-train representation: (Ruymbeek, Michiels, Meerbergen 2022)
- Alternating method (Eisenmann, Nakatsukasa 2022)

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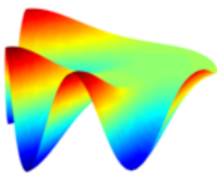

MultiParEig

Version 2.7.0.0 (322 KB) by Bor Plestenjak

Toolbox for multiparameter and singular eigenvalue problems

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This is a joined work with Andrej Muhič, who wrote part of the code, in particular the staircase algorithm for a singular multiparameter eigenvalue problem. If you use the toolbox to solve a singular MEP, please cite: A. Muhič, B. Plestenjak: On the quadratic two-parameter eigenvalue problem and its linearization, Linear Algebra Appl. 432 (2010) 2529-2542.

Toolbox contains numerical methods for multiparameter eigenvalue problems (MEPs) and singular generalized eigenvalue problems

A matrix two-parameter eigenvalue problem (2EP) has the form

A1*x = lambda*B1*x + mu*C1*x,

A2*y = lambda*B2*y + mu*C2*y,

and we are looking for an eigenvalue (lambda,mu) and nonzero eigenvectors x,y. A 2EP is related to a pair of generalized eigenvalue problems

Delta1*z = lambda*Delta0*z,

Delta2*z = mu*Delta0*z,

where Delta0, Delta1 and Delta2 are operator determinants

Delta0 = kron(C2, B1) - kron(B2, C1)

Delta1 = kron(C2, A1) - kron(A2, C1)

Delta2 = kron(A2, B1) - kron(B2, A1)

and z = kron(x,y). The 2EP is nonsingular when Delta0 is nonsingular. This can be generalized to 3EP and MEP.

Requires

MATLAB

Multiprecision examples require Advanpix Multiprecision Computing Toolbox In Matlab before 2014a functions twopareigs, twopareigs_ira and twopareigs_ks run faster if package lapack is installed.

MATLAB Release Compatibility

Created with R2021a
Compatible with any release

Platform Compatibility

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Rectangular MEP

$$M(\boldsymbol{\lambda})\, x := \Big(\sum_{|\boldsymbol{\omega}| \le d} \boldsymbol{\lambda}^{\boldsymbol{\omega}} A_{\boldsymbol{\omega}} \Big)\, x = 0, \qquad \text{(RMEP)}$$

where $\boldsymbol{\omega} = (\omega_1, \dots, \omega_k)$, $|\boldsymbol{\omega}| = \omega_1 + \cdots + \omega_k$, $\boldsymbol{\lambda}^{\boldsymbol{\omega}} = \lambda_1^{\omega_1} \cdots \lambda_k^{\omega_k}$, and $x \in \mathbb{C}^n$.

The key differences with standard MEP:

a) just one equation,

b) $A_{\boldsymbol{\omega}} = A_{\omega_1, \dots, \omega_k}$ are $(n + k - 1) \times n$ rectangular matrices.

We assume full normal rank, i.e., $\mathrm{nrank}(M) := \max_{\boldsymbol{\lambda} \in \mathbb{C}^k} \mathrm{rank}(M(\boldsymbol{\lambda})) = n$

Eigenvalue: $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_k)$ that satisfies (RMEP) for a nonzero x (eigenvector)

$\boldsymbol{\lambda}$ is an eigenvalue iff $\mathrm{rank}(M(\boldsymbol{\lambda})) < n$.

Example

$$\left(\begin{bmatrix}1 & 0\\0 & 1\\1 & 1\end{bmatrix} + \lambda\begin{bmatrix}1 & 2\\1 & 3\\2 & 1\end{bmatrix} + \mu\begin{bmatrix}1 & 0\\0 & 1\\0 & 0\end{bmatrix} + \lambda^2\begin{bmatrix}0 & 1\\1 & 0\\1 & 1\end{bmatrix} + \lambda\,\mu\begin{bmatrix}0 & 1\\2 & 2\\1 & 3\end{bmatrix} + \mu^2\begin{bmatrix}3 & 0\\1 & 1\\1 & 2\end{bmatrix}\right)x = 0$$

(λ, μ) is an eigenvalue, $x \neq 0$ is an eigenvector

This is a quadratic rectangular 2EP

The problem has 12 eigenvalues

$$\begin{aligned}(\lambda_1, \mu_1) &= (2.1783, -2.1234)\\(\lambda_2, \mu_2) &= (1.7620, 0.9830)\\&\vdots\\(\lambda_{11}, \mu_{11}) &= (-1.6590, 0.3378)\\(\lambda_{12}, \mu_{12}) &= (-1.0000, 0.0000)\end{aligned}$$

Related results

$$M(\boldsymbol{\lambda})\,x := \Big(\sum_{|\boldsymbol{\omega}|\le d} \boldsymbol{\lambda}^{\boldsymbol{\omega}} A_{\boldsymbol{\omega}}\Big)\,x = 0, \qquad \text{(RMEP)}$$

where $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_k)$ and $A_{\boldsymbol{\omega}} \in \mathbb{C}^{(n+k-1)\times n}$.

- Khazanov (1998) considers RMEPs, defines eigenvalues and eigenvectors.
- Shapiro and Shapiro (2009) examine linear RMEP ($d = 1$) and show that the problem generically has $\binom{n+k-1}{k}$ eigenvalues.
- Alsubaie (2019): application of linear RMEPs in $\mathcal{H}_2$-optimal model reduction, numerical method for RMEPs based on operator determinants.
- Vermeersch and De Moor (2019, 2022, 2023) and De Moor (2019, 2020): optimal parameters of the ARMA model and the realization of LTI system are eigenvalues of quadratic RMEPs ($d = 2$), numerical method for RMEPs based on block Macaulay matrices.
- Hochstenbach, Košir, P. (2023): a generic RMEP of degree d has $d^k\binom{n+k-1}{k}$ eigenvalues, numerical methods for RMEPs based on MEPs.

Block Macaulay matrices

$$(A + \lambda_1 B_1 + \lambda_2 B_2)x = 0$$

We multiply the equation with monomials $\lambda_1^i \lambda_2^j$ of increasing order, where we add rows in blocks of the same degree. We get a homogeneous system with the block Macaulay matrix

$$\begin{array}{c} \\ 1 \\ \lambda_1 \\ \lambda_2 \\ \lambda_1^2 \\ \vdots \end{array}
\begin{array}{c}
\begin{array}{ccccccccc} 1 & \lambda_1 & \lambda_2 & \lambda_1^2 & \lambda_1\lambda_2 & \lambda_2^2 & \lambda_1^3 & \lambda_1^2\lambda_2 & \dots \end{array} \\
\begin{bmatrix} A & B_1 & B_2 & & & & & & \cdots \\ & A & & B_1 & B_2 & & & & \cdots \\ & & A & & B_1 & B_2 & & & \cdots \\ & & & A & & & B_1 & B_2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & & & & \ddots & \ddots \end{bmatrix}
\end{array}
\begin{bmatrix} x \\ \lambda_1 x \\ \lambda_2 x \\ \lambda_1^2 x \\ \vdots \end{bmatrix} = 0.$$

If a block Macaulay matrix is large enough, than the structure of the nullspace stabilizes and we can compute all eigenvalues from the nullspace.

After some magic (see the PhD defence by Christof Vermeersch later today!), we get matrices $(S_1 Z)^+(S_{\lambda_1} Z)$ and $(S_1 Z)^+(S_{\lambda_2} Z)$ that commute and whose joint eigenvalues are solutions of the RMEP.

Linear RMEP - Algorithm 1

A linear RMEP with k parameters and $A_i \in \mathbb{C}^{(n+k-1)\times n}$ has $\binom{n+k-1}{k}$ eigenvalues

$$(A_0 + \lambda_1 A_1 + \cdots + \lambda_k A_k)\, x = 0 \qquad \text{(RMEP)}$$

We multiply it by random matrices $P_1, \ldots, P_k \in \mathbb{C}^{n\times(n+k-1)}$ and transform it into a MEP with $n \times n$ matrices that has n^k eigenvalues

$$\begin{aligned} (P_1 A_0 + \lambda_1\, P_1 A_1 + \cdots + \lambda_k\, P_1 A_k)\, x_1 &= 0 \\ &\vdots \\ (P_k A_0 + \lambda_1\, P_k A_1 + \cdots + \lambda_k\, P_k A_k)\, x_k &= 0 \end{aligned} \qquad \text{(MEP)}$$

We keep only the eigenvalues for which $\operatorname{rank}(A_0 + \lambda_1 A_1 + \cdots + \lambda_k A_k) < n$.

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We keep only the eigenvalues for which $\operatorname{rank}(A_0 + \lambda_1 A_1 + \cdots + \lambda_k A_k) < n$.

We can apply all numerical methods for MEPs, for instance solve the GEPs

$$\begin{aligned}\Delta_1 z &= \lambda_1 \Delta_0 z\\ &\vdots\\ \Delta_k z &= \lambda_k \Delta_0 z\end{aligned} \qquad (\Delta)$$

Δ_0 is nonsingular in the generic case.

Linear RMEP - Algorithm 2

$$\begin{array}{rcl} (P_1A_0+\lambda_1\,P_1A_1+\cdots+\lambda_k\,P_1A_k)\,x_1 & = & 0 \\ & \vdots & \\ (P_kA_0+\lambda_1\,P_kA_1+\cdots+\lambda_k\,P_kA_k)\,x_k & = & 0 \end{array} \implies \begin{array}{rcl} \Delta_1\,z & = & \lambda_1\,\Delta_0\,z \\ & \vdots & \\ \Delta_k\,z & = & \lambda_k\,\Delta_0\,z \end{array} \quad (\Delta)$$

(Alsubaie 2019) Vectors $z = x \otimes \cdots \otimes x$ span a subspace of dim. $\binom{n+k-1}{k}$ in $\mathbb{C}^{n^k}$.

Can write $z = Tw$ for $w \in \mathbb{C}^{\binom{n+k-1}{k}}$ and $T \in \mathbb{C}^{n^k \times \binom{n+k-1}{k}}$,

$$\text{e.g., } (n=2, k=2): \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \otimes \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1x_1 \\ x_1x_2 \\ x_2x_1 \\ x_2x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}.$$

This enables us to restrict (Δ) to a system of GEPs of size $\binom{n+k-1}{k} \times \binom{n+k-1}{k}$.

$$\begin{array}{rcl} D_1\,w & = & \lambda_1\,D_0\,w \\ & \vdots & \\ D_k\,w & = & \lambda_k\,D_0\,w \end{array}$$

No redundant solutions, can apply all numerical methods for systems of GEPs.

D_0 is nonsingular, smaller matrices than in the block Macaulay method.

Details of the compression

We form operator determinants with $(n+k-1)^k \times n^k$ rectangular matrices

$$\widetilde{\Delta}_0 = \begin{vmatrix} A_1 & \cdots & A_k \\ \vdots & & \vdots \\ A_1 & \cdots & A_k \end{vmatrix}_{\otimes}, \quad \widetilde{\Delta}_i = (-1) \begin{vmatrix} A_1 & \cdots & A_{i-1} & A_0 & A_{i+1} & \cdots & A_k \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ A_1 & \cdots & A_{i-1} & A_0 & A_{i+1} & \cdots & A_k \end{vmatrix}_{\otimes}.$$

Matrices $\Psi_\ell := \widetilde{\Delta}_\ell T$ can have only $\binom{n+k-1}{k}$ different rows (up to sign). If $\Psi_\ell(i_1 \dots i_k, :)$ is the row of Ψ_ℓ corresponding to a multi-index $(i_1, \dots, i_k)$, then:

a) If $i_p = i_q$ for $p \neq q$, then $\Psi_\ell(i_1 \dots i_k, :) = 0$.

b) If $(j_1, \dots, j_k)$ is $\sigma(i_1, \dots, i_k)$, then $\Psi_\ell(j_1 \dots j_k, :) = \mathrm{sgn}(\sigma)\, \Psi_\ell(i_1 \dots i_k, :)$.

By taking only rows with strictly ordered indices we get matrices $D_i = L\, \widetilde{\Delta}_i T$ and restrict (Δ) to a system of GEPs of size $\binom{n+k-1}{k} \times \binom{n+k-1}{k}$.

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By taking only rows with strictly ordered indices we get matrices $D_i = L\,\widetilde{\Delta}_i T$ and restrict (Δ) to a system of GEPs of size $\binom{n+k-1}{k} \times \binom{n+k-1}{k}$.

For $n = 2$ and $k = 2$ we compress the $\widetilde{\Delta}$-matrices from size 9×4 to size 3×3 by:

$$T = \begin{array}{c} \\ 11 \\ 12 \\ 21 \\ 22 \end{array} \begin{array}{c} \begin{array}{ccc} 11 & 12 & 22 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}, \qquad L = \begin{array}{c} \\ 12 \\ 13 \\ 23 \end{array} \begin{array}{c} \begin{array}{ccccccccc} 11 & \color{red}{12} & \color{red}{13} & 21 & 22 & \color{red}{23} & 31 & 32 & 33 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array}.$$

We need to compute only the corresponding $\binom{n+k-1}{k}$ rows of $\widetilde{\Delta}_i$.

Comparison of algorithms for linear RMEPs

Algorithm 1: Transformation to a MEP by multiplications with $P_1, \ldots, P_k$

Algorithm 2: Direct construction of compressed $D_0, \ldots, D_k$.

	Algorithm 1	Algorithm 2
Pros	• Simple construction • Can apply numerical methods for MEPs that do not require Δ-matrices	• Uses D-matrices of optimal size • No redundant solutions • Sparsity is preserved
Cons	• Uses much larger Δ-matrices • Sparsity is lost with random P_i	• D-matrices are needed explicitly • Kronecker structure of $\widetilde{\Delta}$-matrices is lost in multiplication by L and T

Quadratic R2EP - Algorithm 3

We consider a generic quadratic R2EP with $(n+1)\times n$ matrices

$$(A_{00}+\lambda A_{10}+\mu A_{01}+\lambda^2 A_{20}+\lambda\,\mu A_{11}+\mu^2 A_{02})\,x=0 \qquad \text{(QR2EP)}$$

First approach: apply $n\times(n+1)$ matrices P_1, P_2 to get a standard quadratic 2EP

$$\begin{aligned}(P_1A_{00}+\lambda P_1A_{10}+\mu P_1A_{01}+\lambda^2 P_1A_{20}+\lambda\,\mu P_1A_{11}+\mu^2 P_1A_{02})\,x_1 &= 0\\ (P_2A_{00}+\lambda P_2A_{10}+\mu P_2A_{01}+\lambda^2 P_2A_{20}+\lambda\,\mu P_2A_{11}+\mu^2 P_2A_{02})\,x_2 &= 0\end{aligned}$$

This problem has $4n^2$ solutions that include the $2n(n+1)$ solutions of (QR2EP).

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This problem has $4n^2$ solutions that include the $2n(n+1)$ solutions of (QR2EP).

Can apply all methods for quadratic 2EPs, e.g., a linearization to a 2EP

$$(V_{i0}+\lambda V_{i1}+\mu V_{i2})\,u_i=0, \quad i=1,2,$$

where

$$V_{i0}=\begin{bmatrix}P_iA_{00} & P_iA_{10} & P_iA_{01}\\ 0 & -I & 0\\ 0 & 0 & -I\end{bmatrix},\ V_{i1}=\begin{bmatrix}0 & P_iA_{20} & P_iA_{11}\\ I & 0 & 0\\ 0 & 0 & 0\end{bmatrix},\ V_{i2}=\begin{bmatrix}0 & 0 & P_iA_{02}\\ 0 & 0 & 0\\ I & 0 & 0\end{bmatrix},\ u_i=\begin{bmatrix}x_i\\ \lambda x_i\\ \mu x_i\end{bmatrix}$$

This is a singular 2EP with $3n\times 3n$ matrices.

Quadratic R2EP - Algorithm 4

$$(A_{00} + \lambda A_{10} + \mu A_{01} + \lambda^2 A_{20} + \lambda\,\mu A_{11} + \mu^2 A_{02})\,x = 0 \qquad \text{(QR2EP)}$$

$$\begin{aligned}(V_{10} + \lambda V_{11} + \mu V_{12})\,u_1 &= 0\\ (V_{20} + \lambda V_{21} + \mu V_{22})\,u_2 &= 0\end{aligned} \quad \Longrightarrow \quad \begin{aligned}\Delta_1\,z &= \lambda\,\Delta_0\,z\\ \Delta_2\,z &= \mu\,\Delta_0\,z\end{aligned} \qquad (\Delta)$$

A_{ij} is $(n+1)\times n$, V_{ij} is $3n\times 3n$, Δ_i is $9n^2\times 9n^2$, $z = u_1 \otimes u_2$,
$u_i = [x_i^T\ \lambda x_i^T\ \mu x_i^T]^T$ for $i = 1, 2$.

Vectors $[x^T\ \lambda x^T\ \mu x^T]^T \otimes [x^T\ \lambda x^T\ \mu x^T]^T$ span a subspace of dimension $3n(n+1)$ in $\mathbb{C}^{9n^2}$. We can restrict (Δ) to singular GEPs with $3n(n+1)\times 3n(n+1)$ matrices

$$\begin{aligned}\widetilde{D}_1\,u &= \lambda\,\widetilde{D}_0\,u\\ \widetilde{D}_2\,u &= \mu\,\widetilde{D}_0\,u\end{aligned}$$

No redundant solutions, can apply methods for systems of singular GEPs.

Comparison for QR2EP

Block Macaulay method vs. Algorithm 4 for a generic QR2EP

$$(A_{00} + \lambda A_{10} + \mu A_{01} + \lambda^2 A_{20} + \lambda\mu A_{11} + \mu^2 A_{02})\,x = 0$$

with $(n+1) \times n$ matrices for $n = 4, \ldots, 20$.

n	Block Macaulay	Alg. 4 ($\widetilde{D}_i$)	# Eigs
4	180×220	60	40
6	546×630	126	84
8	1224×1368	216	144
10	2310×2530	330	220
12	3900×4212	468	312
14	6090×6510	630	420
16	8976×9520	816	544
18	12654×13338	1026	684
20	17220×18060	1260	840
n	$n(n+1)(2n+1) \times n(n+1)(2n+3)$	$3n(n+1)$	$2n(n+1)$

Matrices $\widetilde{D}_i$ are much smaller than matrices in the block Macaulay method.

Application - ARMA

Let $y_1, \dots, y_m \in \mathbb{R}$. The ARMA$(p, q)$ model is

$$y_k + \sum_{i=1}^{p} \alpha_i \, y_{k-i} = e_k + \sum_{j=1}^{q} \gamma_j \, e_{k-j}, \qquad k = p+1, \dots, m,$$

where p and q are the orders of the autoregressive (AR) and the moving-average (MA) part, respectively. We look for $\alpha_1, \dots, \alpha_p$ and $\gamma_1, \dots, \gamma_q$ that minimize $\|e\|_2$.

(Vermeersch, De Moor 2019): stationary points of an ARMA model are eigenvalues of a quadratic RMEP. We can use the block Macaulay matrices to compute all stationary points including the globally optimal one.

In contrast, state-of-the-art numerical methods for the identification of parameters in ARMA models, based on nonlinear optimization, converge locally without guarantee to find the optimal solution.

ARMA(1,1) via MEP

For $p = q = 1$ we get quadratic R2EP ($\alpha_1 = \alpha$ and $\gamma_1 = \gamma$)

$$(A_{00} + \alpha A_{10} + \gamma A_{01} + \gamma^2 A_{02})\, x = 0 \qquad \text{(ARMA)}$$

with $(3m-1) \times (3m-2)$ matrices A_{ij}.

We introduce a new parameter $\xi = \gamma^2$ and treat the problem as a linear 3EP

$$\begin{aligned} \left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \alpha \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \gamma \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \xi \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right) v &= 0, \\ (A_{00} + \alpha\, A_{10} + \gamma\, A_{01} + \xi\, A_{02})\, x &= 0. \end{aligned}$$

We get singular GEPs of size $(3m-1)(3m-2) \times (3m-1)(3m-2)$

$$\widetilde{D}_1 u = \alpha\, \widetilde{D}_0 u, \quad \widetilde{D}_2 u = \gamma\, \widetilde{D}_0 u, \quad \widetilde{D}_3 u = \xi\, \widetilde{D}_0 u$$

that we solve by the staircase-type algorithm for singular MEPs.

ARMA(1,1) example

We take $y \in \mathbb{R}^{12}$, where

$y = [2.4130,\ 1.0033,\ 1.2378,\ -0.72191,\ -0.81745,\ -2.2918,\ 0.18213,\ 0.073557,\ 0.55248,\ 2.0180,\ 2.6593,\ 1.1791]^T$

and build matrices of size 35×34 of

$$(A_{00} + \alpha A_{10} + \gamma A_{01} + \gamma^2 A_{02})\, x = 0. \qquad \text{(ARMA)}$$

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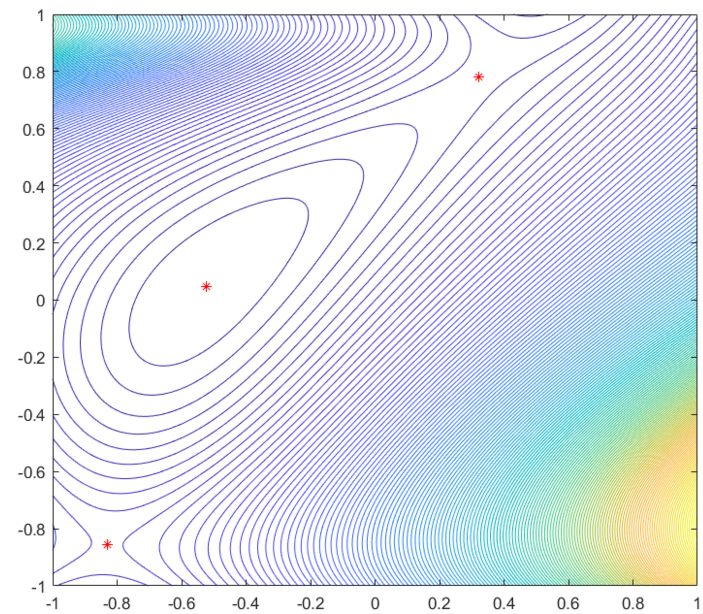
$$(A_{00} + \alpha A_{10} + \gamma A_{01} + \gamma^2 A_{02})\, x = 0. \qquad \text{(ARMA)}$$

From singular GEPs of size 1190×1190

$$\widetilde{D}_1 u = \alpha\, \widetilde{D}_0 u, \quad \widetilde{D}_2 u = \gamma\, \widetilde{D}_0 u, \quad \widetilde{D}_3 u = \xi\, \widetilde{D}_0 u$$

we get 147 eigenvalues of (ARMA). Three of them are real and give the stationary points of the objective function $\|e\|_2^2$.

Type stationary point	α	γ	$\|e\|_2^2$
Saddle point	0.3224	0.7799	17.58
Local minimum	−0.5234	0.0476	13.85
Saddle point	−0.8305	−0.8542	23.78



We estimate that for the same result we need a block Macaulay matrix of size 77385×79764.

ARMA(1,1) comparison

Size of matrices required to compute stationary points for the ARMA(1,1) model from eigenvalues of

$$(A_{00} + \alpha A_{10} + \gamma A_{01} + \gamma^2 A_{02})\, x = 0$$

by MEP approach vs. the block Macaulay approach for $m = 4, 6, \ldots, 20$.

m	# Eigs	$\widetilde{D}_i$	Time (s)	Degree	Macaulay matrix
4	35	110	0.008	19	1881 × 2100
6	63	272	0.027	31	7905 × 8448
8	91	506	0.085	43*	20769 × 21780
10	119	812	0.225	55*	43065 × 44688
12	147	1190	0.595	67*	77385 × 79764
14	175	1640	1.43	79*	126321 × 129600
16	203	2162	3.51	91*	192465 × 196788
18	231	2756	7.41	103*	278409 × 283920
20	259	3422	14.1	115*	386745 × 393588

*: estimate

A block Macaulay method for $m = 8$ needs 41.7 seconds (Vermeersch, De Moor 2023)

Application - LTI realization

Let $y_1, \dots, y_m \in \mathbb{R}$. In the optimal realization problem of autonomous LTI systems (LTI(p)), we look for $\alpha_1, \dots, \alpha_p$ for the best 2-norm approximation of y by $\widehat{y}$ that satisfies

$$\widehat{y}_{k+p} + \alpha_1 \widehat{y}_{k+p-1} + \cdots + \alpha_p \widehat{y}_k = 0, \qquad k = 1, \dots, m-p,$$

where p is the order of the LTI.

De Moor (2019): critical points of an LTI(p) model are eigenvalues of a quadratic RMEP.

Application - LTI(2)

The corresponding quadratic R2EP is

$$(A_{00} + \alpha_1 A_{10} + \alpha_2 A_{01} + (\alpha_1^2 + \alpha_2^2) A_{20} + \alpha_1 \alpha_2 A_{11})\, x = 0 \quad \text{(LTI)}$$

with matrices of size $(3m-4) \times (3m-5)$.

We linearize (LTI) as a four-parameter eigenvalue problem by introducing two new parameters $\xi_1 = \alpha_1\alpha_2$ and $\xi_2 = \alpha_1^2 + \alpha_2^2$ as

$$\begin{aligned}
\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \alpha_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \xi_1 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \xi_2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right) v_1 &= 0, \\
\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \alpha_1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \xi_1 \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} + \xi_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right) v_2 &= 0, \\
(A_{00} + \alpha_1 A_{10} + \alpha_2 A_{01} + \xi_1 A_{11} + \xi_2 A_{20})\, x &= 0.
\end{aligned}$$

We get a system of singular GEPs

$$\widetilde{D}_1 u = \alpha_1 \widetilde{D}_0 u, \quad \widetilde{D}_2 u = \alpha_2 \widetilde{D}_0 u, \quad \widetilde{D}_3 u = \xi_1 \widetilde{D}_0 u, \quad \widetilde{D}_4 u = \xi_2 \widetilde{D}_0 u, \quad (\widetilde{D})$$

where matrices are of size $2(3m-4)(3m-5) \times 2(3m-4)(3m-5)$.

LTI(2) example

We take $z \in \mathbb{R}^{10}$ whose elements satisfy

$$z_k + \alpha_1 z_{k-1} + \alpha_2 z_{k-2} = 0$$

for $\alpha_1 = 0.6$ and $\alpha_2 = -0.25$, and perturb it into `y = z + 0.1*randn(10,1)`.

$y = [0.69582,\ -0.68195,\ -0.24647,\ -0.50437,\ -0.23207,\ 0.34559,\ -0.19628,\ -0.20553,\ -0.17737,\ -0.11543]^T.$

From matrices A_{ij} of size 26×25 of

$$(A_{00} + \alpha_1 A_{10} + \alpha_2 A_{01} + (\alpha_1^2 + \alpha_2^2)A_{20} + \alpha_1\alpha_2 A_{11})\, z = 0$$

we get singular GEPs

$$\widetilde{D}_1 u = \alpha_1\, \widetilde{D}_0 u, \quad \widetilde{D}_2 u = \alpha_2\, \widetilde{D}_0 u, \quad \widetilde{D}_3 u = \xi_1\, \widetilde{D}_0 u, \quad \widetilde{D}_4 u = \xi_2\, \widetilde{D}_0 u$$

with matrices of size 1300×1300.

A staircase-type algorithm returns 1059 eigenvalues. There are 11 real eigenvalues (α_1, α_2) that give critical points of the objective function $\|y - \widehat{y}\|_2^2$, the minimum is obtained at $(\alpha_1, \alpha_2) = (0.60076, -0.26572)$.

We estimate that for the same result we need a block Macaulay matrix of size 29328×30625.

LTI(2) comparison

Size of matrices required to compute critical points for a generic LTI(2) model as eigenvalues of

$$(A_{00} + \alpha_1 A_{10} + \alpha_2 A_{01} + (\alpha_1^2 + \alpha_2^2) A_{20} + \alpha_1 \alpha_2 A_{11})\, x = 0$$

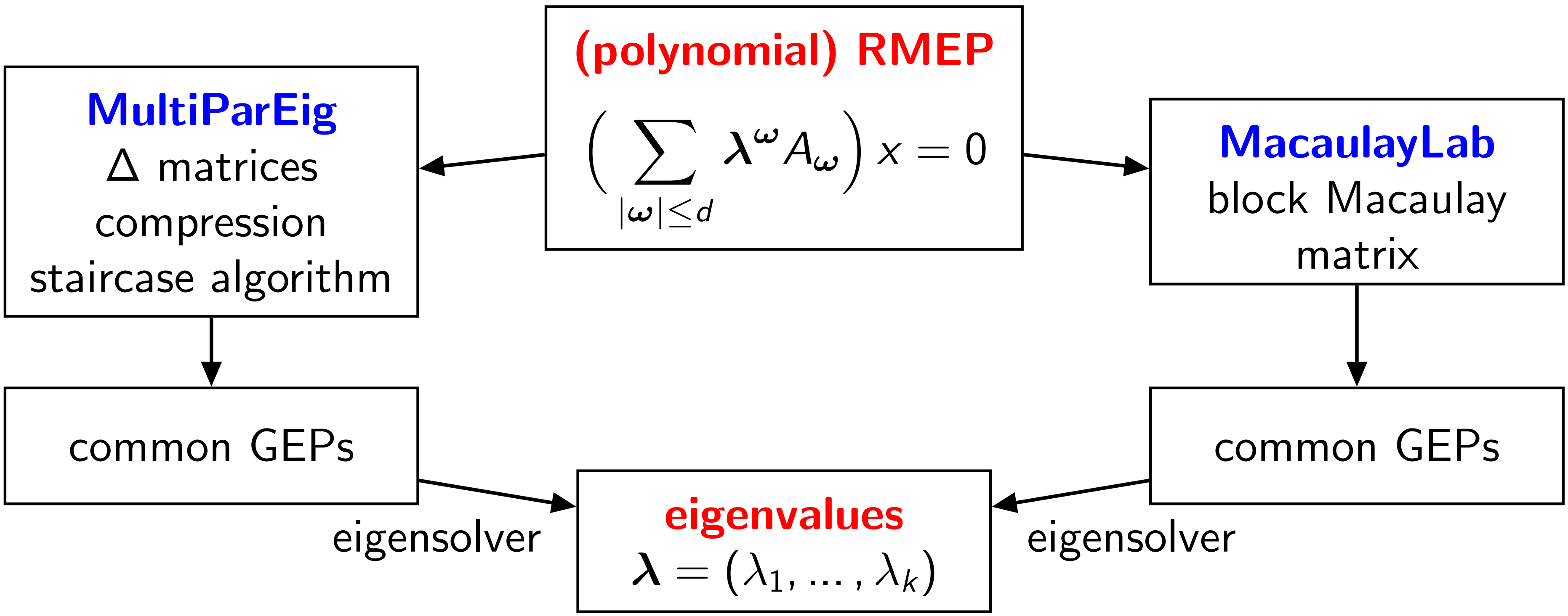
by MEP aproach vs. the block Macaulay approach for $m = 4, 6, \ldots, 12$.

m	# Eigs	$\widetilde{D}_i$	Time (s)	Degree	Macaulay matrix
4	51	112	0.015	12	528 × 637
6	243	364	0.158	24	3864 × 4225
8	579	760	1.18	36	12600 × 13357
10	1059	1300	7.03	48*	29328 × 30625
12	1683	1984	32.5	60*	56640 × 58621

*: estimate

A block Macaulay method for $m = 6$ needs 2.3 seconds (Vermeersch, De Moor 2023)

Common points with block Macaulay matrices approach



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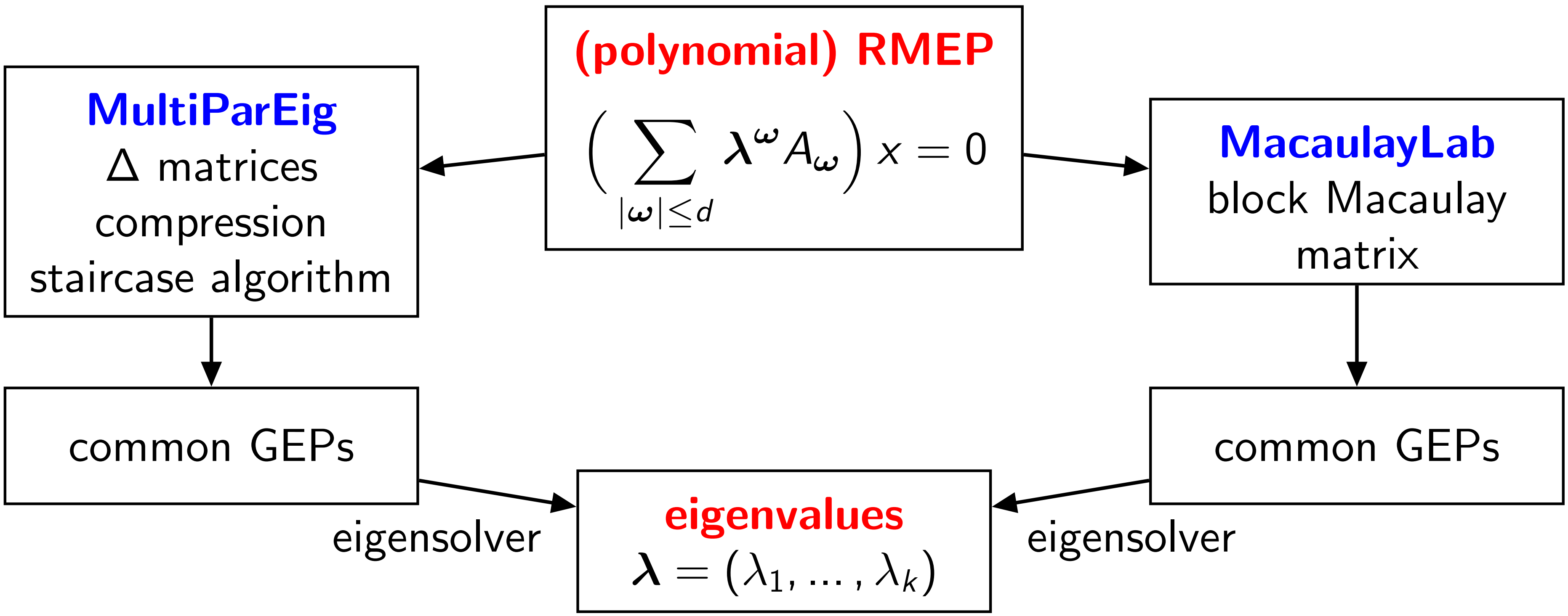

We get two families of k GEPs that have the same eigenvalues. Are these two families related in any way?

For problems with solutions at infinity, it might be interesting to compare the staircase-type method to the transformations applied to the block Macaulay matrix that extract only the part with the affine solutions.

Conclusions

We can "squareify" RMEPs into (square) MEPs.

Operator determinants, compression and staircase algorithm are another way that leads from RMEP to a joint system of GEPs.

Some RMEPs can be solved efficiently with tools for standard MEPs.

New options for finding optimal parameters of ARMA(1,1), ARMA(2,1), and LTI(2) models for small samples.

More details: M.E. Hochstenbach, T. Košir, P.: *On the solution of rectangular multiparameter eigenvalue problems, with applications to finding optimal ARMA and LTI models*, Numer. Linear Algebra Appl. 2023, e2540.

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Thank you for your attention!