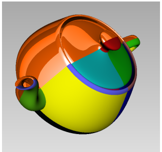
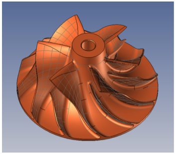
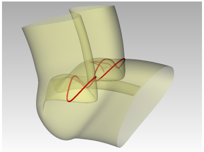
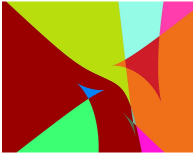
Linear Algebra for non-linear problems

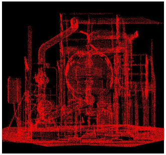
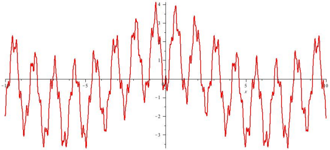
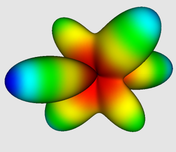
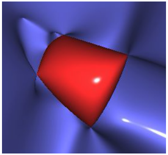
Bernard Mourrain
Inria at Université Côte d'Azur
Bernard.Mourrain@inria.fr

1st December 2023

Finding roots of polynomial equations



...

A primal approach for solving polynomial systems

Given $f_1, \ldots, f_m \in R := \mathbb{K}[x_1, \ldots, x_n]$ with $I := (f_1, \ldots, f_m)$

- **Compute a basis of $\mathcal{A} = R/I$ by linear algebra**

  on the (monomial) multiples of f_i via Grobner Basis, Border Basis or under genericity assumption.

- **Compute the multiplicative structure of $\mathcal{A} = R/I$** (mult. by x_i)

  using normal form reduction via Grobner Basis, Border Basis or Schur complements in resultant matrices.

- **Deduce the roots $\mathcal{V}_{\overline{\mathbb{K}}}(I) = \{\xi_1, \ldots, \xi_r\} \subset \overline{\mathbb{K}}^n$**

  by eigenvalues/eigenvectors or reduction to univariate polynomial solving via change of ordering.

A dual approach

- **Compute a basis of** $I^\perp = \{\lambda \in R^* \mid \forall p \in I, \lambda(p) = 0\} \sim \mathcal{A}^*$
- **Compute the derivation operator by** d_{x_i} **in** $\mathcal{A}^* = I^\perp$
- **Deduce the roots** $\zeta_1, \ldots, \zeta_r$

by eigencomputation.

Multiplication maps

$$\begin{aligned} \mathcal{M}_a : \mathcal{A} &\to \mathcal{A} \\ u &\mapsto a\,u \end{aligned} \qquad \begin{aligned} \mathcal{M}_a^{\mathtt{t}} : \mathcal{A}^* = I^\perp &\to \mathcal{A}^* = I^\perp \\ \Lambda &\mapsto a \star \Lambda = \Lambda \circ \mathcal{M}_a \end{aligned}$$

Multiplication maps

$$\begin{array}{rcl} \mathcal{M}_a : \mathcal{A} & \to & \mathcal{A} \\ u & \mapsto & a\,u \end{array} \qquad \begin{array}{rcl} \mathcal{M}_a^{\mathtt{t}} : \mathcal{A}^* = I^\perp & \to & \mathcal{A}^* = I^\perp \\ \Lambda & \mapsto & a \star \Lambda = \Lambda \circ \mathcal{M}_a \end{array}$$

Theorem

- *The eigenvalues of $\mathcal{M}_a$ are the* **values at the roots** $\{a(\xi_1), \ldots, a(\xi_r)\}$.
- *The eigenvectors* **common** *to all $(\mathcal{M}_a^{\mathtt{t}})_{a\in\mathcal{A}}$ are (up to a scalar) the* **evaluations at the roots** $\mathrm{e}_{\xi_i} : p \mapsto p(\xi_i)$.

[Auzinger-Stetter'88, M'98]

Multiplication maps

$$\begin{array}{rcl} \mathcal{M}_a : \mathcal{A} & \to & \mathcal{A} \\ u & \mapsto & a\,u \end{array} \qquad \begin{array}{rcl} \mathcal{M}_a^{\mathtt{t}} : \mathcal{A}^* = I^{\perp} & \to & \mathcal{A}^* = I^{\perp} \\ \Lambda & \mapsto & a \star \Lambda = \Lambda \circ \mathcal{M}_a \end{array}$$

Theorem

- *The eigenvalues of $\mathcal{M}_a$ are the* **values at the roots** $\{a(\xi_1), \ldots, a(\xi_r)\}$.
- *The eigenvectors* **common** *to all $(\mathcal{M}_a^{\mathtt{t}})_{a\in\mathcal{A}}$ are (up to a scalar) the* **evaluations at the roots** $\mathrm{e}_{\xi_i} : p \mapsto p(\xi_i)$.

[Auzinger-Stetter'88, M'98]

- If the roots are simple, the common eigenvectors are, up to a scalar, **interpolation polynomials** u_i at the roots and idempotent in $\mathcal{A}$.
- In a basis of $\mathcal{A}$, all the matrices M_a $(a \in \mathcal{A})$ are of the form

$$\mathtt{M}_a = \begin{bmatrix} \mathtt{N}_a^1 & & 0 \\ & \ddots & \\ 0 & & \mathtt{N}_a^r \end{bmatrix} \text{ with } \mathtt{N}_a^i = \begin{bmatrix} a(\xi_i) & & \star \\ & \ddots & \\ 0 & & a(\xi_i) \end{bmatrix}$$

Roots from multiplication operators

Matrix of multiplication by x_1 in the basis $B = [1, x_1, x_2, x_1 x_2]$ modulo

> $f_1 = 13\,x_1^2 + 8\,x_1\,x_2 + 4\,x_2^2 - 8\,x_1 - 8\,x_2 + 2$

> $f_2 = x_1^2 + x_1\,x_2 - x_1 - \frac{1}{6}$

Roots from multiplication operators

Matrix of multiplication by x_1 in the basis $B = [1, x_1, x_2, x_1 x_2]$ modulo

> $f_1 = 13\,x_1^2 + 8\,x_1\,x_2 + 4\,x_2^2 - 8\,x_1 - 8\,x_2 + 2$

> $f_2 = x_1^2 + x_1\,x_2 - x_1 - \frac{1}{6}$

$$\begin{aligned}
1 \times x_1 &\equiv x_1, \\
x_1 \times x_1 &\equiv -x_1 x_2 + x_1 + \frac{1}{6}, \qquad x_2 \times x_1 \equiv x_1 x_2, \\
x_1 x_2 \times x_1 &\equiv x_1^2\,x_2 + \frac{1}{9}\,x_1\,f_1 - (\frac{5}{9} + \frac{13}{9}\,x_1 + \frac{4}{9}\,x_2)\,f_2 \\
&\equiv -x_1 x_2 + \frac{55}{54}\,x_1 + \frac{2}{27}\,x_2 + \frac{5}{54}.
\end{aligned}$$

Roots from multiplication operators

Matrix of multiplication by x_1 in the basis $B = [1, x_1, x_2, x_1 x_2]$ modulo

> $f_1 = 13\,x_1^2 + 8\,x_1\,x_2 + 4\,x_2^2 - 8\,x_1 - 8\,x_2 + 2$

> $f_2 = x_1^2 + x_1\,x_2 - x_1 - \frac{1}{6}$

$$
\begin{aligned}
1 \times x_1 &\equiv x_1, \\
x_1 \times x_1 &\equiv -x_1 x_2 + x_1 + \frac{1}{6}, \qquad x_2 \times x_1 \equiv x_1 x_2, \\
x_1 x_2 \times x_1 &\equiv x_1^2\,x_2 + \frac{1}{9}\,x_1\,f_1 - (\frac{5}{9} + \frac{13}{9}\,x_1 + \frac{4}{9}\,x_2)\,f_2 \\
&\equiv -x_1 x_2 + \frac{55}{54}\,x_1 + \frac{2}{27}\,x_2 + \frac{5}{54}.
\end{aligned}
$$

This yields:

$$
M_1 = \begin{bmatrix} 0 & \frac{1}{6} & 0 & \frac{5}{54} \\ 1 & 1 & 0 & \frac{55}{54} \\ 0 & 0 & 0 & \frac{2}{27} \\ 0 & -1 & 1 & -1 \end{bmatrix}.
$$

Computing the roots from the eigenvectors

```
> Eigenvals(M1);
```

$$[-\frac{1}{3}, -\frac{1}{3}, \frac{1}{3}, \frac{1}{3}]$$

```
> Eigenvects(transpose(M1));
```

$$[1, -\frac{1}{3}, \frac{5}{6}, -\frac{5}{18}], [1, \frac{1}{3}, \frac{7}{6}, \frac{7}{18}]$$

Computing the roots from the eigenvectors

```
>  Eigenvals(M1);
```

$$[-\frac{1}{3}, -\frac{1}{3}, \frac{1}{3}, \frac{1}{3}]$$

```
>  Eigenvects(transpose(M1));
```

$$[1, -\frac{1}{3}, \frac{5}{6}, -\frac{5}{18}], [1, \frac{1}{3}, \frac{7}{6}, \frac{7}{18}]$$

As the basis is $B = [1, x_1, x_2, x_1 x_2]$, we deduce the roots:

- $\xi_1 = (-\frac{1}{3}, \frac{5}{6})$,
- $\xi_2 = (\frac{1}{3}, \frac{7}{6})$.

Remark: $v_4\, v_1 = v_2\, v_3$.

Linear functionals: $\lambda \in R^* = \{\lambda : R \to \mathbb{K}, \textbf{linear}\}$

$$\lambda : p = \sum_{\alpha} p_{\alpha} \boldsymbol{x}^{\alpha} \mapsto \langle \lambda | p \rangle = \sum_{\alpha} \lambda_{\alpha} p_{\alpha}$$

Duality, sequences, series

Linear functionals: $\lambda \in R^* = \{\lambda : R \to \mathbb{K}, \textbf{linear}\}$

$$\lambda : p = \sum_{\alpha} p_\alpha \boldsymbol{x}^\alpha \mapsto \langle \lambda | p \rangle = \sum_{\alpha} \lambda_\alpha p_\alpha$$

Multi-index sequences: $\lambda = (\lambda_\alpha)_{\alpha \in \mathbb{N}^n} \in \mathbb{K}^{\mathbb{N}^n}$ indexed by $\alpha \in \mathbb{N}^n$.
The coefficients $\lambda_\alpha = \langle \lambda | \boldsymbol{x}^\alpha \rangle \in \mathbb{K}$, $\alpha \in \mathbb{N}^n$ are the **pseudo-moments** of λ.

Duality, sequences, series

Linear functionals: $\lambda \in R^* = \{\lambda : R \to \mathbb{K}, \textbf{linear}\}$

$$\lambda : p = \sum_{\alpha} p_\alpha \boldsymbol{x}^\alpha \mapsto \langle \lambda | p \rangle = \sum_{\alpha} \lambda_\alpha p_\alpha$$

Multi-index sequences: $\lambda = (\lambda_\alpha)_{\alpha \in \mathbb{N}^n} \in \mathbb{K}^{\mathbb{N}^n}$ indexed by $\alpha \in \mathbb{N}^n$.
The coefficients $\lambda_\alpha = \langle \lambda | \boldsymbol{x}^\alpha \rangle \in \mathbb{K}$, $\alpha \in \mathbb{N}^n$ are the **pseudo-moments** of λ.

Series:

$$\lambda(\boldsymbol{y}) = \sum_{\alpha \in \mathbb{N}^n} \lambda_\alpha \boldsymbol{y}^\alpha \in \mathbb{K}[[y_1, \ldots, y_n]]$$

Linear functionals: $\lambda \in R^* = \{\lambda : R \to \mathbb{K}, \textbf{linear}\}$

$$\lambda : p = \sum_{\alpha} p_\alpha \boldsymbol{x}^\alpha \mapsto \langle \lambda | p \rangle = \sum_{\alpha} \lambda_\alpha p_\alpha$$

Multi-index sequences: $\lambda = (\lambda_\alpha)_{\alpha \in \mathbb{N}^n} \in \mathbb{K}^{\mathbb{N}^n}$ indexed by $\alpha \in \mathbb{N}^n$.
The coefficients $\lambda_\alpha = \langle \lambda | \boldsymbol{x}^\alpha \rangle \in \mathbb{K}$, $\alpha \in \mathbb{N}^n$ are the **pseudo-moments** of λ.

Series:

$$\lambda(\boldsymbol{y}) = \sum_{\alpha \in \mathbb{N}^n} \lambda_\alpha \boldsymbol{y}^\alpha \in \mathbb{K}[[y_1, \ldots, y_n]]$$

Structure of R-module: $\forall p \in R, \lambda \in R^*$, $p \star \lambda : q \mapsto \langle \lambda | p\, q \rangle$:

$$p \star \lambda = (p(y_1^{-1}, \ldots, y_n^{-1}) \lambda(\boldsymbol{y}))_+$$

Examples of basis of $I^\perp$

- If I defines simple roots $\zeta_1, \ldots, \zeta_r$, then

$$I^\perp = \langle \mathsf{e}_{\zeta_1}, \ldots, \mathsf{e}_{\zeta_r} \rangle$$

Examples of basis of $I^\perp$

- If I defines simple roots $\zeta_1, \ldots, \zeta_r$, then

$$I^\perp = \langle \mathrm{e}_{\zeta_1}, \ldots, \mathrm{e}_{\zeta_r} \rangle$$

- **Normal form**

$$\begin{array}{rcl} \mathcal{N} : \mathbb{K}[\boldsymbol{x}] & \longrightarrow & B \\ p & \longmapsto & \mathcal{N}(p) = \sum_{i=1}^{r} \lambda_i(p)\, b_i \end{array}$$

with

- $\mathcal{N} \circ \mathcal{N} = \mathcal{N}$ (projector)
- $\ker(\mathcal{N}) = I$

Examples of basis of $I^\perp$

- If I defines simple roots $\zeta_1, \ldots, \zeta_r$, then

$$I^\perp = \langle \mathrm{e}_{\zeta_1}, \ldots, \mathrm{e}_{\zeta_r} \rangle$$

Normal form

$$\begin{array}{rcl} \mathcal{N} : \mathbb{K}[\boldsymbol{x}] & \longrightarrow & B \\ p & \longmapsto & \mathcal{N}(p) = \sum_{i=1}^{r} \lambda_i(p)\, b_i \end{array}$$

with

- $\mathcal{N} \circ \mathcal{N} = \mathcal{N}$ (projector)
- $\ker(\mathcal{N}) = I$

☞ $\{\lambda_1, \ldots, \lambda_r\}$ **basis of $I^\perp$** (the converse is true)

Example: if $G = \{g_1, \ldots, g_s\}$ is a Grobner (resp. Border) basis of I, then

- $\mathcal{N}(p) = \mathrm{rem}(p, G) = \sum_{\alpha \notin \mathcal{L}(I)} c_\alpha(p)\, \boldsymbol{x}^\alpha$
- $I^\perp = \langle c_\alpha \rangle$

Properties of $I^\perp$

If $I^\perp$ is spanned by the rows of

$$N = \begin{array}{c} \\ \lambda_1 \\ \vdots \\ \vdots \\ \lambda_r \end{array} \begin{array}{c} \begin{array}{ccccccc} \boldsymbol{x}^{\beta_1} & \dots & \dots & \boldsymbol{x}^{\beta_r} & \dots & \boldsymbol{x}^{\alpha} & \dots \end{array} \\ \left[\begin{array}{ccccccc} 1 & 0 & \cdots & 0 & \cdots & \langle \lambda_1 | \boldsymbol{x}^\alpha \rangle & \cdots \\ 0 & 1 & & \vdots & & \vdots & \\ \vdots & & \ddots & 0 & & \vdots & \\ 0 & \cdots & 0 & 1 & \cdots & \langle \lambda_r | \boldsymbol{x}^\alpha \rangle & \cdots \end{array} \right] \end{array}.$$

Properties of $I^\perp$

If $I^\perp$ is spanned by the rows of

$$N = \begin{array}{c} \\ \lambda_1 \\ \vdots \\ \vdots \\ \lambda_r \end{array} \begin{array}{c} \begin{array}{ccccccc} \mathbf{x}^{\beta_1} & \dots & \dots & \mathbf{x}^{\beta_r} & \dots & \mathbf{x}^{\alpha} & \dots \end{array} \\ \left[\begin{array}{ccccccc} 1 & 0 & \cdots & 0 & \cdots & \langle \lambda_1 | \mathbf{x}^{\alpha} \rangle & \cdots \\ 0 & 1 & & \vdots & & \vdots & \\ \vdots & & \ddots & 0 & & \vdots & \\ 0 & \cdots & 0 & 1 & \cdots & \langle \lambda_r | \mathbf{x}^{\alpha} \rangle & \cdots \end{array} \right] \end{array}.$$

then

- $B = \{\mathbf{x}^{\beta_1}, \dots, \mathbf{x}^{\beta_r}\}$ is a basis of $\mathcal{A} = R/I$
- $N[:, x_i \cdot B]^t$ is the matrix M_i of multiplication by x_i in the basis B of $\mathcal{A}$.
- We have the **normal form** $\begin{array}{rcl} \mathcal{N} : R & \to & \langle B \rangle \\ p & \mapsto & \mathcal{N}(p) = \sum_{i=1}^{r} \langle \lambda_i | p \rangle \, \mathbf{x}^{\beta_i} \end{array}$

Properties of $I^\perp$

If $I^\perp$ is spanned by the rows of

$$N = \begin{array}{c} \\ \lambda_1 \\ \vdots \\ \vdots \\ \lambda_r \end{array} \begin{array}{c} \begin{array}{ccccccc} \boldsymbol{x}^{\beta_1} & \dots & \dots & \boldsymbol{x}^{\beta_r} & \dots & \boldsymbol{x}^{\alpha} & \dots \end{array} \\ \left[\begin{array}{ccccccc} 1 & 0 & \cdots & 0 & \cdots & \langle \lambda_1 | \boldsymbol{x}^\alpha \rangle & \cdots \\ 0 & 1 & & \vdots & & \vdots & \\ \vdots & & \ddots & 0 & & \vdots & \\ 0 & \cdots & 0 & 1 & \cdots & \langle \lambda_r | \boldsymbol{x}^\alpha \rangle & \cdots \end{array}\right] \end{array}.$$

then

- $B = \{\boldsymbol{x}^{\beta_1}, \dots, \boldsymbol{x}^{\beta_r}\}$ is a basis of $\mathcal{A} = R/I$
- $N[:, x_i \cdot B]^t$ is the matrix M_i of multiplication by x_i in the basis B of $\mathcal{A}$.
- We have the **normal form** $\begin{array}{rcl} \mathcal{N} : R & \to & \langle B \rangle \\ p & \mapsto & \mathcal{N}(p) = \sum_{i=1}^r \langle \lambda_i | p \rangle \, \boldsymbol{x}^{\beta_i} \end{array}$
- For a general N, B **is a basis of** $\mathcal{A}$ **iff** $N_0 := N[:, B]$ **is invertible.**
- For a general N, $M_i^t = N_0^{-1} N_i$ where $N_i = N[:, x_i \cdot B]$

Properties of $I^\perp$

If $I^\perp$ is spanned by the rows of

$$N = \begin{array}{c} \\ \lambda_1 \\ \vdots \\ \vdots \\ \lambda_r \end{array} \begin{array}{c} \begin{array}{ccccccc} \boldsymbol{x}^{\beta_1} & \cdots & \cdots & \boldsymbol{x}^{\beta_r} & \cdots & \boldsymbol{x}^{\alpha} & \cdots \end{array} \\ \left[\begin{array}{ccccccc} 1 & 0 & \cdots & 0 & \cdots & \langle \lambda_1 | \boldsymbol{x}^\alpha \rangle & \cdots \\ 0 & 1 & & \vdots & & \vdots & \\ \vdots & & \ddots & 0 & & \vdots & \\ 0 & \cdots & 0 & 1 & \cdots & \langle \lambda_r | \boldsymbol{x}^\alpha \rangle & \cdots \end{array}\right] \end{array}.$$

then

- $B = \{\boldsymbol{x}^{\beta_1}, \ldots, \boldsymbol{x}^{\beta_r}\}$ is a basis of $\mathcal{A} = R/I$
- $N[:, x_i \cdot B]^t$ is the matrix M_i of multiplication by x_i in the basis B of $\mathcal{A}$.
- We have the **normal form** $\begin{array}{rcl} \mathcal{N} : R & \rightarrow & \langle B \rangle \\ p & \mapsto & \mathcal{N}(p) = \sum_{i=1}^{r} \langle \lambda_i | p \rangle \, \boldsymbol{x}^{\beta_i} \end{array}$
- For a general N, **B is a basis of $\mathcal{A}$ iff $N_0 := N[:, B]$ is invertible.**
- For a general N, $M_i^t = N_0^{-1} N_i$ where $N_i = N[:, x_i \cdot B]$

☞ **We only need a truncated part of N to solve the equations.**

Truncated normal forms

Resultants based dual

For polynomials $f_1, \ldots, f_m$, consider the resultant map

$$\begin{array}{rrcl} \mathrm{Res}: & V_1 \times \cdots \times V_m & \longrightarrow & V \\ & (q_1, \ldots, q_m) & \longmapsto & q_1 f_1 + \cdots + q_m f_m. \end{array}$$

where $V_i, V \subset R$.

- For f_i dense in degree d_i, $\rho = \sum_i d_i - n + 1$, $V_i \subset R_{\leq \rho - d_i}$.
- For f_i with support in $A_i \subset \mathbb{Z}^n$, $S_i = (\oplus_{j \neq i} A_j + \vec{\epsilon}) \cap \mathbb{Z}^n$, $V_i \subset \langle \boldsymbol{x}^{S_i} \rangle$.

Proposition

If $\mathrm{im}\,\mathrm{Res} = I \cap V$, *then* $N := \mathrm{coker}\,\mathrm{Res} = I^{\perp}_{|V} \subset V^*$.

For $\alpha \gg 0$, $V = R_\alpha$ and $V_i = R_{\alpha - \deg(f_i)}$ and f_i **generic**, $N := I^{\perp}_{|V}$.

Algorithm for solving based on resultant matrices

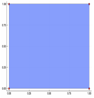
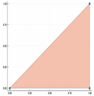
For $f_1, \ldots, f_m \in R$, $V_1, \ldots, V_m, V$ vector spaces of R (e.g. span. by mon.)

$$\begin{array}{rccc} \mathrm{Res}: & V_1 \times \cdots \times V_m & \longrightarrow & V \\ & (q_1, \ldots, q_n) & \longmapsto & q_1 f_1 + \cdots + q_m f_m. \end{array}$$

Roots from the cokernel of a resultant map

- $N \leftarrow (\ker \mathrm{Res}^t)^t$
- $N_{|W} \leftarrow$
  restriction of N to W with $W^+ := W + x_1 \cdot W + \cdots + x_n \cdot W \subset V$
- $\mathtt{Q}, \mathtt{R}, \mathtt{P} \leftarrow \mathit{qrfact}(N_{|W})$
  $N_0 \leftarrow$ first columns in $\mathtt{P}$ of $N_{|W}$ indexed by $B \subset W$
- $N_i \leftarrow$ columns of N corresponding to $x_i \cdot B$
- $M_{x_i} \leftarrow (N_0)^{-1} N_i$
- **return** the roots of $f_1, \ldots, f_m$ from $M_{x_1}, \ldots, M_{x_n}$.

Toric example

$$p_1 = x\,y - y - 3\,x + 3,$$
$$p_2 = x\,y - 5\,x + 4$$

```
> R = toric_matrix([p1,p2])
```

$V_1 = \langle 1, x\rangle,\ V_2 = \langle 1, x\rangle,\ V = \langle 1, x, y, x^2, x\,y, x^2 y\rangle$

$$\mathcal{R}^\top = \begin{array}{c} \\ p_1 \\ x\,p_1 \\ p_2 \\ x\,p_2 \end{array} \begin{array}{cccccc} 1 & x & y & x^2 & x\,y & x^2y \\ \end{array}$$
$$\left[\begin{array}{cccccc} 3 & -3 & -1 & 0 & 1 & 0 \\ 0 & 3 & 0 & -3 & -1 & 1 \\ 4 & -5 & 0 & 0 & 1 & 0 \\ 0 & 4 & 0 & -5 & 0 & 1 \end{array}\right]$$

```
> N = nullspace(R')
```

$$N = \ker \mathcal{R}^\top \approx \begin{bmatrix} 0.45239 & 0.43678 & 0.42117 & 0.40556 & 0.37434 & 0.28068 \\ -0.08256 & -0.05487 & -0.02717 & 0.00052 & 0.05591 & 0.22207 \end{bmatrix}$$

```
> Xi, B = join_eigen(N,L)
```

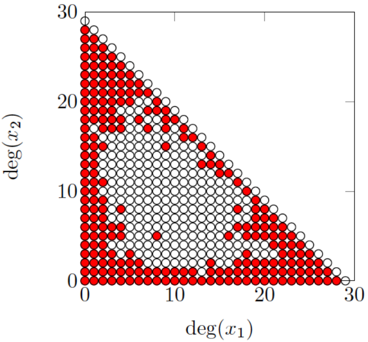
$$B = \{1, x\}$$

$$N_0 \approx \begin{bmatrix} 0.45239 & 0.43678 \\ -0.08256 & -0.05487 \end{bmatrix},\ N_1 \approx \begin{bmatrix} 0.43678 & 0.40556 \\ -0.05487 & 0.00052 \end{bmatrix},\ N_2 \approx \begin{bmatrix} 0.42117 & 0.37434 \\ -0.02717 & 0.05591 \end{bmatrix}$$

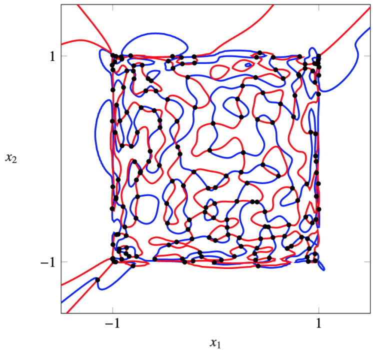
$$\Xi = \begin{bmatrix} eigen(N_0^{-1} N_1) \\ eigen(N_0^{-1} N_2) \end{bmatrix} = \begin{bmatrix} 1.0 & 2.0 \\ 1.0 & 3.0 \end{bmatrix}$$

Example of basis for a generic dense system

A system $f_1, f_2 \in R = \mathbb{R}[x_1, x_2]$ with $\deg(f_i) = 15$, $V = R_{\leq 29}$, $W = R_{\leq 28}$, $\delta = 225$.



A system $f_1, f_2 \in R = \mathbb{R}[x_1, x_2]$ with $\deg(f_i) = 25$, $V = R_{\leq 49}$, $W = R_{\leq 48}$, $\delta = 625$ using representations in Chebyshev basis.



[Telen-M-Van Barel'2018, M-Telen-Van Barel'2019]

Numerical experimentation

$n = 2$, numerical quality and running time.

d	δ	m_1	$m_2 = n_1$	n_2	res	$\delta_{\mathbf{alg}}$	$\delta_{\mathbf{phc}}$	$\delta_{\mathbf{brt}}$
1	1	2	3	1	$1.28 \cdot 10^{-16}$	1	1	1
7	49	56	105	49	$2.06 \cdot 10^{-13}$	49	49	49
13	169	182	351	169	$2.18 \cdot 10^{-13}$	169	169	169
19	361	380	741	361	$5.28 \cdot 10^{-13}$	361	361	361
25	625	650	1,275	625	$1.21 \cdot 10^{-10}$	625	614	625
31	961	992	1,953	961	$5.23 \cdot 10^{-9}$	961	951	961
37	1,369	1,406	2,775	1,369	$4.05 \cdot 10^{-12}$	1,369	1,360	1,368
43	1,849	1,892	3,741	1,849	$1.74 \cdot 10^{-11}$	1,849	1,825	1,845
49	2,401	2,450	4,851	2,401	$1.57 \cdot 10^{-10}$	2,401	2,364	2,163
55	3,025	3,080	6,105	3,025	$1.84 \cdot 10^{-11}$	3,025	2,970	2,487
61	3,721	3,782	7,503	3,721	$3.26 \cdot 10^{-11}$	3,721	3,662	2,260

d	t_M	t_N	t_B	t_S	$t_{\mathbf{alg}}$	$t_{\mathbf{phc}}$	$t_{\mathbf{brt}}$
1	$1.48 \cdot 10^{-4}$	$5.5 \cdot 10^{-5}$	$2.96 \cdot 10^{-4}$	$3.6 \cdot 10^{-5}$	$5.35 \cdot 10^{-4}$	$5.6 \cdot 10^{-2}$	$1.41 \cdot 10^{-2}$
7	$7.88 \cdot 10^{-3}$	$1.68 \cdot 10^{-3}$	$3.76 \cdot 10^{-3}$	$2.78 \cdot 10^{-3}$	$1.61 \cdot 10^{-2}$	0.18	$8.65 \cdot 10^{-2}$
13	$4.65 \cdot 10^{-2}$	$1.03 \cdot 10^{-2}$	$1.66 \cdot 10^{-2}$	$2.81 \cdot 10^{-2}$	0.1	0.84	1.14
19	0.13	$5.69 \cdot 10^{-2}$	$5.34 \cdot 10^{-2}$	0.13	0.37	3.29	8.79
25	0.32	0.18	0.15	0.51	1.16	8.79	33.83
31	0.55	0.51	0.55	1.49	3.1	20.25	98.39
37	0.96	1.52	1.5	3.52	7.5	39.92	258.09
43	1.47	4.05	3.8	8.28	17.6	69.1	504.01
49	2.47	10.46	8.78	17.91	39.62	124.47	891.37
55	3.69	20.51	17.85	34.3	76.34	178.55	1,581.77
61	4.85	36.32	31.26	62.87	135.3	283.87	2,115.66

[Telen-M-Van Barel'18]

Truncated Normal Forms (TNF)

For $V \subset R$, a **Truncated Normal Form** on V w.r.t. I is a map $N : V \to \mathbb{K}^r$ such that $\operatorname{rank} N = r$ and $\ker N = I \cap V$.

Truncated Normal Forms (TNF)

For $V \subset R$, a **Truncated Normal Form** on V w.r.t. I is a map $N : V \to \mathbb{K}^r$ such that rank $N = r$ and ker $N = I \cap V$.

Theorem

Let $V \subset R$ be a finite dimensional and $N : V \to \mathbb{K}^r$ s.t.

1. $1 \in V$
2. $\ker(N) \subset I$
3. $N_{|W}$ *is onto* $\mathbb{K}^r$ *for* $W \subset V$ *s.t.* $W^+ := W + x_1 \cdot W + \cdots + x_n \cdot W \subset V$

Then for any r-dimensional vector subspace $B \subset W$ s.t. $N_{|B}$ is invertible we have:

- (i) $B \simeq R/I$ *(as R-modules),*
- (ii) $V = B \oplus (I \cap V)$ *and* $I = (\ker(N))$*, N is a TNF,*
- (iii) $M_i : b \in B \mapsto N(x_i b) \in B$ *is the multiplication by* x_i *in B modulo I.*

Truncated Normal Forms (TNF)

For $V \subset R$, a **Truncated Normal Form** on V w.r.t. I is a map $N : V \to \mathbb{K}^r$ such that rank $N = r$ and ker $N = I \cap V$.

Theorem

Let $V \subset R$ be a finite dimensional and $N : V \to \mathbb{K}^r$ s.t.

1. $1 \in V$
2. $\ker(N) \subset I$
3. $N_{|W}$ *is onto* $\mathbb{K}^r$ *for* $W \subset V$ *s.t.* $W^+ := W + x_1 \cdot W + \cdots + x_n \cdot W \subset V$

Then for any r-dimensional vector subspace $B \subset W$ s.t. $N_{|B}$ is invertible we have:

(i) $B \simeq R/I$ *(as R-modules),*

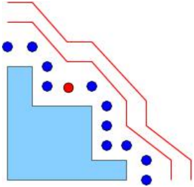
(ii) $V = B \oplus (I \cap V)$ *and* $I = (\ker(N))$, N *is a TNF,*

(iii) $M_i : b \in B \mapsto N(x_i b) \in B$ *is the multiplication by* x_i *in* B *modulo* I.

☞ Compute N by increasing degree, exploiting sparsity,
[Telen-M-Van Barel'18], [M-Telen-Van Barel'19],
cf. also [Batselier-Dreesen-De Moor'14], [Vermeersch-De Moor'23]

Parametric normal forms

Border basis



For B a monomial set,

$$B^+ = B \cup x_1 B \cup \cdots \cup x_n B,$$
$$\partial B = B^+ \setminus B.$$

Definition (Border basis)

A family $F = \{f_\alpha, \alpha \in \partial B\}$ of polynomials of the form

$$f_\alpha = \mathbf{x}^\alpha - \sum_{\mathbf{x}^\beta \in B} c_{\alpha,\beta} \mathbf{x}^\beta, \quad \alpha \in \partial B, \quad c_{\alpha,\beta} \in \mathbb{K},$$

such that $R = \langle B \rangle \oplus (F)$.

Normal form: reduction by F on B.

$$N : p \in R \mapsto \mathrm{red}(p\,; F)$$

☞ Grobner basis as a special case of border basis (B finite).

Parametric border basis (resp. Grobner basis)

Let $I = (h_1, \ldots, h_s) \subset \mathbb{K}[\boldsymbol{x}]$ zero-dimensional.

Let $B = \{\boldsymbol{x}^{\beta_1}, \ldots, \boldsymbol{x}^{\beta_r}\}$ be a set of monomials *containing* 1.

Assume that $g_1, \ldots, g_s$ is a **border basis (resp. Grobner basis)** of I, defining the basis B of $\mathcal{A} = \mathbb{K}[\boldsymbol{x}]/I$. Then we have

$$g_i = \boldsymbol{x}^{\alpha_i} - \sum_{\boldsymbol{x}^\beta \in B_\prec} c_{i,\beta}\, \boldsymbol{x}^\beta \text{ for some } \alpha_i \in \partial B = B^+ \setminus B \text{ and } c_{i,\beta} \in \mathbb{K}.$$

Define

$$G_i = \boldsymbol{x}^{\alpha_i} - \sum_{\boldsymbol{x}^\beta \in B_\prec} \lambda_{i,\beta}\, \boldsymbol{x}^\beta \in \mathbb{K}[\boldsymbol{\lambda}][\boldsymbol{x}]$$

and

$$N : p \in \mathbb{K}[\boldsymbol{x}] \mapsto \mathrm{red}(p, G_1, \ldots, G_s) = \sum_{\boldsymbol{x}^\beta} N_\beta(\lambda)\boldsymbol{x}^\beta \in B_{\boldsymbol{\lambda}} := B \otimes \mathbb{K}[\boldsymbol{\lambda}]$$

Let $M_i : b \in B \mapsto N(x_i \cdot B) \in B_{\boldsymbol{\lambda}} = B \otimes \mathbb{K}[\boldsymbol{\lambda}]$

- $\mathcal{E} = (N_\beta(h_k)$ for $\boldsymbol{x}^\beta \in B$ and $h_1, \ldots, h_m \in \mathbb{K}[x]$ s.t.$(h_k) = (g_l)) \subset \mathbb{K}[\boldsymbol{\lambda}]$
- $\mathcal{C} = ((M_i M_j - M_j M_i)_{b,b'}$ for $b, b' \in B) \subset \mathbb{K}[\boldsymbol{\lambda}]$

Theorem

$$\mathcal{I} := \mathcal{E} + \mathcal{C} = m_{\mathsf{c}} := (\lambda_{i,\beta} - c_{i,\beta})_{i,\beta} \subset \mathbb{K}[\boldsymbol{\lambda}]$$

Let $M_i : b \in B \mapsto N(x_i \cdot B) \in B_{\boldsymbol{\lambda}} = B \otimes \mathbb{K}[\boldsymbol{\lambda}]$

- $\mathcal{E} = (N_\beta(h_k)$ for $\boldsymbol{x}^\beta \in B$ and $h_1, \ldots, h_m \in \mathbb{K}[x]$ s.t.$(h_k) = (g_l)) \subset \mathbb{K}[\boldsymbol{\lambda}]$
- $\mathcal{C} = ((M_i M_j - M_j M_i)_{b,b'}$ for $b, b' \in B) \subset \mathbb{K}[\boldsymbol{\lambda}]$

Theorem

$$\mathcal{I} := \mathcal{E} + \mathcal{C} = m_{\mathsf{c}} := (\lambda_{i,\beta} - c_{i,\beta})_{i,\beta} \subset \mathbb{K}[\boldsymbol{\lambda}]$$

☞ The vanishing of $N(h_k)$ for any set $\{h_k\}$ of generators of $I = (g_l)$ and the commutation relations define uniquely the Grobner (resp. border) basis.

Let $M_i : b \in B \mapsto N(x_i \cdot B) \in B_{\boldsymbol{\lambda}} = B \otimes \mathbb{K}[\boldsymbol{\lambda}]$

- $\mathcal{E} = (N_\beta(h_k)$ for $\boldsymbol{x}^\beta \in B$ and $h_1, \ldots, h_m \in \mathbb{K}[x]$ s.t.$(h_k) = (g_l)) \subset \mathbb{K}[\boldsymbol{\lambda}]$
- $\mathcal{C} = ((M_i M_j - M_j M_i)_{b,b'}$ for $b, b' \in B) \subset \mathbb{K}[\boldsymbol{\lambda}]$

Theorem

$$\mathcal{I} := \mathcal{E} + \mathcal{C} = m_{\mathrm{c}} := (\lambda_{i,\beta} - c_{i,\beta})_{i,\beta} \subset \mathbb{K}[\boldsymbol{\lambda}]$$

☞ The vanishing of $N(h_k)$ for any set $\{h_k\}$ of generators of $I = (g_l)$ and the commutation relations define uniquely the Grobner (resp. border) basis.

☞ **Newton's method on $\mathcal{I}$**

☞ **Numerical improvments of approximate border bases or Gröbner basis, approximate normal forms, p-adic normal forms . . .**

Isolated multiplicity (with J. Hauestein, A. Mantzaflaris, A.Szanto)

System: $f_1 = x_1 - x_2 + x_1^2$, $f_2 = x_1 - x_2 + x_2^2$

Isolated multiplicity (with J. Hauestein, A. Mantzaflaris, A.Szanto)

System: $f_1 = x_1 - x_2 + x_1^2,\ f_2 = x_1 - x_2 + x_2^2$

Parametric multiplication matrices:

$$M_1^t = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & \lambda_1 \\ 0 & 0 & 0 \end{bmatrix}, M_2^t = \begin{bmatrix} 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_3 \\ 0 & 0 & 0 \end{bmatrix}$$

Extended system:

$M_1 M_2 - M_2 M_1 = 0$	$\lambda_1 \lambda_2 - \lambda_3,$
$N(f_1) = 0$	$x_1 - x_2 + {x_1}^2,\quad 1 + 2x_1 - \lambda_2,\quad -1 + \lambda_1,$
$N(f_2) = 0$	$x_1 - x_2 + {x_2}^2,\quad 1 + (-1 + 2x_2)\lambda_2,\quad -1 + 2x_2 + \lambda_2 \lambda_3$

Isolated multiplicity (with J. Hauestein, A. Mantzaflaris, A.Szanto)

System: $f_1 = x_1 - x_2 + x_1^2,\ f_2 = x_1 - x_2 + x_2^2$

Parametric multiplication matrices:

$$M_1^t = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & \lambda_1 \\ 0 & 0 & 0 \end{bmatrix},\ M_2^t = \begin{bmatrix} 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_3 \\ 0 & 0 & 0 \end{bmatrix}$$

Extended system:

$M_1 M_2 - M_2 M_1 = 0$	$\lambda_1 \lambda_2 - \lambda_3,$
$N(f_1) = 0$	$x_1 - x_2 + {x_1}^2,\quad 1 + 2\,x_1 - \lambda_2,\quad -1 + \lambda_1,$
$N(f_2) = 0$	$x_1 - x_2 + {x_2}^2,\quad 1 + (-1 + 2\,x_2)\,\lambda_2,\quad -1 + 2\,x_2 + \lambda_2 \lambda_3$

Numerical improvements:

Iter	$[x_1, x_2, \lambda_1, \lambda_2, \lambda_3]$
0	$[0.1, 0.12, 1.1, 1.25, 1.72]$
1	$[0.0297431315, 0.0351989647, 0.9975178694, 1.0480778978, 1.0227973199]$
2	$[0.0005578682, 0.0008806394, 0.9999134370, 0.9997438194, 0.9996904740]$
3	$[0.0000001981, -0.0000001864, 0.9999999998, 1.0000002375, 1.0000002150]$
4	$[2.084095775\,10^{-14}, -1.9808984139\,10^{-14}, 1.0, 1.0000000000, 1.0000000000]$

Isolated multiplicity (with J. Hauestein, A. Mantzaflaris, A.Szanto)

System: $f_1 = x_1 - x_2 + x_1^2,\ f_2 = x_1 - x_2 + x_2^2$

Parametric multiplication matrices:

$$M_1^t = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & \lambda_1 \\ 0 & 0 & 0 \end{bmatrix},\ M_2^t = \begin{bmatrix} 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_3 \\ 0 & 0 & 0 \end{bmatrix}$$

Extended system:

$M_1 M_2 - M_2 M_1 = 0$	$\lambda_1\lambda_2 - \lambda_3,$
$N(f_1) = 0$	$x_1 - x_2 + {x_1}^2,\quad 1 + 2\,x_1 - \lambda_2,\quad -1 + \lambda_1,$
$N(f_2) = 0$	$x_1 - x_2 + {x_2}^2,\quad 1 + (-1 + 2\,x_2)\,\lambda_2,\quad -1 + 2\,x_2 + \lambda_2\lambda_3$

Numerical improvements:

Iter	$[x_1, x_2, \lambda_1, \lambda_2, \lambda_3]$
0	$[0.1, 0.12, 1.1, 1.25, 1.72]$
1	$[0.0297431315, 0.0351989647, 0.9975178694, 1.0480778978, 1.0227973199]$
2	$[0.0005578682, 0.0008806394, 0.9999134370, 0.9997438194, 0.9996904740]$
3	$[0.0000001981, -0.0000001864, 0.9999999998, 1.0000002375, 1.0000002150]$
4	$[2.084095775\,10^{-14}, -1.9808984139\,10^{-14}, 1.0, 1.0000000000, 1.0000000000]$

☞ **Quadratic convergence:** $\xi = (0,0)$, $N = [1, \partial_1 + \partial_2, \partial_2 + \frac{1}{2}\partial_1^2 + \partial_1\partial_2 + \frac{1}{2}\partial_2^2]$.

Optimization

For $f \in \mathcal{F} \subset \mathcal{C}^0(S)$ where $S \subset \mathbb{R}^n$ is a (compact) domain of $\mathbb{R}^n$.

$$f^* = \inf_{x \in S} f(x)$$

For $f \in \mathcal{F} \subset \mathcal{C}^0(S)$ where $S \subset \mathbb{R}^n$ is a (compact) domain of $\mathbb{R}^n$.

$$f^* = \inf_{x \in S} f(x)$$

which translates into

$\sup \lambda$	$\inf \Lambda(f)$
$s.t.\ \lambda \in \mathbb{R}$	$s.t.\ \Lambda(1) - 1 \in \{0\} = \mathbb{R}^\vee$
$f(x) - \lambda \in \mathrm{Pos}(S)$	$\Lambda \in \mathcal{M}(S) = \mathrm{Pos}(S)^\vee$

where

- $\mathrm{Pos}(S)$ = convex cone of functions of $\mathcal{F}$ **non-negative** on S.
- $\mathcal{M}(S) = \mathrm{Pos}(S)^\vee = \{\Lambda \in \mathcal{F}^* \mid \Lambda(\mathrm{Pos}(S)) \geq 0\}$ dual cone of **measures**.

SoS-Moment relaxation (Lasserre'2001)

For $\mathcal{S}(\mathrm{g}) = \{x \in \mathbb{R}^n \mid g_1(x) \geq 0, \ldots, g_s(x) \geq 0\} \subset \mathbb{R}^n$ with $g_i \in \mathbb{R}[\mathrm{x}]$,

$$
\begin{aligned}
f^{sos,\ell} = \sup\ & \lambda \\
s.t.\ & \lambda \in \mathbb{R} \\
& f(x) - \lambda \in \mathcal{Q}^\ell
\end{aligned}
\qquad (1)
$$

$$
\begin{aligned}
f^{mom,\ell} = \inf\ & \Lambda(f) \\
s.t.\ & \Lambda(1) - 1 \in \{0\} = \mathbb{R}^\vee \\
& \Lambda \in \mathcal{L}^\ell := (\mathcal{Q}^\ell)^\vee
\end{aligned}
\qquad (2)
$$

where

$$\Sigma^{2,\ell} = \{p = \textstyle\sum_i p_i^2, p_i \in \mathbb{R}[\mathrm{x}], \deg(p) \leq \ell\} \subset \mathrm{Pos}(\mathbb{R}^n),$$

$$\mathcal{Q}^\ell = \mathcal{Q}^\ell(\mathrm{g}) = \Sigma^{2,\ell} + g_1 \Sigma^{2,\ell-\deg(g_1)} + \cdots + g_s \Sigma^{2,\ell-\deg(g_s)} \subset \mathrm{Pos}(S)$$

is the **(truncated) quadratic module** and $\mathcal{Q}(\mathrm{g}) = \cup_\ell \mathcal{Q}^\ell(\mathrm{g})$

$$\mathcal{L}^\ell = (\mathcal{Q}^\ell)^\vee = \{\Lambda \in \mathbb{R}[\mathrm{x}]_{2\ell}^* \mid \Lambda(p) \geq 0 \forall p \in \mathcal{Q}^\ell\}$$

Minimizers of f on S

- Compute Λ^* a moment optimum of (2) by convex optimisation (SDP).
- Extract (e.g. using SVD) linearly indep. "rows" $\mathcal{N} = I_{\min}^{\perp}$ from

$$H = [\Lambda^*, x_1 \star \Lambda^*, x_2 \star \Lambda^*, \ldots, x_1^2 \star \Lambda^*, \ldots]$$
$$= [\mathrm{x}^{\alpha} \star \Lambda^*]_{\alpha \in A} = [\Lambda^*(\boldsymbol{x}^{\alpha+\beta})]_{\alpha,\beta \in A}$$
$$= \text{moment matrix of } \Lambda^* \text{ in some degree } (\ell, \ell).$$

- Compute a truncated normal form N from H
- Deduce the minimizers by eigencomputation.

Example of a parallel robot

A fixed plateform $A = [A_1, \ldots, A_6] \subset \mathbb{R}^{3\times 6}$
A moving platform $B = [B_1, \ldots, B_6] \subset \mathbb{R}^{3\times 6}$
connected by extensible arms A_i–B_i.

Displacements:

$$\widehat{B}_i = R\,B_i + T, \quad i = 1, \ldots, 6$$

where R is rotation

$$R = \begin{bmatrix} u_1^2 + u_2^2 - u_3^2 - u_4^2 & -2u_1u_4 + 2u_2u_3 & 2u_1u_3 + 2u_2u_4 \\ 2u_1u_4 + 2u_2u_3 & u_1^2 - u_2^2 + u_3^2 - u_4^2 & -2u_1u_2 + 2u_3u_4 \\ -2u_1u_3 + 2u_2u_4 & 2u_1u_2 + 2u_3u_4 & u_1^2 - u_2^2 - u_3^2 + u_4^2 \end{bmatrix}$$

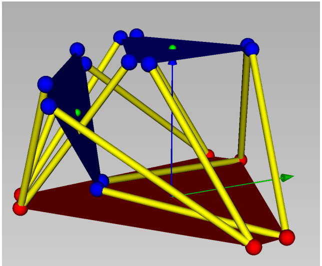
with $u_1^2 + u_2^2 + u_3^2 + u_4^2 = 1$ and $T = [x, y, z]$ the translation.

Length constraints: $m_i \le d_i := \|RB_i + T - A_i\|_2 \le M_i,\ i = 1, \ldots, 6$ with

$$d_i^2 = \|B_i\|^2 + \|A_i\|^2 + \|T\|^2 + 2R\,B_i \cdot T - 2R\,B_i \cdot A_i - 2T \cdot A_i$$

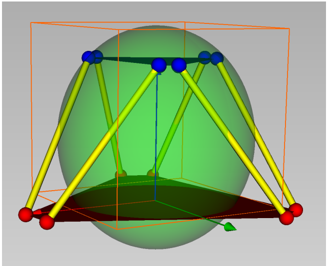
Bounding box

$$\begin{array}{lll} \min_{u_1,u_2,u_3,u_4,x,y,z} & \pm(RC+T)_k, \quad k=1,2,3 \\ \text{s.t.} & m_i^2 \le \|RB_i+T-A_i\|^2 \le M_i^2, \\ & \quad i=1,\ldots,6 \\ & u_1^2+u_2^2+u_3^2+u_4^2=1 \end{array}$$



Minimum enclosing ellipsoid

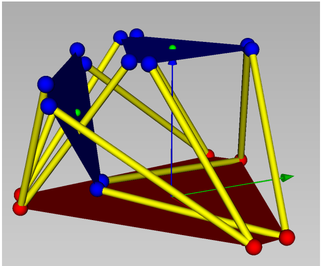
$$\begin{array}{ll} \min_{Z\in\mathbb{S}^4} & \det Z_{[2:4,2:4]} \\ \text{s.t.} & 1-\begin{bmatrix}-1\\ x\end{bmatrix}^T Z \begin{bmatrix}-1\\ x\end{bmatrix} \in Q_d(\mathrm{g}) \\ & Z\succcurlyeq 0 \end{array}$$



[Habibi-Kocvara-M'23]

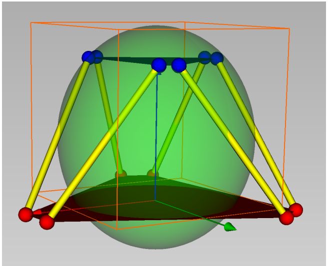
Bounding box

$$\begin{array}{lll} \min_{u_1,u_2,u_3,u_4,x,y,z} & \pm(RC+T)_k, & k=1,2,3 \\ \text{s.t.} & m_i^2 \le \|RB_i + T - A_i\|^2 \le M_i^2, & \\ & \quad i = 1,\dots,6 & \\ & u_1^2+u_2^2+u_3^2+u_4^2 = 1 & \end{array}$$



Minimum enclosing ellipsoid

$$\begin{array}{ll} \min_{Z\in\mathbb{S}^4} & \det Z_{[2:4,2:4]} \\ \text{s.t.} & 1 - \begin{bmatrix} -1 \\ x \end{bmatrix}^T Z \begin{bmatrix} -1 \\ x \end{bmatrix} \in Q_d(\mathbf{g}) \\ & Z \succcurlyeq 0 \end{array}$$



[Habibi-Kocvara-M'23]

- Extensions to compute real radical [Baldi-M' 21]

Roots from moment matrices

Consider I s.t. **$I^\perp$ is generated by a single element σ**:

$$I^\perp = \langle \sigma, x_1 \star \sigma, \ldots, \boldsymbol{x}^\alpha \star \sigma, \ldots \rangle =: \langle\langle \sigma \rangle\rangle$$

where $p \star \sigma : q \mapsto \sigma(p\, q)$ (i.e. $\mathcal{A} = R/I$ Gorenstein Algebra).

Example: If I defines r simple points $\zeta_1, \ldots, \zeta_r$, and $\sigma = \sum_i \mathrm{e}_{\zeta_i}$ then

$$I^\perp = \langle\langle \sigma \rangle\rangle = \langle \mathrm{e}_{\zeta_1}, \ldots, \mathrm{e}_{\zeta_r} \rangle$$

When $\sigma \in (V \cdot V)^*$ with $V = \langle v_1, \ldots, v_s \rangle$, we have

$$H_\sigma^{V,V} : p \in V \mapsto [v_1 \star \sigma(p), \ldots, v_s \star \sigma(p)] \in \mathbb{K}^s$$

Hankel operator with matrix $(\sigma(v_i\, v_j))$.

Flat extensions and truncated normal form

For (monomial) sets $B \subset C$, $B' \subset C'$, $\overline{B} = C \setminus B$, $\overline{B}' = C' \setminus B'$.

Flat extensions and truncated normal form

For (monomial) sets $B \subset C$, $B' \subset C'$, $\overline{B} = C \setminus B$, $\overline{B}' = C' \setminus B'$.

$$\mathsf{H}_\sigma^{\mathsf{C},\mathsf{C}'} = \left[\begin{array}{c|c} \mathsf{H}_\sigma^{\mathsf{B},\mathsf{B}'} & H_\sigma^{B,\overline{B}'} \\ \hline H_\sigma^{\overline{B},B'} & H_\sigma^{\overline{B},\overline{B}'} \end{array}\right]$$

Flat extension when

$$\text{rank } \mathsf{H}_\sigma^{\mathsf{C},\mathsf{C}'} = \text{rank } \mathsf{H}_\sigma^{\mathsf{B},\mathsf{B}'}$$

Flat extensions and truncated normal form

For (monomial) sets $B \subset C$, $B' \subset C'$, $\overline{B} = C \setminus B$, $\overline{B}' = C' \setminus B'$.

$$\mathrm{H}_\sigma^{\mathrm{C},\mathrm{C}'} = \left[\begin{array}{c|c} \mathrm{H}_\sigma^{\mathrm{B},\mathrm{B}'} & H_\sigma^{B,\overline{B}'} \\ \hline H_\sigma^{\overline{B},B'} & H_\sigma^{\overline{B},\overline{B}'} \end{array}\right]$$

Flat extension when

$$\text{rank } \mathrm{H}_\sigma^{\mathrm{C},\mathrm{C}'} = \text{rank } \mathrm{H}_\sigma^{\mathrm{B},\mathrm{B}'}$$

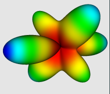
Theorem

Assume $H_\sigma^{B,B'}$ invertible with $|B| = |B'| = r$ and $C \supset B^+$, $C' \supset B'^+$ connected to 1. The following points are equivalent:

- $H_\sigma^{C,C'}$ *is a* **flat extension** *of* $H_\sigma^{B,B'}$
- *The operators* $M_j := H_\sigma^{B,x_j B'} (H_\sigma^{B,B'})^{-1}$ **commute**.
- $H_\sigma^{B,C'}$ *is a* **Truncated Normal Form** *for* $I = \ker(H_\sigma^{C,C'})$.
- $\exists!\, \overline{\sigma} \in \mathrm{R}^*$ *s.t.* $\overline{\sigma}_{|C\cdot C'} = \sigma$ *and* $I = \ker H_{\overline{\sigma}}^{C,R}$ *defines* $\mathcal{A} = R/I$ *of* $\dim \mathcal{A} = r$.

[Laurent-M'09;Brachat-Comon-Tsigaridas'10; Brachat-Bernardi-Comon-M'11. . .]

Symmetric tensor decomposition



$$\psi = (x_0 + 3x_1 - x_2)^4 + (x_0 + x_1 + x_2)^4 - 3\,(x_0 + 2x_1 + 2x_2)^4$$

$$= -{x_0}^4 - 24\,{x_0}^3 x_2 - 8\,{x_0}^3 x_1 - 60\,{x_0}^2 {x_2}^2 - 168\,{x_0}^2 x_1 x_2 - 12\,{x_0}^2 {x_1}^2$$
$$-96\,x_0 {x_2}^3 - 240\,x_0 x_1 x_2^2 - 384\,x_0 x_1^2 x_2 + 16\,x_0 {x_1}^3 - 46\,{x_2}^4 - 200\,x_1 x_2^3$$
$$-228\,{x_1}^2 {x_2}^2 - 296\,x_1^3 x_2 + 34\,{x_1}^4$$

$$\langle \psi, p \rangle_4 = \langle \psi^* | p \rangle \text{ where } \psi^* = e_{(3,-1)} + e_{(1,1)} - 3\,e_{(2,2)} \quad (\textit{by apolarity})$$

$$H^{2,2}_{\psi^*} := \begin{bmatrix} -1 & -2 & -6 & -2 & -14 & -10 \\ -2 & -2 & -14 & 4 & -32 & -20 \\ -6 & -14 & -10 & -32 & -20 & -24 \\ \hline -2 & 4 & -32 & 34 & -74 & -38 \\ -14 & -32 & -20 & -74 & -38 & -50 \\ -10 & -20 & -24 & -38 & -50 & -46 \end{bmatrix}$$

For $B' = \{x_0, x_1, x_2\}, B = x_0 B'$

$$H_\psi^{B, x_0 B'} = \begin{bmatrix} -1 & -2 & -6 \\ -2 & -2 & -14 \\ -6 & -14 & -10 \end{bmatrix}$$

$$H_\psi^{B, x_1 B'} = \begin{bmatrix} -2 & -2 & -14 \\ -2 & 4 & -32 \\ -14 & -32 & -20 \end{bmatrix}$$

$$H_\psi^{B, x_2 B'} = \begin{bmatrix} -6 & -14 & -10 \\ -14 & -32 & -20 \\ -10 & -20 & -24 \end{bmatrix}$$

- The matrix of multiplication by x_2 in $B' = \{x_0, x_1, x_2\}$ is

$$M_2 = (H_\psi^{B, x_0 B'})^{-1} H_\psi^{B, x_2 B'} = \begin{bmatrix} 0 & -2 & -2 \\ 0 & \frac{1}{2} & \frac{3}{2} \\ 1 & \frac{5}{2} & \frac{3}{2} \end{bmatrix}.$$

- The matrix of multiplication by x_2 in $B' = \{x_0, x_1, x_2\}$ is

$$M_{\mathbf{2}} = (H_{\psi}^{B,x_{\mathbf{0}}B'})^{-\mathbf{1}} H_{\psi}^{B,x_{\mathbf{2}}B'} = \begin{bmatrix} 0 & -2 & -2 \\ 0 & \frac{1}{2} & \frac{3}{2} \\ 1 & \frac{5}{2} & \frac{3}{2} \end{bmatrix}.$$

- Its eigenvalues are $[-1, 1, 2]$ and the eigenvectors:

$$U := \begin{bmatrix} 0 & -2 & -1 \\ \frac{1}{2} & \frac{3}{4} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}.$$

that is the polynomials

$$U(x) = \begin{bmatrix} \frac{1}{2}x_1 - \frac{1}{2}x_2 & -2 + \frac{3}{4}x_1 + \frac{1}{4}x_2 & -1 + \frac{1}{2}x_1 + \frac{1}{2}x_2 \end{bmatrix}.$$

- The matrix of multiplication by x_2 in $B' = \{x_0, x_1, x_2\}$ is

$$M_{\mathbf{2}} = (H_{\psi}^{B,x_{\mathbf{0}}B'})^{-\mathbf{1}} H_{\psi}^{B,x_{\mathbf{2}}B'} = \begin{bmatrix} 0 & -2 & -2 \\ 0 & \frac{1}{2} & \frac{3}{2} \\ 1 & \frac{5}{2} & \frac{3}{2} \end{bmatrix}.$$

- Its eigenvalues are $[-1, 1, 2]$ and the eigenvectors:

$$U := \begin{bmatrix} 0 & -2 & -1 \\ \frac{1}{2} & \frac{3}{4} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}.$$

that is the polynomials

$$U(x) = \left[\ \frac{1}{2}x_1 - \frac{1}{2}x_2 \quad -2 + \frac{3}{4}x_1 + \frac{1}{4}x_2 \quad -1 + \frac{1}{2}x_1 + \frac{1}{2}x_2\ \right].$$

- We deduce the weights and the frequencies:

$$H_{\psi}^{[\mathbf{1},x_{\mathbf{1}},x_{\mathbf{2}}],U} = \begin{bmatrix} 1 & 1 & -3 \\ 1 \times 3 & 1 \times 1 & -3 \times 2 \\ 1 \times -1 & 1 \times 1 & -3 \times 2 \end{bmatrix}$$

Weights: $1, 1, -3$;
Frequencies: $(-1, 3), (1, 1), (2, 2)$.

Decomposition:

$$\mu = \mathrm{e}_{(3,-1)} + \mathrm{e}_{(1,1)} - 3\,\mathrm{e}_{(2,2)}$$
$$\psi(\boldsymbol{x}) = (x_0 + 3x_1 - x_2)^4 + (x_0 + x_1 + x_2)^4 - 3(x_0 + 2x_1 + 2x_2)^4$$

Gaussian Mixtures and tensor decomposition

- Compute the **empirical expectation**: $\tilde{\mathbb{E}}(y_k) = \frac{1}{N}\sum_i y_{i,k}$,

$\tilde{\mathbb{E}}(y_k y_l) = \frac{1}{N}\sum_i y_{i,k} y_{i,l}$, $\tilde{\mathbb{E}}(y_k y_l y_m) = \frac{1}{N}\sum_i y_{i,k} y_{i,l} y_{i,m}$,

- Deduce $\tilde{M}_1$, $\tilde{M}_2$, $\tilde{M}_3$.

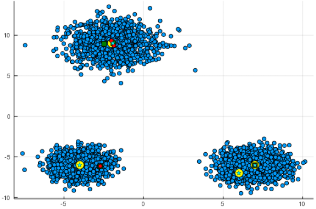
- Decompose $\tilde{M}_3$ and deduce an approximation of the means μ_j, covariance σ_j^2, $j \in [r]$.

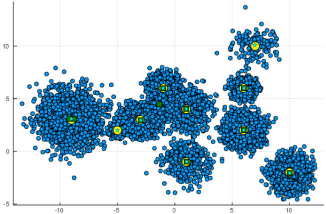
- Compute the **empirical expectation**: $\tilde{\mathbb{E}}(y_k) = \frac{1}{N}\sum_i y_{i,k}$,

$\tilde{\mathbb{E}}(y_k y_l) = \frac{1}{N}\sum_i y_{i,k} y_{i,l}$, $\tilde{\mathbb{E}}(y_k y_l y_m) = \frac{1}{N}\sum_i y_{i,k} y_{i,l} y_{i,m}$,

- Deduce $\tilde{M}_1$, $\tilde{M}_2$, $\tilde{M}_3$.

- Decompose $\tilde{M}_3$ and deduce an approximation of the means μ_j, covariance σ_j^2, $j \in [r]$.





Examples with $n = 6, r = 4$; $n = 30, r = 10$

[Khouja-Mattei-M'22]

Thanks for your attention

TENORS

Tensor modEliNg, geOmetRy and optimiSation

Marie Skłodowska-Curie Doctoral Network

2024-2027

Tensors are nowadays ubiquitous in many domains of applied mathematics, computer science, signal processing, data processing, machine learning and in the emerging area of quantum computing. TENORS aims at fostering cutting-edge research in tensor sciences, stimulating interdisciplinary and intersectoriality knowledge developments between algebraists, geometers, computer scientists, numerical analysts, data analysts, physicists, quantum scientists, and industrial actors facing real-life tensor-based problems.

Partners:

1. Inria, Sophia Antipolis, France (B. Mourrain, A. Mantzaflaris)
2. CNRS, LAAS, Toulouse, France (D. Henrion, V. Magron, M. Skomra)
3. NWO-I/CWI, Amsterdam, the Netherlands (M. Laurent)
4. Univ. Konstanz, Germany (M. Schweighofer, S. Kuhlmann, M. Michałek)
5. MPI, Leipzig, Germany (B. Sturmfels, S. Telen)
6. Univ. Tromsoe, Norway (C. Riener, C. Bordin, H. Munthe-Kaas)
7. Univ. degli Studi di Firenze, Italy (G. Ottaviani)
8. Univ. degli Studi di Trento, Italy (A. Bernardi, A. Oneto, I. Carusotto)
9. CTU, Prague, Czech Republic (J. Marecek)
10. ICFO, Barcelona, Spain (A. Acin)
11. Artelys SA, Paris, France (M. Gabay)

Associate partners:

1. Quandela, France
2. Cambridge Quantum Computing, UK.
3. Bluetensor, Italy.
4. Arva AS, Norway.
5. HSBC Lab., London, UK.

☞ 15 PhD positions (2024-2027)

(recruitment expected around Oct. 2024)
https://tenors-network.eu/

Scientific coord: B. Mourrain
Adm. manager: Linh Nguyen