Solving polynomial systems via determinantal representations

Michiel Hochstenbach

TU Eindhoven

KU Leuven

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Joint work with

Bor Plestenjak	(Ljubljana)
Ada Boralevi	(Torino)
Jan Draisma	(Basel, Eindhoven)
Jasper van Doornmalen	(Eindhoven)

Nice to be here!

NLA top citations

[De Moor Bart]

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Outline

Polynomial systems via determinantal representations

- Zeros of $p(x) = 0$
- Zeros of $p(x, y) = 0, \;\; q(x, y) = 0$

  Determinantal representations
- Two-parameter eigenvalue problem
- Connections with interesting long-standing open problems! 1902!
- Solving singular generalized eigenvalue problems

Related talk by Bor Plestenjak on 1 December

Preview example

$$p(x,y) = 1 + 2x + 3y + 4x^2 + 5xy + 6y^2 + 7x^3 + 8x^2y + 9xy^2 + 10y^3$$

then $p(x,y) = \det(A + x\,B + y\,C)$:

$$\underbrace{\begin{bmatrix} 1 & 3 & 6 & & \\ 2 & 5 & & -1 & \\ 4 & & & & -1 \\ & -1 & & & \\ & & -1 & & \end{bmatrix}}_{A} + x \underbrace{\begin{bmatrix} & & 9 & 1 & \\ & 8 & & & 1 \\ 7 & & & & \\ & & & & \\ & & & & \end{bmatrix}}_{x\,B} + y \underbrace{\begin{bmatrix} & & 10 & & \\ & & & & \\ & & & & \\ 1 & & & & \\ & 1 & & & \end{bmatrix}}_{y\,C}$$

- For $n = 2$ variables (x and y)
- a degree d polynomial
- leads to a size $2d - 1$ uniform determinantal representation

"Uniform": all coefficients affine-linear: of the form $\gamma_0 + \gamma_1 x + \gamma_2 y$

Simple form of main question

$p(x) = x^3 + a_2x^2 + a_1x + a_0$

Companion matrix

$$A = \begin{bmatrix} a_2 & a_1 & a_0 \\ -1 & & \\ & -1 & \end{bmatrix}$$

satisfies $\det(A + xI) = p(x)$ with "perfect size" of A: 3

How about

$p(x, y) = a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{10}x + a_{01}y + a_0$

Do there exist A, B, C with $\det(A + xB + yC) = p(x, y)$?

And what about the minimal size of these matrices ?

Zeros of polynomial in 1 variable

$$p(x) = x^d + a_{d-1}x^{d-1} + \cdots + a_1 x + a_0 = 0$$

Recall: differential equation $y''(t) + y(t) = 0$

Rewrite higher-order ODE to first-order:
Introduce $z = y'$: then $z' = y'' = -y$ and:

$$\begin{bmatrix} y \\ z \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix}$$

For polynomials, practical to introduce:

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{bmatrix} = \begin{bmatrix} 1 \\ x \\ \vdots \\ x^{d-1} \end{bmatrix}$$

or in the opposite order

Zeros of polynomial in 1 variable

$p(x) = x^d + a_{d-1}x^{d-1} + \cdots + a_1 x + a_0 = 0$

Companion matrix approach:

$$\underbrace{\begin{bmatrix} -a_{d-1} & \cdots & -a_1 & -a_0 \\ 1 & & & \\ & \ddots & & \\ & & 1 & \end{bmatrix}}_{A} \begin{bmatrix} x^{d-1} \\ \vdots \\ x \\ 1 \end{bmatrix} = x \begin{bmatrix} x^{d-1} \\ \vdots \\ x \\ 1 \end{bmatrix}$$

Since:

- Zeros of p are eigenvalues of A
- Coefficient of $p(x) = \det(A - xI)$ is ± 1

Conclusion: A is a determinantal representation of p:
$p(x) = \pm \det(A - xI)$

Also the term linearization is frequently used, although in a different sense than in Calculus: here exact, no approximation

Zeros of polynomial in 1 variable

$p(x) = x^d + a_{d-1}x^{d-1} + \cdots + a_1 x + a_0 = 0$

Companion matrix approach:

$$\underbrace{\begin{bmatrix} -a_{d-1} & \cdots & -a_1 & -a_0 \\ 1 & & & \\ & \ddots & & \\ & & 1 & \end{bmatrix}}_{A} \begin{bmatrix} x^{d-1} \\ \vdots \\ x \\ 1 \end{bmatrix} = x \begin{bmatrix} x^{d-1} \\ \vdots \\ x \\ 1 \end{bmatrix}$$

x solution to $p(x) = \det(A - xI) = 0$

$\implies$ Can solve matrix eigenvalue problem $A\,\boldsymbol{u} = x\,\boldsymbol{u}$

Zeros often computed in this way, e.g., Matlab's `roots`

```
% Polynomial roots via a companion matrix
n = length(c);
A = diag(ones(1,n-2,c),-1);
A(1,:)  = -d;
r = eig(A);
```

Determinantal representation in 1 variable

$$\begin{bmatrix} -a_2 & -a_1 & -a_0 \\ 1 & & \\ & 1 & \end{bmatrix} \begin{bmatrix} x^2 \\ x \\ 1 \end{bmatrix} = x \begin{bmatrix} x^2 \\ x \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} a_2 & a_1 & a_0 \\ -1 & & \\ & -1 & \end{bmatrix} \begin{bmatrix} x^2 \\ x \\ 1 \end{bmatrix} = -x \begin{bmatrix} x^2 \\ x \\ 1 \end{bmatrix}$$

$$\det \begin{bmatrix} x + a_2 & a_1 & a_0 \\ -1 & x & \\ & -1 & x \end{bmatrix} = p(x)$$

Other way to see this:

$$\begin{bmatrix} x + a_2 & a_1 & a_0 \\ -1 & x & \\ & -1 & x \end{bmatrix} \begin{bmatrix} 1 & x & \\ & 1 & \\ & & 1 \end{bmatrix} = \begin{bmatrix} x + a_2 & x^2 + a_2 x + a_1 & a_0 \\ -1 & & \\ & -1 & x \end{bmatrix}$$

$$\begin{bmatrix} x + a_2 & x^2 + a_2 x + a_1 & a_0 \\ -1 & & \\ & -1 & x \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & x \\ & & 1 \end{bmatrix} =$$

$$\begin{bmatrix} x + a_2 & x^2 + a_2 x + a_1 & x^3 + a_2 x^2 + a_1 x + a_0 \\ -1 & & \\ & -1 & \end{bmatrix}$$

Determinantal representation in 1 variable

Determinantal representation / linearization of previous slide:

$$\begin{bmatrix} x + a_2 & a_1 & a_0 \\ -1 & x & \\ & -1 & x \end{bmatrix}$$

Slightly more general / flexible:

$$\begin{bmatrix} a_0 & a_1 & a_3 x + a_2 \\ x & -1 & \\ & x & -1 \end{bmatrix}$$

$\det = a_3 x^3 + a_2 x^2 + a_1 x + a_0$

Now:

- We can include the a_3
- With column actions, $p(x)$ appears on position (1,1)

Determinantal representation in 1 variable

Determinantal representation / linearization :

$$\begin{bmatrix} a_0 & a_1 & a_3x + a_2 \\ x & -1 & \\ & x & -1 \end{bmatrix} = \underbrace{\begin{bmatrix} a_0 & a_1 & a_2 \\ & -1 & \\ & & -1 \end{bmatrix}}_{A} + x \cdot \underbrace{\begin{bmatrix} & & a_3 \\ 1 & & \\ & 1 & \end{bmatrix}}_{B}$$

So key to efficient method: construction of A and B with

$p(x) = \det(A + xB)$

Size of A and B $=$ degree of polynomial d

Solving eigenvalue problem takes $\mathcal{O}(d^3)$ flops

"Ideal situation":

- Representation should be of size $\geq d$
- ... and d is also sufficient

Outline

Polynomial systems via determinantal representations

- Zeros of $p(x) = 0$

$\Rightarrow$ Zeros of $p(x, y) = 0, \; q(x, y) = 0$

Determinantal representations

- Two-parameter eigenvalue problem
- Connections with interesting long-standing open problems! 1902!
- Solving singular generalized eigenvalue problems

Zeros of 2 polynomial in 2 variables

$$p(x,y) = a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 + a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{10}x + a_{01}y + a_0$$

$$q(x,y) = b_{30}x^3 + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3 + b_{20}x^2 + b_{11}xy + b_{02}y^2 + b_{10}x + b_{01}y + b_0$$

Bézout's Theorem

System with 2 polynomials of degree d generically has d^2 roots (including multiplicity, roots may be ∞)

Determinantal representation: find:

- A_1, B_1, C_1 with $p(x,y) = \det(A_1 - xB_1 - yC_1)$
- A_2, B_2, C_2 with $q(x,y) = \det(A_2 - xB_2 - yC_2)$

Then this leads to two-parameter eigenvalue problem

$$\begin{aligned} A_1\boldsymbol{u} &= x\,B_1\boldsymbol{u} + y\,C_1\boldsymbol{u} \\ A_2\boldsymbol{v} &= x\,B_2\boldsymbol{v} + y\,C_2\boldsymbol{v} \end{aligned}$$

Size of matrices should be as small as possible!

Big question: is $d \times d$ possible ??

Size of determinantal representation

What is the minimal size of a determinantal representation?

Question solved for $p(x)$:

- Degree of $p(x) = \det(A - xI)$ is $\leq d$ in x
  so size should be $\geq d$
- Companion matrix is a construction of size d that does the job

However, for $p(x, y)$ the situation is already much more complex!

- Dixon (1902):
  $\exists$ **symmetric determinantal representation of size** d
- . . . but no explicit construction . . .
- Open question since 1902!
- **Plestenjak (2017)**: construction of nonsymmetric linearization of size d, involving some computations
- Fast, but roots may be (very) inaccurate for $d > 10$

Size representations

- Dixon (1902!): $\exists$ symmetric linearization of size d
  However, not constructive!
- Quarez (2007): symmetric linearization of size $\frac{1}{4}d^2$
- Plestenjak, H. (2015): nonsymmetric linearization of size $\frac{1}{4}d^2$ or $\frac{1}{6}d^2$ with some minor computations
- Boralevi, Van Doornmalen, Draisma, H., Plestenjak (2017): nonsymmetric linearization, size $2d-1$ without computations
- Open problem if 2 can be improved (without computations), but conjecture is: impossible
- Plestenjak (2017): size d, nonsymmetric, with computations

Determinantal representation in 2 variables

$p(x, y) = a_{10}x + a_{01}y + a_{00}$: already is a 1×1 representation

$p(x, y) = a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{10}x + a_{01}y + a_{00}$

$$\det \begin{bmatrix} a_{10}x + a_{01}y + a_{00} & a_{20}x + a_{11}y & y \\ x & -1 & \\ a_{02}y & & -1 \end{bmatrix} = p(x, y)$$

$$\sim \begin{bmatrix} a_{20}x^2 + a_{11}xy + a_{10}x + a_{01}y + a_{00} & a_{20}x + a_{11}y & y \\ & -1 & \\ a_{02}y & & -1 \end{bmatrix}$$

$$\sim \begin{bmatrix} a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{10}x + a_{01}y + a_{00} & a_{20}x + a_{11}y & \\ & -1 & \\ a_{02}y & & -1 \end{bmatrix}$$

In a similar way we can construct size $2d - 1$ linearization for degree d, without computations!

Just insert the coefficients on certain locations

Construction without computations

$$\begin{bmatrix} a_{10}x + a_{01}y + a_{00} & a_{20}x + a_{11}y & y \\ x & -1 & \\ a_{02}y & & -1 \end{bmatrix} =: A + xB + yC$$

$$A = \begin{bmatrix} a_{00} & & \\ & -1 & \\ & & -1 \end{bmatrix}, \quad B = \begin{bmatrix} a_{10} & a_{20} & \\ 1 & & \\ & & \end{bmatrix}, \quad C = \begin{bmatrix} a_{01} & a_{11} & 1 \\ & & \\ a_{02} & & \end{bmatrix}$$

So, unfortunately, in contrast to Dixon (1902) this linearization:

- is not of size d but $2d - 1$
- is not symmetric

But it is "uniform" (so no computations)

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  Determinantal representations

$\Rightarrow$ Two-parameter eigenvalue problem

- Connections with interesting long-standing open problems! 1902!
- Solving singular generalized eigenvalue problems

Two-parameter eigenvalue problem

So we have rewritten:

$$p(x, y) = 0 \quad \rightarrow \quad \det(A_1 + x\,B_1 + y\,C_1) = 0$$

$$q(x, y) = 0 \quad \rightarrow \quad \det(A_2 + x\,B_2 + y\,C_2) = 0$$

To solve this: two-parameter eigenvalue problem

This increases size from $2d - 1$ to $(2d - 1)^2$

Two-parameter eigenvalue problem

$$\begin{aligned} A_1 \boldsymbol{u} &= \lambda B_1 \boldsymbol{u} + \mu C_1 \boldsymbol{u} \\ A_2 \boldsymbol{v} &= \lambda B_2 \boldsymbol{v} + \mu C_2 \boldsymbol{v} \end{aligned}$$

It is known how to solve this: Atkinson 1972

$$\begin{aligned} \Delta_0 &= B_1 \otimes C_2 - C_1 \otimes B_2 \qquad \text{operator determinants} \\ \Delta_1 &= A_1 \otimes C_2 - C_1 \otimes A_2 \\ \Delta_2 &= B_1 \otimes A_2 - A_1 \otimes B_2 \end{aligned}$$

$$\begin{aligned} &(A_1 \otimes C_2 - C_1 \otimes A_2)(\boldsymbol{u} \otimes \boldsymbol{v}) \\ &= (\lambda B_1 \otimes C_2 - \mu C_1 \otimes C_2 - \lambda C_1 \otimes B_2 + \mu C_1 \otimes C_2)(\boldsymbol{u} \otimes \boldsymbol{v}) \\ &= \lambda (B_1 \otimes C_2 - C_1 \otimes B_2)(\boldsymbol{u} \otimes \boldsymbol{v}) \end{aligned}$$

$$\begin{aligned} &(B_1 \otimes A_2 - A_1 \otimes B_2)(\boldsymbol{u} \otimes \boldsymbol{v}) \\ &= (\lambda B_1 \otimes B_2 + \mu B_1 \otimes C_2 - \lambda B_1 \otimes B_2 - \mu C_1 \otimes B_2)(\boldsymbol{u} \otimes \boldsymbol{v}) \\ &= \mu (B_1 \otimes C_2 - C_1 \otimes B_2)(\boldsymbol{u} \otimes \boldsymbol{v}) \end{aligned}$$

So: $\Delta_1 \boldsymbol{z} = \lambda \Delta_0 \boldsymbol{z}, \ \Delta_2 \boldsymbol{z} = \mu \Delta_0 \boldsymbol{z}, \qquad \boldsymbol{z} = \boldsymbol{u} \otimes \boldsymbol{v}$

Two-parameter eigenvalue problem

Pro and con:

- Decoupling to 2 generalized eigenvalue problems with same eigenvectors:
  - $\Delta_1(\boldsymbol{u} \otimes \boldsymbol{v}) = \lambda \, \Delta_0(\boldsymbol{u} \otimes \boldsymbol{v})$
  - $\Delta_2(\boldsymbol{u} \otimes \boldsymbol{v}) = \mu \, \Delta_0(\boldsymbol{u} \otimes \boldsymbol{v})$
- . . . but of size n^2 !
  Solving these takes $\mathcal{O}(n^6)$ operations
  These costs are common for solving $p(x, y) = 0$, $q(x, y) = 0$

Effect of linearization being "too large"

Example: $p(x,y)$, $q(x,y)$ polynomials of degree 10

Bézout: 100 solutions

- Ideal determinantal representations: 10×10 (Dixon)
  Δ matrices in $\Delta_1 \boldsymbol{w} = \lambda\, \Delta_0 \boldsymbol{w}$ and $\Delta_2 \boldsymbol{w} = \mu\, \Delta_0 \boldsymbol{w}$: 100×100
- $\frac{1}{4}d^2$ determinantal representations: 35×35
  Δ matrices 1225×1225
- $\frac{1}{6}d^2$ determinantal representations: 24×24
  Δ matrices 576×576
- $2d-1$ determinantal representations: 19×19
  Δ matrices 361×361

Recall: eigenvalue problem takes $\mathcal{O}(\ell^3)$ work for size ℓ

So before representations of $\mathcal{O}(d)$, this took $\mathcal{O}(d^{12})$ work!

p degree $d \Rightarrow$ detrep size $\mathcal{O}(d^2) \Rightarrow \Delta$ size $\mathcal{O}(d^4) \Rightarrow$ work $\mathcal{O}(d^{12})$

Or even more because of "iterative shrinking" of matrices

Effect of linearization being “too large”

Apart from the work (100 vs 361), there is an even bigger challenge:

Unless the determinantal representation is of perfect size,
the GEPs $\Delta_1 \boldsymbol{z} = \lambda\, \Delta_0 \boldsymbol{z}$ and $\Delta_2 \boldsymbol{z} = \mu\, \Delta_0 \boldsymbol{z}$ are both singular

I.e.: pencil $\Delta_1 - \lambda \Delta_0$ singular for all $\lambda \in \mathbb{C}$

More about this soon

Sizes for $n = 2$ variables and degree d

Sizes determinantal representations for $p(x, y)$ with degree d

Degree		3	4	5	6	7	8	9	10	11	12
`Lin1`	$\frac{1}{4}d^2$	5	8	11	15	19	24	29	35	41	48
`Lin2`	$\frac{1}{6}d^2$	3	5	8	10	13	17	20	24	29	34
`MinUnif`	$2d - 1$	5	7	9	11	13	15	17	19	21	23

Recall: for $d = 10$ this gives Δ-matrices of size

$35^2 = 1225, \quad 24^2 = 576, \quad 19^2 = 361$

And costs to solve eigenproblem are at least cube of this

Sizes for n variables and degree d

n	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$	$d=8$	$d=9$
2	3	5	7	9	11	13	15	17
3	4	7	10	14	18	22	27	34
4	5	9	14	19	26	34	44	
5	6	11	18	26				
6	7	13	22	33				
7	8	15	27	39				
8	9	17	32					

- Given p in n variables and degree d, this is the smallest known size of linearization A, B, C with $p(x,y) = \det(A - xB - yC)$
- For $n \geq 3$ NP-hard problem (e.g., Turán number) so already for $p(x,y,z) = 0$, $q(x,y,z) = 0$, $r(x,y,z) = 0$

Sizes for n variables and degree d

n	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$	$d=8$	$d=9$
2	3	5	7	9	11	13	15	17
3	4	7	10	14	18	22	27	34
4	5	9	14	19	26	34	44	
5	6	11	18	26				
6	7	13	22	33				
7	8	15	27	39				
8	9	17	32					

Example: p and q, degree $d = 8$ in x and y (so $n = 2$):

- Linearizations of size $2d - 1 = 15$ $(A_1, B_1, C_1, A_2, B_2, C_2)$
- Δ matrices of size $d^2 = 15^2 = 225$
  $\Delta_0 = B_1 \otimes C_2 - C_1 \otimes B_2$, $\Delta_1 = B_1 \otimes C_2 - C_1 \otimes B_2$
- Eigenvalue problem takes $\mathcal{O}((d^2)^3) = \mathcal{O}(d^6)$ work
- Pencil (Δ_1, Δ_0) is singular; size 225 but only 64 solutions
- In past, best linearization size was $\frac{1}{4}d^2$!
  This is already very encouraging!

Difficulty use for $n \geq 3$ variables: $p(x, y, z) = 0, \ldots$

n	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$	$d=8$	$d=9$
2	3	5	7	9	11	13	15	17
3	4	7	**10**	14	18	22	27	34
4	5	**9**	14	19	26	34	44	

Example: 3 variables x, y, z $\quad p(x, y, z)$ degree $d = 4$

2 variables: tensor product $\Delta_0 = B_1 \otimes C_2 - C_1 \otimes B_2$

3 variables: tensor products with terms such as
$B_1 \otimes C_2 \otimes D_3$: size $10^3 = 1000$!

So $\Delta_1 \boldsymbol{u} = \lambda \, \Delta_0 \boldsymbol{u}$ gives 1000 solutions
but the polynomial system has only $4^3 = 64$ roots

Example: 4 variables x, y, z, w $\quad p(x, y, z, w)$ degree $d = 3$

So Δ matrices size $9^4 = 6561$, very expensive! Only $3^4 = 81$ roots!

Experiments

Average milliseconds for Lin1, Lin2, MinUnif, and PHCLab for random bivariate polynomial systems of degree 3 to 15

For Lin1 and MinUnif results are split for real/complex polynomials.

d	Lin1 ($\mathbb{R}$) $\frac{1}{4}d^2$	Lin1 ($\mathbb{C}$) $\frac{1}{4}d^2$	Lin2 $\frac{1}{6}d^2$	PHCLab	MinUnif ($\mathbb{R}$) $2d-1$	MinUnif ($\mathbb{C}$) $2d-1$
3	6	8	4	116	6	7
4	9	11	6	130	12	13
5	20	26	13	151	18	20
6	39	71	28	174	27	27
7	96	160	51	217	36	44
8	205	395	118	264	59	74
9	467	1124	279	329	95	125
10	1424	3412	600	414	147	221
11				538	248	354
12				650	361	530
13				911	592	740
14				1142	842	1148
15				1531	1237	1835

Asymptotic size: work with algebra colleagues

Boralevi, Van Doornmalen, Draisma, H., Plestenjak (2017)

Minimal size of any uniform determinantal representation:

Fixed n, $d \to \infty$: size $\sim \sqrt{d^n}$
previously known upper bound: $\mathcal{O}(d^n)$

Fixed d, $n \to \infty$: size $\sim \sqrt{n^d}$

In particular, for $n = 2$ (2 variables x and y), degree $d \to \infty$, size $\sim d$

The best uniform size we have been able to find is $2d - 1$

Is smaller size: $d \leq \text{size} < 2d - 1$ possible??

E.g.: Degree 10 polynomials, size 19 is best currently known

This leads to $19^2 = 361$ matrices for the MEP, but because of Alsubaie compression, this is reduced to 316

Singular generalized eigenvalue problem (GEP)

Another very interesting aspect:

GEPs $\Delta_1 \boldsymbol{z} = \lambda \, \Delta_0 \boldsymbol{z}$ and $\Delta_2 \boldsymbol{z} = \mu \, \Delta_0 \boldsymbol{z}$ are both singular

I.e.: pencil $\Delta_1 - \lambda \Delta_0$ singular for all $\lambda \in \mathbb{C}$

Introduce normal rank: $\mathrm{nrank}(A, B) = \max_{\zeta \in \mathbb{C}} \mathrm{rank}(A - \zeta B)$
λ is an eigenvalue if $\mathrm{rank}(A - \lambda B) < \mathrm{nrank}(A, B)$

Solution methods:

- `eig` fails
- Staircase method: `guptri`
  (Van Dooren 1979; Demmel, Kagstrom 1993)
  iteratively "cutting away" singular part of pencil,
  may be very time-consuming
- Alternative: rank-completing perturbation
  H., Mehl, Plestenjak (2019)

Singular generalized eigenvalue problem (GEP)

Rank-completing perturbation, main ideas:

- Update $(A, B) \rightarrow (A + E,\ B + F)$, random perturbations
  Generically, new pencil is no longer singular
  However, we perturb all eigenvalues
  Very difficult to see which new eigenvalues corresponds to true eigenvalues

- Update $(A, B) \rightarrow (A + \tau\, U\, D_A\, V^*,\ \ B + \tau\, U\, D_B\, V^*)$
  $U,\ V \in \mathbb{R}^{n \times k}$: rank $k = n - \mathtt{nrank}(A, B)$
  $D_A,\ D_B$: diagonal $k \times k$ (prescribe ourselves)
  random rank-completing perturbation
  Perturbation **just** enough such that new pencil is nonsingular
  and we do not touch original true eigenvalues

Determinantal representations and singular GEP

H., Mehl, Plestenjak (2019, 2023):

- Random rank-completing perturbation of rank $n - r$
- Random projection of dimension r
- Random augmentation

Ex: $p(x, y) = 0$, $q(x, y) = 0$, both degree 10
There are 100 roots, but det. rep. and MEP
$\det(A_i + x\,B_i + y\,C_i) = 0$ are of size 19,
so corresponding GEP $\Delta_1 \boldsymbol{z} = x\,\Delta_0\,\boldsymbol{z}$ is of size $19^2 = 361$

Pair (Δ_1, Δ_0) is of rank 280, so there are 3 options:

- Rank-completing perturbation of rank $361 - 280 = 81$
- Rank projection onto dimension 280
- augmentation to size $361 + 81 = 442$

Many links with KU Leuven group

- De Moor (2019, 2020)
  Vermeersch, De Moor (2019, 2022, 2023)
  Multivariate polynomial system → rectangular MEP
  Approach via block Macaulay matrices
  Interesting applications: ARMA model, LTI model

  Related alternative approach: rectangular MEP → MEP
  Bor Plestenjak, talk 1 December

- Lagauw, De Moor, Mauricio Agudelo (2022): model reduction

Conclusions

- Dixon: $\exists$ "perfect" symmetric linearization of degree d
- ... but proof not constructive: open since 1902!
- Quarez (2012): symmetric linearization size $\frac{1}{4}d^2$
- ... but Δ-matrices $\mathcal{O}(d^4)$ and eigenvalue problem $\mathcal{O}(d^{12+})$
- 2017: nonsymmetric uniform linearization size $2d-1$, no computations; work $\mathcal{O}(d^6)$
- Or "a bit more", since pencil (A, B) is singular
- Competitive with state-of-the-art (Mathematica, SOSTOOLS, PHCpack, Vermeersch–De Moor! ...)
- Classical problem with many different math aspects: algebra, linear algebra, numerics, tensors, $\mathcal{O}(d^{6+})$ work, singular GEP, theory vs. practice (rank decisions), ...

Many fascinating open problems

- For $n = 2$ (2 variables x and y), and degree d,
  is size $2d - 1$ matrices the best we can do ?

  Improvement would have big impact, in view of work $\sim (2d)^6$

- Construction for original Dixon Theorem (1902) is still open !
  How to find symmetric det. representation of size d ?
  Bor Plestenjak has a construction for nonsymmetric of size d,
  but stability is an issue

- Can algorithms exploit symmetry of the matrices ?

- For 4 variables and degree 4, there are only $4^4 = 256$ roots.
  Determinantal representation is only of size 14, quite modest.

  But eigenvalue approach: based on matrices of form
  $\Delta = A \otimes B \otimes C \otimes D$ of size $14^4 = 38416$ ...

Some references

Dixon	Existence size d	1902 !
Dickson		1921
Atkinson	Two-parameter eigenproblems	1972
Quarez	Representation size $\frac{1}{4}d^2$	2012
Plestenjak, H.	Representation size $\frac{1}{4}d^2$ and $\frac{1}{6}d^2$	2016
Boralevi, Van Doornmalen Draisma, H., Plestenjak	Representation size $2d-1$	2017
Plestenjak	Representation size d	2017
Robol, Vandebril, Van Dooren	Various bases	2017
H., Mehl, Plestenjak	Methods for singular generalized eigenvalue problems	2019, 2023
Alsubaie	Model reduction and MEP	2019
De Moor, Vermeersch	ARMA, LTI, rectangular MEP	2019, 2022
De Moor, Lagauw Mauricio Agudelo	Model reduction, rectangular MEP	2022, 2023

Software:

Plestenjak	`BiRoots` (Matlab toolbox)	2016
Plestenjak	`MultiParEig` (Matlab toolbox)	2014–2022