Solving (Overdetermined) Polynomial Equations

'Back to the Roots'

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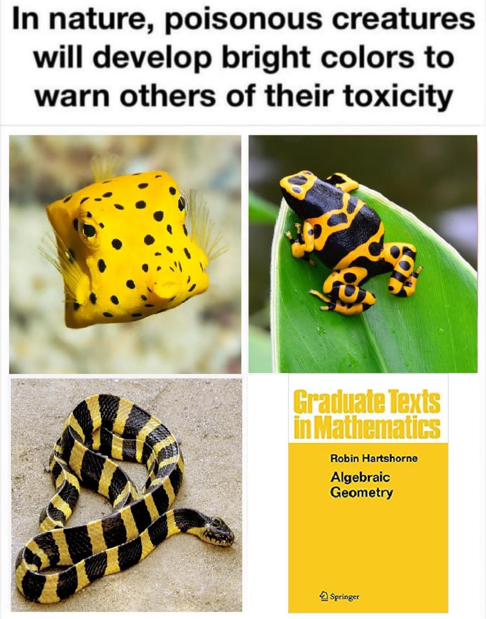

In nature, poisonous creatures will develop bright colors to warn others of their toxicity
Graduate Texts in Mathematics
Robin Hartshorne
Algebraic Geometry
Springer

Outline

The history of polynomial system solving

Univariate polynomials and eigenvalue decompositions

Multivariate polynomial systems

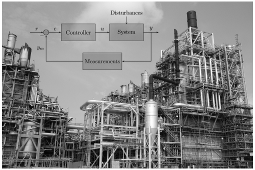
Overdetermined polynomial equations

Recent developments in the Macaulay spirit

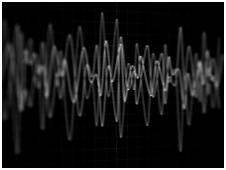
Conclusions and Perspectives

Why Study Polynomial Equations?

- fundamental mathematical objects
- powerful modelling tools
- ubiquitous in Science and Engineering (often *hidden*)



Systems and Control

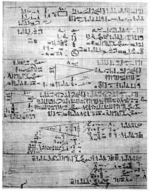


Signal Processing

Computational Biology

Kinematics/Robotics

Polynomial root-finding has a long and rich history...



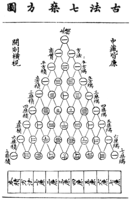
Egypt
(3000BCE-300BCE)

Babylon
(3000BCE-539BCE)

Euclid
(fl. 300BCE)

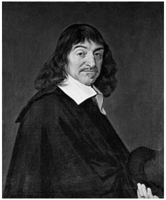
Diophantus
(c200-c284)

Al-Khwarizmi
(c780-c850)

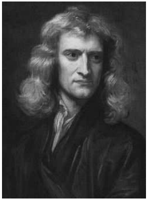


Zhu Shijie
(c1260-c1320)

Pierre de Fermat
(c1601-1665)



René Descartes
(1596-1650)



Isaac Newton
(1643-1727)

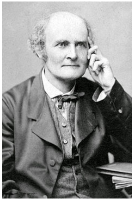
Gottfried Leibniz
(1646-1716)

Etienne Bézout
(1730-1783)

Carl Friedrich Gauss
(1777-1755)

Jean-Victor Poncelet
(1788-1867)

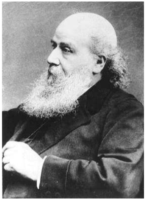
Evariste Galois
(1811-1832)



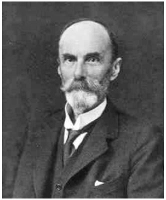
Arthur Cayley
(1821-1895)

Leopold Kronecker
(1823-1891)

Edmond Laguerre
(1834-1886)

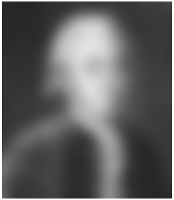
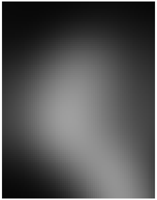
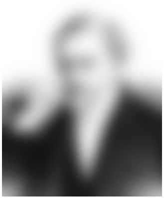
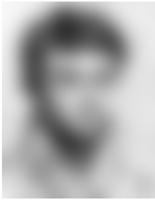
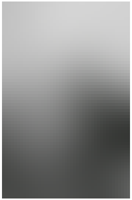


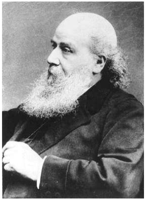
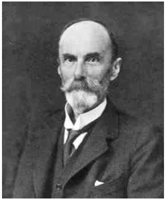
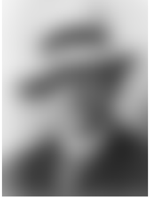
James J. Sylvester
(1814-1897)



Francis S. Macaulay
(1862-1937)

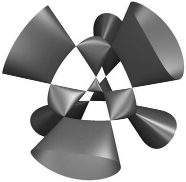
David Hilbert
(1862-1943)

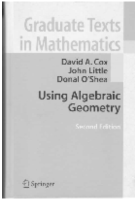






...leading to Algebraic Geometry (and computer algebra)

- large body of literature
- emphasis not (anymore) on *solving* equations
- computer algebra: symbolic manipulations (e.g., Gröbner Bases)
- numerical issues!



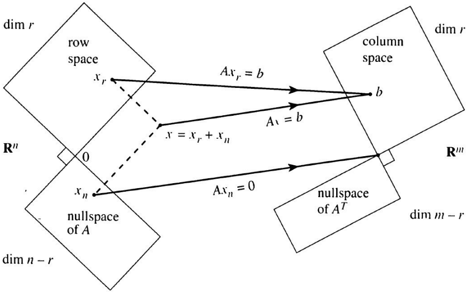


Wolfgang Gröbner
(1899-1980)

Bruno Buchberger

Back to the roots! Let's use linear algebra!?

- comprehensible and accessible language
- intuitive geometric interpretation
- computationally powerful framework
- well-established methods and stable numerics



Outline

Eigenvalue decompositions are at the core of root-finding

Eigenvalue equation

$$Av = \lambda v$$

and eigenvalue decomposition

$$A = V \Lambda V^{-1}$$

Enormous importance in (numerical) linear algebra and apps

- 'understand' the action of matrix A
- at the heart of a multitude of applications: oscillations, vibrations, quantum mechanics, data analytics, graph theory, and **many** more

From eigenvalues to roots . . . and back

Characteristic Polynomial

The eigenvalues of A are the roots of

$$p(\lambda) = |A - \lambda I|$$

Companion Matrix

Solving

$$q(x) = 7x^3 - 2x^2 - 5x + 1 = 0$$

leads to

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1/7 & 5/7 & 2/7 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix} = x \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix}$$

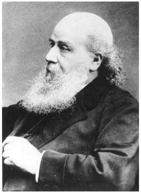
The Sylvester matrix is used for finding common roots of multiple univariate polynomials

Consider two polynomial equations

$$\begin{aligned} f(x) &= x^3 - 6x^2 + 11x - 6 &&= (x-1)(x-2)(x-3) \\ g(x) &= -x^2 + 5x - 6 &&= -(x-2)(x-3) \end{aligned}$$

Common roots if $|S(f,g)| = 0$

$$S(f,g) = \left[\begin{array}{rrrrr} -6 & 11 & -6 & 1 & 0 \\ 0 & -6 & 11 & -6 & 1 \\ \hline -6 & 5 & -1 & 0 & 0 \\ 0 & -6 & 5 & -1 & 0 \\ 0 & 0 & -6 & 5 & -1 \end{array}\right]$$



James Joseph Sylvester

Sylvester's construction can be understood from

$$\begin{array}{l} \\ f(x)=0 \\ x\cdot f(x)=0 \\ g(x)=0 \\ x\cdot g(x)=0 \\ x^2\cdot g(x)=0 \end{array} \quad \begin{array}{c} \begin{array}{ccccc} 1 & x & x^2 & x^3 & x^4 \end{array} \\ \begin{bmatrix} -6 & 11 & -6 & 1 & 0 \\ & -6 & 11 & -6 & 1 \\ -6 & 5 & -1 & & \\ & -6 & 5 & -1 & \\ & & -6 & 5 & -1 \end{bmatrix} \end{array} \begin{bmatrix} 1 & 1 \\ x_1 & x_2 \\ x_1^2 & x_2^2 \\ x_1^3 & x_2^3 \\ x_1^4 & x_2^4 \end{bmatrix} = 0$$

where $x_1 = 2$ and $x_2 = 3$ are the common roots of f and g

The vectors in the Vandermonde-like null space K obey a 'shift structure':

$$\begin{bmatrix} 1 \\ x \\ x^2 \\ x^3 \end{bmatrix} x = \begin{bmatrix} x \\ x^2 \\ x^3 \\ x^4 \end{bmatrix}$$

The Vandermonde-like null space K is not available directly, instead we compute Z, for which $ZV = K$. We now have

$$\begin{aligned} \underline{K} D &= \overline{K} \\ \underline{ZV} D &= \overline{ZV} \end{aligned}$$

leading to the (generalized) eigenvalue problem

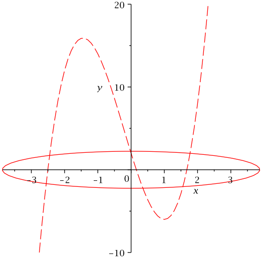
$$\overline{Z} V = \underline{Z} V D$$

Outline

Generalizing the Sylvester matrix to the multivariate case leads to the Macaulay matrix

Consider the system

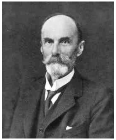
$$\begin{array}{rcll} p(x,y) & = & x^2 + 3y^2 - 15 & = & 0 \\ q(x,y) & = & y - 3x^3 - 2x^2 + 13x - 2 & = & 0 \end{array}$$



Matrix representation of the system: Macaulay matrix M

	1	x	y	x^2	xy	y^2	x^3	x^2y	xy^2	y^3
$p(x,y)$	-15			1		3				
$x \cdot p(x,y)$		-15					1		3	
$y \cdot p(x,y)$			-15					1		3
$q(x,y)$	-2	13	1	-2			-3			

$$
\begin{aligned}
p(x,y) &= x^2 + 3y^2 - 15 &= 0\\
q(x,y) &= y - 3x^3 - 2x^2 + 13x - 2 &= 0
\end{aligned}
$$



Continue to enlarge the Macaulay matrix M:

		1	x	y	x^2	xy	y^2	x^3	x^2y	xy^2	y^3	x^4	x^3y	x^2y^2	xy^3	y^4	x^5	x^4y	x^3y^2	x^2y^3	xy^4	y^5	$\rightarrow$
$d=3$	p	-15			1		3																
	xp		-15					1		3													
	yp			-15					1		3												
	q	-2	13	1	-2			-3															
$d=4$	x^2p				-15							1		3									
	xyp					-15							1		3								
	y^2p						-15							1		3							
	xq		-2		13	1		-2				-3											
	yq			-2		13	1		-2				-3										
$d=5$	x^3p							-15									1		3				
	x^2yp								-15									1		3			
	xy^2p									-15									1		3		
	y^3p										-15									1		3	
	x^2q				-2			13	1				-2				-3						
	xyq					-2			13	1				-2				-3					
	y^2q						-2			13	1				-2				-3				
$\downarrow$								$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$

– Macaulay coefficient matrix M:

$$M = \begin{bmatrix} \times & \times & \times & \times & 0 & 0 & 0 \\ 0 & \times & \times & \times & \times & 0 & 0 \\ 0 & 0 & \times & \times & \times & \times & 0 \\ 0 & 0 & 0 & \times & \times & \times & \times \end{bmatrix}$$

– solutions generate vectors in null space

$$MK = 0$$

– number of solutions m = nullity (provided M large enough)

Multivariate Vandermonde basis for the null space:

$$\left[\begin{array}{c|c|c|c} 1 & 1 & \ldots & 1 \\ \hline x_1 & x_2 & \ldots & x_m \\ y_1 & y_2 & \ldots & y_m \\ \hline x_1^2 & x_2^2 & \ldots & x_m^2 \\ x_1y_1 & x_2y_2 & \ldots & x_my_m \\ y_1^2 & y_2^2 & \ldots & y_m^2 \\ \hline x_1^3 & x_2^3 & \ldots & x_m^3 \\ x_1^2y_1 & x_2^2y_2 & \ldots & x_m^2y_m \\ x_1y_1^2 & x_2y_2^2 & \ldots & x_my_m^2 \\ y_1^3 & y_2^3 & \ldots & y_m^3 \\ \hline x_1^4 & x_2^4 & \ldots & x_4^4 \\ x_1^3y_1 & x_2^3y_2 & \ldots & x_m^3y_m \\ x_1^2y_1^2 & x_2^2y_2^2 & \ldots & x_m^2y_m^2 \\ x_1y_1^3 & x_2y_2^3 & \ldots & x_my_m^3 \\ y_1^4 & y_2^4 & \ldots & y_m^4 \\ \hline \vdots & \vdots & \vdots & \vdots \end{array}\right]$$

Select the 'top' m linear independent rows of K

$$S_1 K$$

$$\begin{bmatrix}
1 & 1 & \dots & 1 \\
x_1 & x_2 & \dots & x_m \\
y_1 & y_2 & \dots & y_m \\
x_1^2 & x_2^2 & \dots & x_m^2 \\
x_1 y_1 & x_2 y_2 & \dots & x_m y_m \\
y_1^2 & y_2^2 & \dots & y_m^2 \\
x_1^3 & x_2^3 & \dots & x_m^3 \\
x_1^2 y_1 & x_2^2 y_2 & \dots & x_m^2 y_m \\
x_1 y_1^2 & x_2 y_2^2 & \dots & x_m y_m^2 \\
y_1^3 & y_2^3 & \dots & y_m^3 \\
x_1^4 & x_2^4 & \dots & x_4^4 \\
x_1^3 y_1 & x_2^3 y_2 & \dots & x_m^3 y_m \\
x_1^2 y_1^2 & x_2^2 y_2^2 & \dots & x_m^2 y_m^2 \\
x_1 y_1^3 & x_2 y_2^3 & \dots & x_m y_m^3 \\
y_1^4 & y_2^4 & \dots & y_m^4 \\
\vdots & \vdots & \vdots & \vdots
\end{bmatrix}$$

Shifting the selected rows gives (shown for 3 columns)

$$
\begin{bmatrix}
1 & 1 & 1 \\
x_1 & x_2 & x_3 \\
y_1 & y_2 & y_3 \\
x_1^2 & x_2^2 & x_3^2 \\
x_1 y_1 & x_2 y_2 & x_3 y_3 \\
y_1^2 & y_2^2 & y_3^2 \\
x_1^3 & x_2^3 & x_3^3 \\
x_1^2 y_1 & x_2^2 y_2 & x_3^2 y_3 \\
x_1 y_1^2 & x_2 y_2^2 & x_3 y_3^2 \\
y_1^3 & y_2^3 & y_3^3 \\
x_1^4 & x_2^4 & x_4^4 \\
x_1^3 y_1 & x_2^3 y_2 & x_3^3 y_3 \\
x_1^2 y_1^2 & x_2^2 y_2^2 & x_3^2 y_3^2 \\
x_1 y_1^3 & x_2 y_2^3 & x_3 y_3^3 \\
y_1^4 & y_2^4 & y_3^4 \\
\vdots & \vdots & \vdots
\end{bmatrix}
\longrightarrow \text{“shift with } x\text{”} \longrightarrow
\begin{bmatrix}
1 & 1 & 1 \\
x_1 & x_2 & x_3 \\
y_1 & y_2 & y_3 \\
x_1^2 & x_2^2 & x_3^2 \\
x_1 y_1 & x_2 y_2 & x_3 y_3 \\
y_1^2 & y_2^2 & y_3^2 \\
x_1^3 & x_2^3 & x_3^3 \\
x_1^2 y_1 & x_2^2 y_2 & x_3^2 y_3 \\
x_1 y_1^2 & x_2 y_2^2 & x_3 y_3^2 \\
y_1^3 & y_2^3 & y_3^3 \\
x_1^4 & x_2^4 & x_4^4 \\
x_1^3 y_1 & x_2^3 y_2 & x_3^3 y_3 \\
x_1^2 y_1^2 & x_2^2 y_2^2 & x_3^2 y_3^2 \\
x_1 y_1^3 & x_2 y_2^3 & x_3 y_3^3 \\
y_1^4 & y_2^4 & y_3^4 \\
\vdots & \vdots & \vdots
\end{bmatrix}
$$

simplified:

$$
\begin{bmatrix}
1 & 1 & 1 \\
x_1 & x_2 & x_3 \\
y_1 & y_2 & y_3 \\
x_1 y_1 & x_2 y_2 & x_3 y_3 \\
x_1^3 & x_2^3 & x_3^3 \\
x_1^2 y_1 & x_2^2 y_2 & x_3^2 y_3
\end{bmatrix}
\begin{bmatrix}
x_1 & & \\
& x_2 & \\
& & x_3
\end{bmatrix}
=
\begin{bmatrix}
x_1 & x_2 & x_3 \\
x_1^2 & x_2^2 & x_3^2 \\
x_1 y_1 & x_2 y_2 & x_3 y_3 \\
x_1^2 y_1 & x_2^2 y_2 & x_3^2 y_3 \\
x_1^4 & x_2^4 & x_4^4 \\
x_1^3 y_1 & x_2^3 y_2 & x_3^3 y_3
\end{bmatrix}
$$

- finding the x-roots: let $D_x = \mathrm{diag}(x_1, x_2, \ldots, x_s)$, then

$$S_1 K D_x = S_x K,$$

where S_1 and S_x select rows from K wrt. shift property

- reminiscent of **Realization Theory**

We have

$$S_1 K D_x = S_x K$$

However, K is not known, instead a basis Z is computed that satisfies

$$ZV = K$$

Which leads to

$$(S_x Z)V = (S_1 Z)VD_x$$

It is possible to shift with y as well...

We find

$$S_1 K D_y = S_y K$$

with D_y diagonal matrix of y-components of roots, leading to

$$(S_y Z)V = (S_1 Z)V D_y$$

Some interesting results:

- same eigenvectors V!
- $(S_y Z)^{-1}(S_1 Z)$ and $(S_x Z)^{-1}(S_1 Z)$ commute

Algorithm

1 Fix a monomial ordering scheme

2 Construct coefficient matrix M to sufficiently large dimensions

3 Compute basis for nullspace of M: nullity s and Z

4 Find s linear independent rows in Z

5 Choose shift function, e.g., x

6 Solve the GEVP

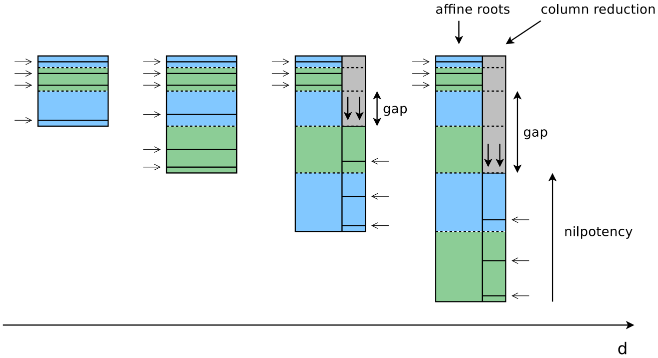
$$(S_2 Z)V = (S_1 Z)VD_x$$

S_1 selects linearly independent rows in Z
S_2 selects rows that are 'hit' by the shift

($S_1 Z$ and $S_2 Z$ can be rectangular as long as $S_1 Z$ contains s linear independent rows)

Roots at infinity? *Mind the Gap!*

- dynamics in the null space of $M(d)$ for increasing degree d
- nilpotency gives rise to a 'gap'
- mechanism to count and separate affine from infinity



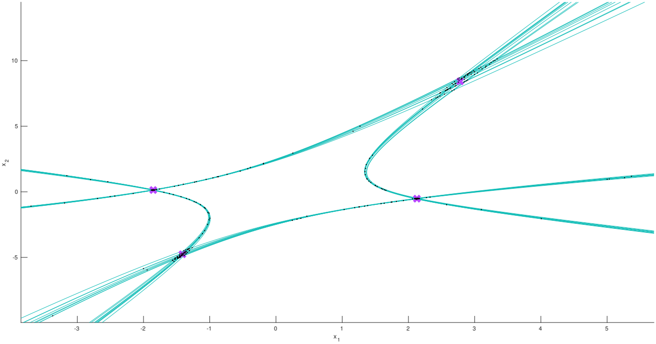
The NLA approach allows for overdetermined systems

Can we 'solve' overdetermined systems of polynomial equations?

Not feasible using (exact) computer algebra methods

System of completely intersecting quadratic bivariate equations

$$x^2 + y^2 - 3xy - x - 6 = 0$$
$$2x^2 - y^2 + 3xy - y - 6 = 0$$



- 8 repetitions of each equation with perturbed coefficients (SNR 40 dB)
- 16 equations in 2 unknowns
- (approximate) rank decisions

Outline

Recent advances in the Macaulay spirit

Computing state-recursion polynomials
Batselier K., Wong N., *"Computing the state recursion polynomials for discrete linear mD systems"*, Automatica, vol. 64, pp.254-261, 2016.

"The CPD appears to be the joint EVD of the multiplication tables"
Vanderstukken J., Stegeman A., De Lathauwer L., *"Systems of polynomial equations, higher-order tensor decompositions and multidimensional harmonic retrieval: A unifying framework."* (two-part paper), KU Leuven ESAT-STADIUS TR 17-133 and TR 17-134, 2017.

Block-shifting with an objective function
Vermeersch C., De Moor B., *"Globally Optimal Least-Squares ARMA Model Identification is an Eigenvalue Problem"*, IEEE Control Systems Letters, 3:4, 1062–1067, 2019.

Adapting the choice of basis for improved numerical stability
Telen S., Mourrain B., Van Barel M., *"Solving Polynomial Systems via Truncated Normal Forms"*, SIAM J Matrix Anal Appl, 39:3, 1421–1447, 2018.

Outline

Conclusions

- bridging the gap between algebraic geometry and (numerical) linear algebra
- finding roots: (numerical) linear algebra (and realization theory)!
- extension to over-constrained systems

Open Problems

Many challenges remain

- exploiting sparsity and structure of M
- efficient (more direct) construction of the eigenvalue problem
- replace SVD by structured low-rank approximation of M

Dreesen P., Batselier K., De Moor B., *"Multidimensional realisation theory and polynomial system solving"*, Int J Control, 91:12, pp. 2692–2704, 2018. (arXiv 1805.02253)

Thank you for listening!

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Solving (Overdetermined) Polynomial Equations

'Back to the Roots'

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