Multiparameter eigenvalue problems

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The eigenvalue problem (EVP) is fundamental and ubiquitous in science and engineering. Its theory, applications and algorithms have been described abundantly. For the **multiparemeter eigenvalue problem (MEVP)**, our theoretical and algorithmic understanding is much less elaborate (early references include [1] [2]). This deficiency is deplorable as applications of the MEVP are also ubiquitous in science and engineering.
In this contribution, we will discuss theory, algorithms and applications for MEVPs, using a combination of insights from (multi-)dimensional system theory (realization algorithms, see e.g. [3]), algebraic geometry (multivariate polynomials, ideals and varieties, see e.g. [4]), operator theory (shift-invariant model spaces, see e.g. [5] [6]) and numerical linear [7] or polynomial [8] algebra. The MEVP is to find the non-trivial eigenvectors $x \in \mathbb{C}^n$ and the eigentuples $\lambda_i \in \mathbb{C}, i = 1, \ldots, p$ in

$$(A_0 + A_1\lambda_1 + \ldots + A_p\lambda_p)x = 0 \ , \tag{57}$$

where the matrices $A_i \in \mathbb{R}^{m\times n}$ with $m \geq n$ contain the problem data. Special cases are the square (Jordan Canonical Form), generalized square (Weierstrass Canonical Form) and rectangular (Kronecker Canonical Form) EVP.
For the sake of clarity of exposition, we consider 5 'prototypical' cases, each of which starts with one or more (multivariate) polynomial 'seed equation(s)', from which new (equivalent) equations are generated by multiplying them with all monomials of increasing degree. With this *forward shift recursion* (FSR), we create structured matrices ('quasi-Toeplitz'), the null spaces of which have special, shift-invariant properties that can be exploited to calculate the solutions of the seed equations via one or several EVPs.
Case 1: Single shift scalar banded Toeplitz matrix: The 'seed equation' is a univariate polynomial $p(\lambda)$ of degree n in a single variable λ. The FSR generates a banded Toeplitz matrix, the null space of which has the structure of a (confluent) Vandermonde matrix, or (in system theory terms) an observability matrix of a linear time-invariant (LTI) system with a single output. By exploiting the shift-invariant structure, the roots of the characteristic equation (the eigenvalues) can be calculated via realization theory.
Case 2: Single shift scalar Sylvester matrix: The 'seed problem' here is to find the common roots of two univariate polynomials $p(\lambda) = 0$ and $q(\lambda) = 0$. The FSR generates a Sylvester matrix. Its nullity reveals the number of common zeros and its null space can be shown to be shift-invariant. The common roots are then obtained via a realization algorithm.
Case 3: Single shift block banded Toeplitz matrix: The 'seed problem' is the polynomial matrix eigenvalue problem $(A_0 + A_1\lambda + A_2\lambda^2 + \ldots + A_p\lambda^p)x = 0$, where the real matrices $A_i \in \mathbb{R}^{m\times n}, m \geq n$ are given. The FSR for this case generates a block banded Toeplitz matrix. Again, its nullity reveals the number of roots, that can be calculated from several EVPs, by exploiting the block shift structure of the null space via realization theory.
Case 4: Multi-shift scalar Macaulay matrix: The 'seed equations' form a set of multivariate polynomials. We want to find their common roots. The FSR generates a quasi-Toeplitz matrix, called a Macaulay matrix. The nullity equals the Bezout number (i.e. the number of affine zeros and zeros at infinity), which can be found by exploiting the multi-shift invariant structure of the null space and applying multi-dimensional realization algorithms to it.
Case 5: Multi-shift block Macaulay matrix: The 'seed problem' is the polynomial multi-parameter eigenvalue problem. As an example, for $p = 2$, with 2 parameters λ_1 and λ_2, it is of the form $(A_{00} + A_{10}\lambda_1 + A_{01}\lambda_2 + A_{20}\lambda_1^2 + A_{11}\lambda_1\lambda_2 + A_{02}\lambda_2^2 + \ldots)x = 0$ with $A_{ij} \in \mathbb{R}^{m\times n}, m \geq n$, with an obvious generalization for p variables. The FSR now generates a block Macaulay matrix, the nullity of which corresponds to the number of solutions. We demonstrate how its null space can be modelled as the observability matrix of a p-dimensional discrete shift invariant state space model with multiple outputs.

For each of these cases, we describe how, starting from the 'seed equation(s)', the FSR generates structured, sparse, quasi-(block)-Toeplitz matrices, the null spaces of which are scalar or vector, single- or multi-shift invariant projective subspaces. They can be 'modelled' as observability matrices of (possibly) singular, autonomous, commutative, (multi-)dimensional discrete shift-invariant dynamical systems [9]. Obtaining the null space is an exercise in linear algebra (e.g. via the SVD), while exploiting the (multi-)shift invariant structure leads to several EVPs, that together deliver all the (common) roots of the seed equation(s).

As a special application we discuss the computation of the global minimum of a multivariate polynomial optimization problem, which corresponds to calculating the minimizing root of a MEVP of the form (57). Important engineering examples include the identification of LTI dynamic models from observed data, where a sum-of-squares of the so-called prediction errors is minimized [10]. There is a rich variety of model classes like ARMA(X), Box-Jenkins, etc., which have been described abundantly in the statistical and engineering literature, the identification of which requires the solution of a nonlinear least squares optimization problem. All known algorithms are heuristic (local minima, convergence behavior, etc.). But a crucial observation (see e.g.[11]) is the fact that all these models and the objective function are multivariate polynomial. As a consequence, one only needs to find the minimizing solution of an MEVP of the form (57), a fact that we consider to be a fundamental breakthrough.

References

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