# Work Package 4B: Computing the global minimum of least squares system identification and $H_2$ model reduction problems


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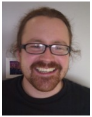

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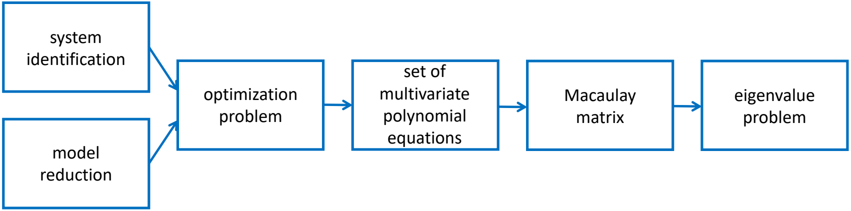
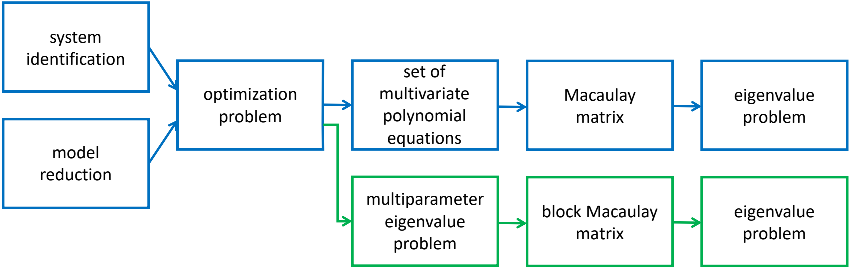
# Strategy

- presented at Selma retreat in Mons, January 2019



# Strategy

- presented at Selma retreat in Mons, January 2019
- improved during Selma project[1]




[1] Bart De Moor, Least squares optimal realisation of autonomous LTI systems is an eigenvalue problem, Communications in Information and systems, vol. 20, no. 2, pp. 163–207, 2020
Christof Vermeersch and Bart De Moor, Globally optimal least-squares ARMA model identification is an eigenvalue problem, IEEE Control Systems Letters (L-CSS), vol. 2, no. 4, pp. 1062–1067, 2019

# Overview

- The multiparameter eigenvalue problem (MEP)
- Solving an MEP using a block Macaulay matrix: simple example
- MEPs from identification problems
- Recent progress
    1. Algorithms
    2. Theory

# The multiparameter eigenvalue problem (MEP)

- Standard eigenvalue problem (SEP)

$$(A - \lambda I)\, v = 0$$

- Generalized eigenvalue problem (GEP)

$$(A - \lambda B)\, v = 0$$

- Polynomial eigenvalue problem (PEP)

$$\left(P_0 + P_1\lambda + \cdots + P_k\lambda^k\right) v = 0$$

- Multiparameter eigenvalue problem (MEP)

  example: $3$-parameter $(\lambda_1, \lambda_2, \lambda_3)$ eigenvalue problem of degree $4$

$$\left(P_{000} + P_{100}\lambda_1 + P_{002}\lambda_3^2 + P_{013}\lambda_2\lambda_3^3\right) v = 0$$

# Solving an MEP using a block Macaulay matrix

- working example

$$(P_{00} + P_{10}\lambda + P_{01}\mu)v = 0$$

$$P_{00} = \begin{pmatrix} 2 & -5 \\ -2 & -1 \\ 5 & -1 \end{pmatrix},\ P_{10} = \begin{pmatrix} 3 & 0 \\ 3 & -1 \\ -3 & 2 \end{pmatrix},\ P_{01} = \begin{pmatrix} 2 & 2 \\ 3 & 2 \\ -2 & -4 \end{pmatrix}$$

find the eigenvalue tuples $(\lambda, \mu)$ and eigenvectors $v$

- three solutions (one real, two complex, no solutions at infinity)

| | | |
|---|---|---|
| $\lambda_1 = 3.4536$ | $\mu_1 = 1.1169$ | $v_1 = \begin{pmatrix} 0.1862 \\ 0.9825 \end{pmatrix}$ |
| $\lambda_2 = -0.2268 + 1.4608i$ | $\mu_2 = 0.4415 - 0.7775i$ | $v_2 = \begin{pmatrix} -0.4946 + 0.5971i \\ -0.5972 + 0.2053i \end{pmatrix}$ |
| $\lambda_3 = -0.2268 - 1.4608i$ | $\mu_3 = 0.4415 + 0.7775i$ | $v_3 = \begin{pmatrix} -0.4946 - 0.5971i \\ -0.5972 - 0.2053i \end{pmatrix}$ |

- create extra equations

$$
\begin{aligned}
(P_{00} + P_{10}\lambda + P_{01}\mu)v &= 0 \\
\lambda(P_{00} + P_{10}\lambda + P_{01}\mu)v &= 0 \\
\mu(P_{00} + P_{10}\lambda + P_{01}\mu)v &= 0
\end{aligned}
$$

- result: block-Macaulay matrix $\mathcal{M}$

$$
\underbrace{\begin{pmatrix} P_{00} & P_{10} & P_{01} & 0 & 0 & 0 \\ 0 & P_{00} & 0 & P_{10} & P_{01} & 0 \\ 0 & 0 & P_{00} & 0 & P_{10} & P_{01} \end{pmatrix}}_{\mathcal{M}} \begin{pmatrix} v \\ \lambda v \\ \mu v \\ \lambda^2 v \\ \lambda\mu v \\ \mu^2 v \end{pmatrix} = 0
$$

- null space of $\mathcal{M}$ has dimension $3$
- vectors $k_1$, $k_2$ and $k_3$ span the null space of $\mathcal{M}$

$$k_1 = \begin{pmatrix} v_1 \\ \lambda_1 v_1 \\ \mu_1 v_1 \\ \lambda_1^2 v_1 \\ \lambda_1 \mu_1 v_1 \\ \mu_1^2 v_1 \end{pmatrix} \quad k_2 = \begin{pmatrix} v_2 \\ \lambda_2 v_2 \\ \mu_2 v_2 \\ \lambda_2^2 v_2 \\ \lambda_2 \mu_2 v_2 \\ \mu_2^2 v_2 \end{pmatrix} \quad k_3 = \begin{pmatrix} v_3 \\ \lambda_3 v_3 \\ \mu_3 v_3 \\ \lambda_3^2 v_3 \\ \lambda_3 \mu_3 v_3 \\ \mu_3^2 v_3 \end{pmatrix}$$

- calculate a basis for null space of $\mathcal{M}$: columns of $\Gamma$

- columns of $\Gamma$ are linear combinations of vectors $k_1, k_2, k_3$:

$$\underset{\text{calculated}}{\Gamma} = \underbrace{\begin{pmatrix} v_1 & v_2 & v_3 \\ \lambda_1 v_1 & \lambda_2 v_2 & \lambda_3 v_3 \\ \mu_1 v_1 & \mu_2 v_2 & \mu_3 v_3 \\ \lambda_1^2 v_1 & \lambda_2^2 v_2 & \lambda_3^2 v_3 \\ \lambda_1 \mu_1 v_1 & \lambda_2 \mu_2 v_2 & \lambda_3 \mu_3 v_3 \\ \mu_1^2 v_1 & \mu_2^2 v_2 & \mu_3^2 v_3 \end{pmatrix} T}_{\text{unknown}} \quad (T \text{ is nonsingular})$$

- $\Gamma$ has special structure

$$\Gamma = \begin{pmatrix} C \\ CA_1 \\ CA_2 \\ CA_1^2 \\ CA_1 A_2 \\ CA_2^2 \end{pmatrix} \quad \text{where} \quad \left\{ \begin{array}{rcl} \underset{2\times 3}{C} & = & \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} T \\ \underset{3\times 3}{A_1} & = & T^{-1} \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix} T \\ \underset{3\times 3}{A_2} & = & T^{-1} \begin{pmatrix} \mu_1 & & \\ & \mu_2 & \\ & & \mu_3 \end{pmatrix} T \end{array} \right.$$

- from $\Gamma$ we can find $C$: the first block row of $\Gamma$
- from $\Gamma$ we can find $A_1$ and $A_2$ by solving linear equations
  - selection of block rows

$$\underbrace{\begin{pmatrix} C \\ CA_1 \\ CA_2 \\ CA_1^2 \\ CA_1A_2 \\ CA_2^2 \end{pmatrix}}_{\Gamma} \xrightarrow{\cdot A_1} \underbrace{\begin{pmatrix} C \\ CA_1 \\ CA_2 \\ CA_1^2 \\ CA_1A_2 \\ CA_2^2 \end{pmatrix}}_{\Gamma} \qquad \underbrace{\begin{pmatrix} C \\ CA_1 \\ CA_2 \\ CA_1^2 \\ CA_1A_2 \\ CA_2^2 \end{pmatrix}}_{\Gamma} \xrightarrow{\cdot A_2} \underbrace{\begin{pmatrix} C \\ CA_1 \\ CA_2 \\ CA_1^2 \\ CA_1A_2 \\ CA_2^2 \end{pmatrix}}_{\Gamma}$$

  - solving equations

$$\begin{pmatrix} C \\ CA_1 \\ CA_2 \end{pmatrix} A_1 = \begin{pmatrix} CA_1 \\ CA_1^2 \\ CA_1A_2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} C \\ CA_1 \\ CA_2 \end{pmatrix} A_2 = \begin{pmatrix} CA_2 \\ CA_1A_2 \\ CA_2^2 \end{pmatrix}$$

- recall:

$$C = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} T$$

$$A_1 = T^{-1} \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix} T$$

$$A_2 = T^{-1} \begin{pmatrix} \mu_1 & & \\ & \mu_2 & \\ & & \mu_3 \end{pmatrix} T$$

- the eigenvalues of $A_1$ and $A_2$ give the solutions $(\lambda_i, \mu_i)$ $(i = 1, 2, 3)$ of the MEP
- the eigenvectors of $A_1$ are equal to the eigenvectors of $A_2$ and form the matrix $T$
- the eigenvectors of the MEP can be found as

$$\begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} = CT^{-1}$$

# MEPs from identification problems

- big MEPs
- not only linear terms in the MEP
- solutions at infinity
- solution set at infinity, dimension $> 0$

example: first order ARMA model with parameters $\alpha$ and $\beta$:

$$\left(P_{00} + P_{10}\alpha + P_{01}\beta + P_{20}\alpha^2 + P_{11}\alpha\beta + P_{02}\beta^2\right) v = 0$$

size of the matrices: $(3N-1) \times (3N-2)$, where $N$ is the number of output measurements

# Recent progress: Algorithms

1. recursive and sparse algorithms to compute the null space of the block Macaulay matrix[2]

   $\rightarrow$ faster
   
   $\rightarrow$ less memory

   example: identification of order $1$ ARMA model from $N = 8$ data points

   - size of matrices in MEP: $23 \times 22$
   - size of block Macaulay matrix: $20\,769 \times 21\,780$
   - memory usage: factor $150\,000$ reduction
     - standard approach: $3.62$ GB
     - sparse adaptation: $24.28$ kB
   - computation time: factor $725$ faster
     - standard approach: $30\,225$ s ($8.5$ hours)
     - sparse-recursive: $42$ s

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[2] Christof Vermeersch and Bart De Moor, Two Double Recursive Block Macaulay Matrix Algorithms to Solve Multiparameter Eigenvalue Problems, accepted

2. avoid computation of null space, work directly in column space of block Macaulay matrix[3]

[3] Christof Vermeersch and Bart De Moor, A column space based approach to solve systems of multivariate polynomial equations, IFAC-PapersOnLine, vol. 54, no. 9, pp. 137–144, July 2021
Christof Vermeersch and Bart De Moor, Two Complementary Block Macaulay Algorithms to Solve Multiparameter Eigenvalue Problems, submitted

# Recent progress: Theory

1. different types of eigenvalue problems and their connection to shift-invariant spaces and realization theory[4], see poster Christof Vermeersch
2. study of multidimensional (mD) systems
    - an mD system has more than one independent variable, e.g., time and place
    - connection with our Selma work:
      $\Gamma$ of the MEP example is the observability matrix of a $2$-dimensional LTI system with commuting system matrices $A_1$ and $A_2$ and output matrix $C$
    - results:
        - $2$D descriptor systems: condition for system so that non-trivial state sequence exists[5]
        - observability
        - mD realization: find commutative system from output data

[4] Katrien De Cock and Bart De Moor, Multiparameter Eigenvalue Problems and Shift-Invariance, IFAC PapersOnLine, vol. 54, no. 9, pp. 159–165, July 2021

[5] Bob Vergauwen and Bart De Moor, Two-dimensional descriptor systems, IFAC PapersOnLine vol. 54, no. 9, pp. 151–158, July 2021

- ongoing, see poster Lukas Vanpoucke
  - relation to PDEs
  - least squares identification
  - open problem: minimal parameterization of mD difference equations

3. globally optimal LS identification of multiple-output systems
   tools: functional analysis, operator theory, behavioral framework
   - inner functions
   - Beurling-Lax theorem
   - isometric state space models
   - norm preserving behavioral models
4. globally optimal $H_2$ model reduction
   - by solving an MEP[6]
     proof-of-concept: order $2 \to$ order $1$
   - improvements, ongoing, see poster Sibren Lagauw
     based on Walsh's theorem: order $7 \to$ order $3$

[6] Oscar M. Agudelo, Christof Vermeersch and Bart De Moor, Globally Optimal $H_2$-Norm Model Reduction: A Numerical Linear Algebra Approach, IFAC PapersOnLine vol. 54, no. 9, pp. 564–571, July 2021