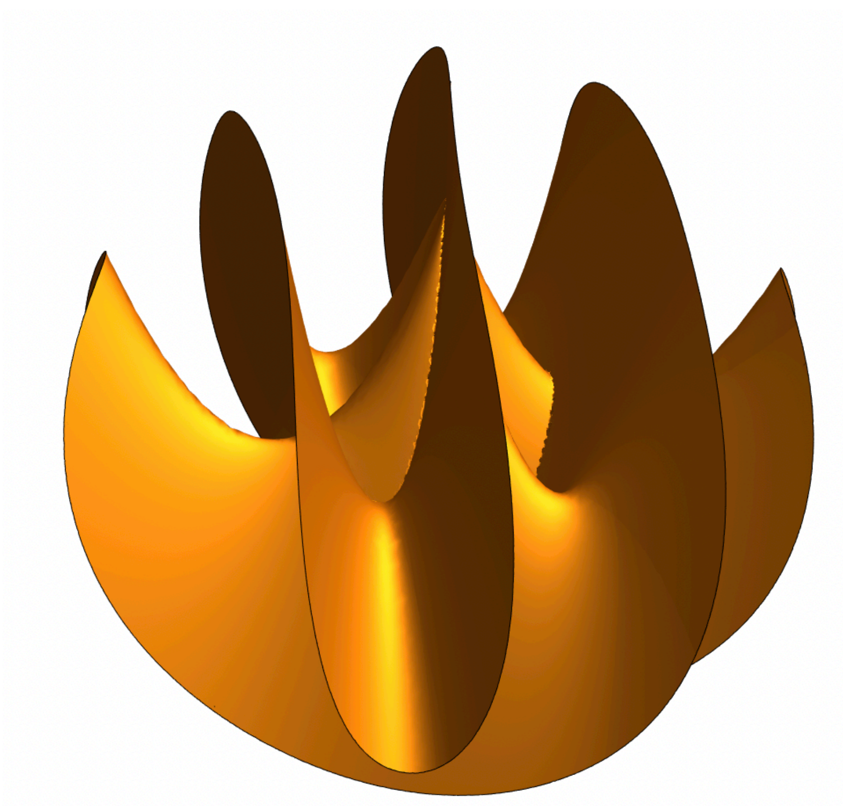
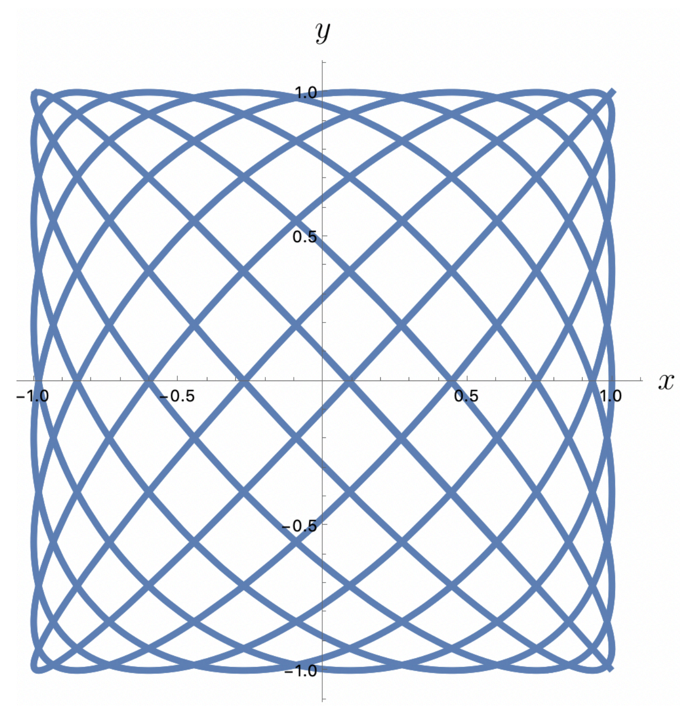
Max-Planck-Institut für Mathematik in den Naturwissenschaften

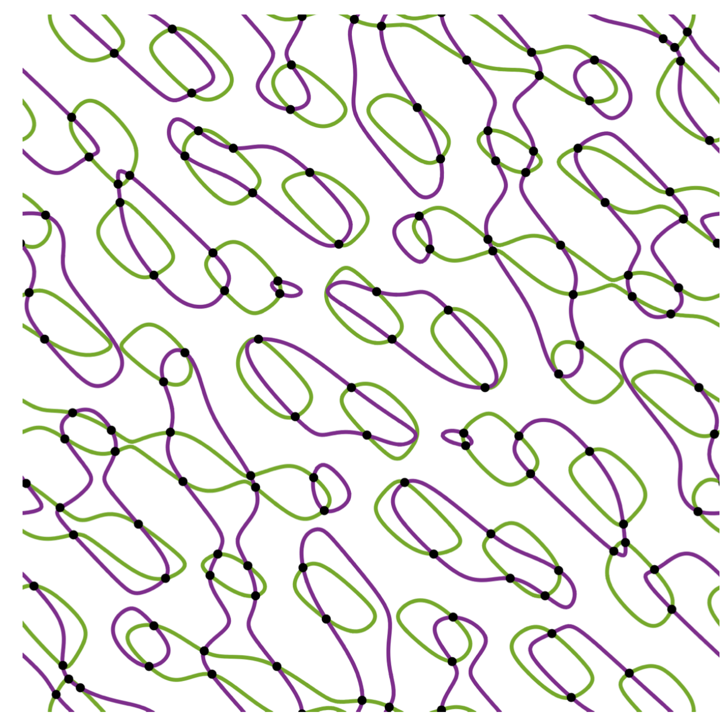
Chebyshev Varieties

Simon Telen

Back to the Roots seminar
KU Leuven, May 28, 2024







Joint work with Zaïneb Bel-Afia and Chiara Meroni

Solving polynomial equations

$$a + b \cdot t^5 + c \cdot t^7 = 0$$

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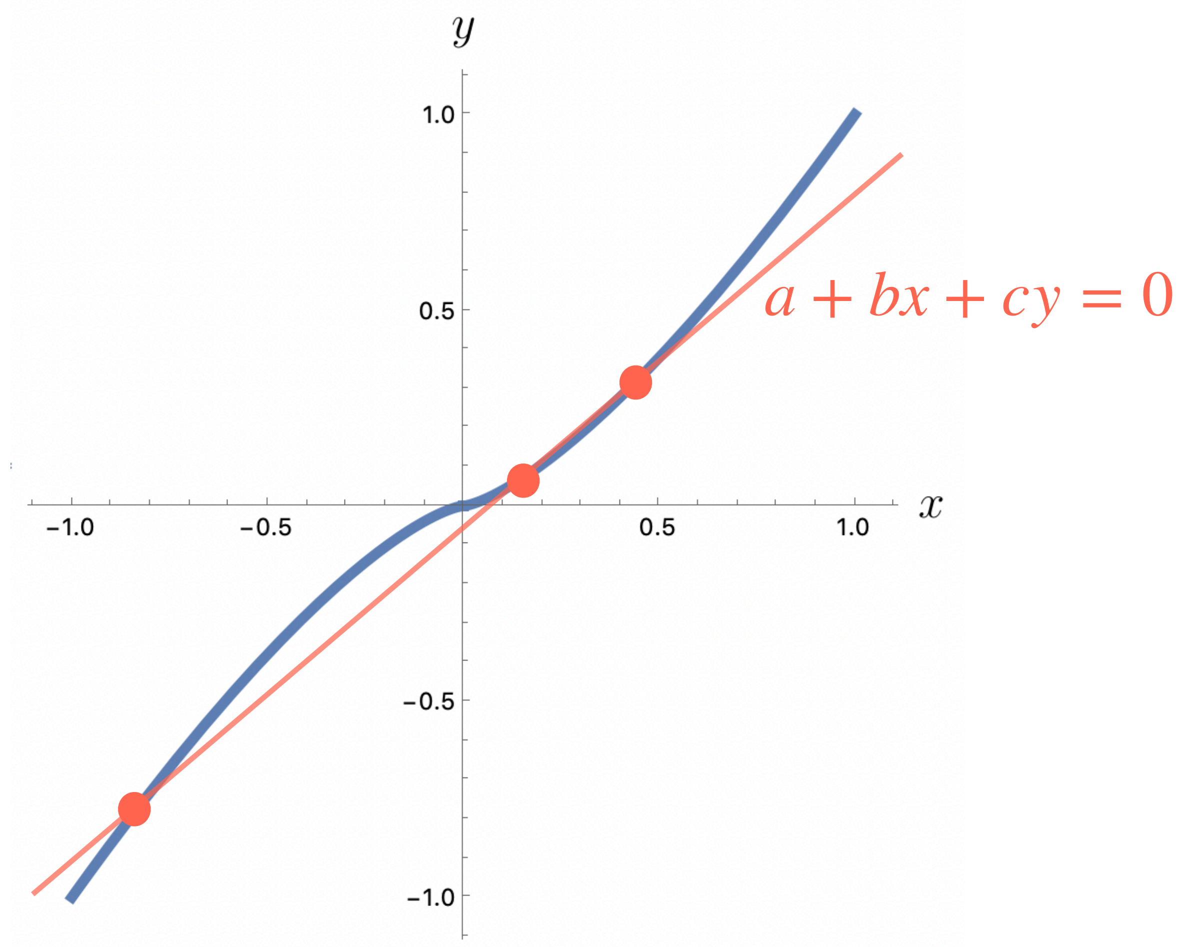

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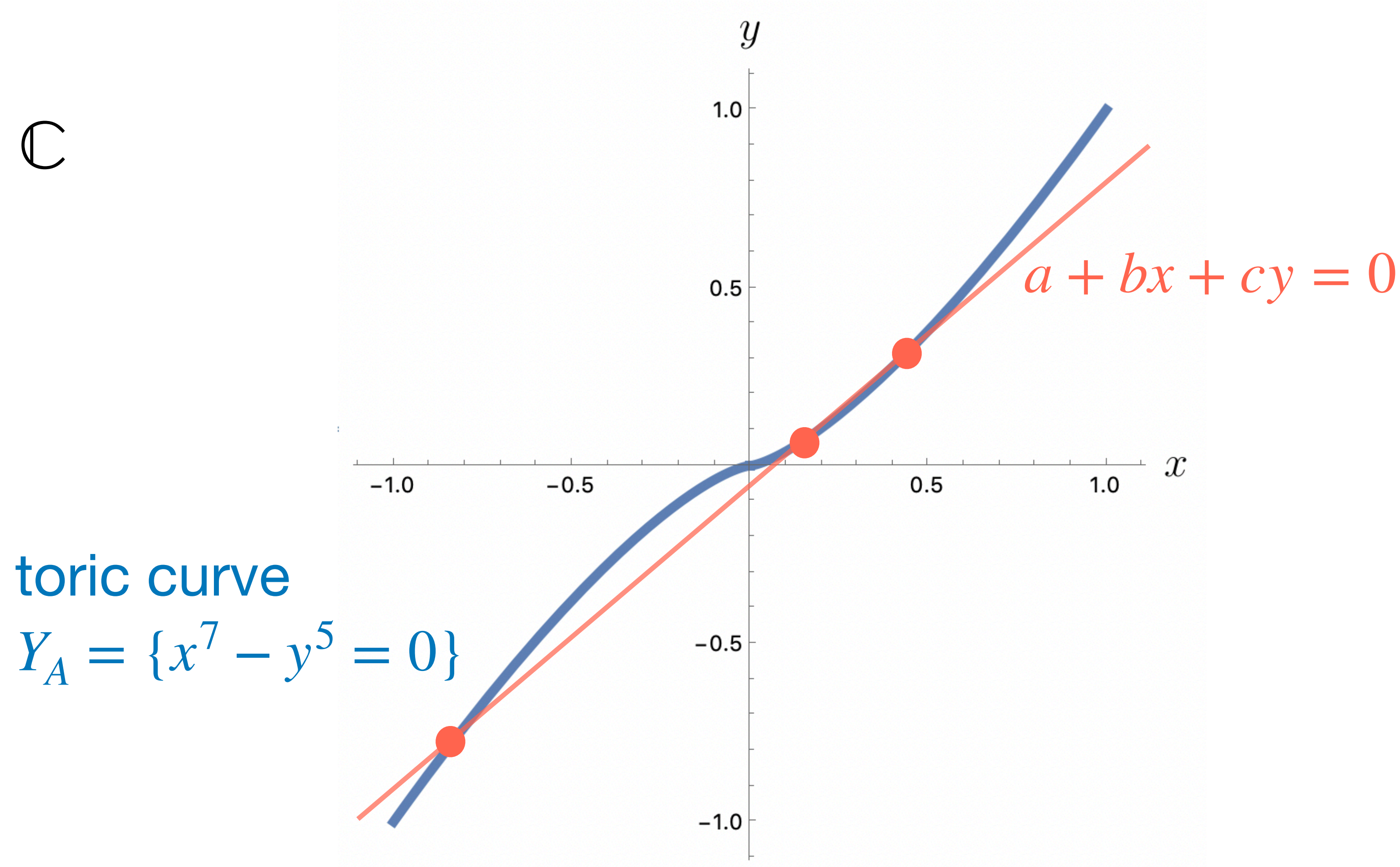

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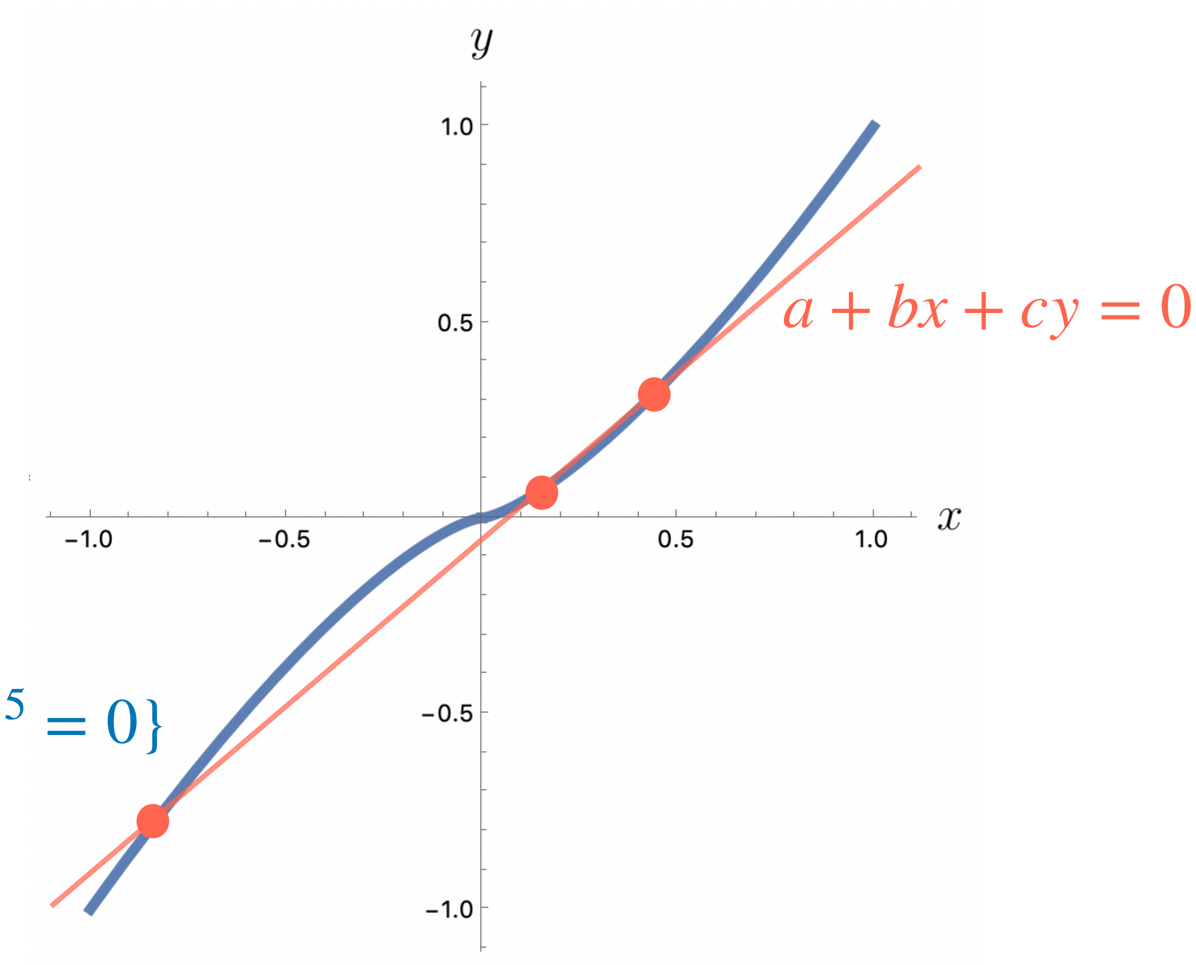

Different geometry for each sparsity pattern

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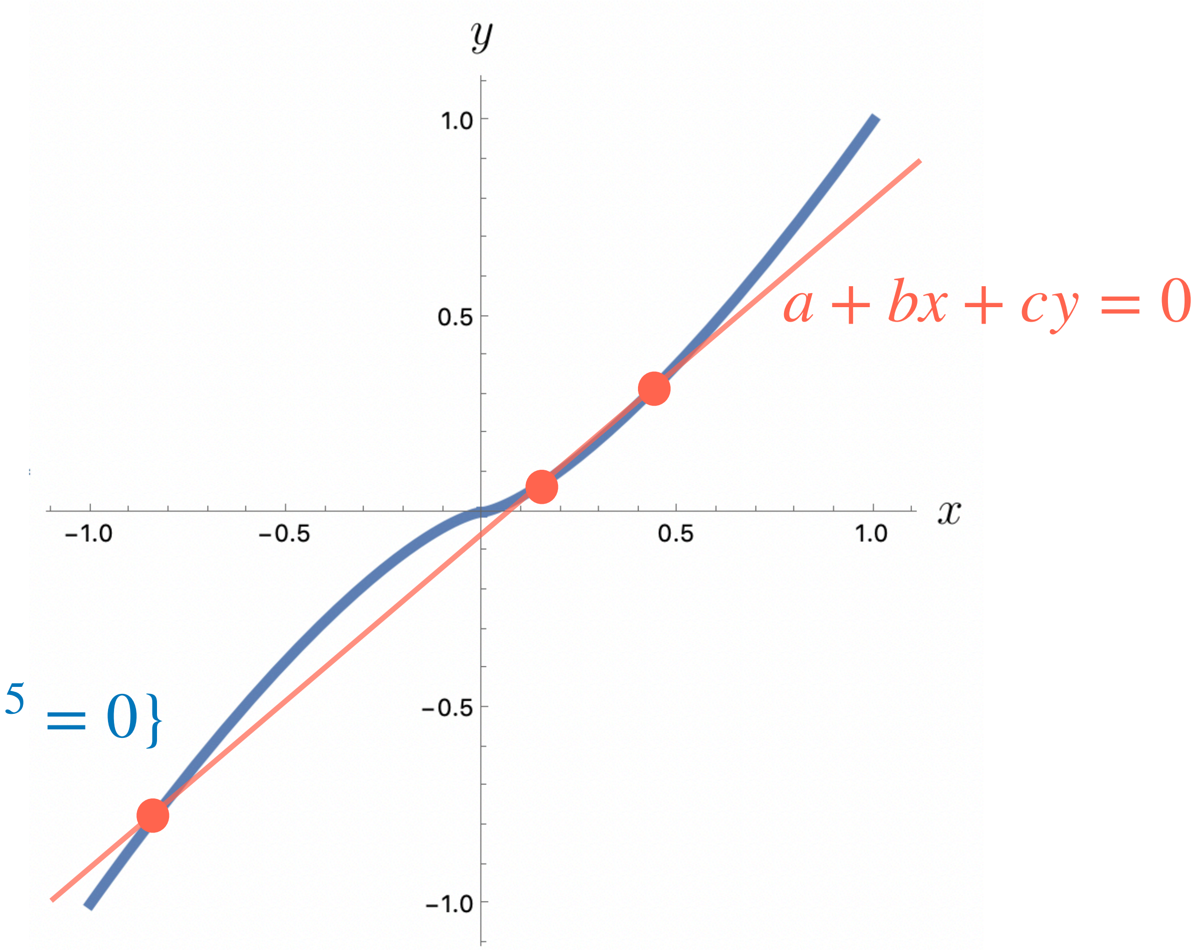
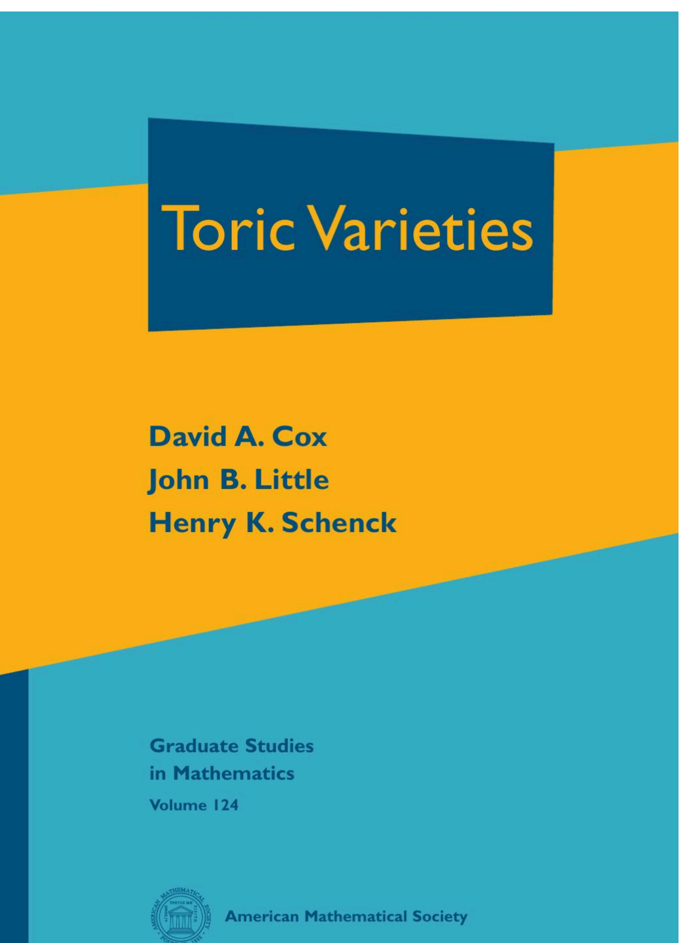


toric curve
$Y_A = \{x^7 - y^5 = 0\}$



Different geometry for each sparsity pattern

Solving polynomial equations

$$0 = a_0 + a_1 t_1 t_2 + a_2 t_1^2 t_2 + a_3 t_1 t_2^2$$

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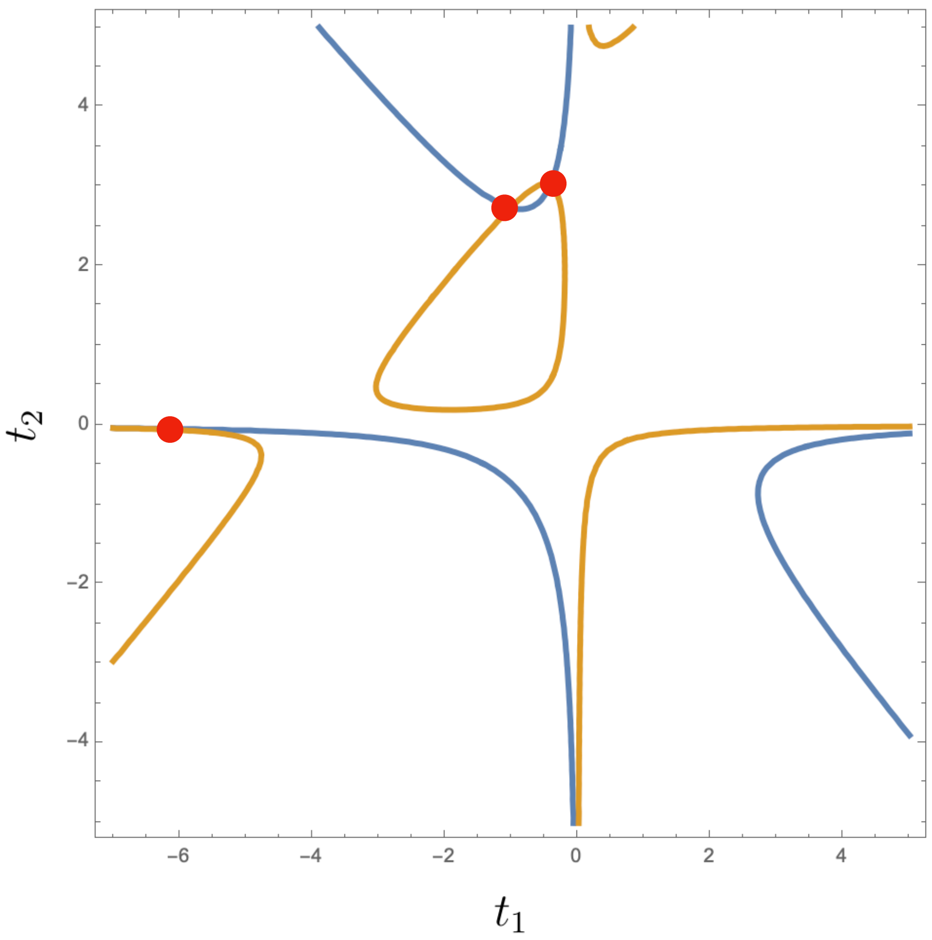

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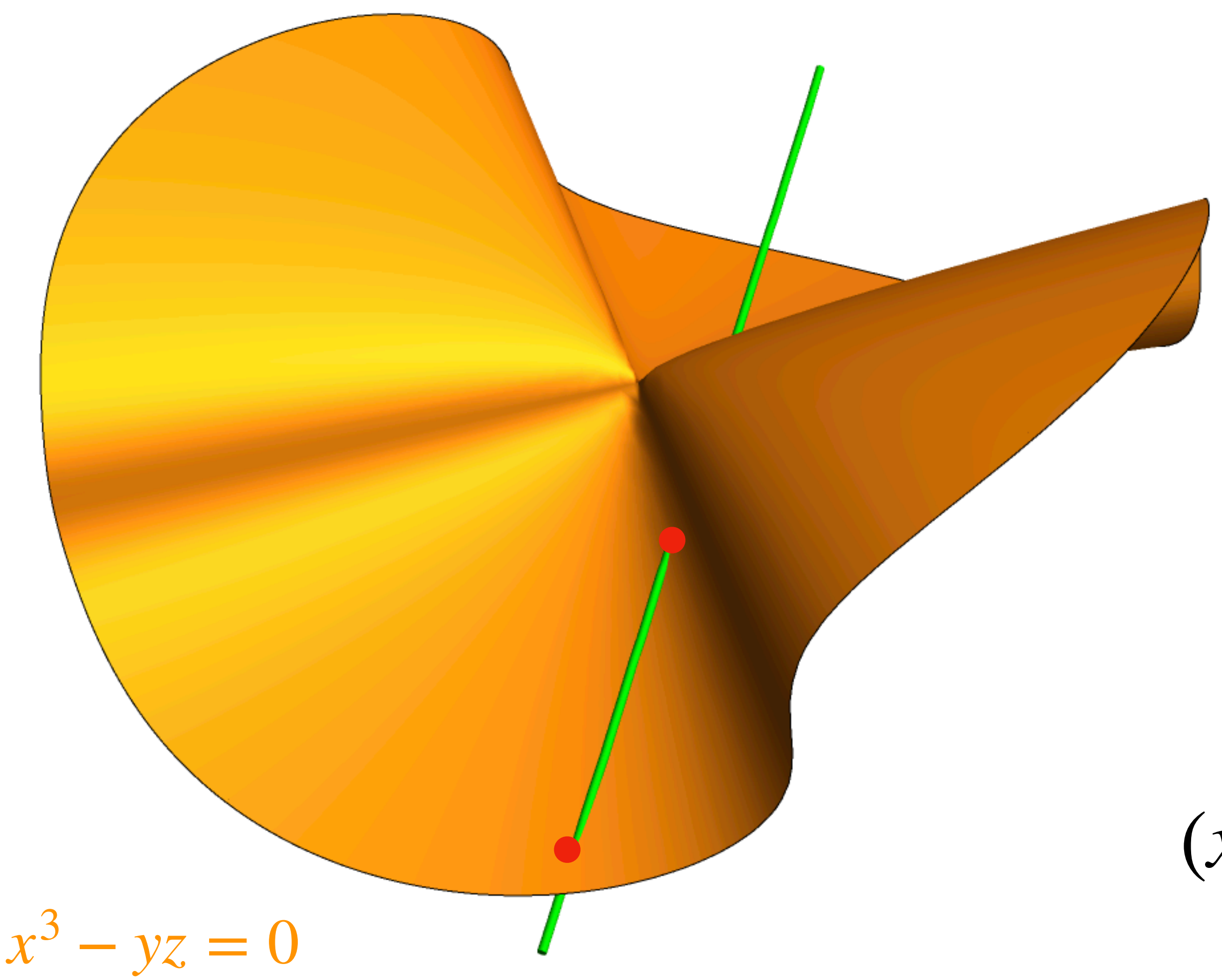

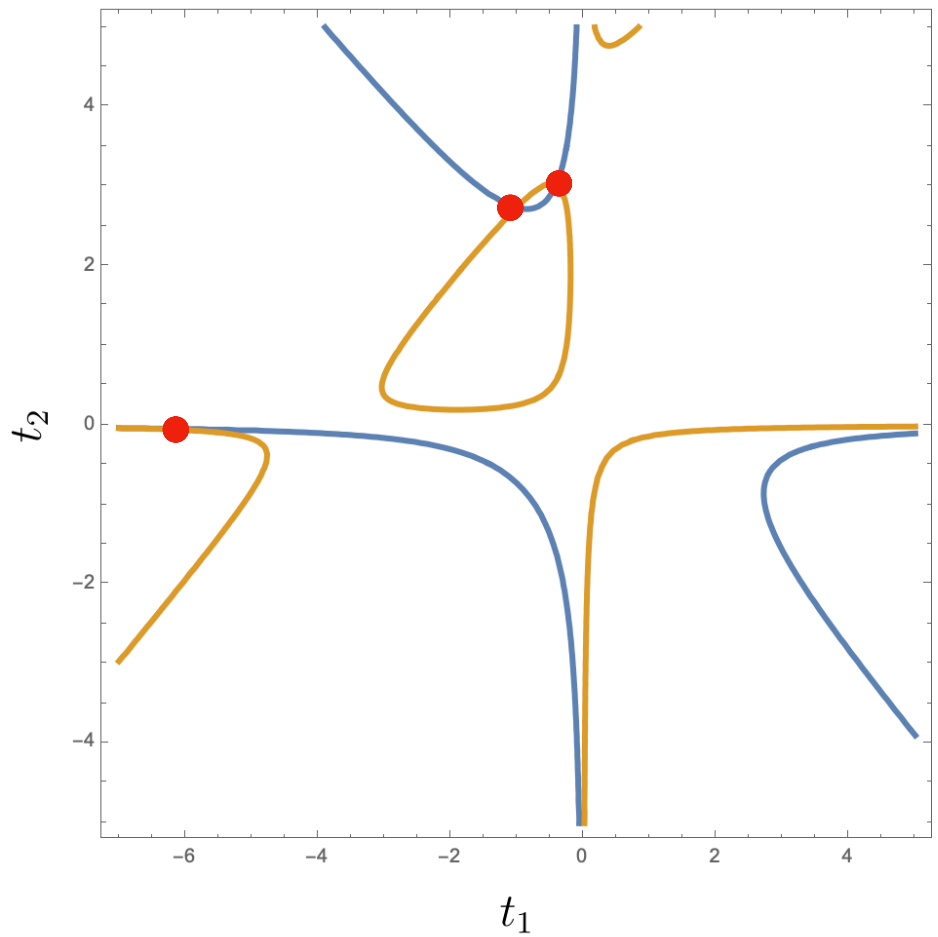

$$0 = a_0 + a_1 x + a_2 y + a_3 z$$
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$$(x, y, z) = (t_1 t_2, t_1^2 t_2, t_1 t_2^2) \text{ for some } (t_1, t_2)$$

Number of solutions

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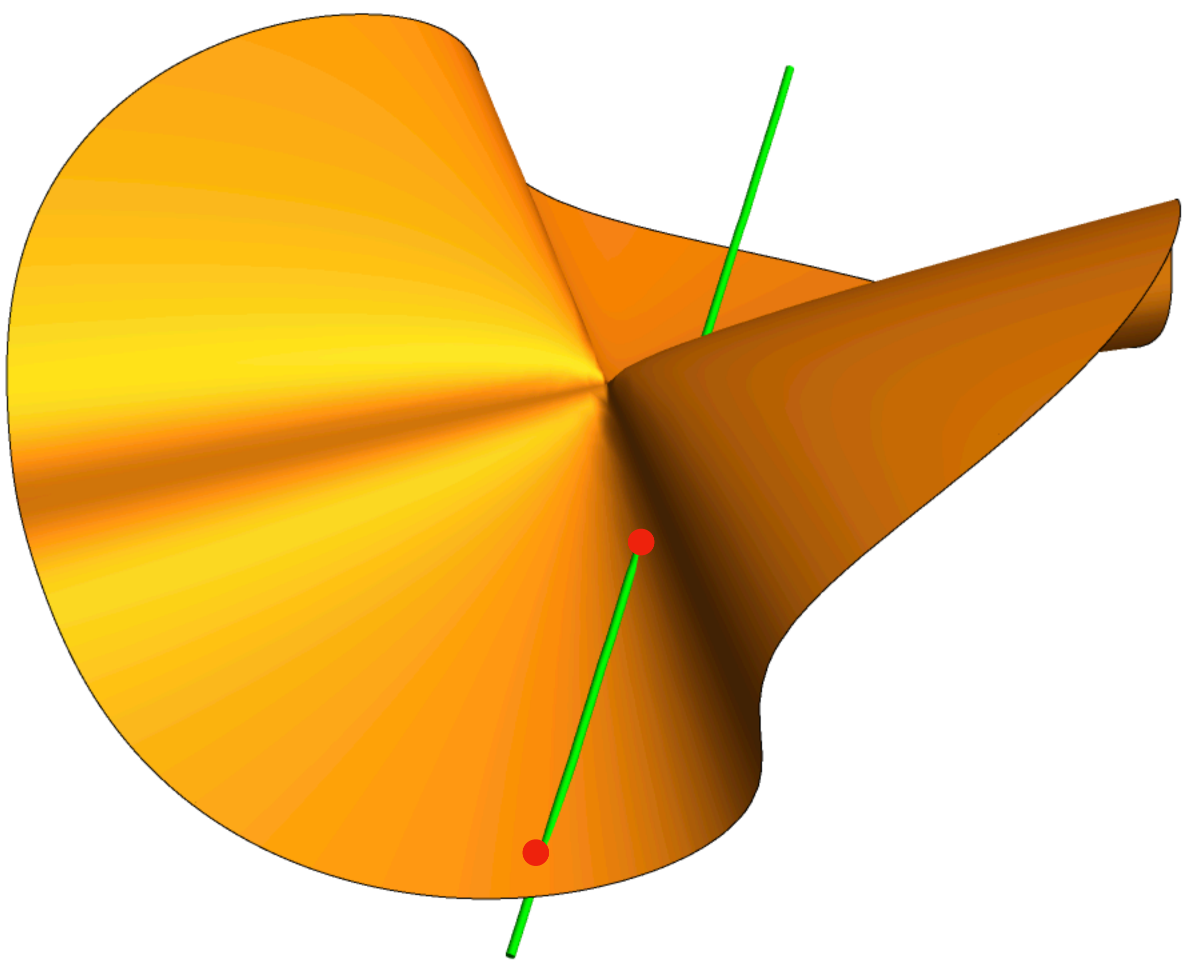

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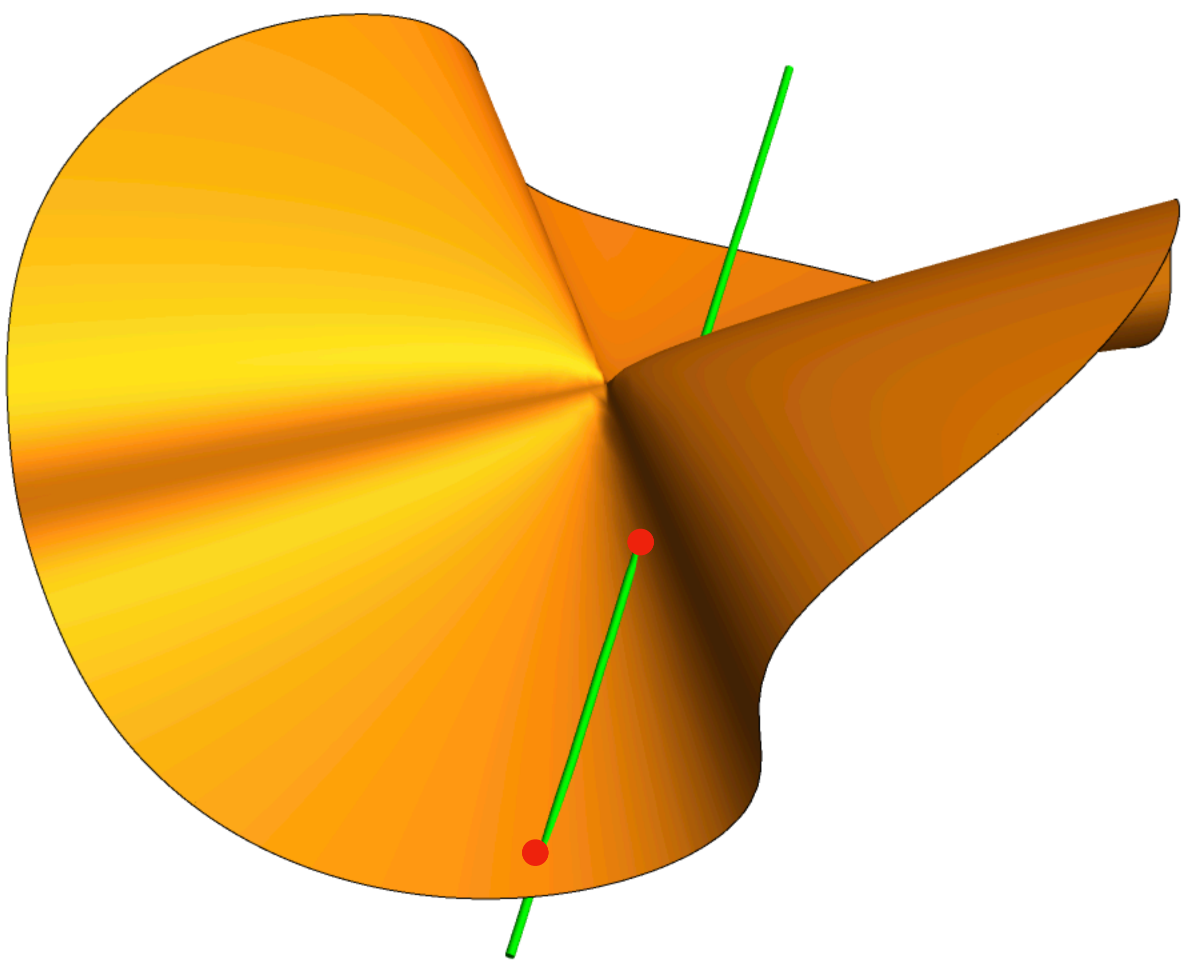

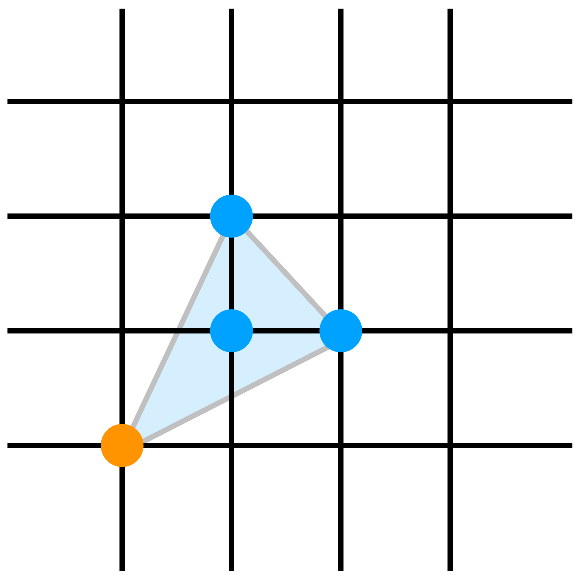

$$Y_A = \overline{\{(t_1 t_2, t_1^2 t_2, t_1 t_2^2) \;:\; (t_1, t_2) \in (\mathbb{C}^*)^2\}}$$

Theorem (Kushnirenko): $\deg(Y_A) = \mathrm{vol}(A)$

number of complex solutions is
bounded by the volume

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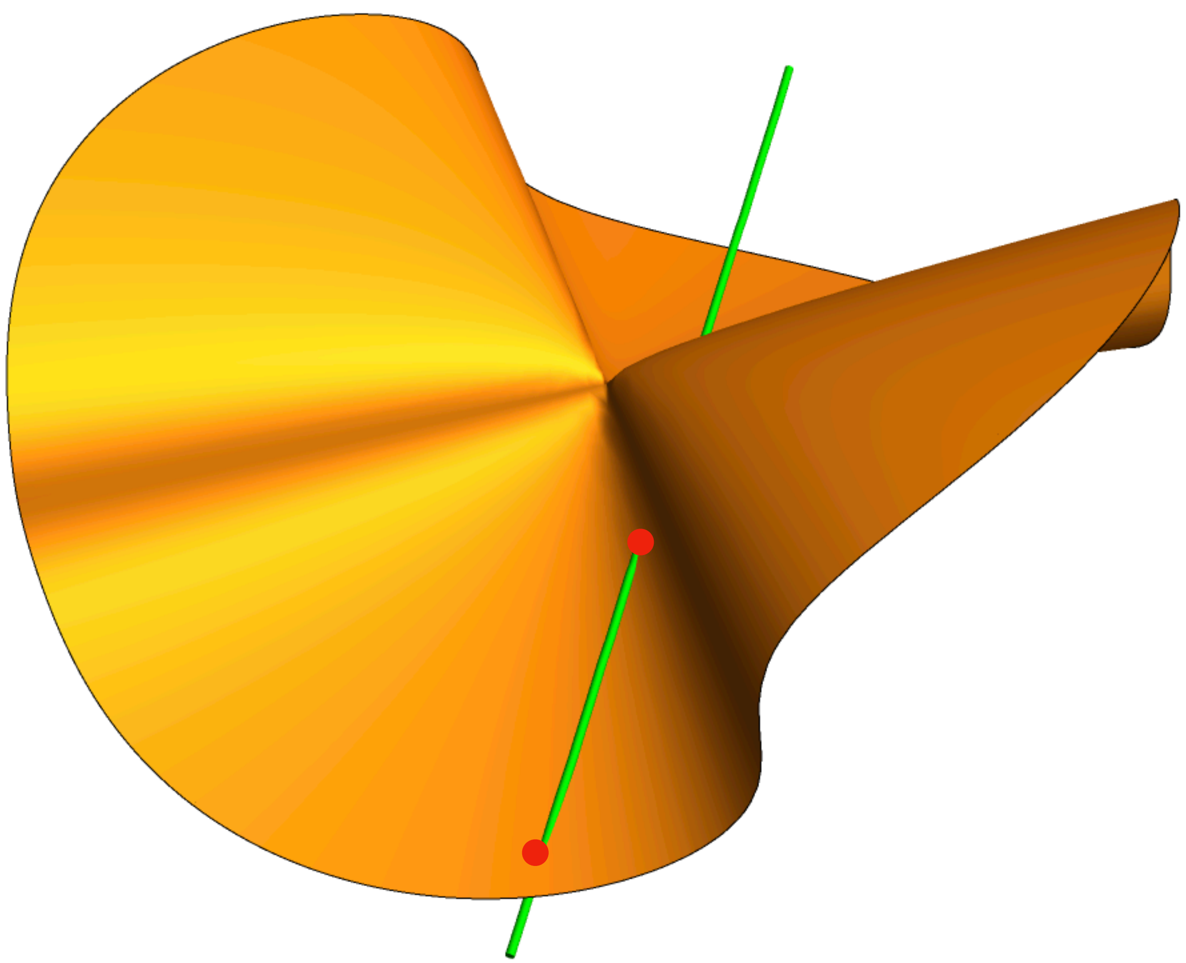

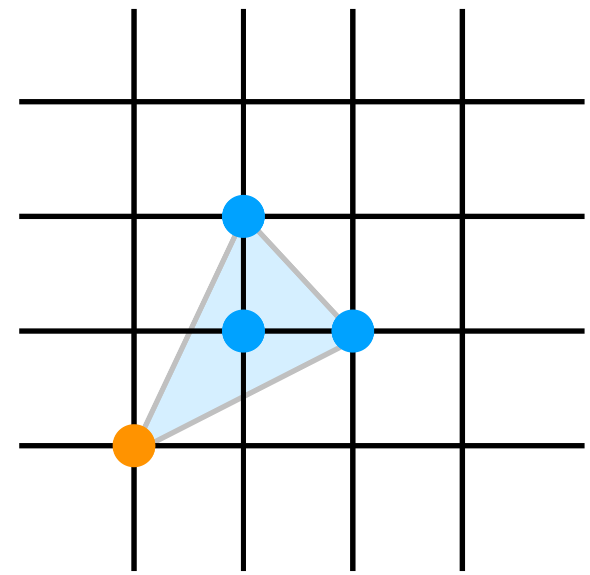

"Newton polytope"

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Monomial sparsity in algorithms

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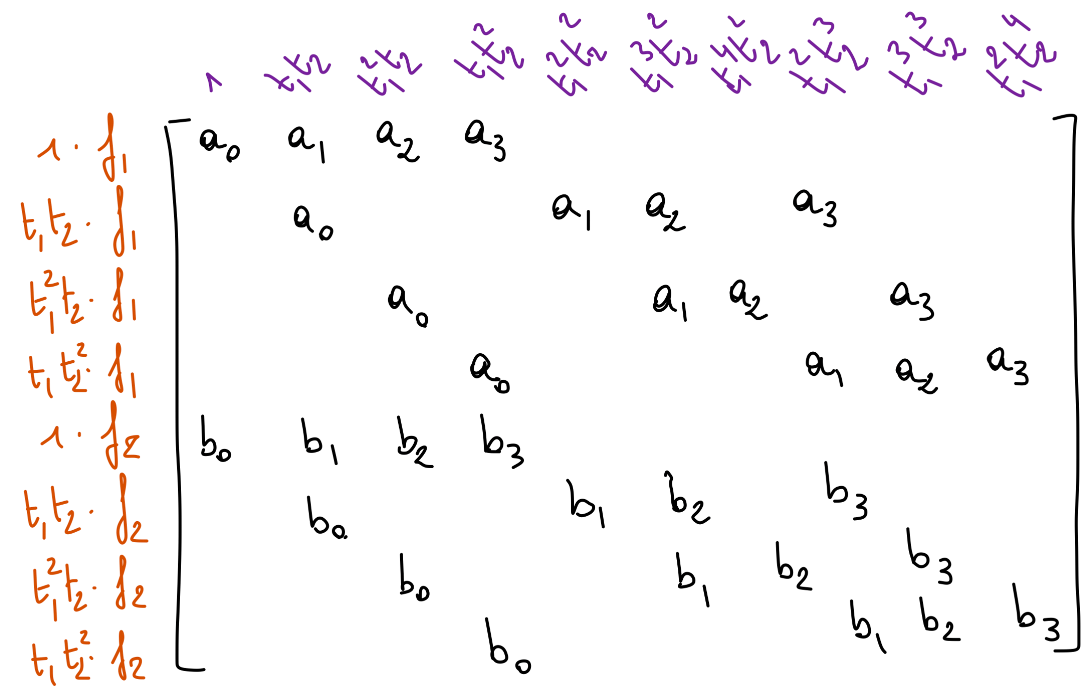

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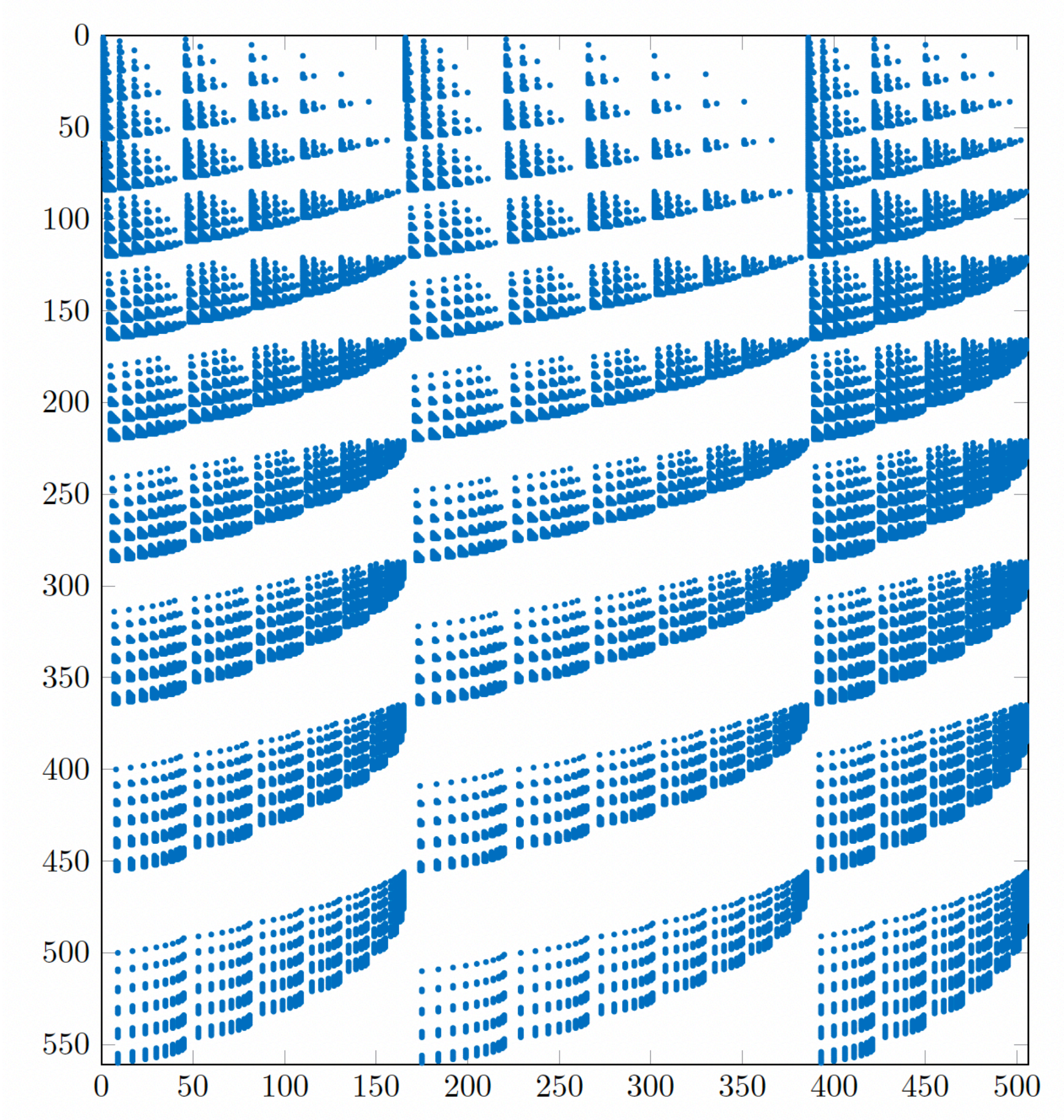

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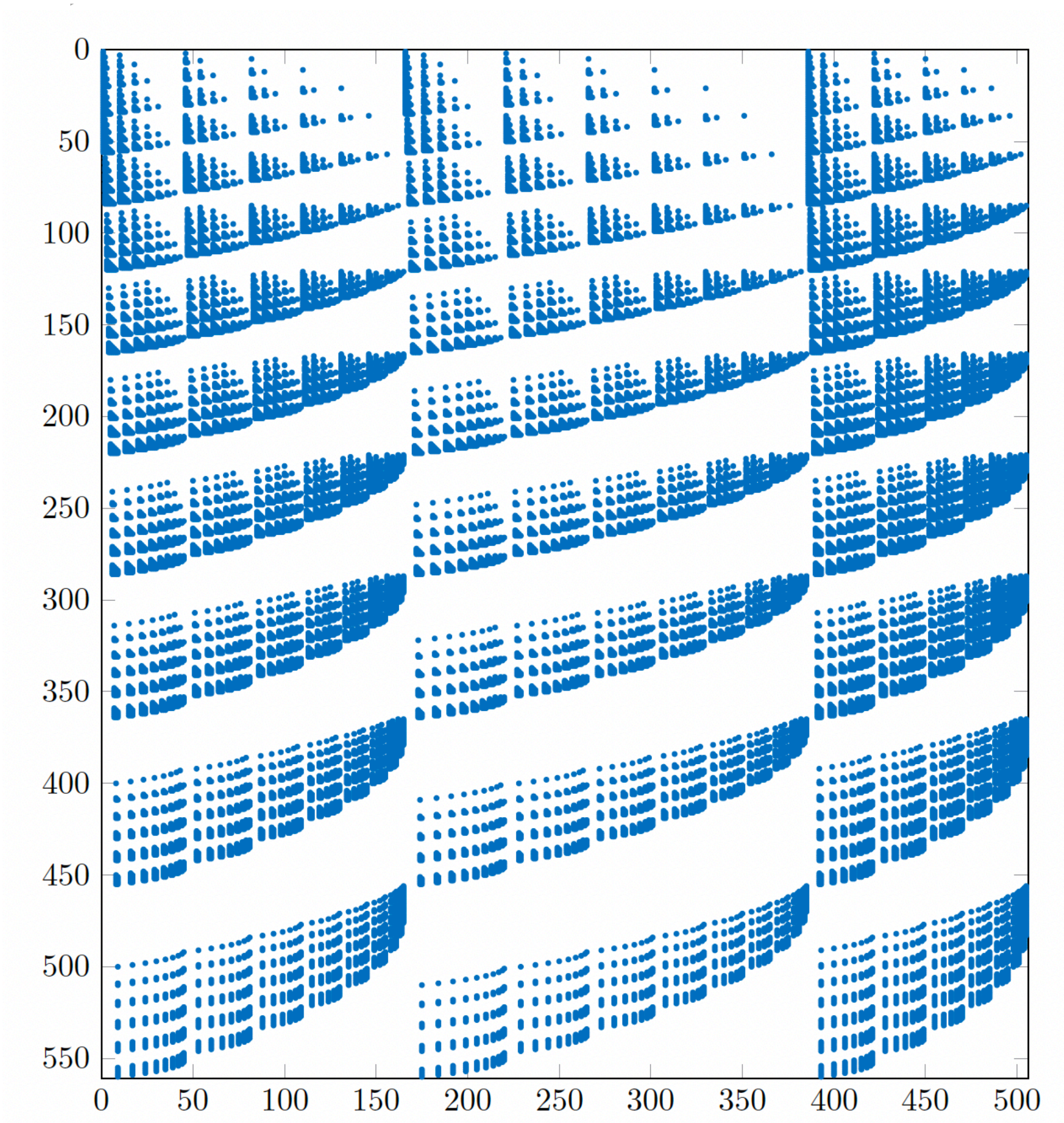

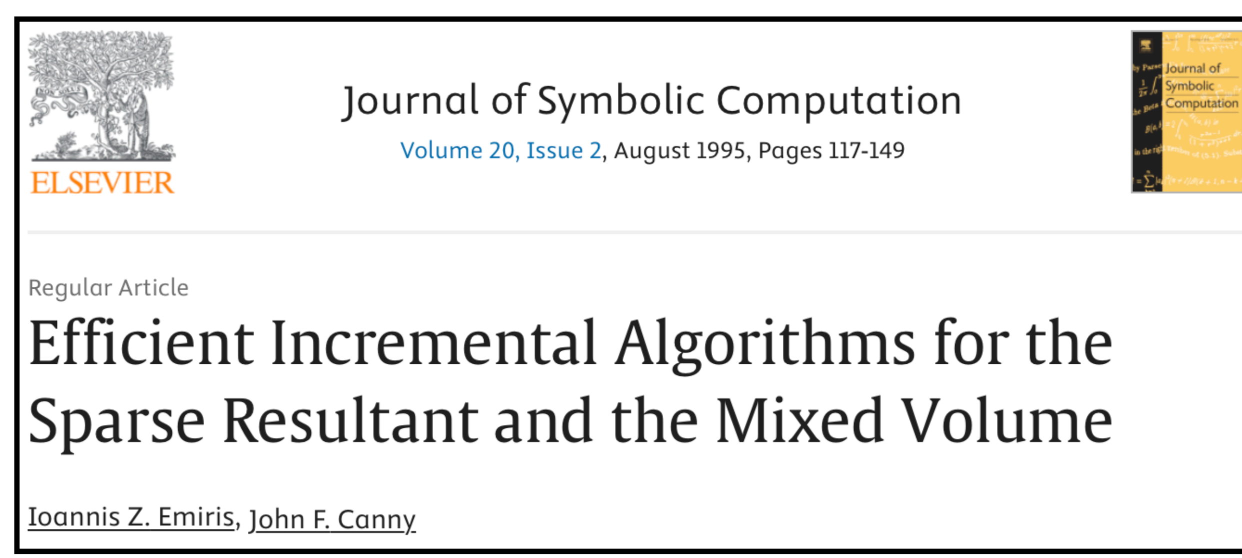
ELSEVIER
Journal of Symbolic Computation
Volume 20, Issue 2, August 1995, Pages 117-149

Regular Article

Efficient Incremental Algorithms for the Sparse Resultant and the Mixed Volume

Ioannis Z. Emiris, John F. Canny

sparse resultants, Gröbner bases, normal forms …

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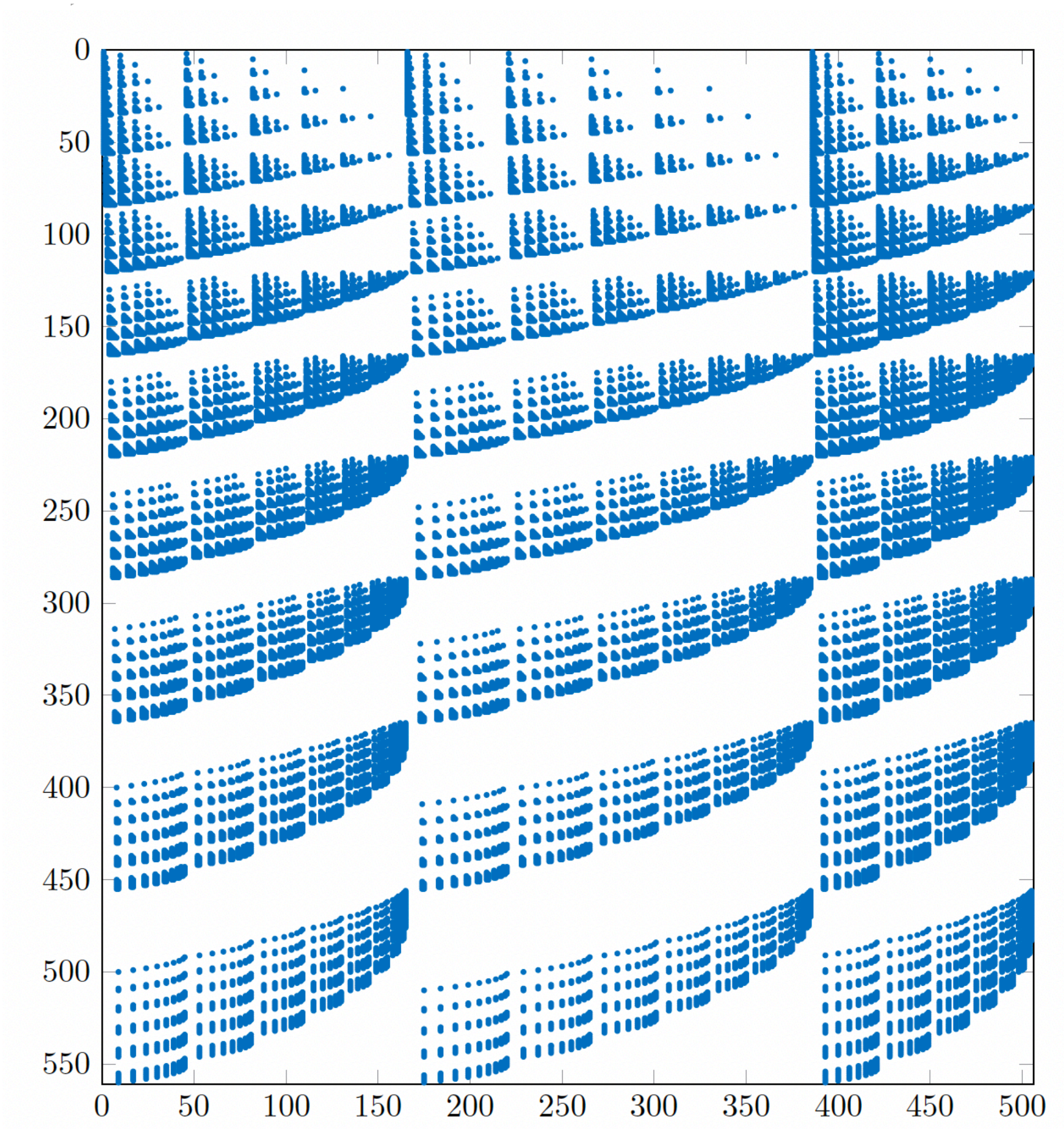

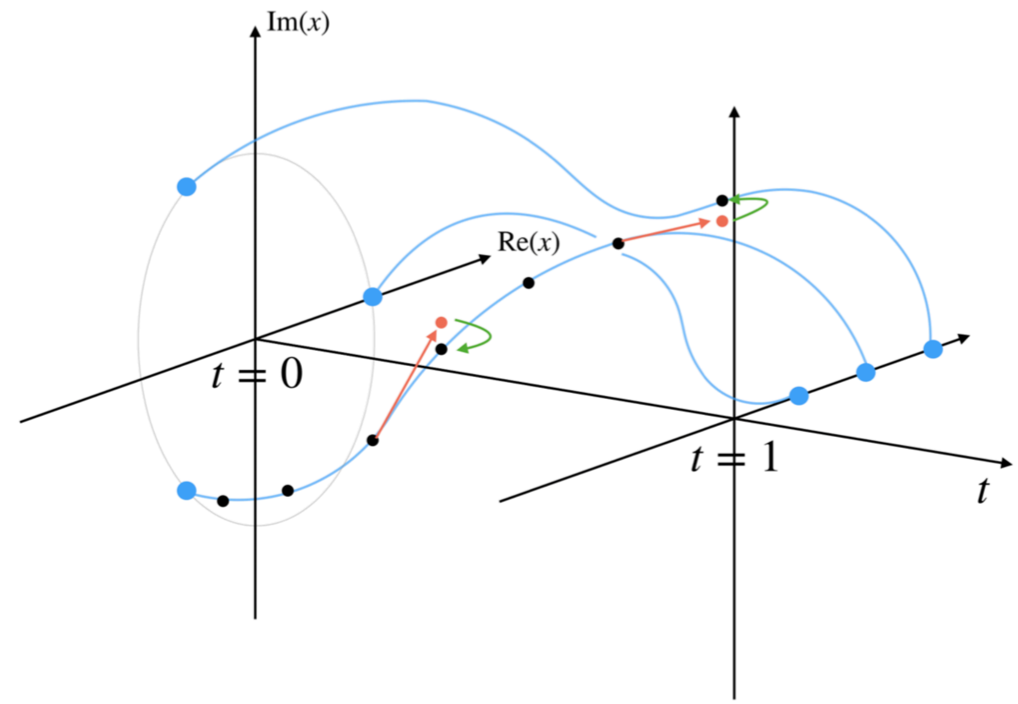

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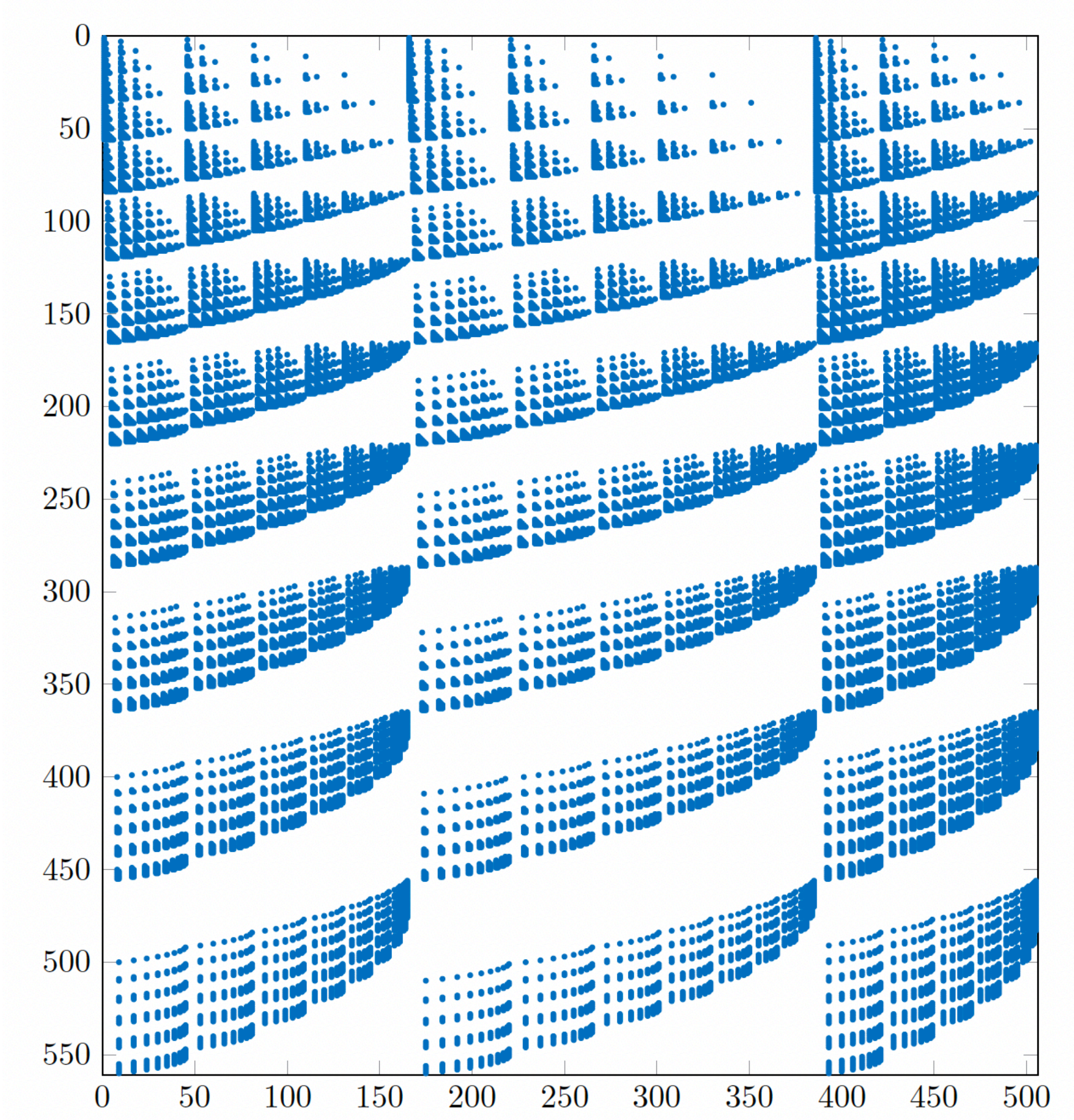

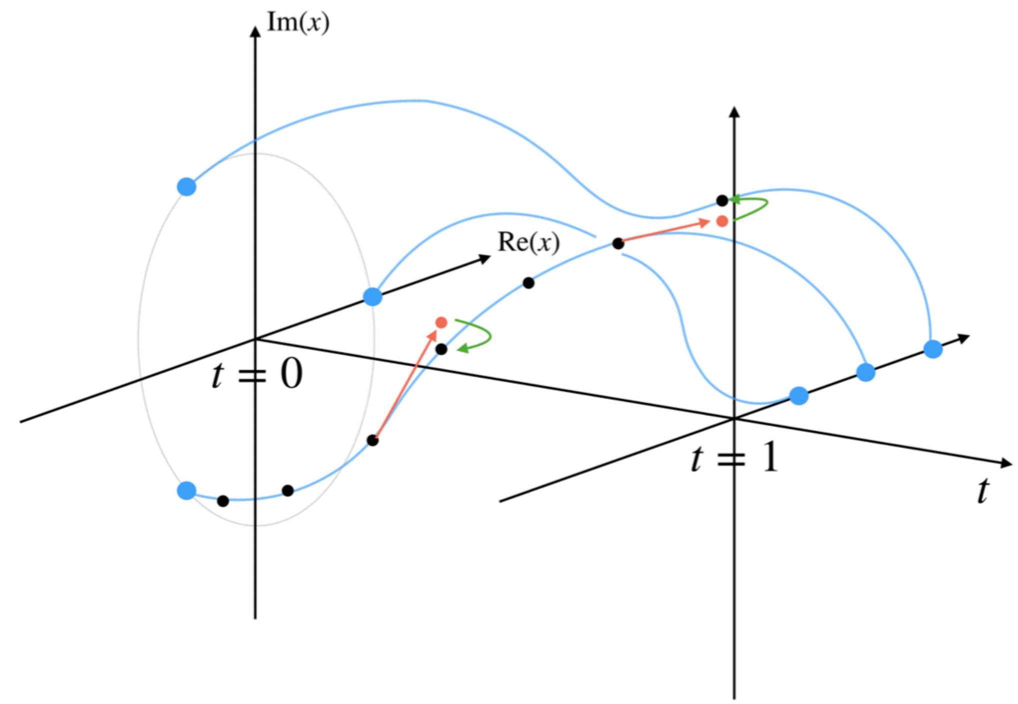

A Polyhedral Method for Solving Sparse Polynomial Systems

Birkett Huber, Bernd Sturmfels

Mathematics of Computation, Vol. 64, No. 212 (Oct., 1995), pp. 1541-1555 (15 pages)

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Numerische Mathematik

Computing the common zeros of two bivariate functions via Bézout resultants

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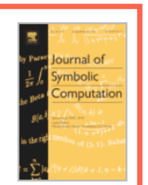

Truncated normal forms for solving polynomial systems: Generalized and efficient algorithms

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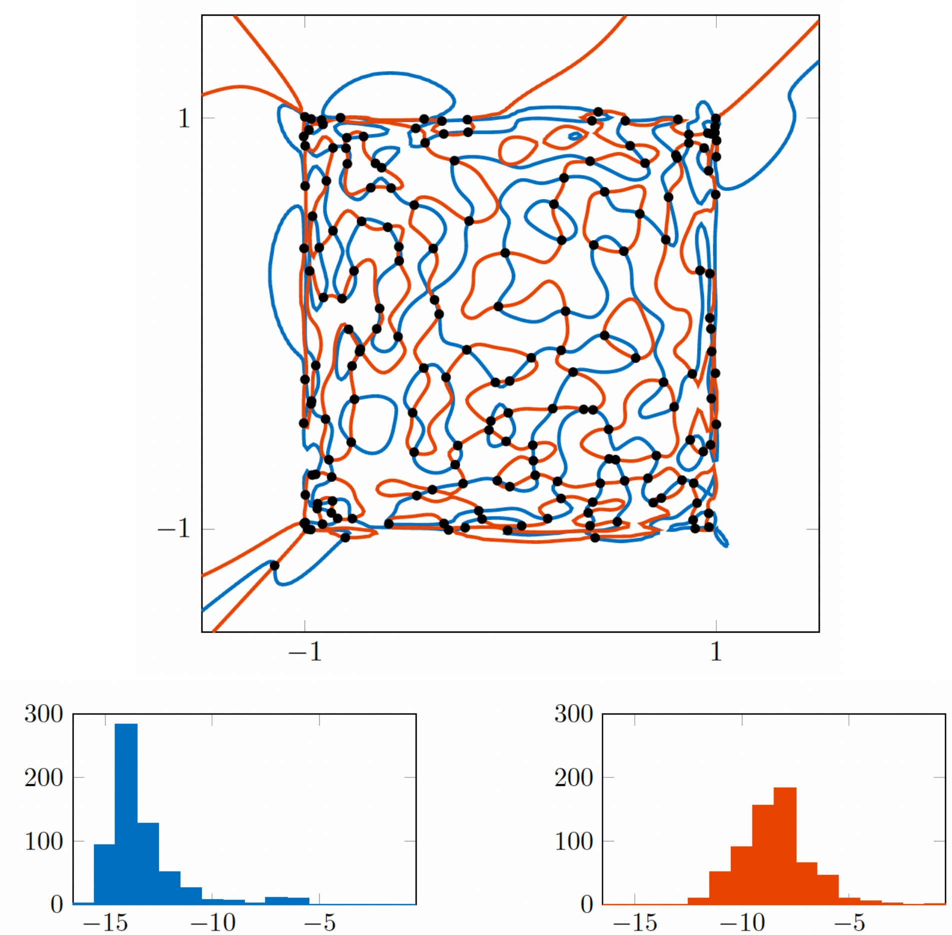

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[Submitted on 4 Jan 2024]

Chebyshev Subdivision and Reduction Methods for Solving Multivariable Systems of Equations

Erik Parkinson, Kate Wall, Jane Slagle, Daniel Treuhaft, Xander de la Bruere, Samuel Goldrup, Timothy Keith, Peter Call, Tyler J. Jarvis

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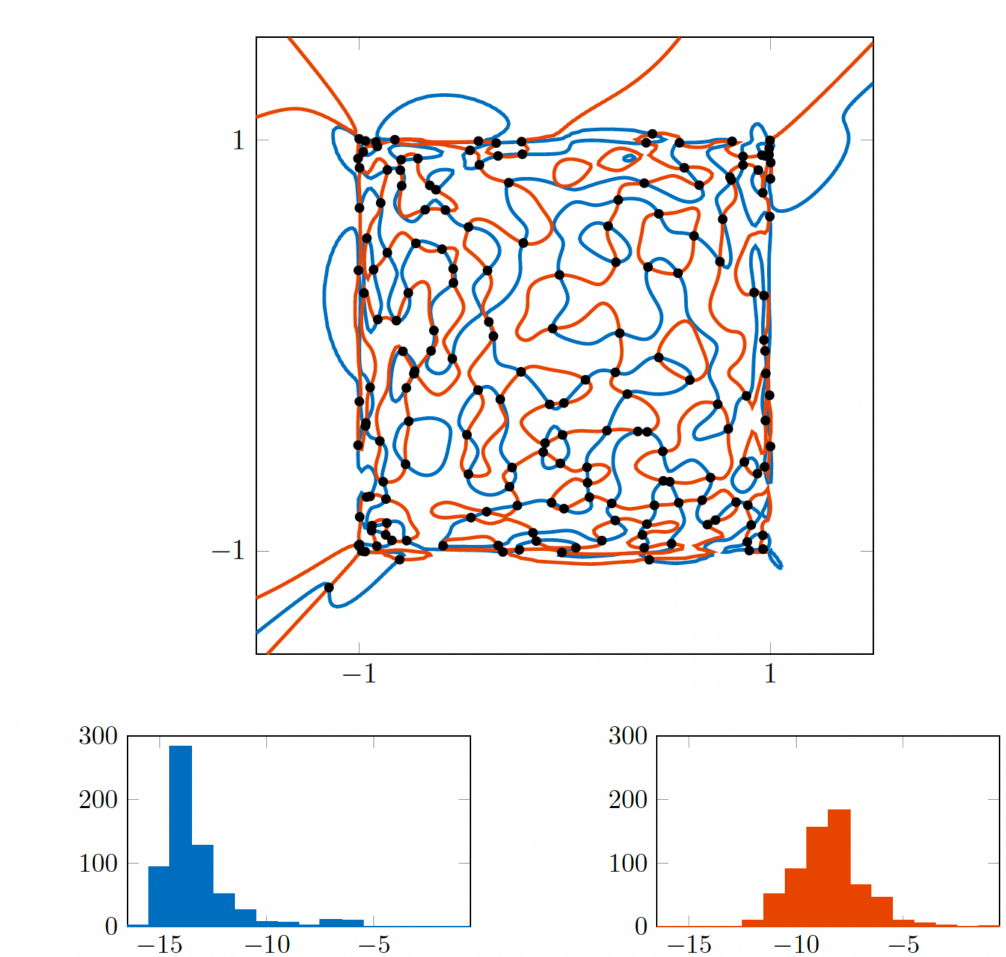
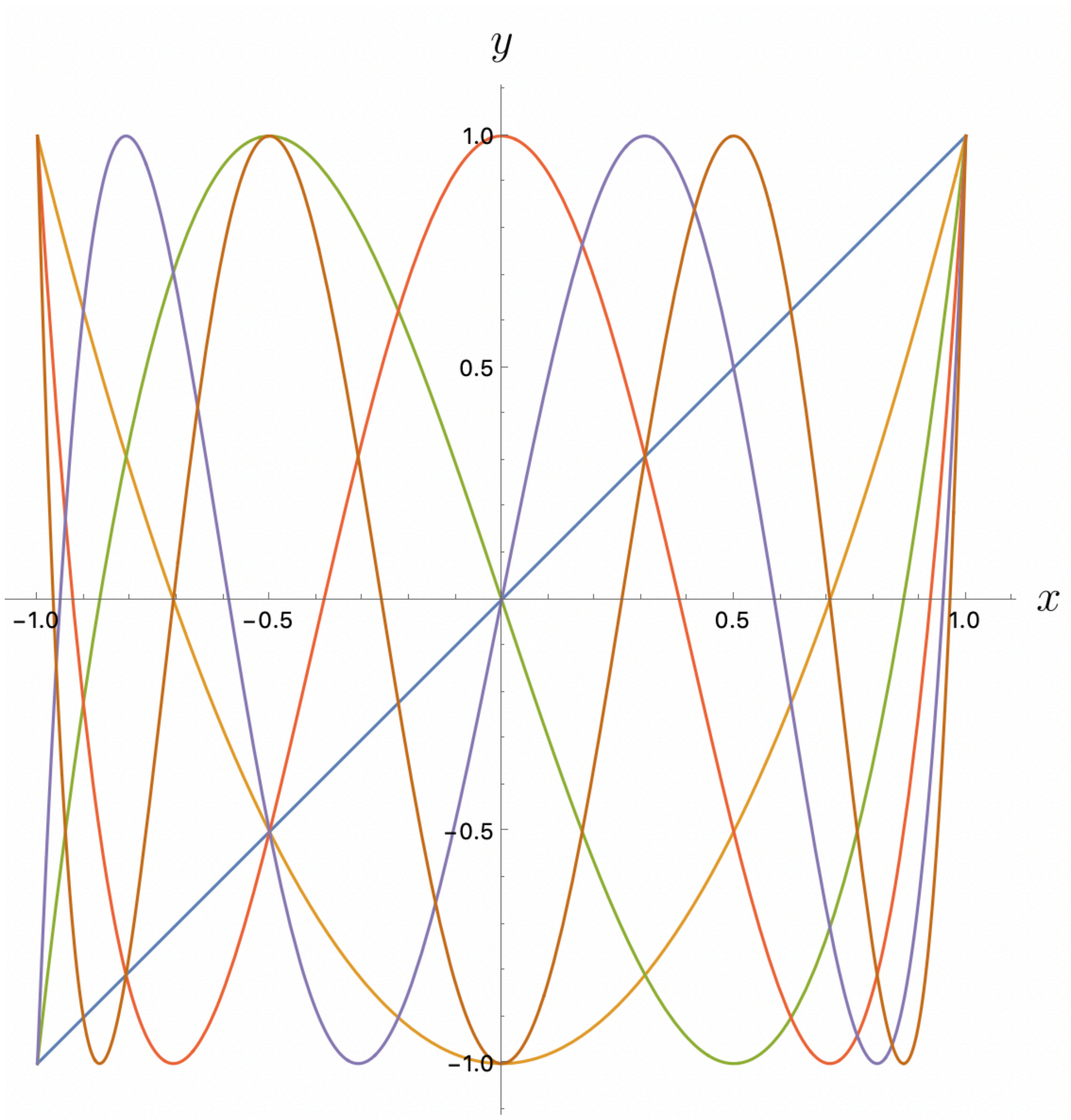


Chebyshev polynomials

$$T_n(x) = \cos(n \cdot \arccos(x)), \quad x \in [-1,1]$$

$$T_0(x) = 1$$
$$T_1(x) = x$$
$$T_2(x) = 2x^2 - 1$$
$$T_3(x) = 4x^3 - 3x$$
$$T_4(x) = 8x^4 - 8x^2 + 1$$
$$T_5(x) = 16x^5 - 20x^3 + 5x$$
$$T_6(x) = 32x^6 - 48x^4 + 18x^2 - 1$$



Chebyshev plane curves

$$a + b \cdot t^5 + c \cdot t^7 = 0$$

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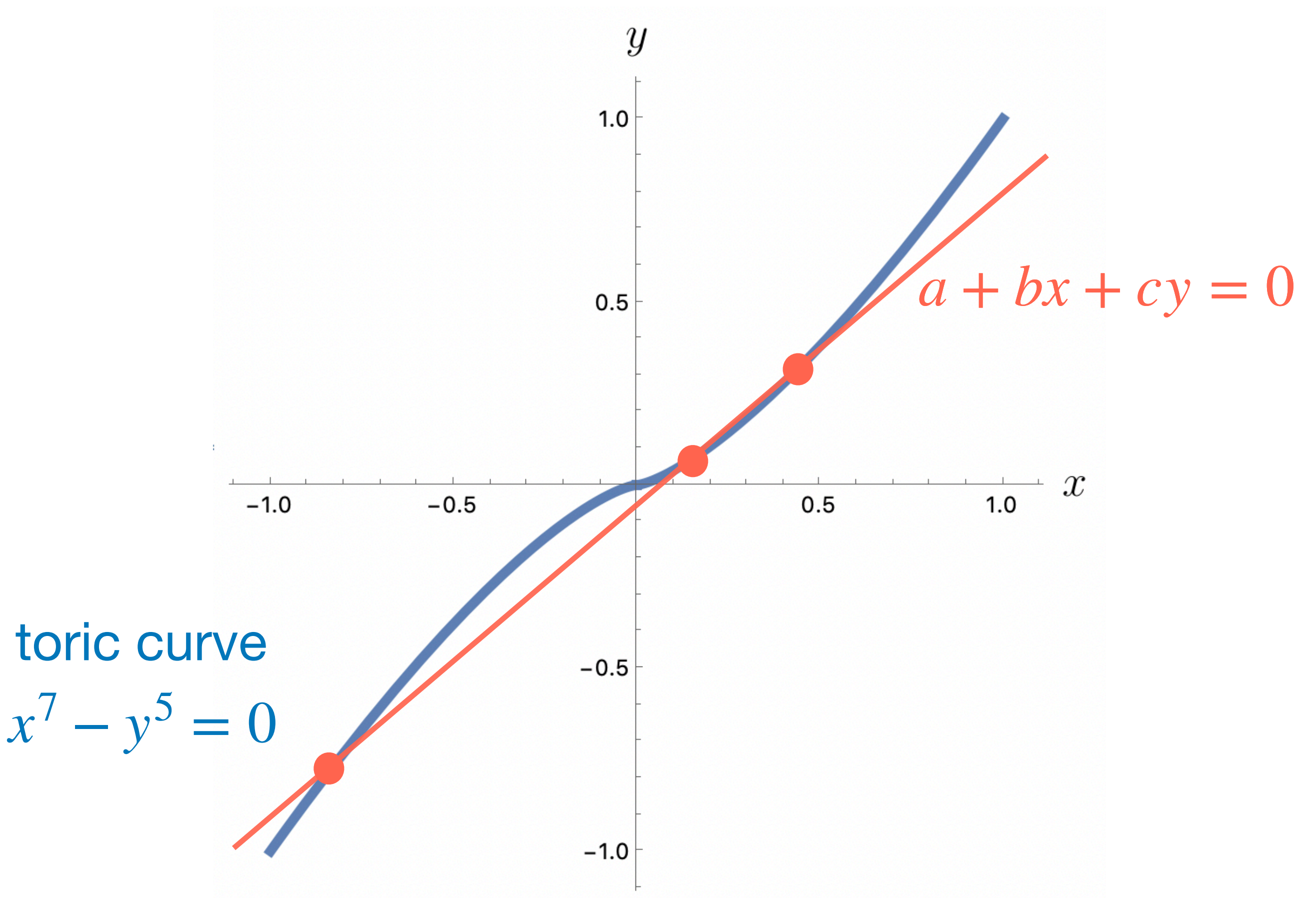

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$$a + b \cdot T_5(t) + c \cdot T_7(t) = 0$$

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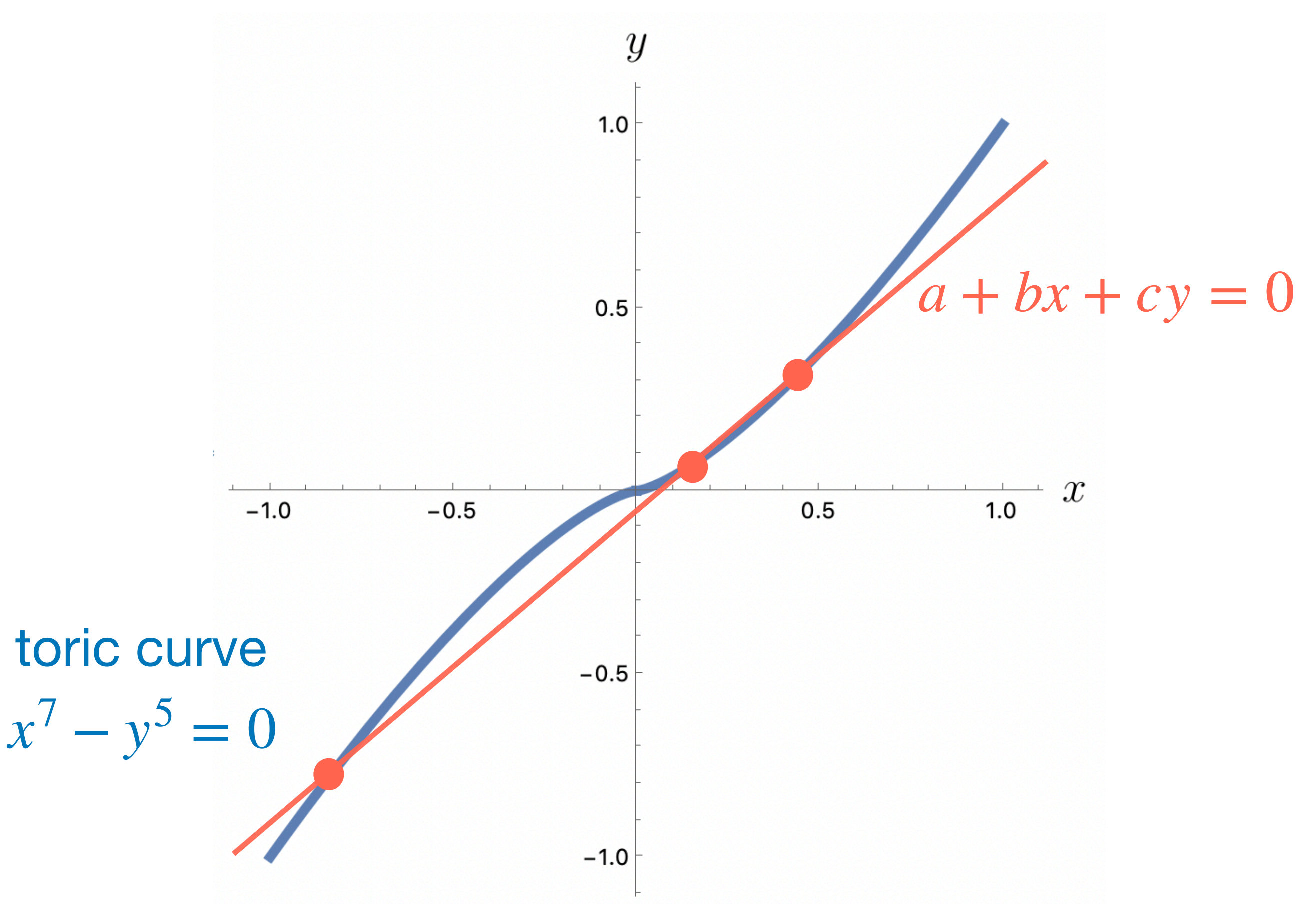

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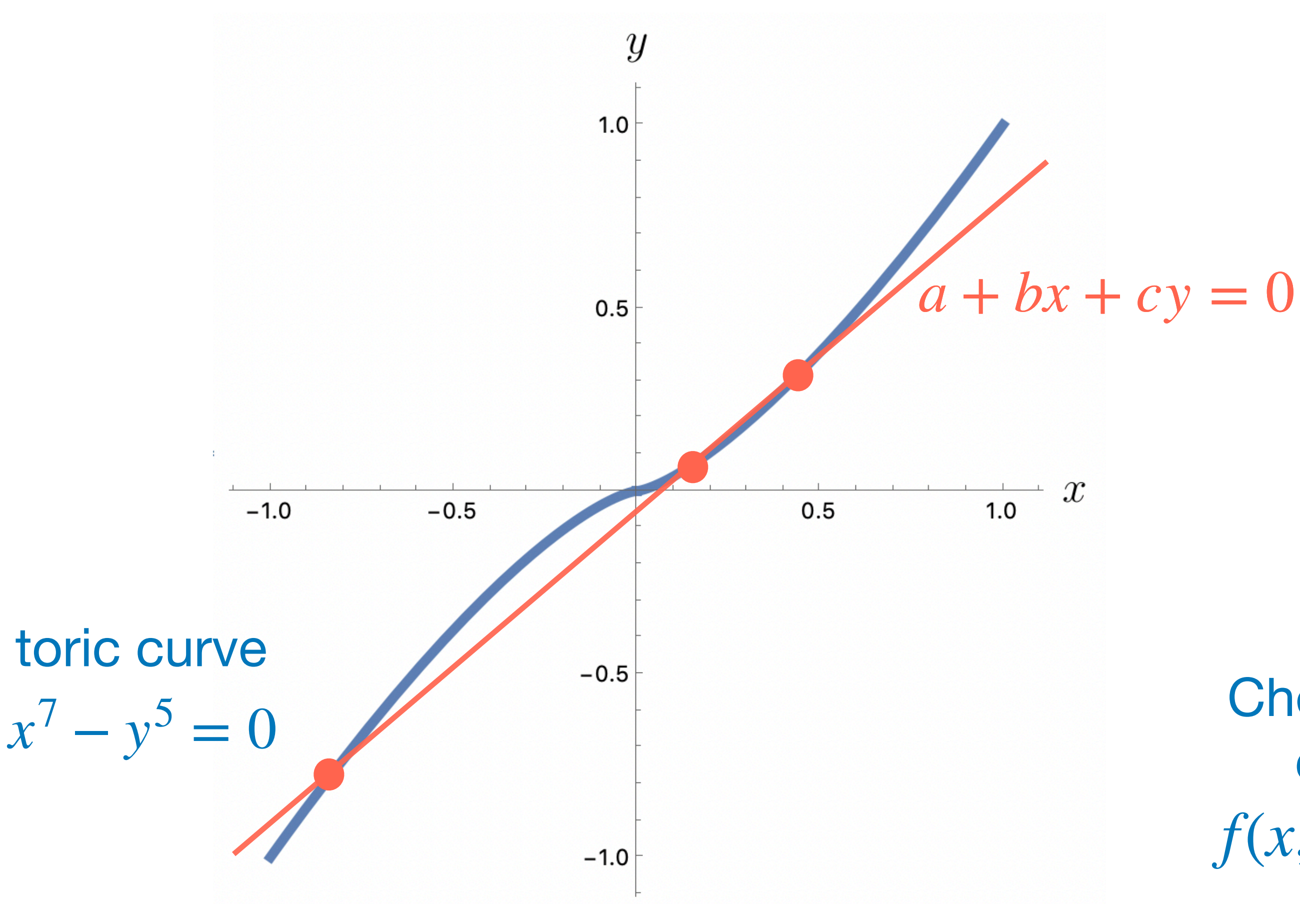

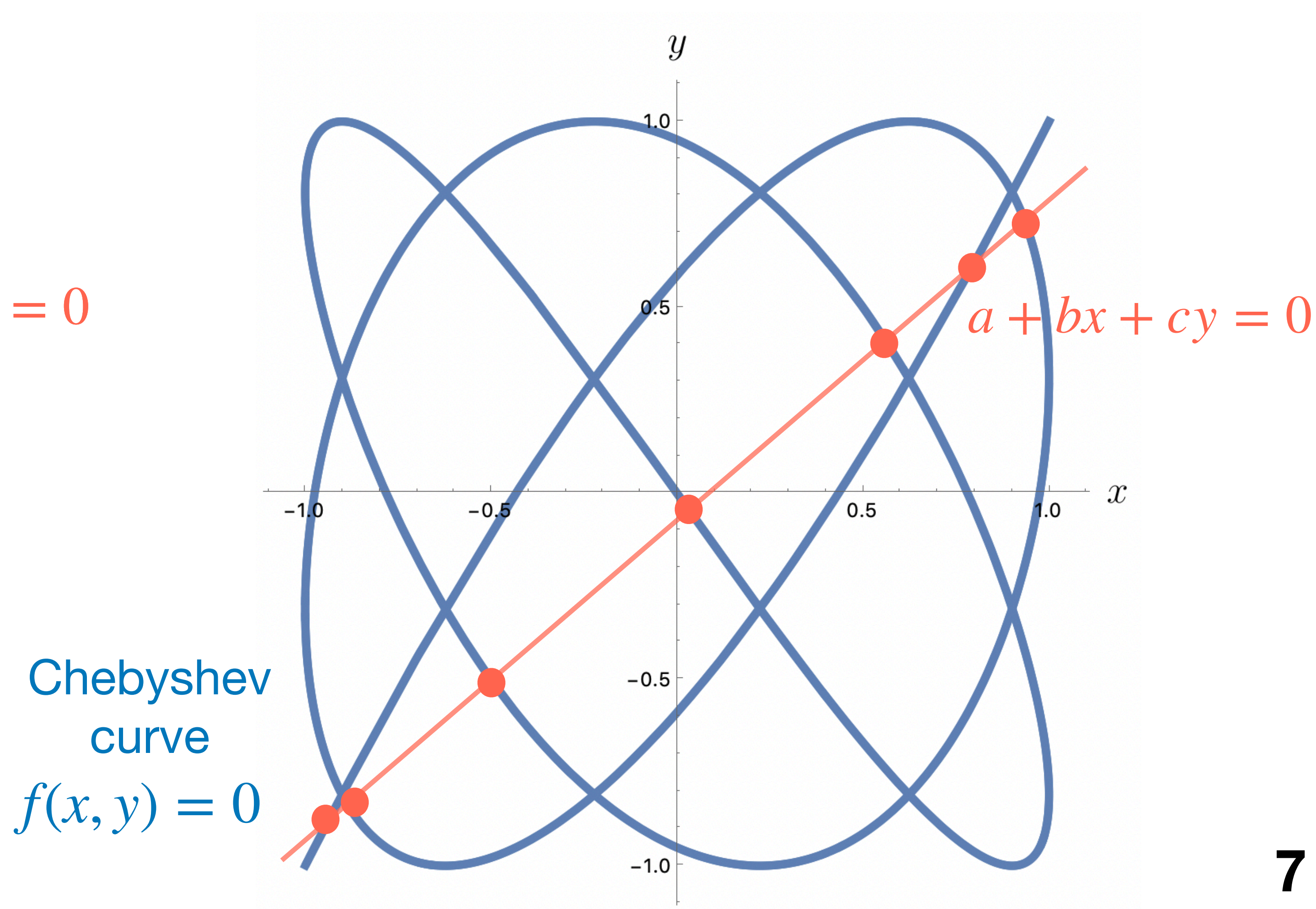

Chebyshev plane curves

$$T_n(\cos(\theta)) = \cos(n \cdot \theta)$$

$$\int_{-1}^{1} T_m(t) \cdot T_n(t) \cdot \frac{\mathrm{d}t}{\sqrt{1 - t^2}} = \delta_{mn}\pi$$

Chebyshev plane curves

$$T_n(\cos(\theta)) = \cos(n \cdot \theta) \qquad \int_{-1}^{1} T_m(t) \cdot T_n(t) \cdot \frac{\mathrm{d}t}{\sqrt{1-t^2}} = \delta_{mn}\pi$$

$$T_0(t) = 1, \qquad T_1(t) = t, \qquad T_{n+1}(t) = 2t \cdot T_n(t) - T_{n-1}(t)$$

$$T_n(T_m(t)) = T_m(T_n(t)) = T_{mn}(t), \qquad 2T_m(t)T_n(t) = T_{m+n}(t) + T_{|m-n|}(t)$$

Chebyshev plane curves

$$T_n(\cos(\theta)) = \cos(n \cdot \theta) \qquad \int_{-1}^{1} T_m(t) \cdot T_n(t) \cdot \frac{\mathrm{d}t}{\sqrt{1-t^2}} = \delta_{mn}\pi$$

$$T_0(t) = 1, \qquad T_1(t) = t, \qquad T_{n+1}(t) = 2t \cdot T_n(t) - T_{n-1}(t)$$

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$$\mathscr{C}_{m,n} = \{(T_m(t), T_n(t)) \, : \, t \in \mathbb{C}\} \subset \mathbb{C}^2$$

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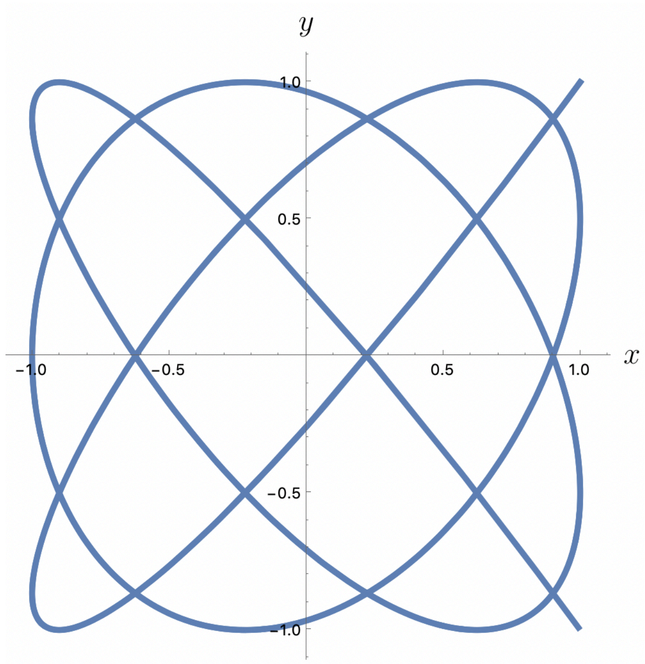
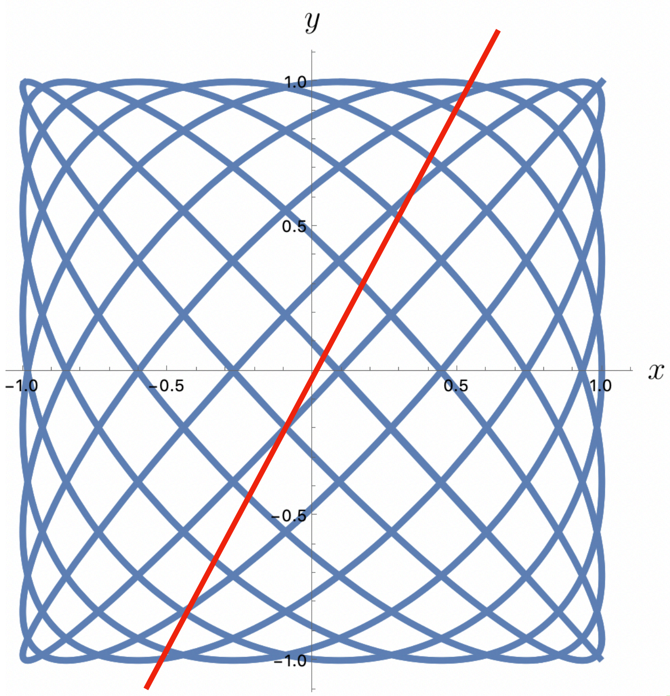
Theorem. $\mathscr{C}_{m,n} \subset \{(x,y) \in \mathbb{C}^2 : T_n(x) - T_m(y) = 0\}$

If $\gcd(m,n) = 1$ then $T_n(x) - T_m(y)$ is irreducible

Freudenburg-Freudenburg, Curves defined by Chebyshev polynomials (2009)

Chebyshev plane curves





Theorem. The Chebyshev curves

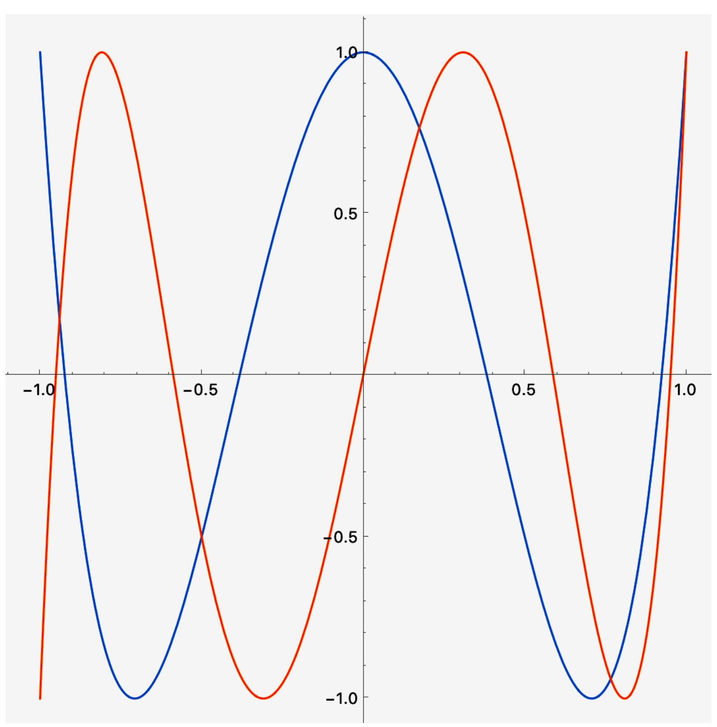
$$\mathscr{C}_{m,m+1} \cap \mathbb{R}^2$$

are hyperbolic with respect to 0.

$$a \cdot T_m(t) + b \cdot T_{m+1}(t) = 0$$

has only real roots.

Proof:



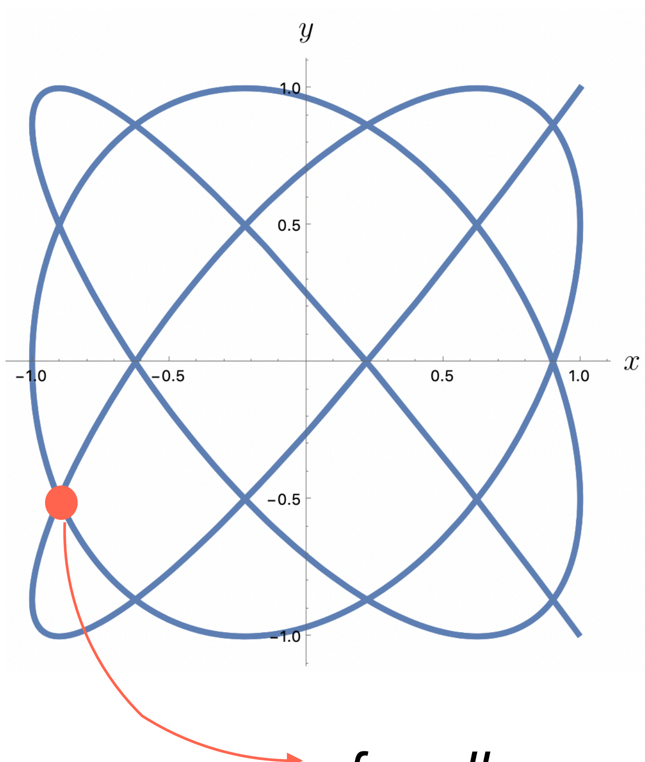
$$a \cdot T_4(t) + b \cdot T_5(t) = 0$$

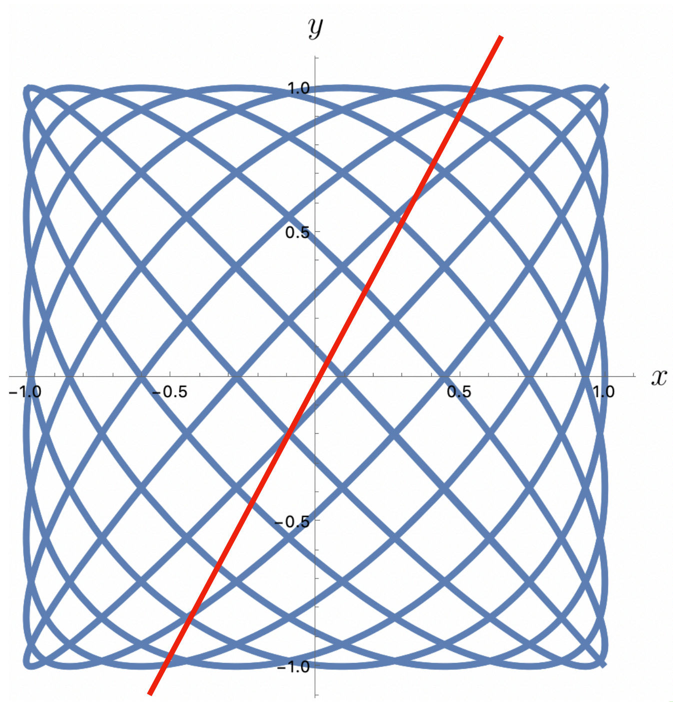
changes sign between any two roots of $T_5(t)$

$\Rightarrow$ 4 real roots

$\Rightarrow$ 5 real roots $\square$

Chebyshev plane curves





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for all m, n, all singularities are nodes, and they can be listed [Freudenburg x 2]

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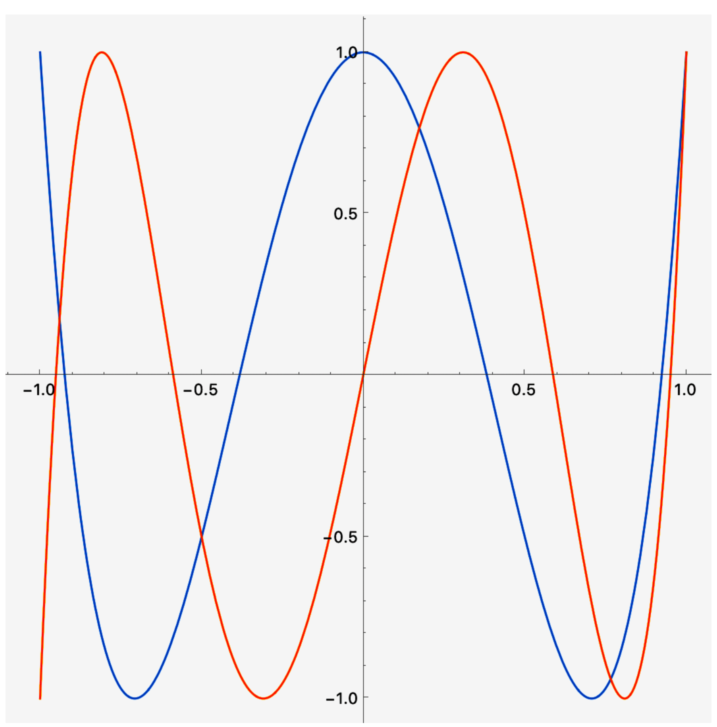

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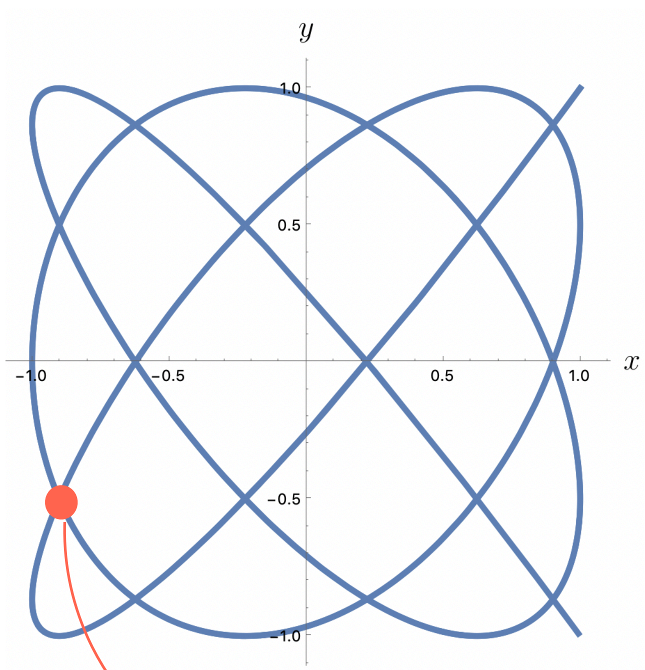

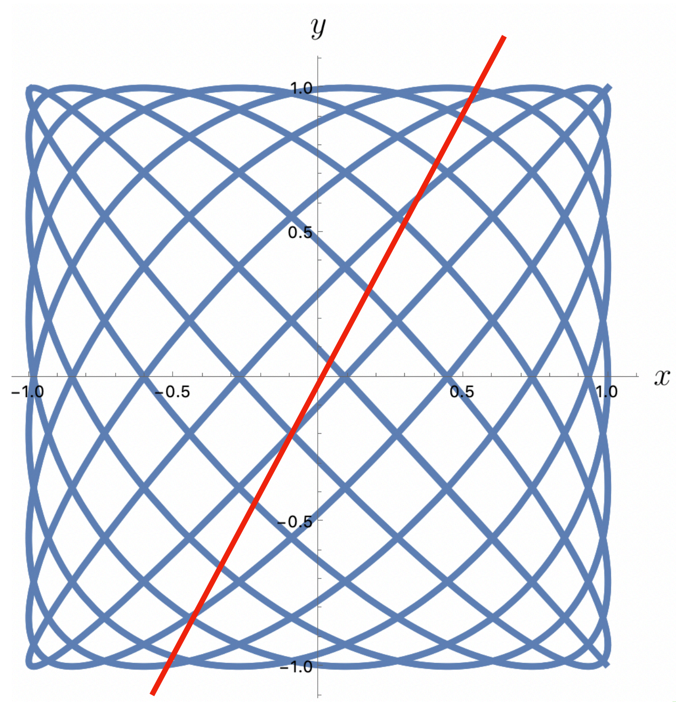

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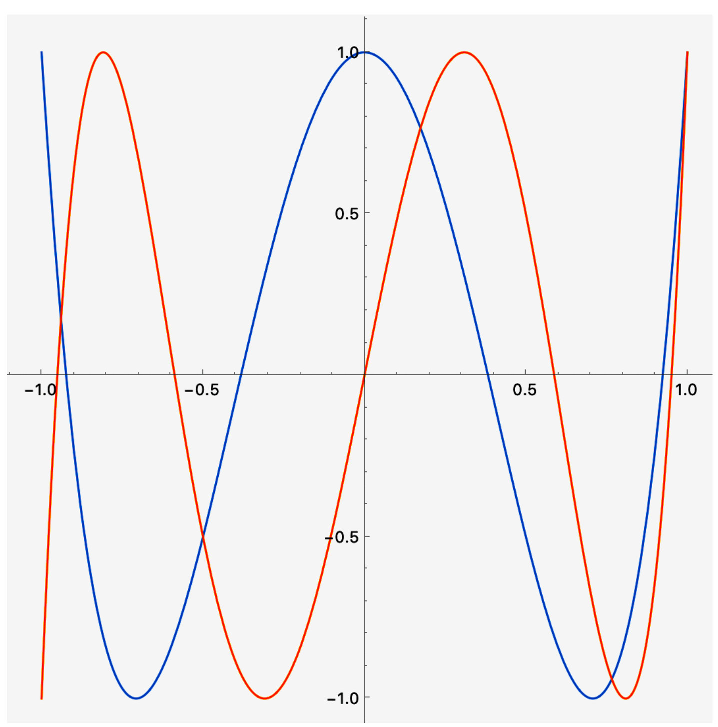

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Chebyshev space curves

Theorem If m, n, p are pairwise coprime, then $t \mapsto (T_m(t), T_n(t), T_p(t))$ parametrizes a smooth curve $\mathscr{C}_{m,n,p}$

$\exists P(x, y, z)$ s.t. $P(T_m(t), T_n(t), T_p(t)) = t$ and

$$\mathscr{C}_{m,n,p} = \{(x, y, z) \in \mathbb{C}^3 \,:\, x - T_m(P) = y - T_n(P) = z - T_p(P) = 0\}$$

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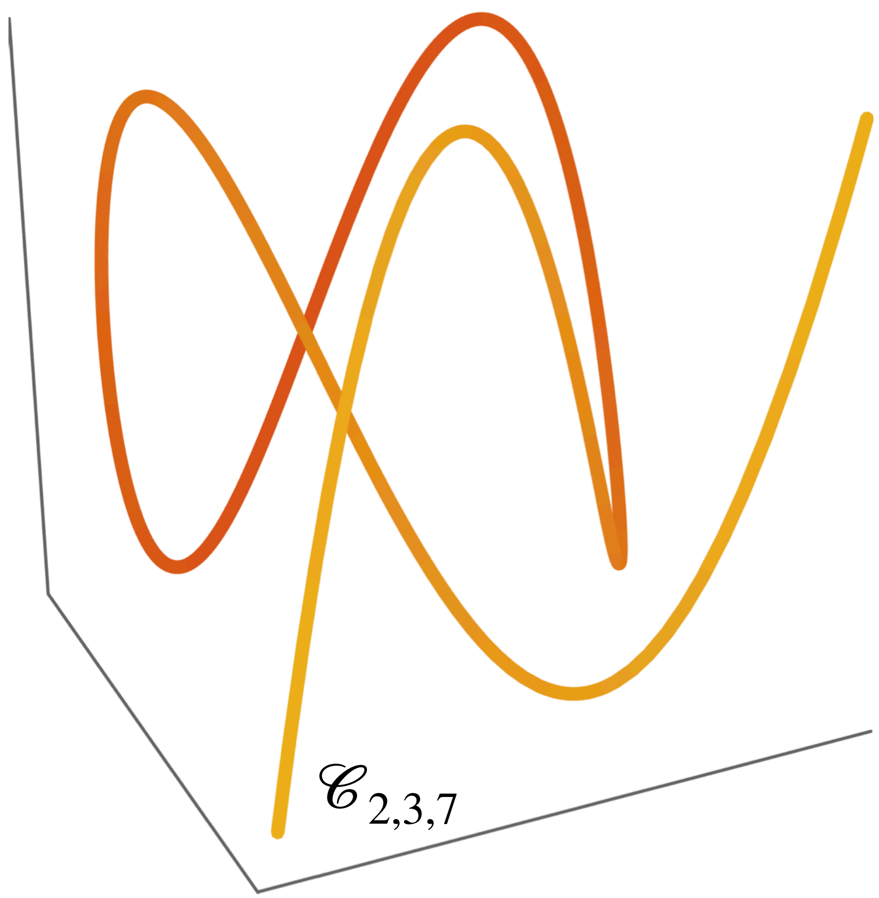

$$(m, n, p) = (2,3,7)$$

$$P(x, y, z) = 2T_4(x)T_1(z) - T_5(y)$$

$$= -16y^5 + 16x^4z + 20y^3 - 16x^2z - 5y + 2z$$

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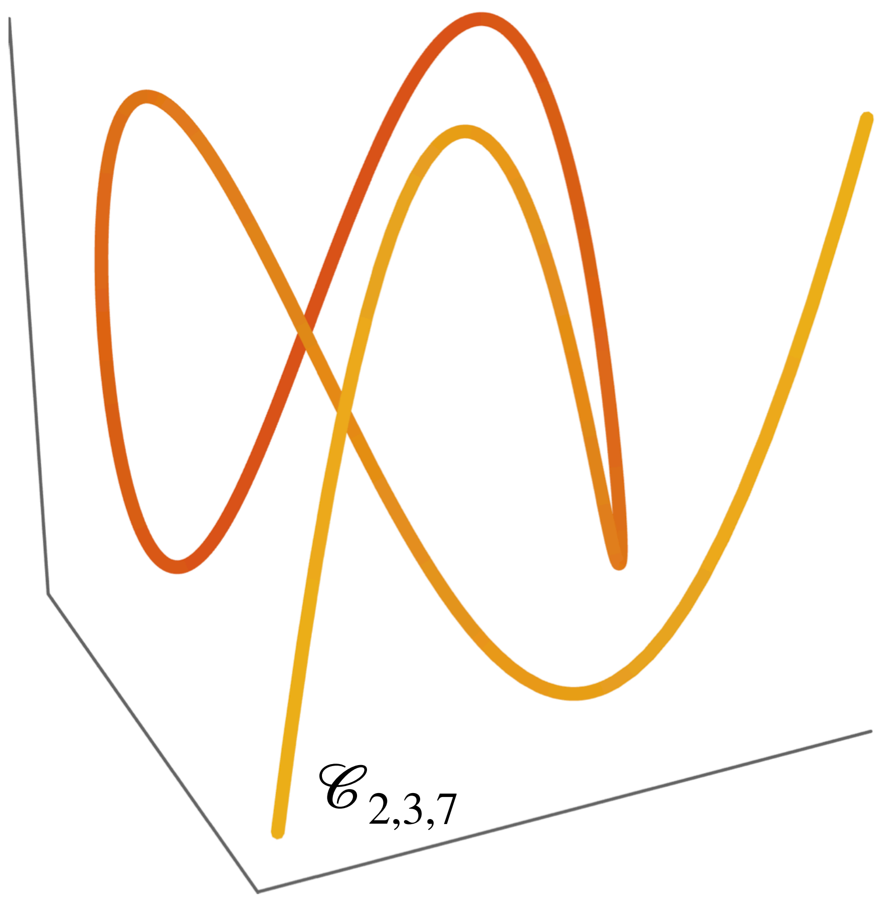

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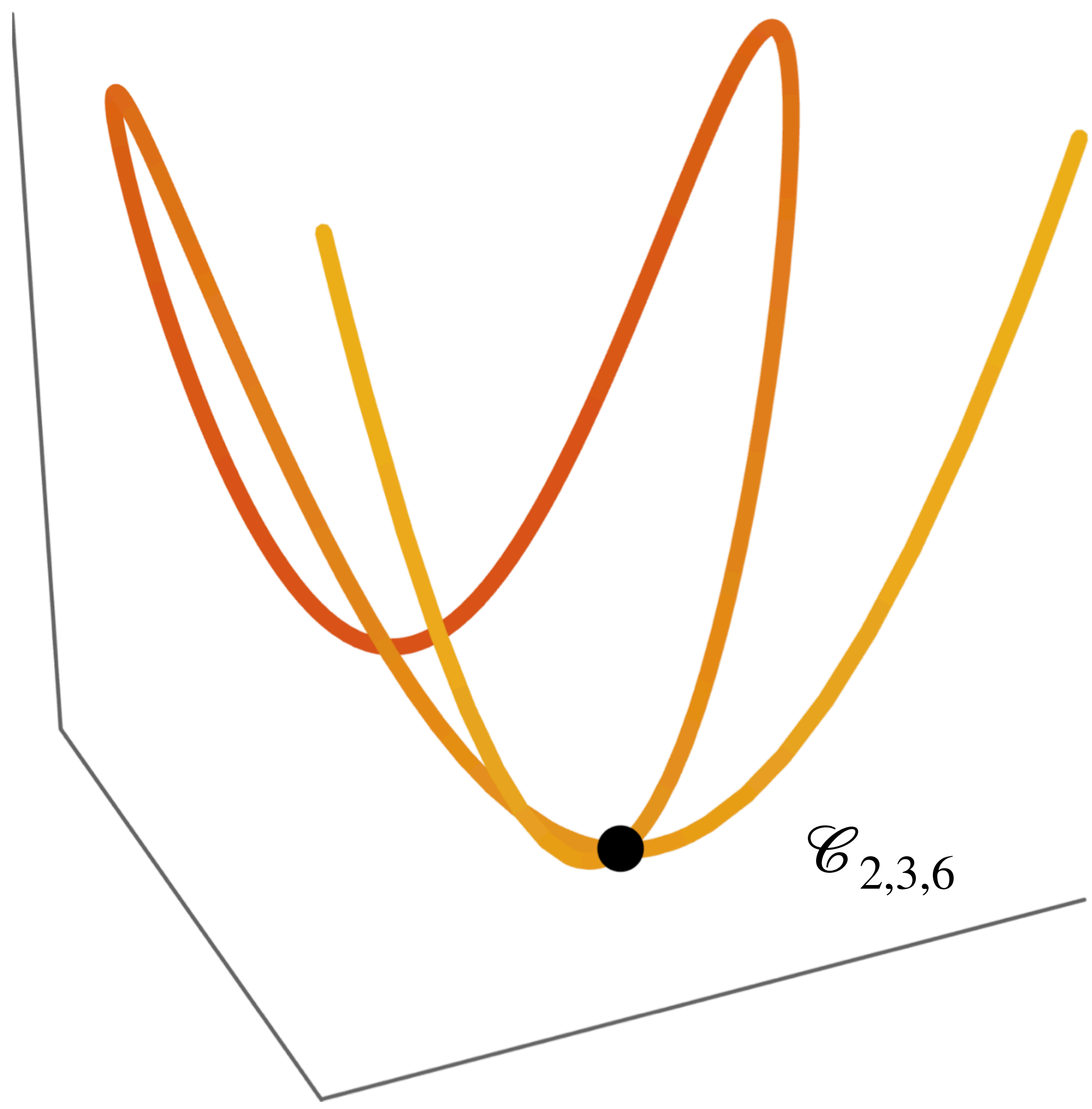
$$P(T_2(t), T_3(t), T_7(t)) \ = \ t$$

Different choices give the same ideal:

$$\tilde{P}(x, y, z) \ = \ 2T_{25}(x)T_7(z) - T_{33}(y)$$
$$= -4294967296y^{33} + 2147483648x^{25}z^7 + \ldots - 33y$$

Chebyshev space curves

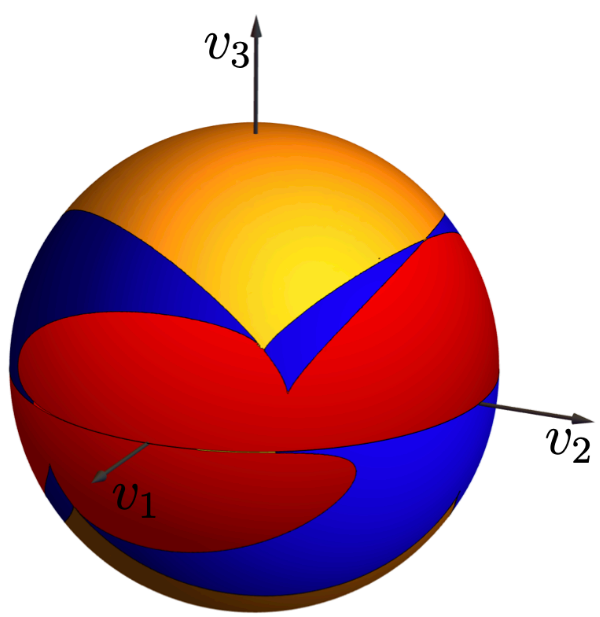


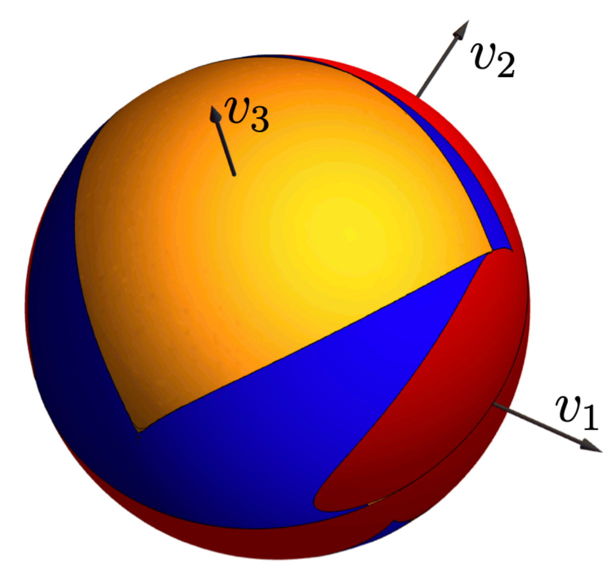
Chebyshev space curves

Theorem. Let $0 < m < n$. A polynomial of the form $f = \alpha T_m(t) + \beta T_n(t)$ has at least m real roots.

Q: What is the minimal/expected number of real roots of $v_1 T_2(t) + v_2 T_3(t) + v_3 T_7(t)$?

$$
\begin{aligned}
v_3 \cdot (&2048 v_1^4 v_2^5 + 2304 v_1^2 v_2^7 + 27648 v_2^9 + 25000 v_1^8 v_3 + 28125 v_1^6 v_2^2 v_3 + 378460 v_1^4 v_2^4 v_3 - 26112 v_1^2 v_2^6 v_3 \\
&- 27648 v_2^8 v_3 - 481250 v_1^6 v_2 v_3^2 - 5797820 v_1^4 v_2^3 v_3^2 + 3930304 v_1^2 v_2^5 v_3^2 + 119808 v_2^7 v_3^2 + 153125 v_1^6 v_3^3 \\
&+ 23852220 v_1^4 v_2^2 v_3^3 - 19302080 v_1^2 v_2^4 v_3^3 + 1480192 v_2^6 v_3^3 - 29985060 v_1^4 v_2 v_3^4 + 34354880 v_1^2 v_2^3 v_3^4 \\
&- 1229312 v_2^5 v_3^4 + 12850152 v_1^4 v_3^5 - 77561904 v_1^2 v_2^2 v_3^5 + 1229312 v_2^4 v_3^5 + 114556512 v_1^2 v_2 v_3^6 \\
&+ 22588608 v_2^3 v_3^6 - 58353904 v_1^2 v_3^7 - 22588608 v_2^2 v_3^7 - 52706752 v_2 v_3^8 + 52706752 v_3^9) \; = \; 0.
\end{aligned}
$$





Chebyshev surfaces

Chebyshev surfaces

Surfaces in three-space: $\begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \end{pmatrix}$

$(t_1 t_2^2 , t_1 t_2 , t_1^2 t_2^3)$ $(T_1(t_1)T_2(t_2) , T_1(t_1)T_1(t_2) , T_2(t_1)T_3(t_2))$ $(\cos(t_1 + 2t_2) , \cos(t_1 + t_2), \cos(2t_1 + 3t_2))$

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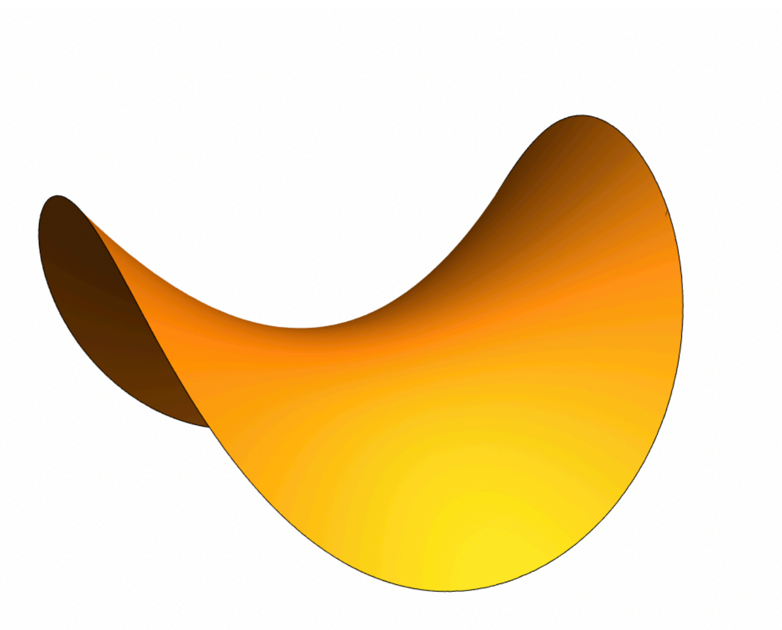
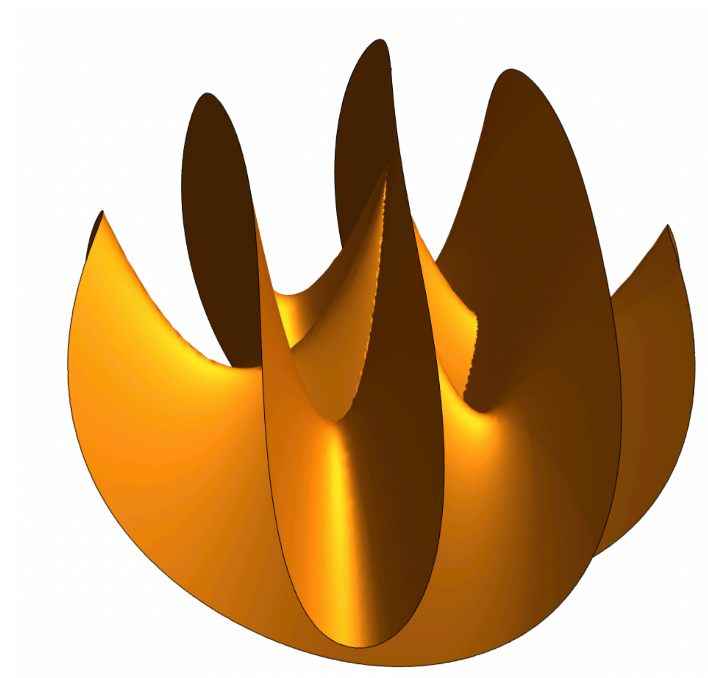


$(T_1(t_1)T_2(t_2) , T_1(t_1)T_1(t_2) , T_2(t_1)T_3(t_2))$



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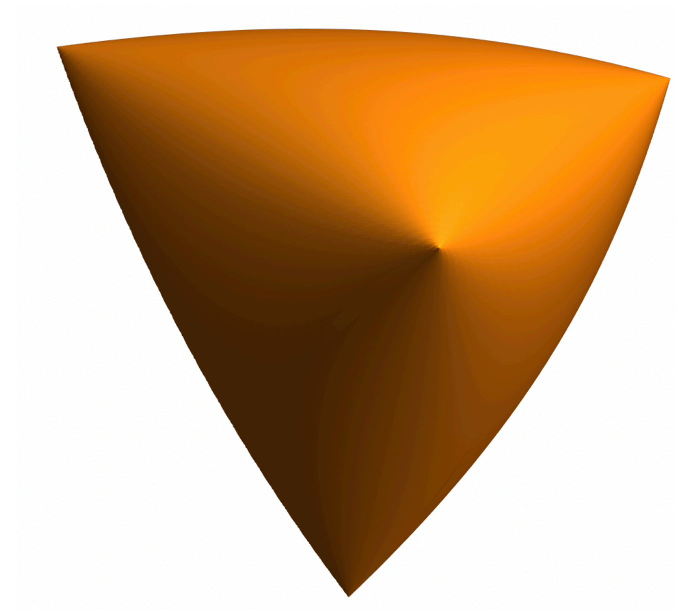

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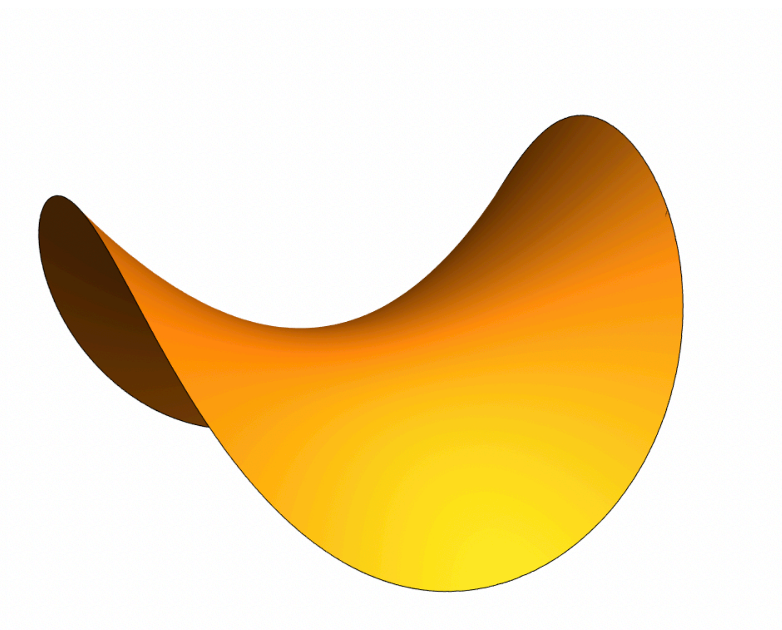

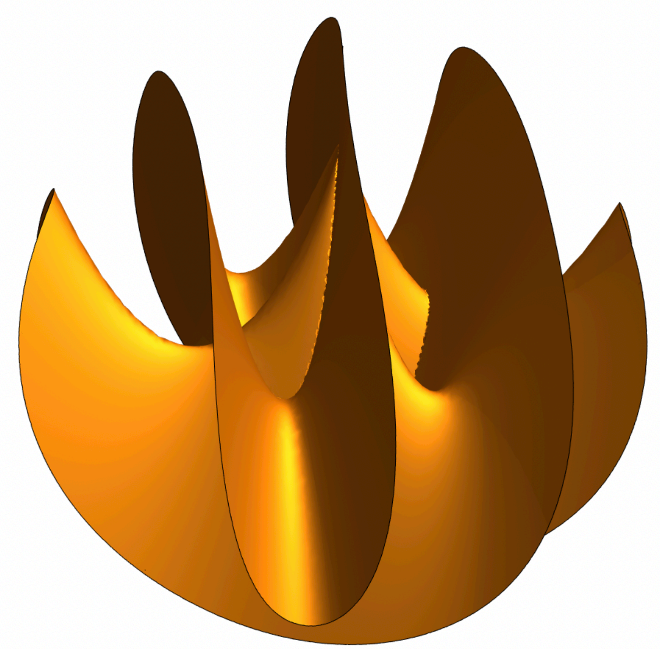

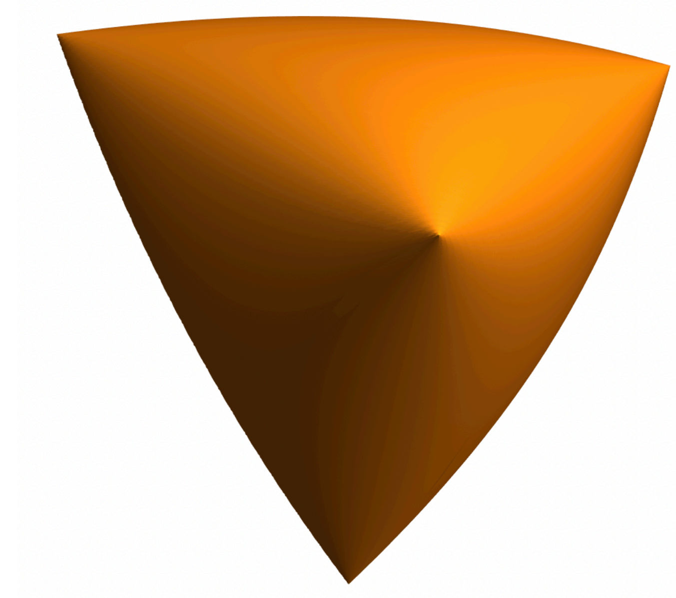

$xy - z = 0$

$$-6x^4y + x^3z - x^2y\left(-48y^4 + 22y^2 - 3\right)$$
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$$\det \begin{pmatrix} 1 & x & y \\ x & 1 & z \\ y & z & 1 \end{pmatrix} = 0$$

Tensor product basis

$$A = (a_1 \quad a_2 \quad \cdots \quad a_n) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \in \mathbb{N}^{m \times n}$$

$$\mathscr{T}_{a_j,\otimes}(t_1, \ldots, t_m) = T_{a_{1j}}(t_1) \cdot T_{a_{2j}}(t_2) \cdot \cdots \cdot T_{a_{mj}}(t_m)$$

$$\mathscr{X}_{A,\otimes} = \overline{\{(\mathscr{T}_{a_1,\otimes}(t), \ldots, \mathscr{T}_{a_n,\otimes}(t)) : t \in \mathbb{C}^m\}} \subset \mathbb{C}^n$$

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$m = 1 \implies$ *Chebyshev space curves*

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Example. $\begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \end{pmatrix} \longrightarrow (T_1(t_1)T_2(t_2), T_1(t_1)T_1(t_2), T_2(t_1)T_3(t_2))$

$$-6x^4y + x^3z - x^2y\left(-48y^4 + 22y^2 - 3\right)$$
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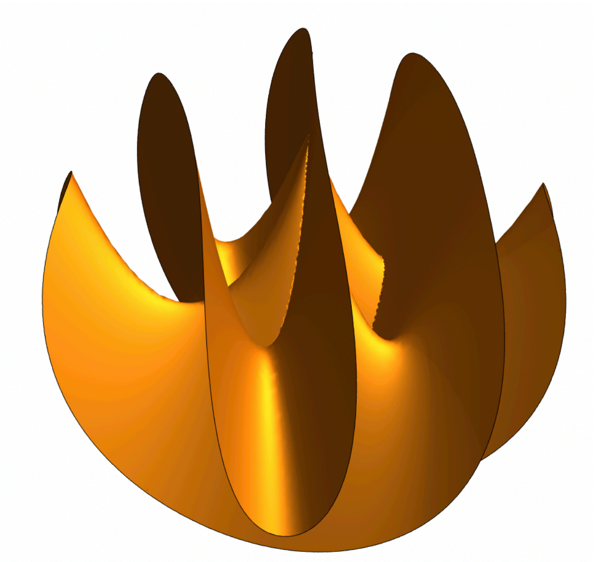

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$$f_i(t) = c_{i0} + c_{i1}\mathcal{T}_{a_1,\otimes}(t) + \cdots + c_{in}\mathcal{T}_{a_n,\otimes}(t) = 0, \quad i = 1,\ldots,m$$

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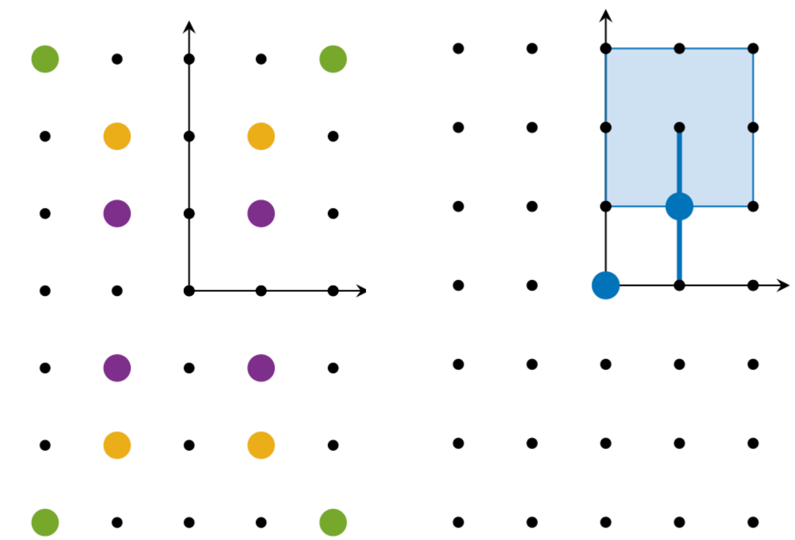

To state a degree bound, we define two polytopes:

$$B_j = \{-a_{1j}, a_{1j}\} \times \cdots \times \{-a_{mj}, a_{mj}\} \subset \mathbb{Z}^m$$

$$P_B = \mathrm{Conv}(B_1, \ldots, B_n)$$

$$P_C = \mathrm{Conv}(\mathrm{Newt}(\mathcal{T}_{a_1,\otimes}) \cup \ldots \cup \mathrm{Newt}(\mathcal{T}_{a_n,\otimes}) \cup 0)$$

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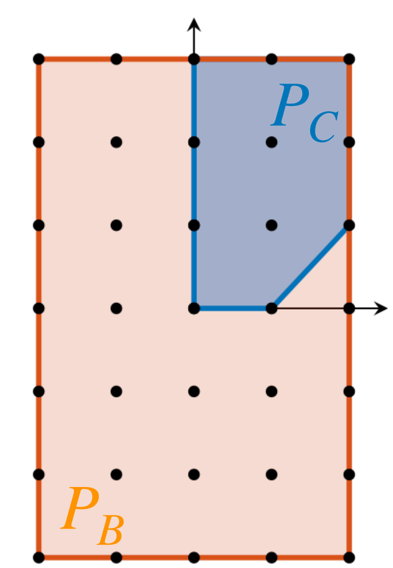

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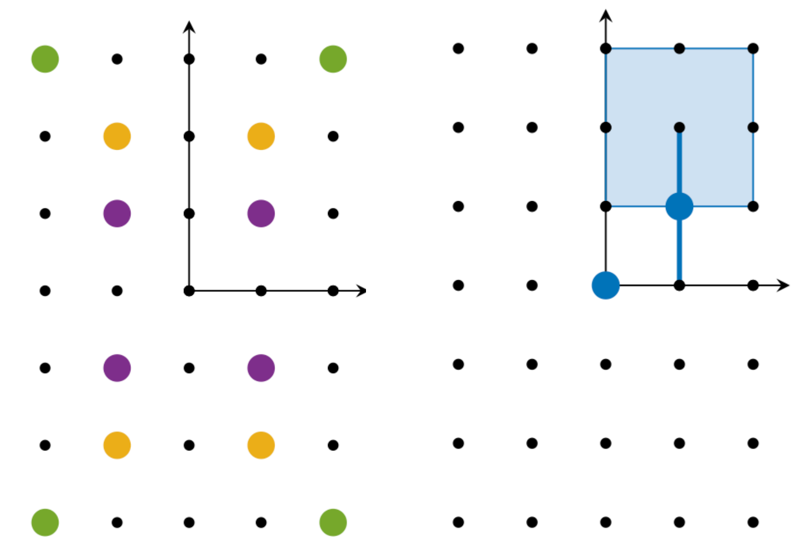

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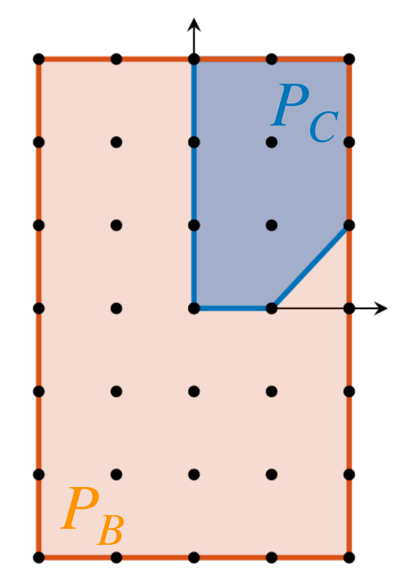
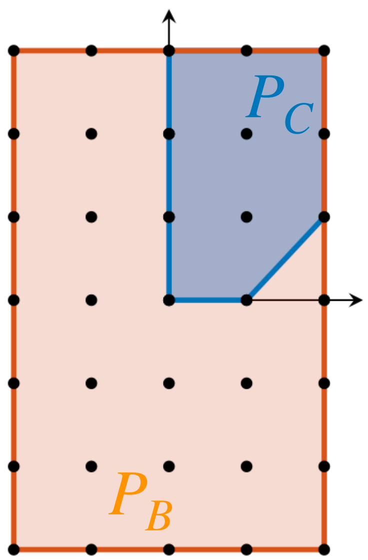


Theorem. $\deg \mathcal{X}_{A,\otimes} \leq m!\,\mathrm{vol}(P_C) \leq m! \cdot 2^{-m} \cdot \mathrm{vol}(P_B)$

Tensor product basis



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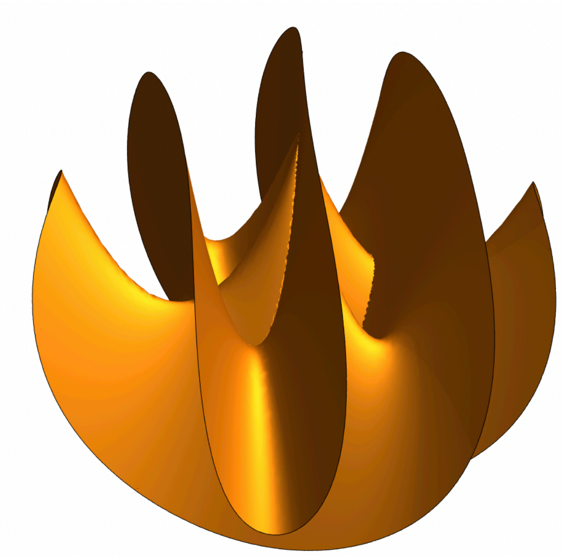
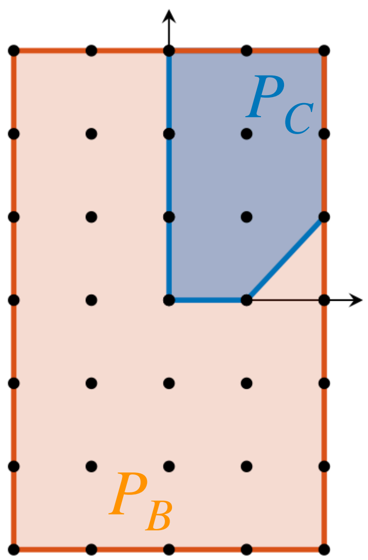
$$7 < 11 < 12$$



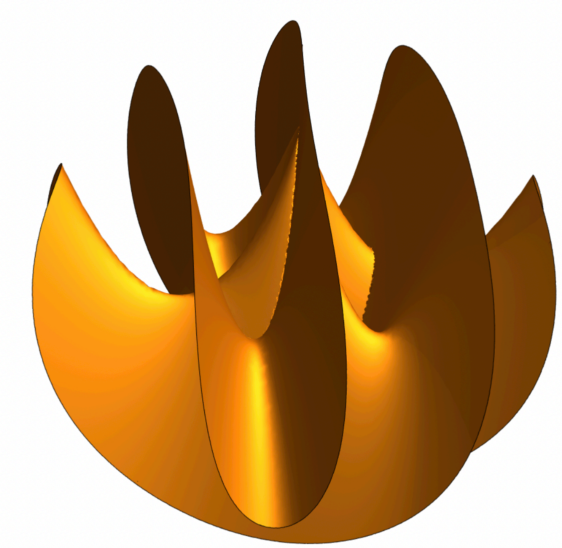
Tensor product basis



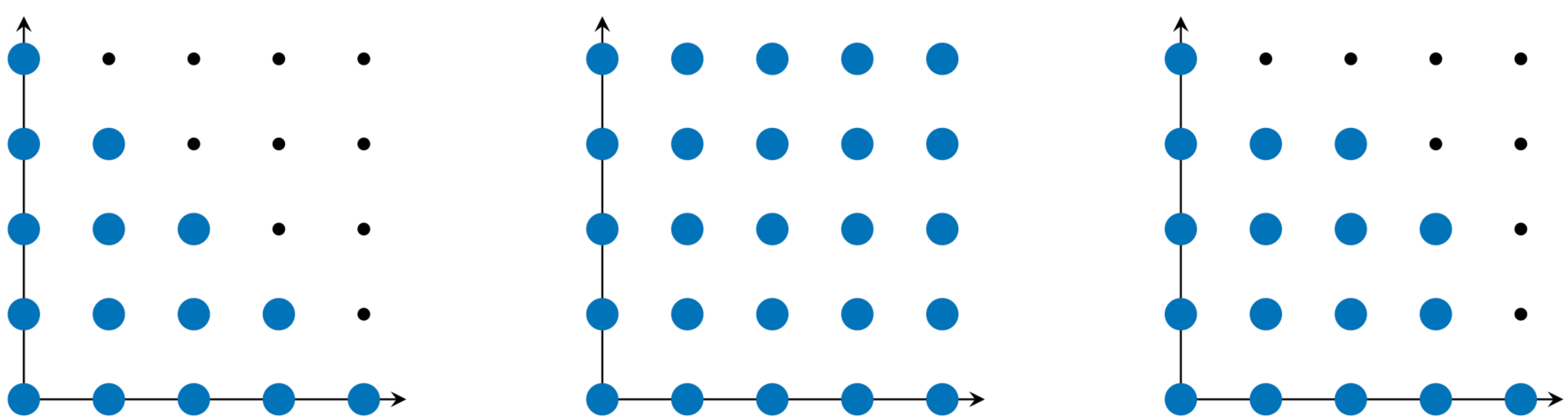
$$-6x^4y + x^3z - x^2y\left(-48y^4 + 22y^2 - 3\right)$$
$$-xy^2\left(20y^2 - 3\right)z + y^3\left(-16y^4 + 8y^2 + 2z^2 - 1\right) \;=\; 0$$

$$\deg \mathscr{X}_{A,\otimes} \;\le\; m!\,\mathrm{vol}(P_C) \;\le\; m! \cdot 2^{-m} \cdot \mathrm{vol}(P_B)$$

$$7 < 11 < 12$$



Theorem. Both bounds are equalities if A is *sufficiently dense*



L. N. Trefethen. Cubature, approximation, and isotropy in the hypercube. *SIAM Review*, 59(3):469–491, 2017.

Tensor product basis

Example 6.4. Let A be the matrix of all tuples a_j of Euclidean degree $k = 30$ (see Remark 4.1). Using the Julia package `Oscar.jl` [21], we compute that $\delta = 2! \cdot \mathrm{vol}(P_A) = 1396$. We set up the system (21) with real coefficients $c_{i,j}, c_{i,0}$ drawn from a standard normal distribution, and use the outlined eigenvalue algorithm to solve it. The matrix M has size 1560×2953. Its (numerical) rank is 1557. Among the 1396 complex (approximate) solutions, 382 are (approximately) real, and 338 are contained in the square $[-1, 1]^2$, see Figure 10. ◇

$$c_{10} + c_{11}\mathscr{T}_{a_1,\otimes}(t) + \cdots + c_{1n}\mathscr{T}_{a_n,\otimes}(t) = c_{20} + c_{21}\mathscr{T}_{a_1,\otimes}(t) + \cdots + c_{2n}\mathscr{T}_{a_n,\otimes}(t) = 0$$

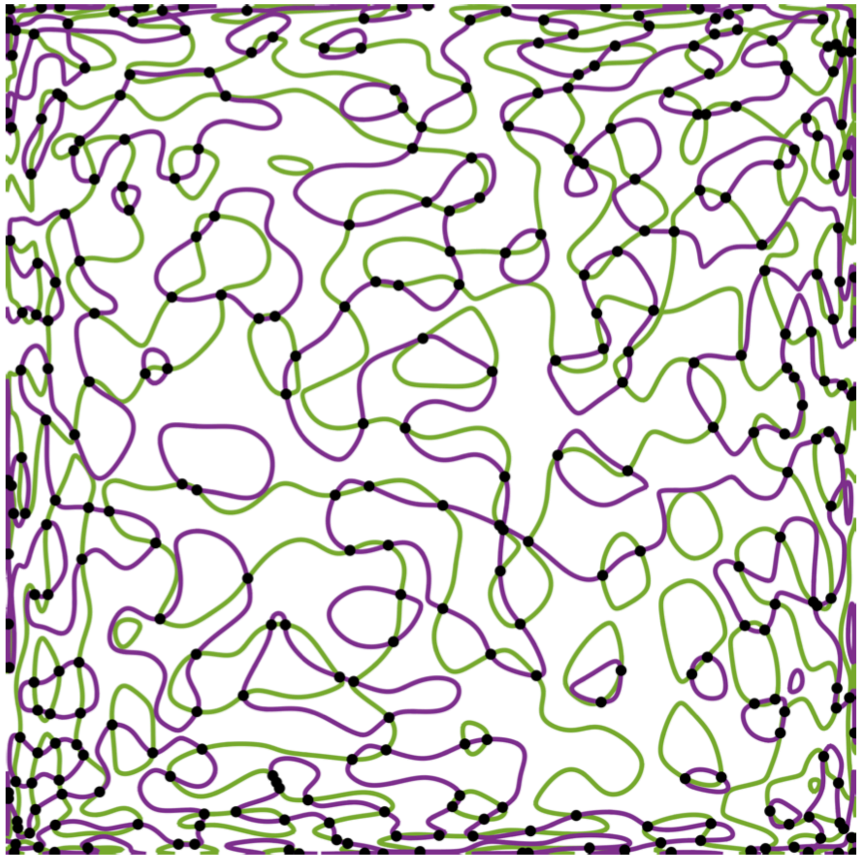


Figure 10: The curves from Example 6.4 in $[-1, 1]^2$.

Cosines of linear spaces

$$A = (a_1 \quad a_2 \quad \cdots \quad a_n) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \in \mathbb{N}^{m \times n}$$

$$\mathcal{T}_{a_j,\cos}(u_1, \ldots, u_m) = \cos(a_j \cdot u) = \cos(a_{1j}u_1 + \cdots + a_{mj}u_m)$$

$$\mathcal{X}_{A,\cos} = \overline{\{(\mathcal{T}_{a_1,\cos}(u), \ldots, \mathcal{T}_{a_n,\cos}(u)) : u \in \mathbb{C}^m\}} \subset \mathbb{C}^n$$

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$m = 1 \implies$ *Chebyshev space curves*

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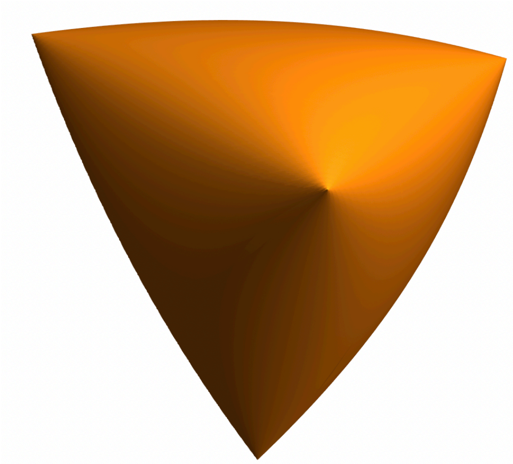
$m=1 \implies$ *Chebyshev space curves*

Example. $\begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \end{pmatrix} \longrightarrow (\cos(u_1+2u_2), \cos(u_1+u_2), \cos(2u_1+3u_2))$

$$\det\begin{pmatrix} 1 & x & y \\ x & 1 & z \\ y & z & 1 \end{pmatrix} = 1-x^2-y^2-z^2+2xyz = 0$$



Cosines of linear spaces

Theorem. The Chebyshev variety $\mathscr{X}_{A,\cos}$ is irreducible of dimension $\mathrm{rank}(A)$.
It is obtained as the closure of the projection to $\mathbb{C}^n$ of

$$\mathscr{Y} \;=\; \{(x,u) \in \mathbb{C}^n \times (\mathbb{C}\backslash\{0\})^n \;:\; u \in Y_A, u_j^2 - 2x_j u_j + 1 = 0\}$$

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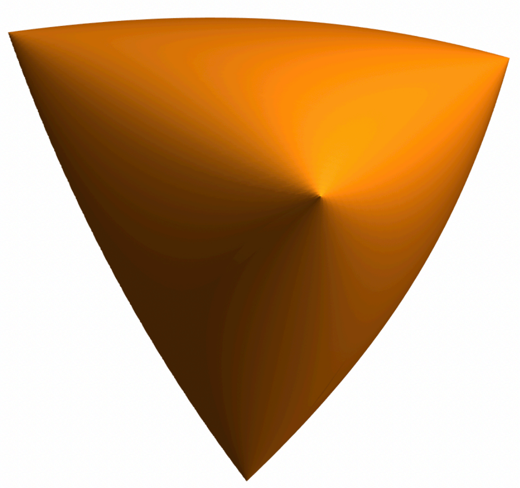
Example: $A \;=\; \begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \end{pmatrix}$

```
f = u1 u2 - u3;
f000 = (f /. {u1 → x + Sqrt[x ^ 2 - 1], u2 → y + Sqrt[y ^ 2 - 1], u3 → z + Sqrt[z ^ 2 - 1]});
f001 = (f /. {u1 → x + Sqrt[x ^ 2 - 1], u2 → y + Sqrt[y ^ 2 - 1], u3 → z - Sqrt[z ^ 2 - 1]});
f010 = (f /. {u1 → x + Sqrt[x ^ 2 - 1], u2 → y - Sqrt[y ^ 2 - 1], u3 → z + Sqrt[z ^ 2 - 1]});
f011 = (f /. {u1 → x + Sqrt[x ^ 2 - 1], u2 → y - Sqrt[y ^ 2 - 1], u3 → z - Sqrt[z ^ 2 - 1]});
f100 = (f /. {u1 → x - Sqrt[x ^ 2 - 1], u2 → y + Sqrt[y ^ 2 - 1], u3 → z + Sqrt[z ^ 2 - 1]});
f101 = (f /. {u1 → x - Sqrt[x ^ 2 - 1], u2 → y + Sqrt[y ^ 2 - 1], u3 → z - Sqrt[z ^ 2 - 1]});
f110 = (f /. {u1 → x - Sqrt[x ^ 2 - 1], u2 → y - Sqrt[y ^ 2 - 1], u3 → z + Sqrt[z ^ 2 - 1]});
f111 = (f /. {u1 → x - Sqrt[x ^ 2 - 1], u2 → y - Sqrt[y ^ 2 - 1], u3 → z - Sqrt[z ^ 2 - 1]});
Simplify[f000 * f001 * f010 * f011 * f100 * f101 * f110 * f111]
```

Out[411]= $16\left(-1 + x^2 + y^2 - 2\,x\,y\,z + z^2\right)^2$



Cosines of linear spaces

Let $P_{A,\cos} = \mathrm{Conv}(A \cup -A)$

Theorem. Under mild assumptions on A, we have $2\deg \mathcal{X}_{\mathrm{A,cos}} = m!\,\mathrm{vol}\,P_{A,\cos}$

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$$f_i(u) \;=\; c_{i0} + c_{i1}\cos(a_1 \cdot u) \;+\; \cdots \;+\; c_{in}\cos(a_n \cdot u) = 0, \quad i = 1,\ldots,m$$

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$$v = e^{\sqrt{-1}u}$$

$$f_i(v) = c_{i0} + c_{i1} \frac{v^{a_1} + v^{-a_1}}{2} + \cdots + c_{in} \frac{v^{a_n} + v^{-a_n}}{2} = 0$$

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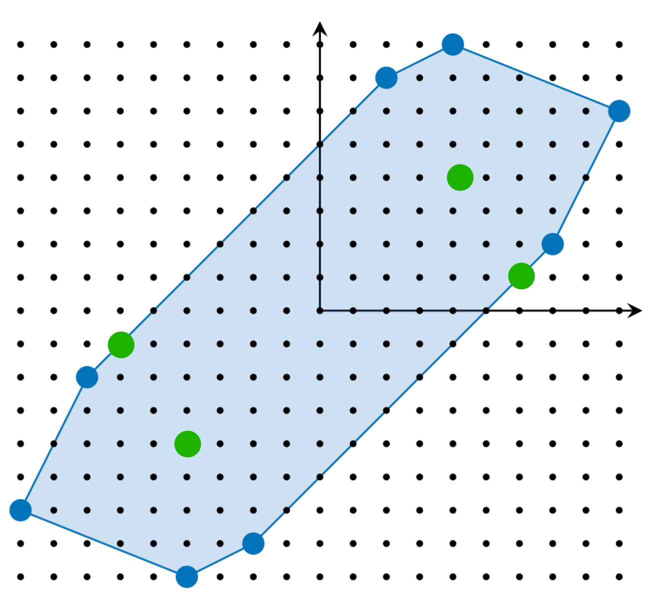
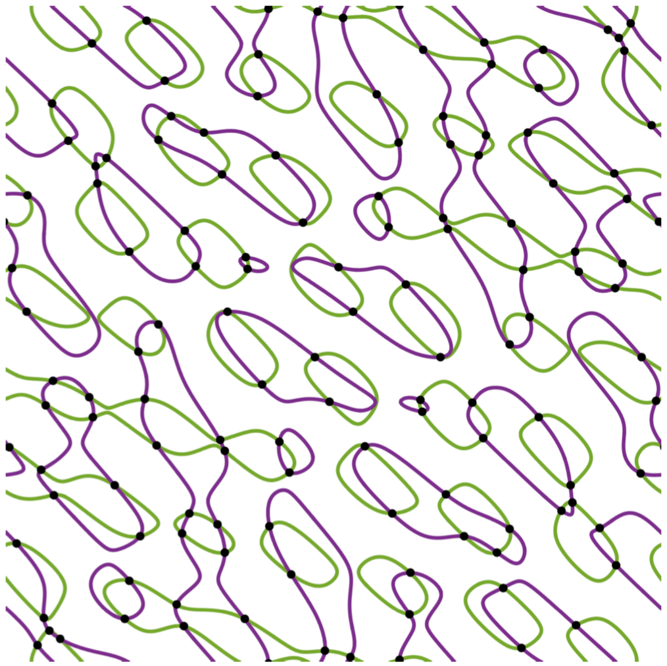
Example.

$$A = \begin{pmatrix} 4 & 4 & 6 & 7 & 9 & 2 \\ 8 & 4 & 1 & 2 & 6 & 7 \end{pmatrix}$$

$\deg \mathscr{X}_{A,\cos} = 129$

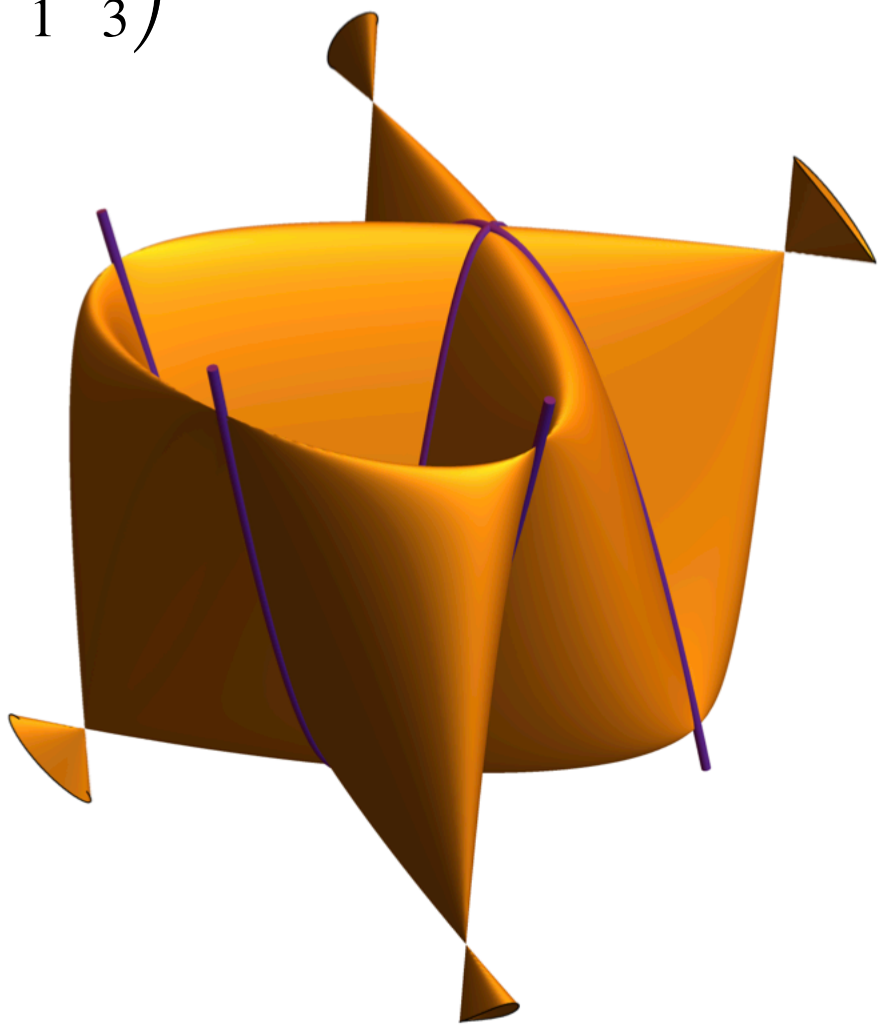
64 pairs of real solutions

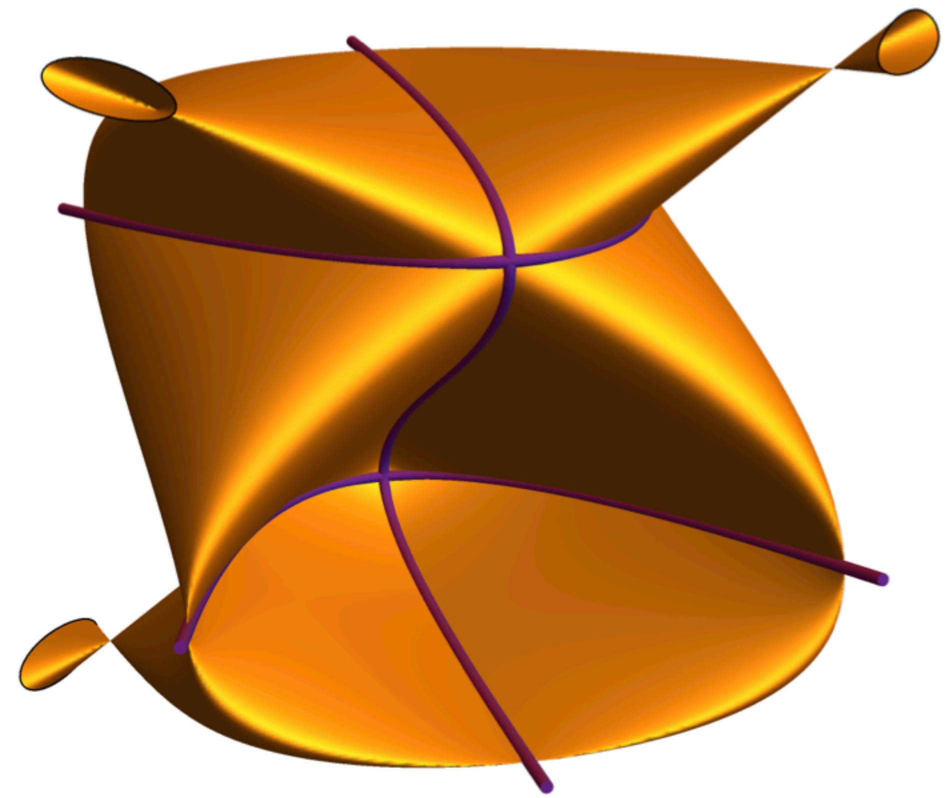




Cosines of linear spaces

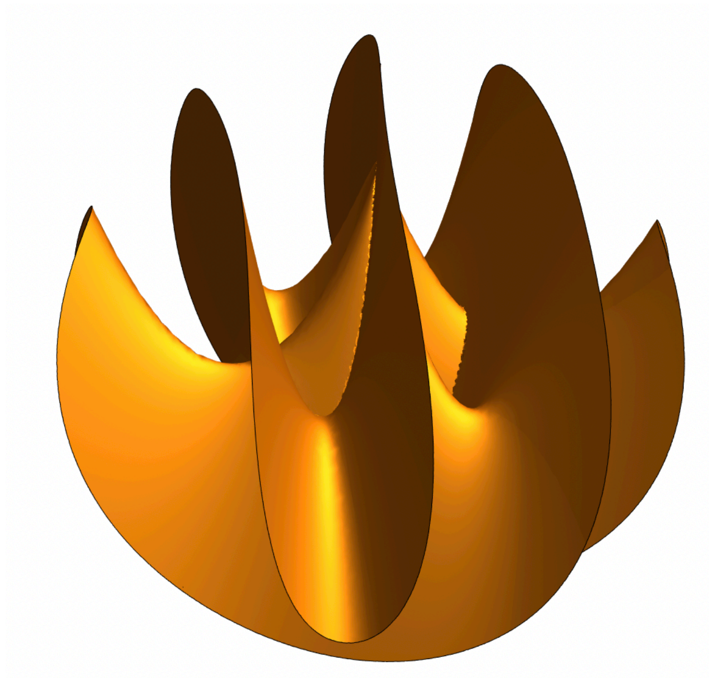
$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{pmatrix}$$

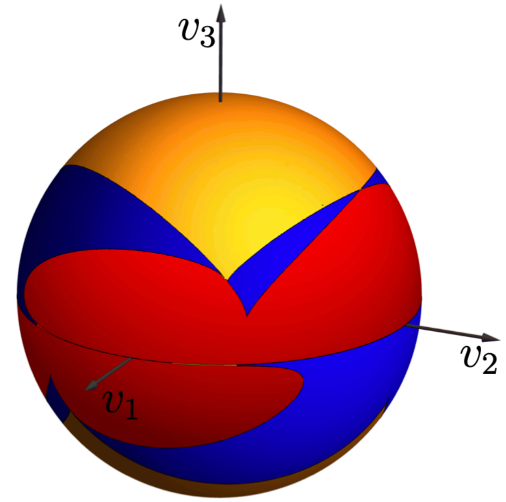


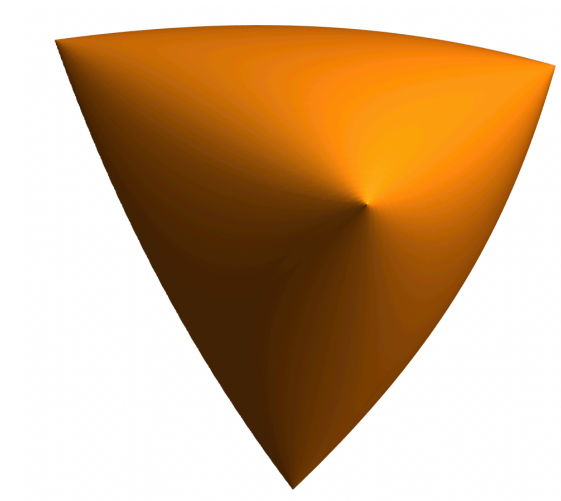


$$4x_1^4 - 16x_1^2x_2^3x_3 + 12x_1^2x_2x_3 - 4x_1^2 + 16x_2^6 - 24x_2^4 + 8x_2^3x_3 + 9x_2^2 - 6x_2x_3 + x_3^2 = 0$$

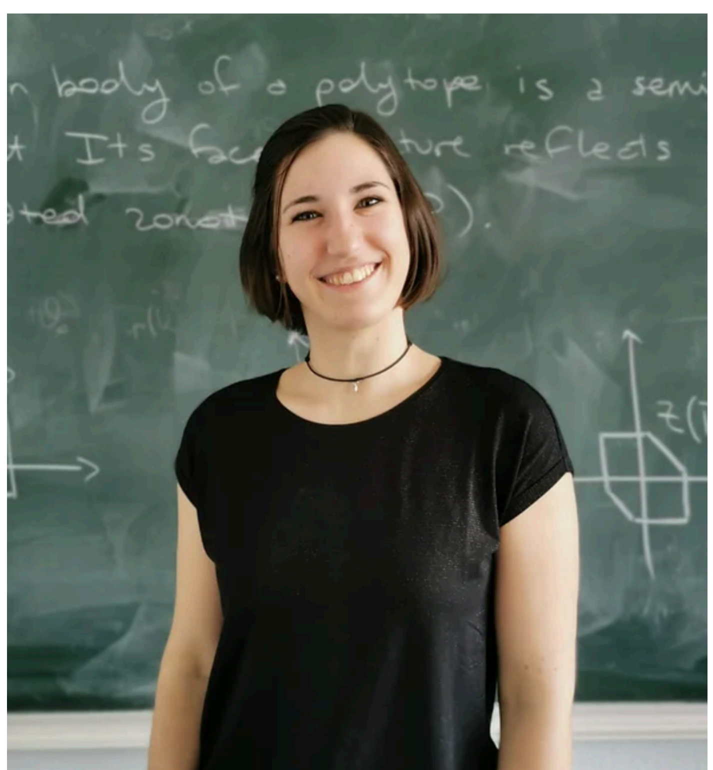
$$(1,1,1), \quad (-1,-1,-1), \quad (1,-1,1), \quad (-1,1,-1),$$
$$\{x_1 = 0,\ 4x_2^3 - 3x_2 = -x_3\},\ \{2x_2 = -1,\ 2x_1^2 - 1 = x_3\},\ \{2x_2 = 1,\ 2x_1^2 - 1 = -x_3\}$$







Thank you!



Mathematics > Algebraic Geometry

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Chebyshev Varieties

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