A new symbolic-numeric method to solve the multiparameter eigenvalue problem

Matías Bender

Inria - CMAP, École Polytechnique

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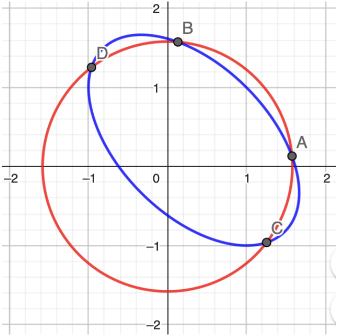
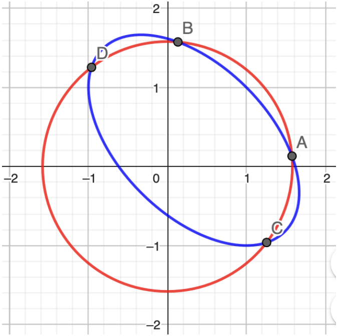
Solving polynomial systems



$$\begin{cases} 2x^2 + 2y^2 - 5 = 0, \\ x^2 + xy + y^2 - x - y - 1 = 0 \end{cases}$$

Solving polynomial systems



$$\begin{cases} \color{red}{\mathbf{2\,x^2 + 2\,y^2 - 5}} = 0, \\ \color{blue}{\mathbf{x^2 + xy + y^2 - x - y - 1}} = 0 \end{cases}$$

Exact solution (Symbolic)

$$\begin{cases} 4\,x^4 - 8\,x^3 - 2\,x^2 + 8\,x - 1 = 0, \\ y = 2\,x^3 - 2\,x^2 - 3\,x + 2 \end{cases}$$

Approximate solution (Numeric)

$$\Big\{(1.575665992, 0.1314407890),$$
$$(0.1314407890, 1.575665992),$$
$$(1.254847886, -0.9619546675),$$
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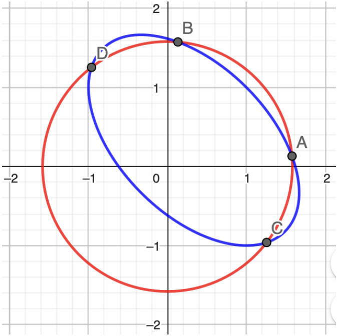
Solving polynomial systems



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Symbolic-numeric method

1. Use symbolic formulation to linearize problem.
2. Solve using numerical linear algebra.

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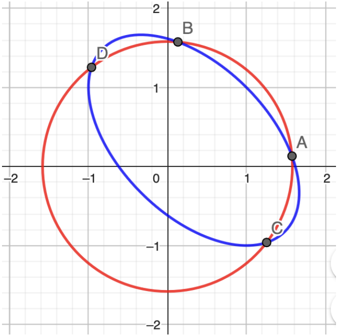

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Symbolic-numeric method

1. Use symbolic formulation to linearize problem. ← Today
2. Solve using numerical linear algebra.

- Joint work with Jean-Charles Faugère, Angelos Mantzaflaris, and Elias Tsigaridas.
    - “Koszul-type determinantal formulas for families of mixed multilinear systems” [arXiv:2105.13188]
    - “Bilinear systems with two supports: Koszul resultant matrices, eigenvalues, and eigenvectors” [arXiv:1805.05060]
- New symbolic-numeric algorithm to solve mixed multilinear systems.
    - For example, the Multiparameter Eigenvalue Problem (MEP)
- We introduce new Macaulay and Koszul-type matrices.

The resultant

Projective resultant

Necessary and sufficient condition for a homogeneous system in $(f_0, \dots, f_n) \in \mathbb{C}[x_0, \dots, x_n]^{n+1}$ to have solutions in $\mathbb{P}^n$.

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Example : Resultant of linear forms = Determinant

The system $\begin{cases} a_1\, x + a_2\, y + a_3\, z = 0 \\ b_1\, x + b_2\, y + b_3\, z = 0 \\ c_1\, x + c_2\, y + c_3\, z = 0 \end{cases}$ has a solution over $\mathbb{P}^2$

$$\Updownarrow$$

$$\texttt{det} \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} = 0.$$

The resultant

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Necessary and sufficient condition for a homogeneous system in $(f_0, \dots, f_n) \in \mathbb{C}[x_0, \dots, x_n]^{n+1}$ to have solutions in $\mathbb{P}^n$.

Example : Resultant of linear forms = Determinant

Example : Resultant of binary forms = Det of Sylvester matrix

$$\begin{cases} a_1\, x^2 + a_2\, x\, y + a_3\, y^2 = 0 \\ b_1\, x^3 + b_2\, x^2\, y + b_3\, x\, y^2 + b_4\, y^3 = 0 \end{cases} \quad \text{has a solution over } \mathbb{P}^1$$

$$\Updownarrow$$

$$\mathtt{det} \begin{pmatrix} a_1 & a_2 & a_3 & 0 & 0 \\ 0 & a_1 & a_2 & a_3 & 0 \\ 0 & 0 & a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 & b_4 & 0 \\ 0 & b_1 & b_2 & b_3 & b_4 \end{pmatrix} = 0.$$

Sylvester-type formulas

Classical way of computing resultant → Sylvester-type formula

$$(g_0, \dots, g_n) \mapsto \sum_{i=0}^{n} g_i\, f_i$$

Macaulay resultant matrix [Macaulay, 1916]

$$\begin{cases} f_1 := & \boldsymbol{a_1}\, x^2 + \boldsymbol{a_2}\, x\, y + \boldsymbol{a_3}\, x\, z + \\ & \boldsymbol{a_4}\, y^2 + \boldsymbol{a_5}\, y\, z + \boldsymbol{a_6}\, z^2 \\ f_2 := & \boldsymbol{b_1}\, x + \boldsymbol{b_2}\, y + \boldsymbol{b_3}\, z \\ f_3 := & \boldsymbol{c_1}\, x + \boldsymbol{c_2}\, y + \boldsymbol{c_3}\, z \end{cases}$$

$$\begin{array}{c|cccccc} & x^2 & x\,y & x\,z & y^2 & y\,z & z^2 \\ \hline f_1 & \boldsymbol{a_1} & \boldsymbol{a_2} & \boldsymbol{a_3} & \boldsymbol{a_4} & \boldsymbol{a_5} & \boldsymbol{a_6} \\ x\, f_2 & \boldsymbol{b_1} & \boldsymbol{b_2} & \boldsymbol{b_3} & & & \\ y\, f_2 & & \boldsymbol{b_1} & & \boldsymbol{b_2} & \boldsymbol{b_3} & \\ z\, f_2 & & & \boxed{\boldsymbol{b_1}} & & \boldsymbol{b_2} & \boldsymbol{b_2} \\ y\, f_3 & & \boldsymbol{c_1} & & \boldsymbol{c_2} & \boldsymbol{c_3} & \\ z\, f_3 & & & \boldsymbol{c_1} & & \boldsymbol{c_2} & \boldsymbol{c_3} \end{array}$$

$$\texttt{Determinant} = \texttt{Resultant} \cdot \underbrace{\boxed{\texttt{ExtraFactor}}}_{\text{Minor of the matrix}}$$

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$$(g_0, \dots, g_n) \mapsto \sum_{i=0}^{n} g_i \, f_i$$

Macaulay resultant matrix [Macaulay, 1916]

$$\left\{ \begin{aligned} f_1 &:= a_1 \, x^2 + a_2 \, x \, y + a_3 \, x \, z + \\ & a_4 \, y^2 + a_5 \, y \, z + a_6 \, z^2 \\ f_2 &:= b_1 \, x + b_2 \, y + b_3 \, z \\ f_3 &:= c_1 \, x + c_2 \, y + c_3 \, z \end{aligned} \right.$$

$$\begin{array}{c|cccccc} & x^2 & x\,y & x\,z & y^2 & y\,z & z^2 \\ \hline f_1 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ x\,f_2 & b_1 & b_2 & b_3 & & & \\ y\,f_2 & & b_1 & & b_2 & b_3 & \\ z\,f_2 & & & \boxed{b_1} & & b_2 & b_2 \\ y\,f_3 & & c_1 & & c_2 & c_3 & \\ z\,f_3 & & & c_1 & & c_2 & c_3 \end{array}$$

$$\texttt{Determinant} = \texttt{Resultant} \cdot \underbrace{\boxed{\texttt{ExtraFactor}}}_{\text{Minor of the matrix}}$$

Determinantal formula → `ExtraFactor` is a constant.

Solving via Sylvester-type formulas

- We want to compute the two solutions $\alpha, \beta \in \mathbb{P}^2$ of

$$\begin{cases} f_1 := & \mathbf{1}\, x^2 + \mathbf{-1}\, x\, y + \mathbf{4}\, x\, z + \mathbf{-2}\, y^2 + \mathbf{-5}\, y\, z + \mathbf{3}\, z^2 \\ f_2 := & \mathbf{1}\, x + \mathbf{-1}\, y + \mathbf{-1}\, z \end{cases} .$$

Solving via Sylvester-type formulas

- We want to compute the two solutions $\alpha, \beta \in \mathbb{P}^2$ of

$$\begin{cases} f_1 := & 1\,x^2 + -1\,x\,y + 4\,x\,z + -2\,y^2 + -5\,y\,z + 3\,z^2 \\ f_2 := & 1\,x + -1\,y + -1\,z \end{cases}.$$

- Introduce $f_0 := -1\,x + 2\,y + 1\,z$ and consider a Sylvester-type formula.

$$\left(\begin{array}{c|c} M_{1,1} & M_{1,2} \\ \hline M_{2,1} & M_{2,2} \end{array}\right) = \begin{array}{c|cccc|cc} & x^2 & x\,y & x\,z & y^2 & y\,z & z^2 \\ \hline f_1 & 1 & -1 & 4 & -2 & -5 & 3 \\ x\,f_2 & 1 & -1 & -1 & & & \\ y\,f_2 & & 1 & & -1 & -1 & \\ z\,f_2 & & & 1 & & -1 & -1 \\ \hline y\,f_0 & & -1 & & 2 & 1 & \\ z\,f_0 & & & -1 & & 2 & 1 \end{array}$$

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- Schur complement of $M_{2,2} \leftrightarrow$ Multiplication map of $\frac{f_0}{z}$ in $\mathbb{C}[x, y, z]/\langle f_1, f_2 \rangle$

$$\widetilde{M}_{2,2} = M_{2,2} - M_{2,1}\, M_{1,1}^{-1}\, M_{1,2} = \begin{pmatrix} 0 & 4 \\ 1 & 0 \end{pmatrix}$$

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- Eigenvalues of $\widetilde{M}_{2,2}$ [Lazard, 1981]

$$\frac{f_0}{z}(\alpha) = 2 \qquad \text{and} \qquad \frac{f_0}{z}(\beta) = -2.$$

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- Eigenvectors of $\widetilde{M}_{2,2}$ [Auzinger & Stetter, 1988]

$$\begin{pmatrix} \alpha_y \\ \alpha_z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} \beta_y \\ \beta_z \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

Problems

We want to compute the two solutions a similar system

$$\begin{cases} f_1 := & \mathbf{1}\,x^2 + \mathbf{-1}\,x\,y + \mathbf{4}\,x\,z + \mathbf{-2}\,y^2 + \mathbf{-5}\,y\,z + \mathbf{3}\,z^2 \\ f_2 := & \boxed{\mathbf{0}}\,x + \mathbf{-1}\,y + \mathbf{-1}\,z \end{cases} .$$

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The submatrix $M_{1,1}$ not invertible $\implies$ we cannot compute Schur complement.

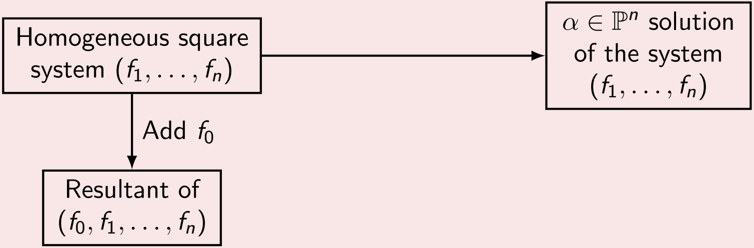
Why? Because the `ExtraFactor` vanishes.

Solving polynomial systems using the resultant

Homogeneous square system $(f_1, \dots, f_n)$

$\alpha \in \mathbb{P}^n$ solution of the system $(f_1, \dots, f_n)$

Solving polynomial systems using the resultant



Homogeneous square system $(f_1, \ldots, f_n)$

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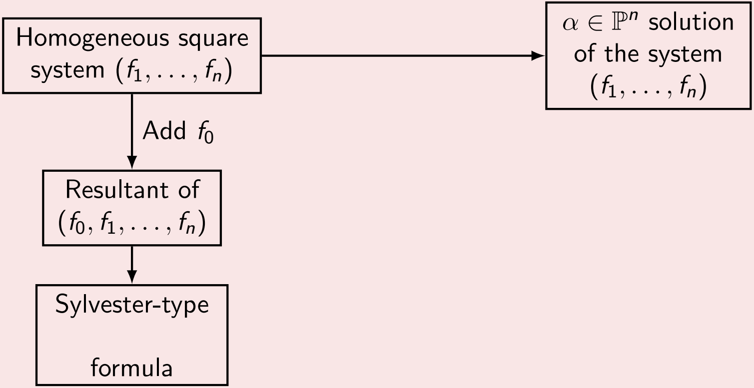
Add f_0

Resultant of $(f_0, f_1, \ldots, f_n)$

Solving polynomial systems using the resultant



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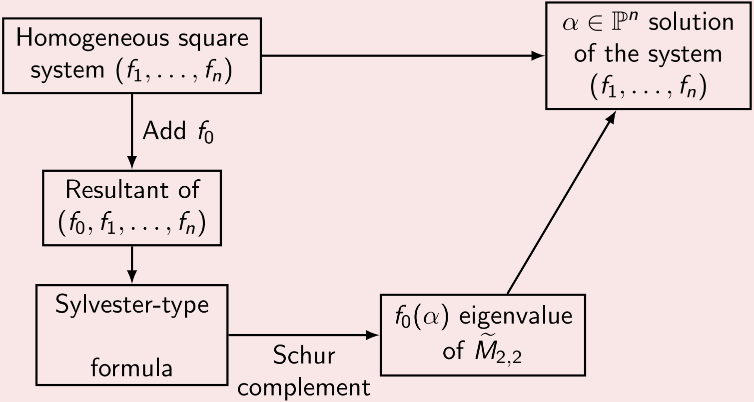
Sylvester-type

formula

Solving polynomial systems using the resultant



Homogeneous square system $(f_1, \dots, f_n)$

Add f_0

Resultant of $(f_0, f_1, \dots, f_n)$

Sylvester-type formula

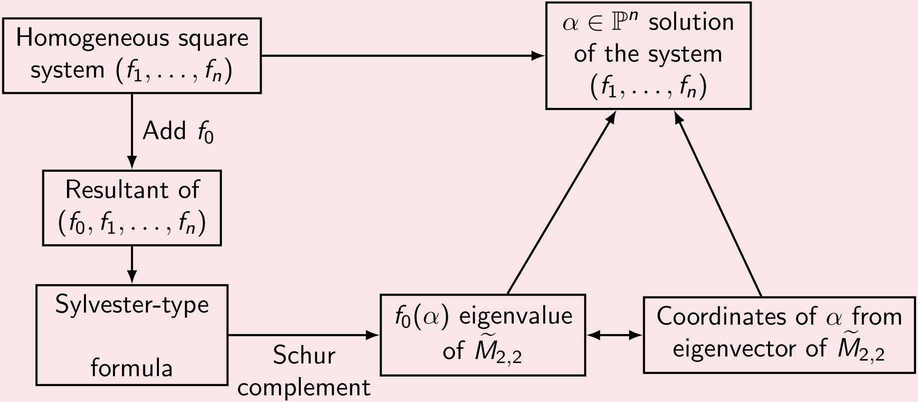
Schur complement

$f_0(\alpha)$ eigenvalue of $\widetilde{M}_{2,2}$

$\alpha \in \mathbb{P}^n$ solution of the system $(f_1, \dots, f_n)$

$$\begin{bmatrix} M_{1,1} & M_{1,2} \\ M_{2,1} & M_{2,2} \end{bmatrix} \rightarrow \begin{bmatrix} M_{1,1} & M_{1,2} \\ 0 & \widetilde{M}_{2,2} \end{bmatrix}$$

Solving polynomial systems using the resultant



Homogeneous square system $(f_1, \ldots, f_n)$

Add f_0

Resultant of $(f_0, f_1, \ldots, f_n)$

Sylvester-type formula

Schur complement

$f_0(\alpha)$ eigenvalue of $\widetilde{M}_{2,2}$

Coordinates of α from eigenvector of $\widetilde{M}_{2,2}$

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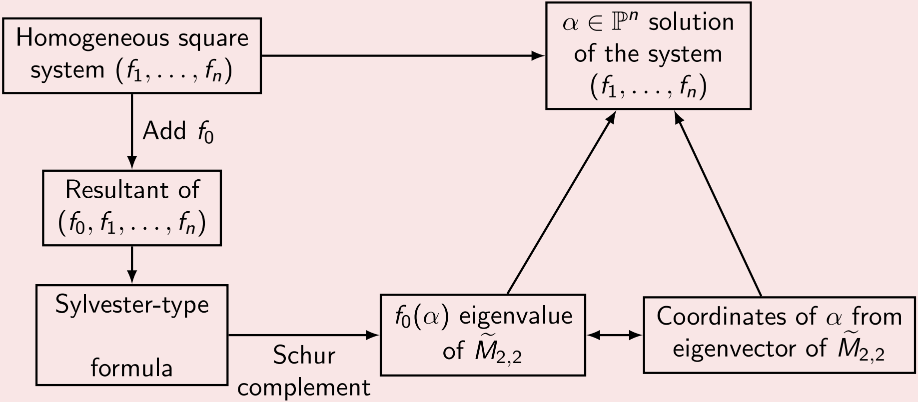
Solving polynomial systems using the resultant



Homogeneous square system $(f_1, \ldots, f_n)$

Add f_0

Resultant of $(f_0, f_1, \ldots, f_n)$

Sylvester-type formula

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Coordinates of α from eigenvector of $\widetilde{M}_{2,2}$

$\alpha \in \mathbb{P}^n$ solution of the system $(f_1, \ldots, f_n)$

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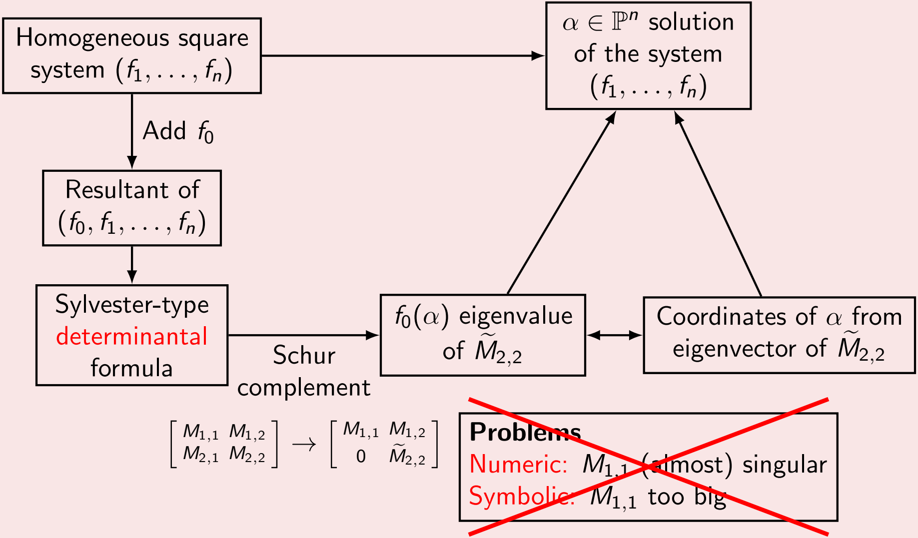
Problems

Numeric: $M_{1,1}$ (almost) singular

Symbolic: $M_{1,1}$ too big

Solving polynomial systems using the resultant



Homogeneous square system $(f_1, \ldots, f_n)$

Add f_0

Resultant of $(f_0, f_1, \ldots, f_n)$

Sylvester-type determinantal formula

Schur complement

$$\begin{bmatrix} M_{1,1} & M_{1,2} \\ M_{2,1} & M_{2,2} \end{bmatrix} \to \begin{bmatrix} M_{1,1} & M_{1,2} \\ 0 & \widetilde{M}_{2,2} \end{bmatrix}$$

$f_0(\alpha)$ eigenvalue of $\widetilde{M}_{2,2}$

Coordinates of α from eigenvector of $\widetilde{M}_{2,2}$

$\alpha \in \mathbb{P}^n$ solution of the system $(f_1, \ldots, f_n)$

~~**Problems**~~

~~Numeric: $M_{1,1}$ (almost) singular~~

~~Symbolic: $M_{1,1}$ too big~~

The Koszul complex

Vanishing of the resultant equivalent to exact chain complex.

Koszul complex in $R := \mathbb{C}[x, y, z]$

$$\begin{cases} f_1 := \mathbf{1}\, x^2 + \mathbf{-1}\, x\, y + \mathbf{4}\, x\, z + \mathbf{-2}\, y^2 + \mathbf{-5}\, y\, z + \mathbf{3}\, z^2 \\ f_2 := \mathbf{0}\, x + \mathbf{-1}\, y + \mathbf{-1}\, z \\ f_0 := \mathbf{-1}\, x + \mathbf{2}\, y + \mathbf{1}\, z \end{cases} .$$

$$\begin{bmatrix} f_0 & -f_2 & f_1 \end{bmatrix} \qquad \begin{bmatrix} -f_2 & f_1 & 0 \\ -f_0 & 0 & f_1 \\ 0 & -f_0 & f_2 \end{bmatrix} \qquad \begin{bmatrix} f_1 \\ f_2 \\ f_0 \end{bmatrix}$$

$$K_\bullet : 0 \to R(-4) \xrightarrow{\delta_2} R(-3)^2 \oplus R(-2) \xrightarrow{\delta_1} R(-2) \oplus R(-1)^2 \xrightarrow{\delta_0} R \to 0$$

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Vanishing of the resultant equivalent to exact chain complex.

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Resultant does not vanish if and only if ...

- Koszul complex exact, that is, $(\forall i)\, \mathtt{Im}(\delta_{i+1}) = \mathtt{Ker}(\delta_i)$.

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Linearization!

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$$(K_\bullet)_3 : 0 \to R_0^2 \oplus R_1 \xrightarrow{\begin{bmatrix} 1 & -0 & 1 & 3 & 4 & 1 & -5 & -1 & -2 & & & & & & \\ -1 & 1 & -2 & & & & & & & 3 & 4 & 1 & -5 & -1 & -2 \\ & & & -1 & 1 & & -2 & & & -1 & 0 & & -1 & & \\ & & & & -1 & 1 & & -2 & & & -1 & 0 & & -1 & \\ & & & & & & -1 & 1 & -2 & & & & -1 & 0 & -1 \end{bmatrix}} R_1 \oplus R_2^2$$

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$$(K_\bullet)_2 : 0 \to R_0 \xrightarrow{\begin{bmatrix} 0 & \boxed{-1} & 1 & -2 & \mathbf{-1} & \mathbf{0} & \mathbf{-1} \end{bmatrix}} R_0 \oplus R_1^2 \xrightarrow[\text{Macaulay Matrix}]{\begin{bmatrix} \mathbf{3} & \mathbf{4} & \mathbf{1} & \mathbf{-5} & \mathbf{-1} & \mathbf{-2} \\ \cancel{\mathbf{-1}} & \cancel{\mathbf{0}} & & \cancel{\mathbf{-1}} & & \\ & \mathbf{-1} & \mathbf{0} & & \mathbf{-1} & \\ & & & \mathbf{-1} & \mathbf{0} & \mathbf{-1} \\ 1 & -1 & & 2 & & \\ & 1 & -1 & & 2 & \\ & & & 1 & -1 & 2 \end{bmatrix}} R_2$$

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Nice intro: "Resultant as Determinant of Koszul Complex" by Anokhina, Morozov and Shakirov [arXiv 0812.5013]

Determinantal formulas

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$$(K_\bullet)_5 : 0 \to R_0 \xrightarrow{\begin{bmatrix} b_1 & b_2 & b_3 & b_4 & -a_1 & -a_2 & -a_3 \end{bmatrix}} R_2 \oplus R_3 \xrightarrow{\begin{bmatrix} a_1 & a_2 & a_3 & & & \\ & a_1 & a_2 & a_3 & & \\ & & a_1 & a_2 & a_3 & \\ & & & a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 & b_4 & & \\ & b_1 & b_2 & b_3 & b_4 & \\ & & b_1 & b_2 & b_3 & b_4 \end{bmatrix}} R_5 \to 0$$

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$$(K_\bullet)_4 : 0 \to 0 \to R_1 \oplus R_2 \xrightarrow[\text{Sylvester matrix}]{\begin{bmatrix} a_1 & a_2 & a_3 & & \\ & a_1 & a_2 & a_3 & \\ & & a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 & b_4 & \\ & b_1 & b_2 & b_3 & b_4 \end{bmatrix}} R_4 \to 0$$

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Multihomogeneous systems

(Affine) Generalized Eigenvalue Problem

$$\left(\begin{bmatrix} \mathbf{2} & \mathbf{6} \\ \mathbf{-1} & \mathbf{20} \end{bmatrix} + \lambda \begin{bmatrix} \mathbf{-2} & \mathbf{4} \\ \mathbf{0} & \mathbf{20} \end{bmatrix}\right) \begin{bmatrix} 1 \\ w \end{bmatrix} = 0$$

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(Affine) Generalized Eigenvalue Problem

$$\left(\begin{bmatrix} \mathbf{2} & \mathbf{6} \\ \mathbf{-1} & \mathbf{20} \end{bmatrix} + \lambda \begin{bmatrix} \mathbf{-2} & \mathbf{4} \\ \mathbf{0} & \mathbf{20} \end{bmatrix} \right) \begin{bmatrix} 1 \\ w \end{bmatrix} = 0$$

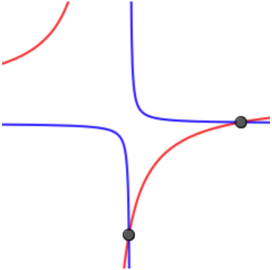
$$\begin{cases} \mathbf{2} + \mathbf{6}\, w \mathbf{-2}\, \lambda + \mathbf{4}\, \lambda\, w = 0 \\ \mathbf{-1} + \mathbf{20}\, w + \mathbf{0}\, \lambda + \mathbf{20}\, \lambda\, w = 0 \end{cases}$$

Multihomogeneous systems

(Affine) Generalized Eigenvalue Problem

$$\left(\begin{bmatrix} 2 & 6 \\ -1 & 20 \end{bmatrix} + \lambda \begin{bmatrix} -2 & 4 \\ 0 & 20 \end{bmatrix}\right)\begin{bmatrix} 1 \\ w \end{bmatrix} = 0$$

$$\begin{cases} 2 + 6\,w - 2\,\lambda + 4\,\lambda\,w = 0 \\ -1 + 20\,w + 0\,\lambda + 20\,\lambda\,w = 0 \end{cases}$$



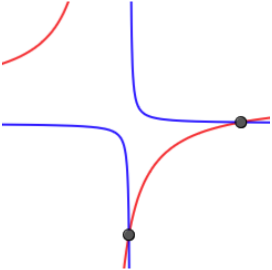
Two solutions

Multihomogeneous systems

(Affine) Generalized Eigenvalue Problem

$$\left(\begin{bmatrix} \mathbf{2} & \mathbf{6} \\ \mathbf{-1} & \mathbf{20} \end{bmatrix} + \lambda \begin{bmatrix} \mathbf{-2} & \mathbf{4} \\ \mathbf{0} & \mathbf{20} \end{bmatrix}\right) \begin{bmatrix} 1 \\ w \end{bmatrix} = 0$$

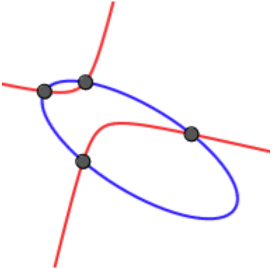
$$\begin{cases} \mathbf{2} + \ \mathbf{6}\, w \mathbf{-2}\, \lambda + \mathbf{4}\, \lambda\, w = 0 \\ \mathbf{-1} + \mathbf{20}\, w + \mathbf{0}\, \lambda + \mathbf{20}\, \lambda\, w = 0 \end{cases}$$



Two solutions

Generic polyns of degree 2 in $\mathbb{C}[\lambda, w]$

$$\begin{cases} \underline{\lambda^2 \mathbf{-} w^2} + \mathbf{4}\, \lambda\, w \mathbf{-2}\lambda + \mathbf{6} w + \mathbf{2} = 0 \\ \underline{\lambda^2 + \mathbf{2}\, w^2} + \mathbf{2}\lambda\, w + \mathbf{2} w \mathbf{-} \lambda + w \mathbf{-20} = 0 \end{cases}$$



Four solutions

Multihomogeneous systems

Generalized Eigenvalue Problem

$$\left(\begin{bmatrix} \mathbf{2} & \mathbf{6} \\ \mathbf{-1} & \mathbf{20} \end{bmatrix} + \lambda \cdot \begin{bmatrix} \mathbf{-2} & \mathbf{4} \\ \mathbf{0} & \mathbf{20} \end{bmatrix} \right) \cdot \begin{bmatrix} 1 \\ w \end{bmatrix} = 0$$

$$\left\{ \begin{array}{l} \mathbf{2} + \mathbf{6}\, w \mathbf{-2}\, \lambda + \mathbf{4}\, \lambda\, w = 0 \\ \mathbf{-1} + \mathbf{20}\, w + \mathbf{0}\, \lambda + \mathbf{20}\, \lambda\, w = 0 \end{array} \right. \quad \in$$

Multihomogeneous systems

Generalized Eigenvalue Problem

$$\left(\mu \cdot \begin{bmatrix} \mathbf{2} & \mathbf{6} \\ \mathbf{-1} & \mathbf{20} \end{bmatrix} + \lambda \cdot \begin{bmatrix} \mathbf{-2} & \mathbf{4} \\ \mathbf{0} & \mathbf{20} \end{bmatrix}\right) \cdot \begin{bmatrix} v \\ w \end{bmatrix} = 0$$

$$\left\{ \begin{array}{r} \mathbf{2}\,\mu\,v + \ \ \mathbf{6}\,\mu\,w \mathbf{-2}\,\lambda\,v + \mathbf{4}\,\lambda\,w = 0 \\ \mathbf{-1}\,\mu\,v + \mathbf{20}\,\mu\,w + \mathbf{0}\,\lambda\,v + \mathbf{20}\,\lambda\,w = 0 \end{array} \right. \quad \in \mathbb{C}[\mu, \lambda]_1 \otimes \mathbb{C}[v, w]_1$$

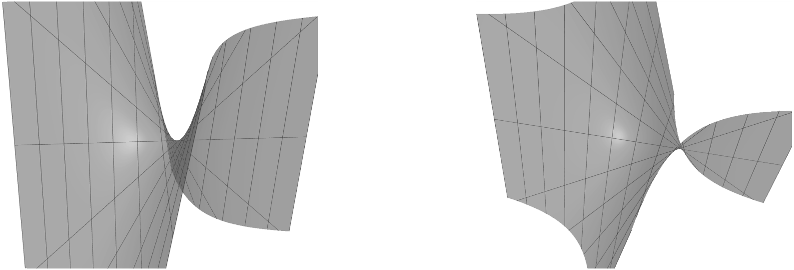
Multihomogeneous systems

Generalized Eigenvalue Problem

$$\left(\mu \cdot \begin{bmatrix} \mathbf{2} & \mathbf{6} \\ \mathbf{-1} & \mathbf{20} \end{bmatrix} + \lambda \cdot \begin{bmatrix} \mathbf{-2} & \mathbf{4} \\ \mathbf{0} & \mathbf{20} \end{bmatrix}\right) \cdot \begin{bmatrix} v \\ w \end{bmatrix} = 0$$

$$\begin{cases} \mathbf{2}\,\mu\, v + \ \mathbf{6}\,\mu\, w \mathbf{-2}\,\lambda\, v + \mathbf{4}\,\lambda\, w = 0 \\ \mathbf{-1}\,\mu\, v + \mathbf{20}\,\mu\, w + \mathbf{0}\,\lambda\, v + \mathbf{20}\,\lambda\, w = 0 \end{cases} \in \mathbb{C}[\mu,\lambda]_1 \otimes \mathbb{C}[v,w]_1$$

$$\boxed{\text{We look for solutions in } \mathbb{P}^1 \times \mathbb{P}^1}$$



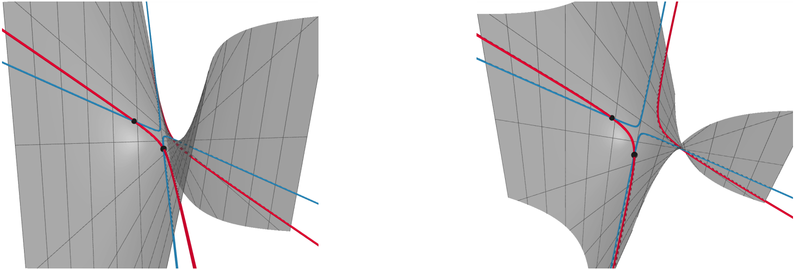
Multihomogeneous systems

Generalized Eigenvalue Problem

$$\left(\mu \cdot \begin{bmatrix} \mathbf{2} & \mathbf{6} \\ \mathbf{-1} & \mathbf{20} \end{bmatrix} + \lambda \cdot \begin{bmatrix} \mathbf{-2} & \mathbf{4} \\ \mathbf{0} & \mathbf{20} \end{bmatrix}\right) \cdot \begin{bmatrix} v \\ w \end{bmatrix} = 0$$

$$\begin{cases} \mathbf{2}\,\mu\, v + \ \mathbf{6}\,\mu\, w \mathbf{-2}\,\lambda\, v + \mathbf{4}\,\lambda\, w = 0 \\ \mathbf{-1}\,\mu\, v + \mathbf{20}\,\mu\, w + \mathbf{0}\,\lambda\, v + \mathbf{20}\,\lambda\, w = 0 \end{cases} \in \mathbb{C}[\mu, \lambda]_1 \otimes \mathbb{C}[v, w]_1$$

We look for solutions in $\mathbb{P}^1 \times \mathbb{P}^1$



Multiprojective resultant

Multiprojective space $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r} \leftrightarrow$ Multihomogeneous polynomials.

Multiprojective resultant

Necessary and sufficient cond. for a multihomogeneous sys. $(f_0, \ldots, f_{n_1+\cdots+n_r}) \in (\mathbb{C}[x_{1,0} \ldots x_{1,n_1}] \otimes \cdots \otimes \mathbb{C}[x_{r,0} \ldots x_{r,n_r}])^{n_1+\cdots+n_r+1}$ to have sols. in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$.

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Weyman complex [Weyman, 1994]

- Complex parameterized by vector $\boldsymbol{m}$. Its determinant is the resultant.
- Strategy → Find $\boldsymbol{m}$ st Weyman complex gives determinantal formula.

Multiprojective resultant

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Multiprojective resultant

Necessary and sufficient cond. for a multihomogeneous sys. $(f_0, \ldots, f_{n_1+\cdots+n_r}) \in (\mathbb{C}[x_{1,0} \ldots x_{1,n_1}] \otimes \cdots \otimes \mathbb{C}[x_{r,0} \ldots x_{r,n_r}])^{n_1+\cdots+n_r+1}$ to have sols. in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$.

Weyman complex [Weyman, 1994]

- Complex parameterized by vector $\boldsymbol{m}$. Its determinant is the resultant.
- Strategy → Find $\boldsymbol{m}$ st Weyman complex gives determinantal formula.

Sylvester-type determinantal formulas

- Unmixed case (same multidegree)

  [Sturmfels, Zelevinsky, 1994], [Weyman, Zelevinsky, 1994], [Dickenstein, Emiris, 2003], [Emiris, Mantzaflaris, 2012], [Emiris, Mantzaflaris, Tsigaridas, 2021]

- Mixed case (different multidegree)

  [Busé Mantzaflaris Tsigaridas, 2017], [B. Faugére Mantzaflaris Tsigaridas, 2021]

Applications : Multiparameter Eigenvalue Problem

Generalized Eigenvalue Problem

$$\left(\begin{bmatrix} -7 & -3 \\ -8 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 12 & 2 \\ 13 & 1 \end{bmatrix} \right) \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} = 0$$

.

Applications : Multiparameter Eigenvalue Problem

- Generalization of the Generalized Eigenvalue Problem

$$\left(\lambda_0 \begin{bmatrix} -7 & -3 \\ -8 & -2 \end{bmatrix} + \lambda_1 \begin{bmatrix} 12 & 2 \\ 13 & 1 \end{bmatrix} + \lambda_2 \begin{bmatrix} -7 & -1 \\ -7 & -1 \end{bmatrix}\right) \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} = 0 \quad .$$

$$\left(\lambda_0 \begin{bmatrix} -11 & -3 \\ 4 & 1 \end{bmatrix} + \lambda_1 \begin{bmatrix} 7 & -1 \\ 1 & 2 \end{bmatrix} + \lambda_2 \begin{bmatrix} -4 & 0 \\ -1 & -1 \end{bmatrix}\right) \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} = 0$$

Applications : Multiparameter Eigenvalue Problem

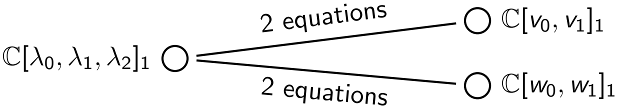
- Generalization of the Generalized Eigenvalue Problem
- Applications in physics (Sturm-Liouville theory)

$$\left(\lambda_0 \begin{bmatrix} -7 & -3 \\ -8 & -2 \end{bmatrix} + \lambda_1 \begin{bmatrix} 12 & 2 \\ 13 & 1 \end{bmatrix} + \lambda_2 \begin{bmatrix} -7 & -1 \\ -7 & -1 \end{bmatrix}\right) \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} = 0$$

$$\left(\lambda_0 \begin{bmatrix} -11 & -3 \\ 4 & 1 \end{bmatrix} + \lambda_1 \begin{bmatrix} 7 & -1 \\ 1 & 2 \end{bmatrix} + \lambda_2 \begin{bmatrix} -4 & 0 \\ -1 & -1 \end{bmatrix}\right) \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} = 0 \; .$$

Applications : Multiparameter Eigenvalue Problem

- Generalization of the Generalized Eigenvalue Problem
- Applications in physics (Sturm-Liouville theory)
- It is a square star multilinear system



$\mathbb{C}[\lambda_0, \lambda_1, \lambda_2]_1$

2 equations

$\mathbb{C}[v_0, v_1]_1$

2 equations

$\mathbb{C}[w_0, w_1]_1$

$$\left(\lambda_0 \begin{bmatrix} -7 & -3 \\ -8 & -2 \end{bmatrix} + \lambda_1 \begin{bmatrix} 12 & 2 \\ 13 & 1 \end{bmatrix} + \lambda_2 \begin{bmatrix} -7 & -1 \\ -7 & -1 \end{bmatrix}\right) \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} = 0 \quad .$$

$$\left(\lambda_0 \begin{bmatrix} -11 & -3 \\ 4 & 1 \end{bmatrix} + \lambda_1 \begin{bmatrix} 7 & -1 \\ 1 & 2 \end{bmatrix} + \lambda_2 \begin{bmatrix} -4 & 0 \\ -1 & -1 \end{bmatrix}\right) \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} = 0$$

Applications : Multiparameter Eigenvalue Problem

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- Applications in physics (Sturm-Liouville theory)
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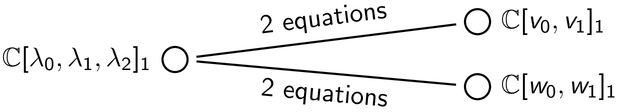


$\mathbb{C}[\lambda_0, \lambda_1, \lambda_2]_1$

2 equations

$\mathbb{C}[v_0, v_1]_1$

2 equations

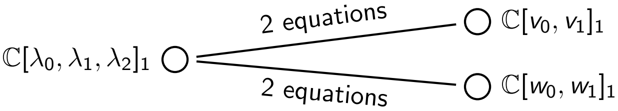
$\mathbb{C}[w_0, w_1]_1$

$$\begin{bmatrix} (-7\,\lambda_0 + 12\,\lambda_1 - 7\,\lambda_2) & (-3\,\lambda_0 + 2\,\lambda_1 - \lambda_2) \\ (-8\,\lambda_0 + 13\,\lambda_1 - 7\,\lambda_2) & (-2\,\lambda_0 + \lambda_1 - \lambda_2) \end{bmatrix} \cdot \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} = 0 \quad .$$

$$\begin{bmatrix} (-11\,\lambda_0 + 7\,\lambda_1 - 4\,\lambda_2) & (-3\,\lambda_0 - \lambda_1) \\ (4\,\lambda_0 + \lambda_1 - \lambda_2) & (\lambda_0 + 2\,\lambda_1 - \lambda_2) \end{bmatrix} \cdot \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} = 0$$

Applications : Multiparameter Eigenvalue Problem

- Generalization of the Generalized Eigenvalue Problem
- Applications in physics (Sturm-Liouville theory)
- It is a square star multilinear system



$$\begin{cases} \left(-7\,\lambda_0 + 12\,\lambda_1 - 7\,\lambda_2\right)\, v_0 + \left(-3\,\lambda_0 + 2\,\lambda_1 - \lambda_2\right) v_1 = 0 \\ \left(-8\,\lambda_0 + 13\,\lambda_1 - 7\,\lambda_2\right)\, v_0 + \left(-2\,\lambda_0 + \lambda_1 - \lambda_2\right) v_1 = 0 \\ \left(-11\,\lambda_0 + 7\,\lambda_1 - 4\,\lambda_2\right)\, w_0 + \left(-3\,\lambda_0 - \lambda_1\right)\, w_1 = 0 \\ \left(4\,\lambda_0 + \lambda_1 - \lambda_2\right)\, w_0 + \left(\lambda_0 + 2\,\lambda_1 - \lambda_2\right)\, w_1 = 0 \end{cases} .$$

Applications : Multiparameter Eigenvalue Problem

- Generalization of the Generalized Eigenvalue Problem
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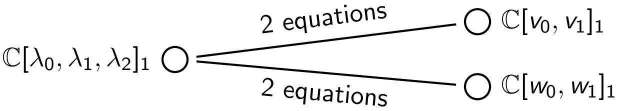

$\mathbb{C}[\lambda_0, \lambda_1, \lambda_2]_1$

2 equations

$\mathbb{C}[v_0, v_1]_1$

2 equations

$\mathbb{C}[w_0, w_1]_1$

$$\begin{cases} f_1 := (-7\,\lambda_0 + 12\,\lambda_1 - 7\,\lambda_2)\; v_0 + (-3\,\lambda_0 + 2\,\lambda_1 - \lambda_2)\, v_1 \\ f_2 := (-8\,\lambda_0 + 13\,\lambda_1 - 7\,\lambda_2)\; v_0 + (-2\,\lambda_0 + \lambda_1 - \lambda_2)\, v_1 \\ f_3 := (-11\,\lambda_0 + 7\,\lambda_1 - 4\,\lambda_2)\; w_0 + (-3\,\lambda_0 - \lambda_1)\; w_1 \\ f_4 := (4\,\lambda_0 + \lambda_1 - \lambda_2)\; w_0 + (\lambda_0 + 2\,\lambda_1 - \lambda_2)\; w_1 \end{cases} .$$

Applications : Multiparameter Eigenvalue Problem

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[1] [B., Faugére, Mantzaflaris, & Tsigaridas, 2021]

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$$\begin{cases} f_1 := (-7\,\lambda_0 + 12\,\lambda_1 - 7\,\lambda_2)\ v_0 + (-3\,\lambda_0 + 2\,\lambda_1 - \lambda_2)\, v_1 \\ f_2 := (-8\,\lambda_0 + 13\,\lambda_1 - 7\,\lambda_2)\ v_0 + (-2\,\lambda_0 + \lambda_1 - \lambda_2)\, v_1 \\ f_3 := (-11\,\lambda_0 + 7\,\lambda_1 - 4\,\lambda_2)\ w_0 + (-3\,\lambda_0 - \lambda_1)\ w_1 \\ f_4 := (4\,\lambda_0 + \lambda_1 - \lambda_2)\ w_0 + (\lambda_0 + 2\,\lambda_1 - \lambda_2)\ w_1 \end{cases} .$$

To solve, we add linear $f_0 := -\lambda_0 + 5\,\lambda_1 - 3\,\lambda_2 \in \mathbb{C}[\boldsymbol{\lambda}]_1$.

Weyman complex[1] → Sylvester-type formula

$$\delta : (\mathbb{C}[\boldsymbol{v}]_1 \otimes \mathbb{C}[\boldsymbol{w}]_1) \times (\mathbb{C}[\boldsymbol{w}]_1)^2 \times (\mathbb{C}[\boldsymbol{v}]_1)^2 \quad \rightarrow (\mathbb{C}[\boldsymbol{\lambda}]_1 \otimes \mathbb{C}[\boldsymbol{v}]_1 \otimes \mathbb{C}[\boldsymbol{w}]_1)$$

$$(\qquad g_0, \qquad g_1, g_2 \qquad g_3, g_4 \quad) \mapsto \qquad \textstyle\sum_{i=0}^{4} g_i\, f_i$$

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Weyman complex[1] → Sylvester-type formula

$$\begin{array}{lllll} \delta : (\mathbb{C}[\boldsymbol{v}]_1 \otimes \mathbb{C}[\boldsymbol{w}]_1) & \times(\mathbb{C}[\boldsymbol{w}]_1)^2 & \times(\mathbb{C}[\boldsymbol{v}]_1)^2 & \rightarrow & (\mathbb{C}[\boldsymbol{\lambda}]_1 \otimes \mathbb{C}[\boldsymbol{v}]_1 \otimes \mathbb{C}[\boldsymbol{w}]_1) \\ (\qquad g_0, & g_1, g_2 & g_3, g_4 &) \mapsto & \sum_{i=0}^{4} g_i\, f_i \end{array}$$

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[1] [B., Faugére, Mantzaflaris, & Tsigaridas, 2021]

Applications : Multiparameter Eigenvalue Problem

$$\delta : (\mathbb{C}[\boldsymbol{v}]_1 \otimes \mathbb{C}[\boldsymbol{w}]_1) \times (\mathbb{C}[\boldsymbol{w}]_1)^2 \times (\mathbb{C}[\boldsymbol{v}]_1)^2 \quad \to (\mathbb{C}[\boldsymbol{\lambda}]_1 \otimes \mathbb{C}[\boldsymbol{v}]_1 \otimes \mathbb{C}[\boldsymbol{w}]_1)$$

$$(\qquad g_0, \qquad\qquad g_1, g_2 \qquad g_3, g_4 \qquad) \mapsto \qquad \textstyle\sum_{i=0}^4 g_i \, f_i$$

$$\begin{bmatrix} M_{1,1} & M_{1,2} \\ M_{2,1} & M_{2,2} \end{bmatrix} = \left[\begin{array}{c|cccccccc|cccc}
w_0 \, f_1 & -7 & & -1 & & 12 & & 2 & & -7 & & -3 & \\
w_1 \, f_1 & & -7 & & -1 & & 12 & & 2 & & -7 & & -3 \\
w_0 \, f_2 & -7 & & -1 & & 13 & & 1 & & -8 & & -2 & \\
w_1 \, f_2 & & -7 & & -1 & & 13 & & 1 & & -8 & & -2 \\
v_0 \, f_3 & -4 & & & & 7 & -1 & & & -11 & -3 & & \\
v_1 \, f_3 & & & -4 & & & & 7 & -1 & & & -11 & -3 \\
v_0 \, f_4 & -1 & -1 & & & 1 & 2 & & & 4 & 1 & & \\
v_1 \, f_4 & & & -1 & -1 & & & 1 & 2 & & & 4 & 1 \\
\hline
v_0 \, w_0 \, f_0 & -3 & & & & 5 & & & & -1 & & & \\
v_0 \, w_1 \, f_0 & & -3 & & & & 5 & & & & -1 & & \\
v_1 \, w_0 \, f_0 & & & -3 & & & & 5 & & & & -1 & \\
v_1 \, w_1 \, f_0 & & & & -3 & & & & 5 & & & & -1
\end{array}\right] .$$

Applications : Multiparameter Eigenvalue Problem

$$\begin{bmatrix} M_{1,1} & M_{1,2} \\ M_{2,1} & M_{2,2} \end{bmatrix} = \left[\begin{array}{c|cccccccc|cccc}
w_0 f_1 & -7 & & -1 & & 12 & & 2 & & -7 & & -3 & \\
w_1 f_1 & & -7 & & -1 & & 12 & & 2 & & -7 & & -3 \\
w_0 f_2 & -7 & & -1 & & 13 & & 1 & & -8 & & -2 & \\
w_1 f_2 & & -7 & & -1 & & 13 & & 1 & & -8 & & -2 \\
v_0 f_3 & -4 & & & & 7 & -1 & & & -11 & -3 & & \\
v_1 f_3 & & & -4 & & & & 7 & -1 & & & -11 & -3 \\
v_0 f_4 & -1 & -1 & & & 1 & 2 & & & 4 & 1 & & \\
v_1 f_4 & & & -1 & -1 & & & 1 & 2 & & & 4 & 1 \\
\hline
v_0 w_0 f_0 & 0 & & & & 1 & & & & 0 & & & \\
v_0 w_1 f_0 & & 0 & & & & 1 & & & & 0 & & \\
v_1 w_0 f_0 & & & 0 & & & & 1 & & & & 0 & \\
v_1 w_1 f_0 & & & & 0 & & & & 1 & & & & 0
\end{array}\right].$$

We consider $f_0 = \lambda_1$ and compute "multiplication map" wrt f_0

(Schur complement of $M_{2,2}$)

$$\widetilde{M}_{2,2} = \begin{bmatrix} 4 & \frac{3}{2} & 1 & \frac{1}{2} \\ -6 & -\frac{5}{2} & 0 & -\frac{1}{2} \\ 3 & \frac{3}{2} & 2 & \frac{1}{2} \\ -6 & -\frac{7}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

Applications : Multiparameter Eigenvalue Problem

$$\begin{bmatrix} M_{1,1} & M_{1,2} \\ M_{2,1} & M_{2,2} \end{bmatrix} = \left[\begin{array}{r|rrrrrrrr|rrrr}
w_0\,f_1 & -7 & & -1 & & 12 & & 2 & & -7 & & -3 & \\
w_1\,f_1 & & -7 & & -1 & & 12 & & 2 & & -7 & & -3 \\
w_0\,f_2 & -7 & & -1 & & 13 & & 1 & & -8 & & -2 & \\
w_1\,f_2 & & -7 & & -1 & & 13 & & 1 & & -8 & & -2 \\
v_0\,f_3 & -4 & & & & 7 & -1 & & & -11 & -3 & & \\
v_1\,f_3 & & & -4 & & & & 7 & -1 & & & -11 & -3 \\
v_0\,f_4 & -1 & -1 & & & 1 & 2 & & & 4 & 1 & & \\
v_1\,f_4 & & & -1 & -1 & & & 1 & 2 & & & 4 & 1 \\
\hline
v_0\,w_0\,f_0 & 0 & & & & 1 & & & & 0 & & & \\
v_0\,w_1\,f_0 & & 0 & & & & 1 & & & & 0 & & \\
v_1\,w_0\,f_0 & & & 0 & & & & 1 & & & & 0 & \\
v_1\,w_1\,f_0 & & & & 0 & & & & 1 & & & & 0
\end{array}\right].$$

We consider $f_0 = \lambda_1$ and compute "multiplication map" wrt f_0

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and solve the system...

Atkinson's Delta method [Atkinson, 1965]

$$\left(\lambda_0 \overbrace{\begin{bmatrix} -7 & -3 \\ -8 & -2 \end{bmatrix}}^{A_1} + \lambda_1 \overbrace{\begin{bmatrix} 12 & 2 \\ 13 & 1 \end{bmatrix}}^{B_1} + \lambda_2 \overbrace{\begin{bmatrix} -7 & -1 \\ -7 & -1 \end{bmatrix}}^{C_1}\right) \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} = 0$$

$$\left(\lambda_0 \underbrace{\begin{bmatrix} -11 & -3 \\ 4 & 1 \end{bmatrix}}_{A_2} + \lambda_1 \underbrace{\begin{bmatrix} 7 & -1 \\ 1 & 2 \end{bmatrix}}_{B_2} + \lambda_2 \underbrace{\begin{bmatrix} -4 & 0 \\ -1 & -1 \end{bmatrix}}_{C_2}\right) \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} = 0 .$$

$$\Delta_0 = B_1 \otimes C_2 - C_1 \otimes B_2$$
$$\Delta_1 = C_1 \otimes A_2 - A_1 \otimes C_2$$

The eigenvalues of pencil (Δ_0, Δ_1) are the λ_1-coordinates of solutions.

Atkinson's Delta method [Atkinson, 1965]

$$\begin{aligned}
\left(\lambda_0 \overbrace{\begin{bmatrix} -7 & -3 \\ -8 & -2 \end{bmatrix}}^{A_1} + \lambda_1 \overbrace{\begin{bmatrix} 12 & 2 \\ 13 & 1 \end{bmatrix}}^{B_1} + \lambda_2 \overbrace{\begin{bmatrix} -7 & -1 \\ -7 & -1 \end{bmatrix}}^{C_1}\right) \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} = 0 \\
\left(\lambda_0 \underbrace{\begin{bmatrix} -11 & -3 \\ 4 & 1 \end{bmatrix}}_{A_2} + \lambda_1 \underbrace{\begin{bmatrix} 7 & -1 \\ 1 & 2 \end{bmatrix}}_{B_2} + \lambda_2 \underbrace{\begin{bmatrix} -4 & 0 \\ -1 & -1 \end{bmatrix}}_{C_2}\right) \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} = 0
\end{aligned} .$$

$$\begin{aligned}
\Delta_0 &= B_1 \otimes C_2 - C_1 \otimes B_2 \\
\Delta_1 &= C_1 \otimes A_2 - A_1 \otimes C_2
\end{aligned}$$

The eigenvalues of pencil (Δ_0, Δ_1) are the λ_1-coordinates of solutions.

Are they multiplication maps?

$$\Delta_0^{-1} \Delta_1 = \begin{bmatrix} 4 & \frac{3}{2} & 1 & \frac{1}{2} \\ -6 & -\frac{5}{2} & 0 & -\frac{1}{2} \\ 3 & \frac{3}{2} & 2 & \frac{1}{2} \\ -6 & -\frac{7}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

Atkinson's Delta method [Atkinson, 1965]

$$\begin{gathered}\left(\lambda_0 \overbrace{\begin{bmatrix}-7 & -3\\ -8 & -2\end{bmatrix}}^{A_1} + \lambda_1 \overbrace{\begin{bmatrix}12 & 2\\ 13 & 1\end{bmatrix}}^{B_1} + \lambda_2 \overbrace{\begin{bmatrix}-7 & -1\\ -7 & -1\end{bmatrix}}^{C_1}\right)\begin{bmatrix}v_0\\ v_1\end{bmatrix} = 0\\ \left(\lambda_0 \underbrace{\begin{bmatrix}-11 & -3\\ 4 & 1\end{bmatrix}}_{A_2} + \lambda_1 \underbrace{\begin{bmatrix}7 & -1\\ 1 & 2\end{bmatrix}}_{B_2} + \lambda_2 \underbrace{\begin{bmatrix}-4 & 0\\ -1 & -1\end{bmatrix}}_{C_2}\right)\begin{bmatrix}w_0\\ w_1\end{bmatrix} = 0\end{gathered}\ .$$

$$\begin{aligned}\Delta_0 &= B_1 \otimes C_2 - C_1 \otimes B_2\\ \Delta_1 &= C_1 \otimes A_2 - A_1 \otimes C_2\end{aligned}$$

The eigenvalues of pencil (Δ_0, Δ_1) are the λ_1-coordinates of solutions.
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&\left(\lambda_0 \underbrace{\begin{bmatrix} -11 & -3 \\ 4 & 1 \end{bmatrix}}_{A_2} + \lambda_1 \underbrace{\begin{bmatrix} 7 & -1 \\ 1 & 2 \end{bmatrix}}_{B_2} + \lambda_2 \underbrace{\begin{bmatrix} -4 & 0 \\ -1 & -1 \end{bmatrix}}_{C_2}\right) \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} = 0
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Atkinson's Delta method is a sort of Cramer rule for MEP.

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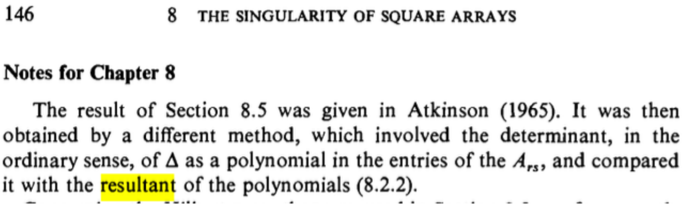


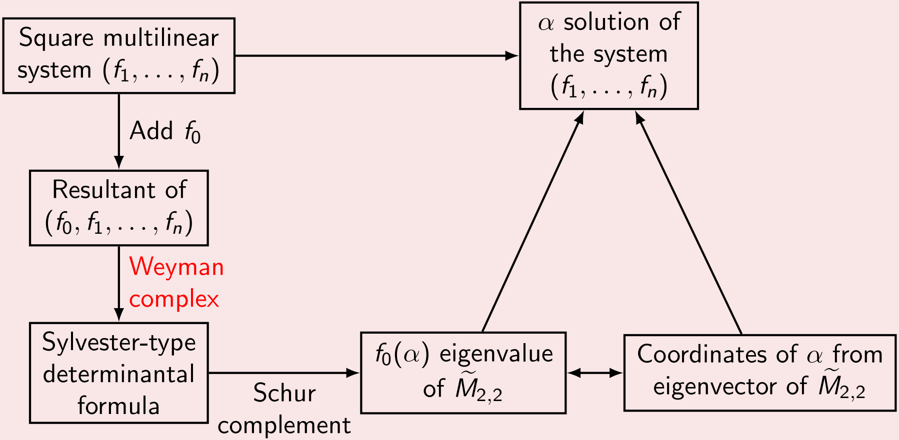
146 8 THE SINGULARITY OF SQUARE ARRAYS

Notes for Chapter 8

The result of Section 8.5 was given in Atkinson (1965). It was then obtained by a different method, which involved the determinant, in the ordinary sense, of Δ as a polynomial in the entries of the A_{rs}, and compared it with the resultant of the polynomials (8.2.2).

Atkinson, F.V. (1972) Multiparameter eigenvalue problems (Vol. 1) NY Academic Press

Solving polynomial systems using the resultant



Square multilinear system $(f_1, \ldots, f_n)$

Add f_0

Resultant of $(f_0, f_1, \ldots, f_n)$

Weyman complex

Sylvester-type determinantal formula

Schur complement

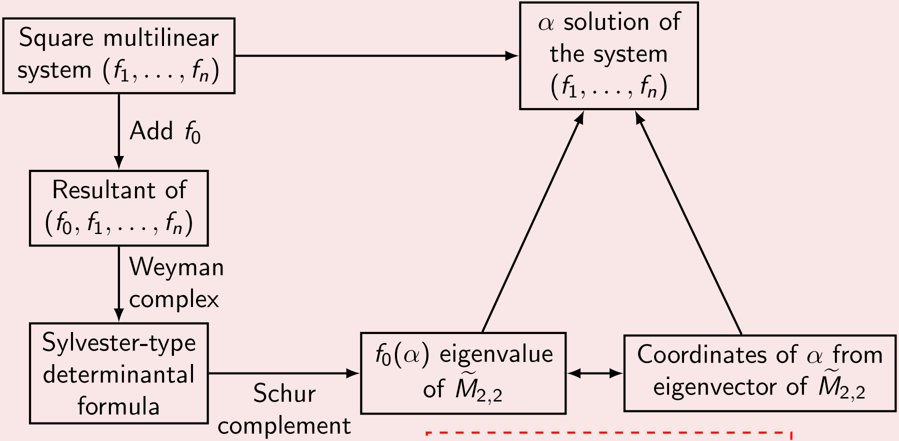
$f_0(\alpha)$ eigenvalue of $\widetilde{M}_{2,2}$

Coordinates of α from eigenvector of $\widetilde{M}_{2,2}$

α solution of the system $(f_1, \ldots, f_n)$

$$\begin{bmatrix} M_{1,1} & M_{1,2} \\ M_{2,1} & M_{2,2} \end{bmatrix} \rightarrow \begin{bmatrix} M_{1,1} & M_{1,2} \\ 0 & \widetilde{M}_{2,2} \end{bmatrix}$$

Solving polynomial systems using the resultant



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$$\begin{bmatrix} M_{1,1} & M_{1,2} \\ M_{2,1} & M_{2,2} \end{bmatrix} \to \begin{bmatrix} M_{1,1} & M_{1,2} \\ 0 & \widetilde{M}_{2,2} \end{bmatrix}$$

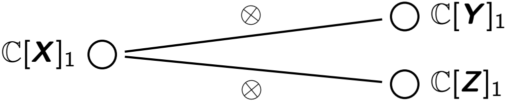
Great, but...

Sylvester-type determ. formulas does not exist in general.

Can we generalize the scheme ?

Example : Solving bilinear system with two supports

- Over $\mathbb{P}^{n_x} \times \mathbb{P}^{n_y} \times \mathbb{P}^{n_z}$, we want to solve bilinear system $(f_1, \ldots, f_n)$:
  - $f_1, \ldots, f_r \in \mathbb{C}[\boldsymbol{X}]_1 \otimes \mathbb{C}[\boldsymbol{Y}]_1$,
  - $f_{r+1}, \ldots, f_n \in \mathbb{C}[\boldsymbol{X}]_1 \otimes \mathbb{C}[\boldsymbol{Z}]_1$



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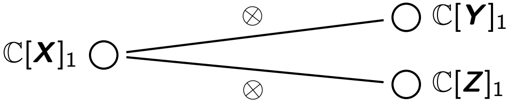

- We introduce a trilinear polynomial $f_0 \in \mathbb{C}[\boldsymbol{X}]_1 \otimes \mathbb{C}[\boldsymbol{Y}]_1 \otimes \mathbb{C}[\boldsymbol{Z}]_1$.

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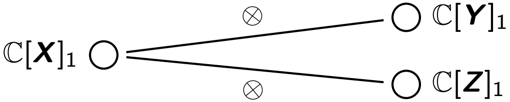

- We introduce a trilinear polynomial $f_0 \in \mathbb{C}[\boldsymbol{X}]_1 \otimes \mathbb{C}[\boldsymbol{Y}]_1 \otimes \mathbb{C}[\boldsymbol{Z}]_1$.
- Weyman complex → Koszul-type formula for the resultant.
  - The elements in the matrix are ± the coefficients of the polynomials.

Example : Solving bilinear system with two supports

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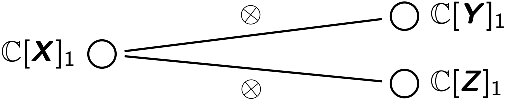


- We introduce a trilinear polynomial $f_0 \in \mathbb{C}[\boldsymbol{X}]_1 \otimes \mathbb{C}[\boldsymbol{Y}]_1 \otimes \mathbb{C}[\boldsymbol{Z}]_1$.
- Weyman complex → Koszul-type formula for the resultant.
  - The elements in the matrix are ± the coefficients of the polynomials.

Number of solutions of $(f_1, \ldots, f_n)$	Size of the Koszul-type matrix
$\binom{\boldsymbol{r}}{\boldsymbol{n_y}}\binom{\boldsymbol{n-r}}{\boldsymbol{n_z}}$	$(n_x + 1)\binom{\boldsymbol{r}}{\boldsymbol{n_y}}\binom{\boldsymbol{n-r}}{\boldsymbol{n_z}}\frac{r \cdot (n-r) - n_y \cdot n_z + n + 1}{(r - n_y + 1)(n - r - n_z + 1)}$

Example : Koszul-type formula

$$\left\{\begin{array}{l} f_1 := 7\,x_0y_0 + -8\,x_0y_1 + -1\,x_1y_0 + 2\,x_1y_1 \\ f_2 := -5\,x_0y_0 + 7\,x_0y_1 + -1\,x_1y_0 + -1\,x_1y_1 \\ f_3 := -6\,x_0z_0 + 9\,x_0z_1 + -1\,x_1z_0 + -2\,x_1z_1 \end{array}\right. \qquad \begin{array}{l} \left.\begin{array}{l} \\ \\ \end{array}\right\} \in \mathbb{C}[\boldsymbol{X},\boldsymbol{Y}] \\ \quad\ \in \mathbb{C}[\boldsymbol{X},\boldsymbol{Z}] \end{array}$$

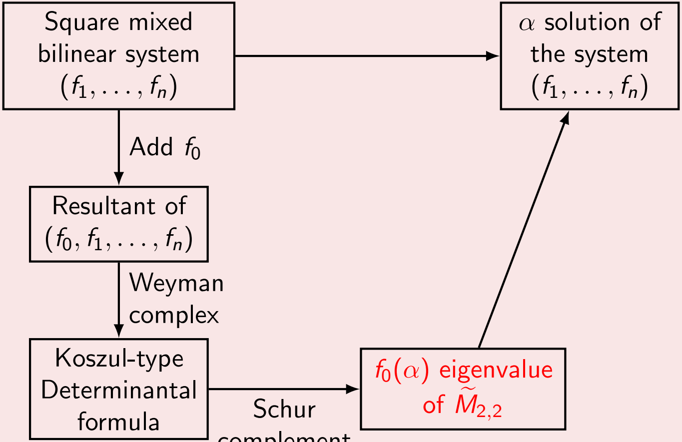
Example : Koszul-type formula

$$\left\{\begin{array}{l} \left.\begin{array}{l} f_0 := \mathbf{3}\,x_0 y_0 z_0 + \mathbf{-1} x_0 y_0 z_1 + \mathbf{-4}\,x_0 y_1 z_0 + \mathbf{2}\,x_0 y_1 z_1 \\ \qquad + \mathbf{1} x_1 y_0 z_0 + \mathbf{2}\,x_1 y_0 z_1 + \mathbf{2}\,x_1 y_1 z_0 + \mathbf{-2}\,x_1 y_1 z_1 \end{array}\right\} \in \mathbb{C}[\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}] \\ \left.\begin{array}{l} f_1 := \mathbf{7}\,x_0 y_0 + \mathbf{-8}\,x_0 y_1 + \mathbf{-1}\,x_1 y_0 + \mathbf{2}\,x_1 y_1 \\ f_2 := \mathbf{-5}\,x_0 y_0 + \mathbf{7}\,x_0 y_1 + \mathbf{-1}\,x_1 y_0 + \mathbf{-1}\,x_1 y_1 \end{array}\right\} \in \mathbb{C}[\boldsymbol{X}, \boldsymbol{Y}] \\ f_3 := \mathbf{-6}\,x_0 z_0 + \mathbf{9}\,x_0 z_1 + \mathbf{-1}\,x_1 z_0 + \mathbf{-2}\,x_1 z_1 \qquad \in \mathbb{C}[\boldsymbol{X}, \boldsymbol{Z}] \end{array}\right.$$

Example : Koszul-type formula

$$\begin{cases} \left.\begin{aligned} f_0 := {} & 3\, x_0 y_0 z_0 + -1 x_0 y_0 z_1 + -4\, x_0 y_1 z_0 + 2\, x_0 y_1 z_1 \\ & +1 x_1 y_0 z_0 + 2\, x_1 y_0 z_1 + 2\, x_1 y_1 z_0 + -2\, x_1 y_1 z_1 \end{aligned}\right\} \in \mathbb{C}[\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}] \\ \left.\begin{aligned} f_1 := {} & 7\, x_0 y_0 + -8\, x_0 y_1 + -1\, x_1 y_0 + 2\, x_1 y_1 \\ f_2 := {} & -5\, x_0 y_0 + 7\, x_0 y_1 + -1\, x_1 y_0 + -1\, x_1 y_1 \end{aligned}\right\} \in \mathbb{C}[\boldsymbol{X}, \boldsymbol{Y}] \\ f_3 := -6\, x_0 z_0 + 9\, x_0 z_1 + -1\, x_1 z_0 + -2\, x_1 z_1 \qquad \in \mathbb{C}[\boldsymbol{X}, \boldsymbol{Z}] \end{cases}$$

$$M = \begin{bmatrix} & & & 5 & -7 & 1 & 1 & & & \\ & & & 7 & -8 & -1 & 2 & & & \\ & -1 & & & & & & -1 & -5 & 7 \\ 7 & & -1 & & & & & -1 & & -5 \\ & 1 & & & & & & -2 & -7 & 8 \\ 8 & & -2 & & & & & 1 & & -7 \\ & 2 & & 9 & & -2 & & -2 & -1 & 2 \\ 2 & & -2 & & 9 & & -2 & 2 & & -1 \\ & 1 & & -6 & & -1 & & 2 & 3 & -4 \\ -4 & & 2 & & -6 & & -1 & 1 & & 3 \end{bmatrix}$$



Square mixed bilinear system $(f_1, \ldots, f_n)$

α solution of the system $(f_1, \ldots, f_n)$

Add f_0

Resultant of $(f_0, f_1, \ldots, f_n)$

Weyman complex

Koszul-type Determinantal formula

Schur complement

$f_0(\alpha)$ eigenvalue of $\widetilde{M}_{2,2}$

$$\begin{bmatrix} M_{1,1} & M_{1,2} \\ M_{2,1} & M_{2,2} \end{bmatrix} \to \begin{bmatrix} M_{1,1} & M_{1,2} \\ 0 & \widetilde{M}_{2,2} \end{bmatrix}$$

Example : Generalized Eigenvalue Criterion

$$\begin{cases} \boldsymbol{f_0} := & \boxed{\mathbf{3}}\, \boldsymbol{x_0 y_0 z_0} + \mathbf{-1} x_0 y_0 z_1 + \mathbf{-4}\, x_0 y_1 z_0 + \mathbf{2}\, x_0 y_1 z_1 \\ & +\mathbf{1} x_1 y_0 z_0 + \mathbf{2}\, x_1 y_0 z_1 + \mathbf{2}\, x_1 y_1 z_0 + \mathbf{-2}\, x_1 y_1 z_1 \\ \boldsymbol{f_1} := & \mathbf{7}\, x_0 y_0 + \mathbf{-8}\, x_0 y_1 + \mathbf{-1}\, x_1 y_0 + \mathbf{2}\, x_1 y_1 \\ \boldsymbol{f_2} := & \mathbf{-5}\, x_0 y_0 + \mathbf{7}\, x_0 y_1 + \mathbf{-1}\, x_1 y_0 + \mathbf{-1}\, x_1 y_1 \\ \boldsymbol{f_3} := & \mathbf{-6}\, x_0 z_0 + \mathbf{9}\, x_0 z_1 + \mathbf{-1}\, x_1 z_0 + \mathbf{-2}\, x_1 z_1 \end{cases}$$

$$\left[\begin{array}{c|c} M_{1,1} & M_{1,2} \\ \hline M_{2,1} & \boldsymbol{M_{2,2}} \end{array}\right] = \left[\begin{array}{cccccccc|cc} 0 & 0 & 0 & \mathbf{5} & \mathbf{-7} & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{7} & \mathbf{-8} & \mathbf{-1} & \mathbf{2} & 0 & 0 & 0 \\ 0 & \mathbf{-1} & 0 & 0 & 0 & 0 & 0 & \mathbf{-1} & \mathbf{-5} & \mathbf{7} \\ \mathbf{7} & 0 & \mathbf{-1} & 0 & 0 & 0 & 0 & \mathbf{-1} & 0 & \mathbf{-5} \\ 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & \mathbf{-2} & \mathbf{-7} & \mathbf{8} \\ \mathbf{8} & 0 & \mathbf{-2} & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & \mathbf{-7} \\ 0 & \mathbf{2} & 0 & \mathbf{9} & 0 & \mathbf{-2} & 0 & \mathbf{-2} & \mathbf{-1} & \mathbf{2} \\ \mathbf{2} & 0 & \mathbf{-2} & 0 & \mathbf{9} & 0 & \mathbf{-2} & \mathbf{2} & 0 & \mathbf{-1} \\ \hline 0 & \mathbf{1} & 0 & \mathbf{-6} & 0 & \mathbf{-1} & 0 & \mathbf{2} & \boxed{\mathbf{3}} & \mathbf{-4} \\ \mathbf{-4} & 0 & \mathbf{2} & 0 & \mathbf{-6} & 0 & \mathbf{-1} & \mathbf{1} & 0 & \boxed{\mathbf{3}} \end{array}\right]$$

Example : Generalized Eigenvalue Criterion

$$\begin{cases} f_0 := & \boxed{3}\, \boldsymbol{x_0 y_0 z_0} + -1 x_0 y_0 z_1 + -4\, x_0 y_1 z_0 + 2\, x_0 y_1 z_1 \\ & +1 x_1 y_0 z_0 + 2\, x_1 y_0 z_1 + 2\, x_1 y_1 z_0 + -2\, x_1 y_1 z_1 \\ f_1 := & 7\, x_0 y_0 + -8\, x_0 y_1 + -1\, x_1 y_0 + 2\, x_1 y_1 \\ f_2 := & -5\, x_0 y_0 + 7\, x_0 y_1 + -1\, x_1 y_0 + -1\, x_1 y_1 \\ f_3 := & -6\, x_0 z_0 + 9\, x_0 z_1 + -1\, x_1 z_0 + -2\, x_1 z_1 \end{cases}$$

$$\left[\begin{array}{c|c} M_{1,1} & M_{1,2} \\ \hline M_{2,1} & \boldsymbol{M_{2,2}} \end{array}\right] = \left[\begin{array}{cccccccc|cc} 0 & 0 & 0 & 5 & -7 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 7 & -8 & -1 & 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & -5 & 7 \\ 7 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & -5 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -2 & -7 & 8 \\ 8 & 0 & -2 & 0 & 0 & 0 & 0 & 1 & 0 & -7 \\ 0 & 2 & 0 & 9 & 0 & -2 & 0 & -2 & -1 & 2 \\ 2 & 0 & -2 & 0 & 9 & 0 & -2 & 2 & 0 & -1 \\ \hline 0 & 1 & 0 & -6 & 0 & -1 & 0 & 2 & \boxed{3} & -4 \\ -4 & 0 & 2 & 0 & -6 & 0 & -1 & 1 & 0 & \boxed{3} \end{array}\right]$$

$$\widetilde{M}_{2,2} := \left(M_{2,2} - M_{2,1} \cdot M_{1,1}^{-1} \cdot M_{1,2}\right) = \begin{bmatrix} 5 & -2 \\ 4 & -1 \end{bmatrix}$$

Example : Generalized Eigenvalue Criterion

$$\begin{cases} f_0 := & \boxed{3}\,\mathbf{x_0 y_0 z_0} + -1 x_0 y_0 z_1 + -4\, x_0 y_1 z_0 + 2\, x_0 y_1 z_1 \\ & +1 x_1 y_0 z_0 + 2\, x_1 y_0 z_1 + 2\, x_1 y_1 z_0 + -2\, x_1 y_1 z_1 \\ f_1 := & 7\, x_0 y_0 + -8\, x_0 y_1 + -1\, x_1 y_0 + 2\, x_1 y_1 \\ f_2 := & -5\, x_0 y_0 + 7\, x_0 y_1 + -1\, x_1 y_0 + -1\, x_1 y_1 \\ f_3 := & -6\, x_0 z_0 + 9\, x_0 z_1 + -1\, x_1 z_0 + -2\, x_1 z_1 \end{cases}$$

$$\left[\begin{array}{c|c} M_{1,1} & M_{1,2} \\ \hline M_{2,1} & \mathbf{M_{2,2}} \end{array}\right] = \left[\begin{array}{cccccccc|cc} 0 & 0 & 0 & 5 & -7 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 7 & -8 & -1 & 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & -5 & 7 \\ 7 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & -5 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -2 & -7 & 8 \\ 8 & 0 & -2 & 0 & 0 & 0 & 0 & 1 & 0 & -7 \\ 0 & 2 & 0 & 9 & 0 & -2 & 0 & -2 & -1 & 2 \\ 2 & 0 & -2 & 0 & 9 & 0 & -2 & 2 & 0 & -1 \\ \hline 0 & 1 & 0 & -6 & 0 & -1 & 0 & 2 & \boxed{3} & -4 \\ -4 & 0 & 2 & 0 & -6 & 0 & -1 & 1 & 0 & \boxed{3} \end{array}\right]$$

$$\widetilde{M}_{2,2} := \left(M_{2,2} - M_{2,1} \cdot M_{1,1}^{-1} \cdot M_{1,2}\right) = \begin{bmatrix} 5 & -2 \\ 4 & -1 \end{bmatrix}$$

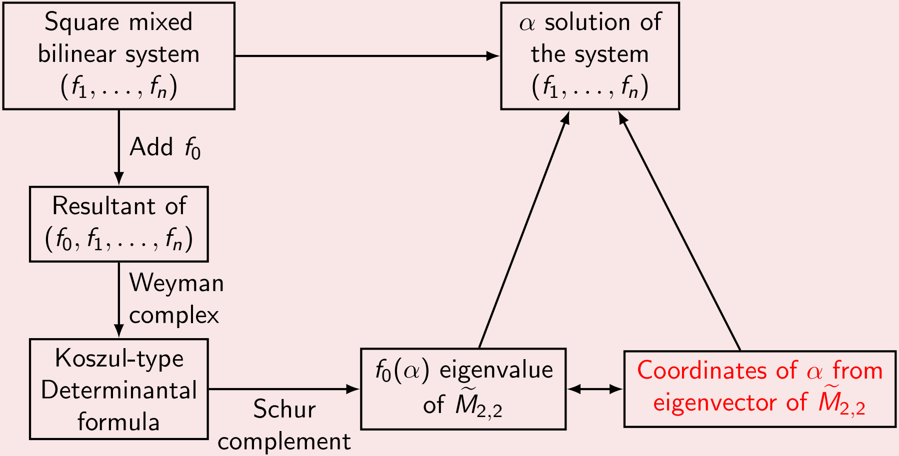
(f_1, f_2, f_3) has 2 solutions

$$\left.\begin{array}{l} (1:1\ ;\ 1:1\ ;\ 1:1) \\ (1:3\ ;\ 1:2\ ;\ 1:3) \end{array}\right\} \in \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$$

Eigenvalues of $\widetilde{M}_{2,2}$

$$\frac{f_0}{x_0 y_0 z_0}((1:1\ ;\ 1:1\ ;\ 1:1)) = 3$$

$$\frac{f_0}{x_0 y_0 z_0}((1:3\ ;\ 1:2\ ;\ 1:3)) = 1$$



Square mixed bilinear system $(f_1, \ldots, f_n)$

α solution of the system $(f_1, \ldots, f_n)$

Add f_0

Resultant of $(f_0, f_1, \ldots, f_n)$

Weyman complex

Koszul-type Determinantal formula

Schur complement

$f_0(\alpha)$ eigenvalue of $\widetilde{M}_{2,2}$

Coordinates of α from eigenvector of $\widetilde{M}_{2,2}$

$$\begin{bmatrix} M_{1,1} & M_{1,2} \\ M_{2,1} & M_{2,2} \end{bmatrix} \rightarrow \begin{bmatrix} M_{1,1} & M_{1,2} \\ 0 & \widetilde{M}_{2,2} \end{bmatrix}$$

Example : Generalization of the eigenvector criterion

$$\frac{f_0}{x_0 y_0 z_0}((1:3\ ;\ 1:2\ ;\ 1:3)) = 1 \qquad \bar{\boldsymbol{v}} := \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 5 & -2 \\ 4 & -1 \end{bmatrix} \cdot \bar{\boldsymbol{v}} = 1 \cdot \bar{\boldsymbol{v}}$$

We can not recover $(1:3\ ;\ 1:2\ ;\ 1:3)$ from $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

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We can not recover $(1:3\ ;\ 1:2\ ;\ 1:3)$ from $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

We extend $\boldsymbol{\bar{v}} \to \boldsymbol{v}$ s.t.

$$\left[\begin{array}{c|c} M_{1,1} & M_{1,2} \\ \hline M_{2,1} & M_{2,2} \end{array}\right] \cdot \boldsymbol{v} = \frac{f_0}{\mathfrak{m}}(\alpha) \cdot \left[\begin{array}{c} 0 \\ \hline \boldsymbol{\bar{v}} \end{array}\right] \qquad \begin{bmatrix} 1 \\ 2 \end{bmatrix} \to \begin{bmatrix} 4 & = & 1\cdot 2\cdot 2 \\ 3 & = & 3\cdot 1\cdot 1 \\ 12 & = & 3\cdot 2\cdot 2 \\ 1 & = & 1\cdot 1\cdot 1 \\ 2 & = & 1\cdot 1\cdot 2 \\ 3 & = & 3\cdot 1 \\ 6 & = & 3\cdot 1\cdot 2 \\ 6 & = & 3\cdot 2 \\ 1 & = & 1\cdot 1 \\ 2 & = & 1\cdot 2 \end{bmatrix}$$

$$\begin{cases} (1\cdot \partial x_0 + 3\cdot \partial x_1) \otimes \left(1\cdot 1\cdot \partial y_0^2 + 1\cdot 2\cdot \partial y_0 \partial y_1 + 2\cdot 2\cdot \partial {y_1}^2\right) \otimes 1 \\ (1\cdot \partial x_0 + 3\cdot \partial x_1) \otimes (1\cdot \partial y_0 + 2\cdot \partial y_1) \otimes 1 \end{cases}$$

Solving Mixed Square Multilinear Systems

Let $(f_1, \ldots, f_n) \in (\mathbb{C}[\boldsymbol{X}_1] \otimes \cdots \otimes \mathbb{C}[\boldsymbol{X}_A] \otimes \mathbb{C}[\boldsymbol{Y}_1] \otimes \cdots \otimes \mathbb{C}[\boldsymbol{Y}_B])^n$.

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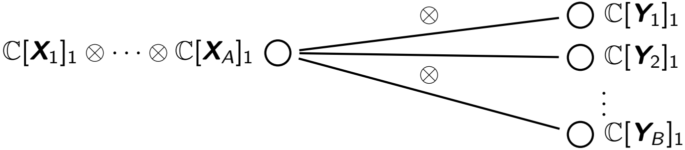
- **Star multilinear system**: For every f_k, there is j_k such that

$$f_k \in \mathbb{C}[\boldsymbol{X}_1]_1 \otimes \dots \mathbb{C}[\boldsymbol{X}_A]_1 \otimes \mathbb{C}[\boldsymbol{Y}_{j_k}]_1.$$



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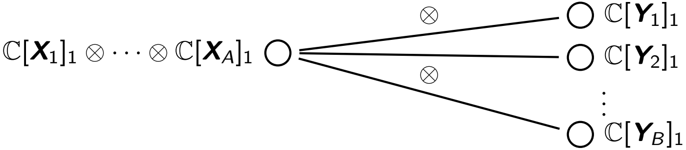
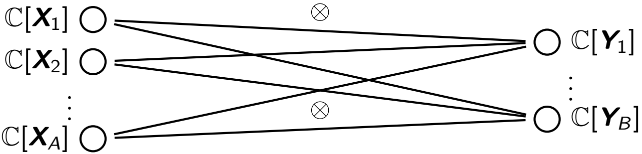


- **Bipartite bilinear system**: For every f_k, there are i_k and j_k such that

$$f_k \in \mathbb{C}[\boldsymbol{X}_{i_k}]_1 \otimes \mathbb{C}[\boldsymbol{Y}_{j_k}]_1.$$



Summing-up

Tools

- Weyman complex → Determinantal formula
- Sylvester- and Koszul-type formulas
- Eigenvalues/Eigenvectors
  (Evaluation of the solutions/coordinates of the solutions)

Results

- Sylvester- and Koszul-type determinantal formula for the resultant
- Extension of the Eigenvalue and Eigenvector criteria
- Applications to the Multiparameter Eigenvalue Problem

Questions

- Can we exploit structure of Koszul-type formulas?
- What can we say numerically?

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Thank you!

[arXiv:1805.05060]
[arXiv:2105.13188]

Sylvester–type formulas via Weyman complex (I)

Consider $X = \mathbb{P}^1 \times \mathbb{P}^1$ and $R = \bigoplus_{(a,b)\in\mathbb{Z}^2} \mathbb{C}[x_0, x_1]_a \otimes \mathbb{C}[y_0, y_1]_b$.

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We want $(n, m) \in \mathbb{N}^2$ such that the map δ, at that degree, is surjective:

$$\delta : (g_0, g_1, g_2) \in R(-1, -1)^3 \mapsto \sum_i f_i \, g_i \in R$$

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The Koszul complex associated to δ is not exact,

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$$0 \to \mathcal{O}_X(\quad -3, \quad -3) \to \mathcal{O}_X(\quad -2, \quad -2)^3 \to \mathcal{O}_X(\quad -1, \quad -1)^3 \to \mathcal{O}_X \qquad \to 0.$$

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Via sheaf cohomology, transform this complex into exact complex of vect. spaces.

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Fix $r \in \mathbb{N}$ st, for all $t \neq r$ and i, $H^t(X, \mathcal{K}_i \otimes O_X(n, m)) = 0$. We get exact complex,

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- **Künneth formula:** $H^i(X, \mathcal{O}_X(a, b)) \simeq \bigoplus_{j+k=i} H^j(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a)) \otimes H^k(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(b))$.

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For $n = 3, m = 3$, we can consider $r = 0$ and obtain exact complex

$$0 \to H^0(X, \mathcal{O}(0, 0)) \to H^0(X, \mathcal{O}(1, 1))^3 \to H^0(X, \mathcal{O}(2, 2))^3 \xrightarrow{\delta} H^0(X, \mathcal{O}(3, 3)) \to 0$$

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$$H^r(X, \mathcal{O}_X(n-1, m-1))^3 \to H^r(X, \mathcal{O}_X(n, m)) \to 0.$$

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- **Künneth formula:** $H^i(X, \mathcal{O}_X(a, b)) \simeq \bigoplus_{j+k=i} H^j(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a)) \otimes H^k(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(b))$.
- **Serre duality:** $H^j(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a)) \simeq \begin{cases} \mathbb{C}[x_0, x_1]_a & \text{if } j = 0 \\ \mathbb{C}[\partial x_0, \partial x_1]_{-2-a} & \text{if } j = 1 \\ 0 & \text{Otherwise} \end{cases}$.

For $n = 3, m = 3$, we can consider $r = 0$ and obtain exact complex

$$0 \to R_{(0,0)} \to R^3_{(1,1)} \to R^3_{(2,2)} \xrightarrow{\delta} R_{(3,3)} \to 0$$

Sylvester–type formulas via Weyman complex (II)

Consider $X = \mathbb{P}^1 \times \mathbb{P}^1$ and $R = \bigoplus_{(a,b)\in\mathbb{Z}^2} \mathbb{C}[x_0, x_1]_a \otimes \mathbb{C}[y_0, y_1]_b$.

Fix $r \in \mathbb{N}$ st, for all $t \neq r$ and i, $H^t(X, \mathcal{K}_i \otimes O_X(n, m)) = 0$. We get exact complex,

$$0 \to H^r(X, \mathcal{O}_X(n-3, m-3)) \to H^r(X, \mathcal{O}_X(n-2, m-2))^3 \to$$
$$H^r(X, \mathcal{O}_X(n-1, m-1))^3 \to H^r(X, \mathcal{O}_X(n, m)) \to 0.$$

We can compute easily the cohomologies of the sheaves $\mathcal{O}_X(a, b)$,

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For $n = 2, m = 2$, we can consider $r = 0$ and obtain exact complex

$$0 \to 0 \to R^3_{(0,0)} \to R^3_{(1,1)} \xrightarrow{\delta} R_{(2,2)} \to 0$$

Sylvester–type formulas via Weyman complex (II)

Consider $X = \mathbb{P}^1 \times \mathbb{P}^1$ and $R = \bigoplus_{(a,b)\in\mathbb{Z}^2} \mathbb{C}[x_0, x_1]_a \otimes \mathbb{C}[y_0, y_1]_b$.

Fix $r \in \mathbb{N}$ st, for all $t \neq r$ and i, $H^t(X, \mathcal{K}_i \otimes O_X(n, m)) = 0$. We get exact complex,

$$0 \to H^r(X, \mathcal{O}_X(n-3, m-3)) \to H^r(X, \mathcal{O}_X(n-2, m-2))^3 \to H^r(X, \mathcal{O}_X(n-1, m-1))^3 \to H^r(X, \mathcal{O}_X(n, m)) \to 0.$$

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For $n = 2, m = 1$, we can consider $r = 0$ and obtain exact complex

$$0 \to 0 \to 0 \to R^3_{(1,0)} \xrightarrow{\delta} R_{(2,1)} \to 0$$

Determinantal formula!!

Koszul-type formulas via Weyman complex

Consider $X = \mathbb{P}^1 \times \mathbb{P}^1$, $R = \mathbb{C}[X]$, $f_0, f_1, f_2 \in R_{1,1}$ with no common zeros.

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For $n = -1, m = 2$, we consider $r = 1$ and obtain exact complex

$$0 \to H^1(X, \mathcal{O}_X(-4,-1)) \to H^1(X, \mathcal{O}_X(-3,0))^3 \to H^1(X, \mathcal{O}_X(-2,1))^3$$
$$\to H^1(X, \mathcal{O}_X(-1,2)) \to 0$$

Koszul-type formulas via Weyman complex

Consider $X = \mathbb{P}^1 \times \mathbb{P}^1$, $R = \mathbb{C}[X]$, $f_0, f_1, f_2 \in R_{1,1}$ with no common zeros.

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For $n = -1, m = 2$, we consider $r = 1$ and obtain exact complex

$$0 \to 0 \to (\mathbb{C}[\partial x_0, \partial x_1]_1 \otimes \mathbb{C}[y_0, y_1]_0)^3 \xrightarrow{\mu} (\mathbb{C}[\partial x_0, \partial x_1]_0 \otimes \mathbb{C}[y_0, y_1]_1)^3$$
$$\to 0 \to 0$$

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Let E be a 3-dimensional vector space with basis e_0, e_1, e_2 and identify $(\mathbb{C}[\partial x_0, \partial x_1]_1 \otimes \mathbb{C}[y_0, y_1]_0)^3 \simeq \mathbb{C}[\partial x_0, \partial x_1]_1 \otimes \mathbb{C}[y_0, y_1]_0 \otimes \bigwedge^2 E$ and $(\mathbb{C}[\partial x_0, \partial x_1]_0 \otimes \mathbb{C}[y_0, y_1]_1)^3 \simeq \mathbb{C}[\partial x_0, \partial x_1]_0 \otimes \mathbb{C}[y_0, y_1]_1 \otimes \bigwedge^1 E$.

Koszul-type formulas via Weyman complex

Consider $X = \mathbb{P}^1 \times \mathbb{P}^1$, $R = \mathbb{C}[X]$, $f_0, f_1, f_2 \in R_{1,1}$ with no common zeros.

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For $n = -1, m = 2$, we consider $r = 1$ and obtain exact complex

$$0 \rightarrow 0 \rightarrow (\mathbb{C}[\partial x_0, \partial x_1]_1 \otimes \mathbb{C}[y_0, y_1]_0)^3 \xrightarrow{\mu} (\mathbb{C}[\partial x_0, \partial x_1]_0 \otimes \mathbb{C}[y_0, y_1]_1)^3 \rightarrow 0 \rightarrow 0$$

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For $f_i = \sum_{(j,k) \in \{0,1\}^2} c^{(i)}_{j,k} \, x_j \, y_k$, we have that

$\mu(\partial x_0 \otimes 1 \otimes e_0 \wedge e_1) = 1 \otimes (c^{(0)}_{0,0} \, y_0 + c^{(0)}_{0,1} \, y_1) \otimes e_1 - 1 \otimes (c^{(1)}_{0,0} \, y_0 + c^{(1)}_{0,1} \, y_1) \otimes e_0$

This is the end, beautiful friend