Studying group actions for fun and profit

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Outline

Groups and group actions

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Throughout the talk I will assume that

- Γ is a finite
- the field k is "sufficiently" algebraically closed for Γ.

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such that $F(g,\cdot) : \Omega \to \Omega$ is a bijection for every $g \in \Gamma$, and $F(g, F(h,\omega)) = F(gh,\omega)$.

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- if we are given a homomorphism $\rho\colon \Gamma \to \mathrm{GL}(V)$ to the group of invertible matrices, then Γ acts on V via

$$(g, v) \mapsto \rho(g)v.$$

Pair (V, ρ) is also known as **linear representation** of Γ.

The basic example

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Notes:

- Γ preserves the degree
- $R = \mathbb{R}[x_1,\dots,x_4]$ is an infinite-dimensional (linear) representation of Γ.

Invariance and Equivariance

Invariant subspaces I

Example (continued)

Let

$$V = \Big(\langle \mathbb{R}[x_1, \ldots, x_4] \rangle \Big)_{0 \leq deg(m) \leq (2)}$$

an **invariant** (i.e preserved by the action of Γ), **15**-dimensional vector space.

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with the complement of $V^{\Gamma} < V$ is spanned by

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- Given any $f \in R$ define the multiplication map

$$\begin{aligned} \mathcal{A}_f \colon R/I &\to R/I \\ [r] &\mapsto [r \cdot f] \end{aligned}$$

- Since $\mathcal{A}_f$ is a **linear map** for any choice of $\overline{g} = \{[g_1], \ldots [g_d]\}$ a linear independent basis for R/I we can realize $\mathcal{A}_f$ as a **matrix** $A_{f,\overline{g}} = A_f$.

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- each **eigen pair** (λ, v) of A_f corresponds to $(f(p), \overline{g}(p))$ for some $p \in Z(I)$.

Misleading quote of the day

The structure of simplifications that can be derived from group symmetry does not depend on particular choices of R, I, etc.

Equivariant maps

Definition

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- With a particular choice of basis the equivariance condition now reads

$$A_f\rho(g)v = \rho(g)A_f v \quad \text{for all } v \in R/I,\ g \in \Gamma,$$

i.e. A_f commutes with all matrices defined by ρ!

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- Hence

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An equivariant map L will not split an irreducible subspace!

Example (continued)

Recall:

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a 15 dimensional vector space with permutation action of $\Gamma = \mathrm{Sym}(4)$.

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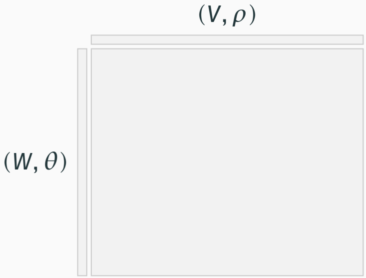

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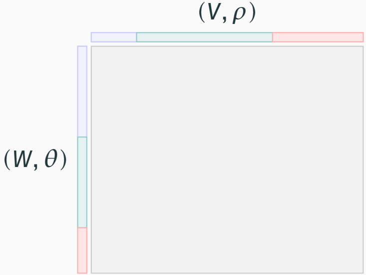

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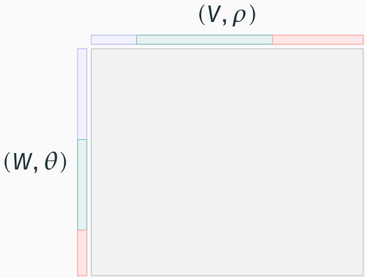

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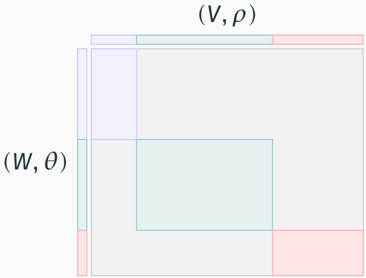

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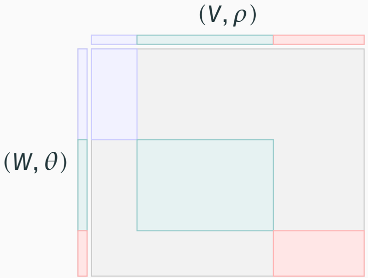
- If $L_{i,j}\colon V_i \to W_j$, then by Schur $L_{i,j} = 0$ when $i \neq j$.



Projections to isotypical components (V_i or W_j) can be expressed in a base-free (hence matrix-free!) form as elements of **group algebra**.

Group algebra and projections

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Group algebra $\mathbb{R}[G]$

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Projections onto isotypical subspaces live in $\mathbb{R}[G]$ in a matrix-free form.

$$V_3 = \begin{bmatrix} x_1x_2 - x_1x_4 - x_2x_3 + x_3x_4 \\ x_1x_3 - x_1x_4 - x_2x_3 + x_2x_4 \end{bmatrix} \longleftrightarrow p_3 = \frac{1}{12}\begin{pmatrix} 2() - (2,4,3) - (2,3,4) + 2(1,2)(3,4) - \\ (1,3,2) - (1,4,2) - (1,4,3) + (1,3)(2,4) - \\ (1,2,3) + 2(1,4)(2,3) - (1,2,4) - (1,3,4) \end{pmatrix}$$

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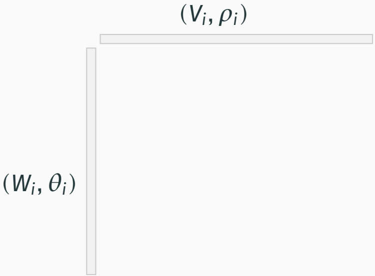

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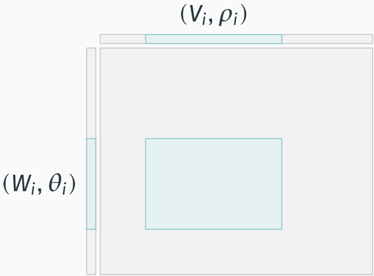

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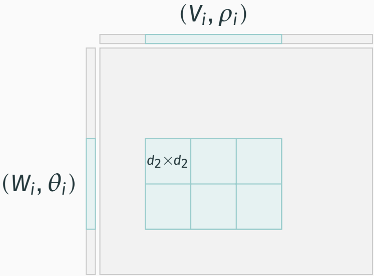

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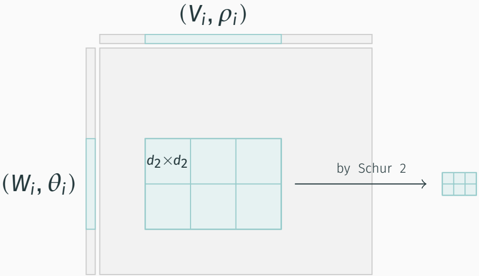

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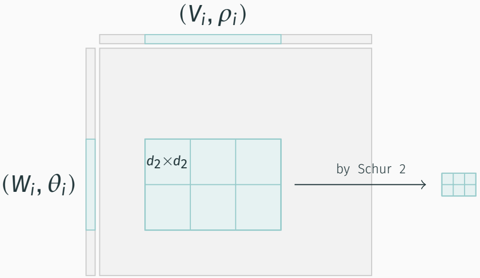

$$V_3' = \frac{1}{2}\begin{pmatrix} x_1x_2 - x_1x_4 - x_2x_3 + x_3x_4 + \\ x_1x_3 - x_1x_4 - x_2x_3 + x_2x_4 \end{pmatrix} \longleftrightarrow q_3 \cdot p_3 \quad \text{where } q_3 = \frac{1}{2}\left(() + (3,4)\right)$$

Example: SymbolicWedderburn.jl

```
# [ ... ]
symmetry_adapted_basis(Rational{Int}, G, VariablePermutation(), basis
  [, semisimple=false])
```

Simple blocks when acting on basis:

$$V_1' = \begin{bmatrix} 1 \\ x_1 + x_2 + x_3 + x_4 \\ x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4 \\ x_1^2 + x_2^2 + x_3^2 + x_4^2 \end{bmatrix} \quad V_2' = \begin{bmatrix} \frac{1}{3}(3x_1 - x_2 - x_3 - x_4) \\ \frac{1}{3}(3x_1^2 - x_2^2 - x_3^2 - x_4^2) \\ x_1x_2 + x_1x_3 + x_1x_4 - x_2x_3 - x_2x_4 - x_3x_4 \end{bmatrix}$$

$$V_3' = \left[\frac{1}{2}(2x_1x_2 - x_1x_3 - x_1x_4 - x_2x_3 - x_2x_4 + 2x_3x_4) \right]$$

Reduction: $15 \times 15 \to (4 \times 4, 9 \times 9, 2 \times 2) \to (4 \times 4, 3 \times 3, 1 \times 1)$-psd constraints.

Large scale example

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Estimate the spectral gap of the group Laplacian for $\mathrm{Aut}(F_5)$

If $\Delta^2 - \lambda\Delta \geqslant 0$ then $(0, \lambda)$ is not in the spectrum.

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- symmetry group: $S_2 \wr S_5$ (3840 elements)
- After symmetrization:
  - 29-blocks (largest: 58×58) ($13\,232$ variables in total)
  - $7\,230$ constraints
- Solvable in 20 minutes to $\varepsilon \sim 10^{-12}$!

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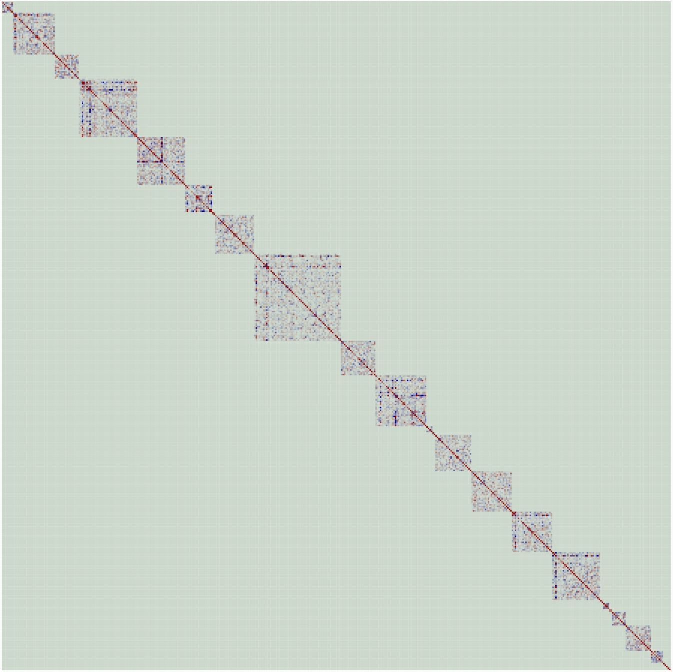

diagonalized psd (⊂ 448 × 448)
Original psd constraint (4 641 × 4 641)