Analytical Performance Evaluation of Multidimensional HOSVD-based Subspace Estimation Techniques

Martin Haardt, Florian Römer, and Hanna Becker

Ilmenau University of Technology
Communications Research Laboratory
P.O. Box 10 05 65
D-98684 Ilmenau, Germany
E-Mail: **haardt@ieee.org**

Homepage: http://www.tu-ilmenau.de/crl

Outline

- Introduction
  - ⇒ Tensor-based algorithms vs. matrix-based algorithms
  - ⇒ Application example: *R*-D harmonic retrieval
- Subspace estimation
  - ⇒ HOSVD-based enhanced subspace estimate
  - ⇒ Perturbation analysis of matrix-based techniques
  - ⇒ Extension to the tensor case
- Analytical performance evaluation for Tensor-ESPRIT-type algorithms
- Simulation results
- Conclusions

Outline

- **Introduction**
  - ⇒ Tensor-based algorithms vs. matrix-based algorithms
  - ⇒ Application example: *R*-D harmonic retrieval
- Subspace estimation
  - ⇒ HOSVD-based enhanced subspace estimate
  - ⇒ Perturbation analysis of matrix-based techniques
  - ⇒ Extension to the tensor case
- Analytical performance evaluation for Tensor-ESPRIT-type algorithms
- Simulation results
- Conclusions

Ilmenau University of Technology
Communications Research Laboratory

Why tensors?

- Tensor-based signal processing techniques offer **fundamental advantages** over their matrix-based counterparts
  - ⇒ **Identifiability**
    - the tensor **rank** can largely **exceed** its **dimensions**
    - more sources than sensors can be identified
  - ⇒ **Uniqueness**
    - bilinear (matrix) decomposition: requires constraints for uniqueness, such as orthogonality (SVD)
    - trilinear/multilinear (tensor) decomposition: **essentially unique** up to permutation and scaling
      - – columns of mixing matrix can be identified individually
      - – blind source separation

Ilmenau University of Technology
Communications Research Laboratory

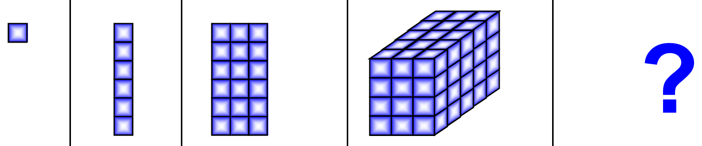
Why tensors (cont.)?

- Tensor-based signal processing offers **fundamental advantages** over matrix-based techniques
  - ⇒ **Multilinear rank reduction**
    - More efficient **denoising**: exploiting the **structure**, therefore more noise is suppressed
    - many applications, e.g., chemometrics, psychometrics, computer vision, watermarking, data mining, array processing, ICA, …
  - ⇒ **Improved subspace estimate**
    - multidimensional subspace-based parameter estimation schemes: can be improved by using the multilinear rank reduction
    - yields an **improved subspace estimate**, therefore a higher accuracy
    - many applications, e.g., channel modeling, surveillance RADAR, microwave imaging, positioning, blind channel estimation, …
    - **goal of this talk: quantify** this improvement **analytically**
    - for simplicity: **2-D case** only, generalization to *R*-D straightforward

What is a tensor?

- Strictly speaking: An **outer** (tensor) **product** of *R* linear spaces.
  - ⇒ like a matrix is an outer product of two linear spaces
  - ⇒ engineers typically work with **coordinate representations**
    - are obtained by **fixing** the **bases** of all spaces
  - ⇒ for simplicity, we assimilate tensors with their coordinate representations
    - ***R*-way arrays**

Scalars	Vectors	Matrices	Order-3-tensors	Order-4-tensors	…
$x \in \mathbb{C}$	$\boldsymbol{x} \in \mathbb{C}^{I_1}$	$\boldsymbol{X} \in \mathbb{C}^{I_1 \times I_2}$	$\boldsymbol{\mathcal{X}} \in \mathbb{C}^{I_1 \times I_2 \times I_3}$	$\boldsymbol{\mathcal{X}} \in \mathbb{C}^{I_1 \times I_2 \times I_3 \times I_4}$	

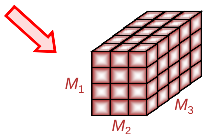


$R = 0$	$R = 1$	$R = 2$	$R = 3$	$R = 4$	

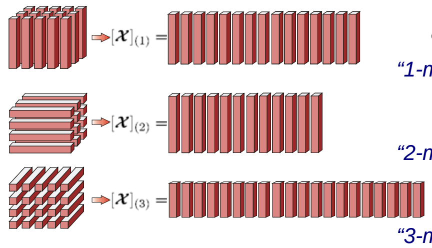
Tensor algebra

- 3-D tensor = 3-way array

$\mathcal{X} \in \mathbb{C}^{M_1 \times M_2 \times M_3}$



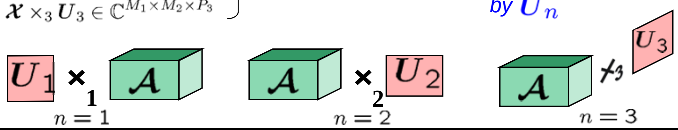
- Unfoldings $[\mathcal{X}]_{(n)}$ $\quad n\text{-rank}(\mathcal{X}) = \text{rank}([\mathcal{X}]_{(n)})$



- n-mode products between $\mathcal{X} \in \mathbb{C}^{M_1 \times M_2 \times M_3}$ and $U_n \in \mathbb{C}^{P_n \times M_n}$

$$\left.\begin{array}{l} \mathcal{Y} = \mathcal{X} \times_1 U_1 \in \mathbb{C}^{P_1 \times M_2 \times M_3} \\ \mathcal{Y} = \mathcal{X} \times_2 U_2 \in \mathbb{C}^{M_1 \times P_2 \times M_3} \\ \mathcal{Y} = \mathcal{X} \times_3 U_3 \in \mathbb{C}^{M_1 \times M_2 \times P_3} \end{array}\right\} \Leftrightarrow [\mathcal{Y}]_{(n)} = U_n \cdot [\mathcal{X}]_{(n)}$$

i.e., all the n-mode vectors multiplied from the left-hand-side by U_n



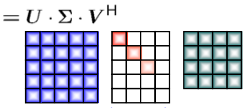
Higher-Order SVD (HOSVD)

- **Singular Value Decomposition**

$X \in \mathbb{C}^{M \times N}$, rank$(X) = r$

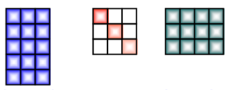
"Full SVD"

$$X = U \cdot \Sigma \cdot V^{\mathrm{H}}$$



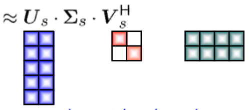
"Economy size SVD"

$$X = U_s \cdot \Sigma_s \cdot V_s^{\mathrm{H}}$$



Low-rank approximation

$$X \approx U_s \cdot \Sigma_s \cdot V_s^{\mathrm{H}}$$



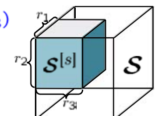
- **Higher-Order SVD (orth. Tucker3) [LMV00]**

$\mathcal{X} \in \mathbb{C}^{I_1 \times I_2 \times I_3}$, $n-$rank$(\mathcal{X}) = r_n$, $n = 1, 2, 3$

"Full HOSVD"

$$\mathcal{X} = \mathcal{S} \times_1 U_1 \times_2 U_2 \times_3 U_3$$

$(I_1 \times I_2 \times I_3)$ $\quad (I_1 \times I_1)\,(I_2 \times I_2)\,(I_3 \times I_3)$



"Economy size HOSVD"

$$\mathcal{X} = \mathcal{S}^{[s]} \times_1 U_1^{[s]} \times_2 U_2^{[s]} \times_3 U_3^{[s]}$$

$(r_1 \times r_2 \times r_3)$ $\quad (I_1 \times r_1)\,(I_2 \times r_2)\,\ (I_3 \times r_3)$

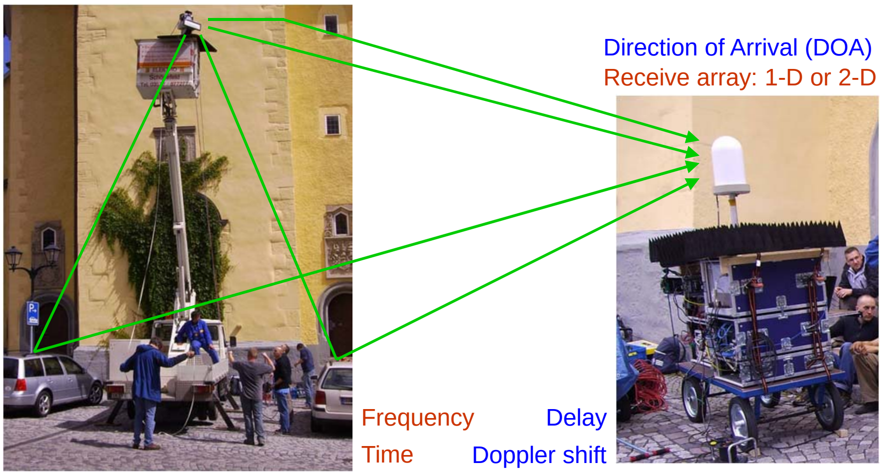
Low-rank approximation (truncated HOSVD)

$$\mathcal{X} \approx \mathcal{S}^{[s]} \times_1 U_1^{[s]} \times_2 U_2^{[s]} \times_3 U_3^{[s]}$$

$(r_1' \times r_2' \times r_3')$ $\quad (I_1 \times r_1')\ (I_2 \times r_2')\ (I_3 \times r_3')$

L. de Lathauwer, B. de Moor, and J. Vanderwalle, "A multilinear singular value decomposition", *SIAM J. Matrix Anal. Appl.*, vol. 21, no. 4, 2000.

Example for *R*-D harmonic retrieval: Channel Sounding



Ilmenau University of Technology
Communications Research Laboratory

Existing Approaches

High-resolution parameter estimation

- Maximum-Likelihood
  - ⇒ SAGE *[Fessler et al. 1994]*
  - ⇒ Extensions *[Fleury et al. 1999, Pederson et al. 2000, Thomä et al. 2004]*
- Subspace-based
  - ⇒ MUSIC *[Schmidt 1979]*,
  - ⇒ Root MUSIC *[Barabell 1983]*
  - ⇒ ESPRIT *[Roy et al. 1986]*, *R*-D Unitary ESPRIT *[Haardt et al. 1998]*
  - ⇒ RARE (Rank reduction estimator) *[Pesavento et al. 2004]*
  - ⇒ MDF (Multidimensional folding) *[Mokios et al. 2004]*
  - (many more)

***R*-D Standard Tensor-ESPRIT**
***R*-D Unitary Tensor-ESPRIT**

Enhanced signal subspace estimation

Ilmenau University of Technology
Communications Research Laboratory

Channel Sounding

R-D parameter estimation

Spatial dimensions RX ⇒ Direction of Arrival

Spatial dimensions TX ⇒ Direction of Departure

Frequency ⇒ Delay

Time ⇒ Doppler shift

Model: *superposition of d undamped exponentials sampled on an R-dimensional grid and observed at N subsequent time instances.*

R-D measurements (*R*-D harmonic retrieval)

$$x_{m_1,m_2,\ldots,m_R,n} = \sum_{i=1}^{d} \left(\prod_{r=1}^{R} e^{j\cdot(m_r-1)\cdot\mu_i^{(r)}} \right) \cdot s_i(n) + n_{m_1,m_2,\ldots,m_R,n}$$

$m_r = 1, 2, \ldots, M_r$

$r = 1, 2, \ldots, R$

$n = 1, 2, \ldots, N$

Spatial frequencies
⇒ one to one mapping to physical parameters

Ilmenau University of Technology
Communications Research Laboratory

Channel Sounding (*R* = 2)

2-D measurements

$$x_{m_1,m_2,n} = \sum_{i=1}^{d} e^{j\cdot(m_1-1)\cdot\mu_i^{(1)}} \cdot e^{j\cdot(m_2-1)\cdot\mu_i^{(2)}} \cdot s_i(n) + n_{m_1,m_2,n}$$

$m_1 = 1, 2, \ldots, M_1$

$m_2 = 1, 2, \ldots, M_2$

$n = 1, 2, \ldots, N$

Spatial frequencies
⇒ one to one mapping to physical parameters

Ilmenau University of Technology
Communications Research Laboratory

Data model for 2-D harmonic retrieval

Matrix case

$$M = M_1 \cdot M_2$$

$$\underbrace{\boldsymbol{X}}_{M\times N} = \underbrace{\boldsymbol{A}}_{M\times d} \cdot \underbrace{\boldsymbol{S}}_{d\times N} + \underbrace{\boldsymbol{N}}_{M\times N}$$

$$\boldsymbol{A} = \boldsymbol{A}^{(1)} \diamond \boldsymbol{A}^{(2)}$$

$$\underbrace{\boldsymbol{A}^{(r)}}_{M_r\times d} = \left[\boldsymbol{a}^{(r)}\left(\mu_1^{(r)}\right), \boldsymbol{a}^{(r)}\left(\mu_2^{(r)}\right), \ldots, \boldsymbol{a}^{(r)}\left(\mu_d^{(r)}\right)\right]$$

$$\boldsymbol{a}^{(r)}\left(\mu_i^{(r)}\right) = \begin{bmatrix} 1 \\ \mathrm{e}^{\jmath\mu_i^{(r)}} \\ \mathrm{e}^{2\jmath\mu_i^{(r)}} \\ \vdots \\ \mathrm{e}^{(M_r-1)\jmath\mu_i^{(r)}} \end{bmatrix}$$

⇒ in case of uniform linear arrays: Vandermonde structured array steering vectors

Ilmenau University of Technology
Communications Research Laboratory

Data model for 2-D harmonic retrieval

Matrix case

$$M = M_1 \cdot M_2$$

$$\underbrace{\boldsymbol{X}}_{M\times N} = \underbrace{\boldsymbol{A}}_{M\times d} \cdot \underbrace{\boldsymbol{S}}_{d\times N} + \underbrace{\boldsymbol{N}}_{M\times N}$$

$$\boldsymbol{A} = \boldsymbol{A}^{(1)} \diamond \boldsymbol{A}^{(2)}$$

$$\boldsymbol{X} = [\boldsymbol{\mathcal{X}}]_{(3)}^{\mathrm{T}}$$

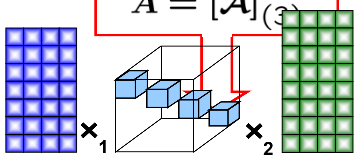
$$\underbrace{\boldsymbol{A}^{(r)}}_{M_r\times d} = \left[\boldsymbol{a}^{(r)}\left(\mu_1^{(r)}\right), \boldsymbol{a}^{(r)}\left(\mu_2^{(r)}\right), \ldots, \boldsymbol{a}^{(r)}\left(\mu_d^{(r)}\right)\right]$$

Tensor case

$$\boldsymbol{\mathcal{X}} = \underbrace{\boldsymbol{\mathcal{A}}} \times_3 \boldsymbol{S}^{\mathrm{T}} + \boldsymbol{\mathcal{N}}$$

$\boldsymbol{\mathcal{X}}$: $M_1 \times M_2 \times N$; $\boldsymbol{\mathcal{A}}$: $M_1 \times M_2 \times d$

$$\boldsymbol{A} = [\boldsymbol{\mathcal{A}}]_{(3)}^{\mathrm{T}}$$

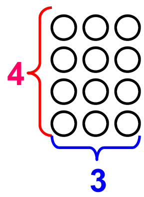


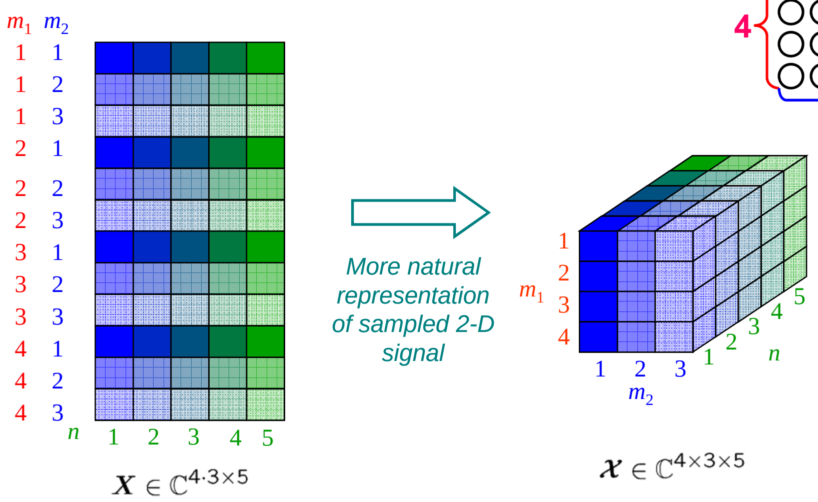
$$\boldsymbol{\mathcal{A}} = \boldsymbol{\mathcal{I}}_{3,d} \times_1 \boldsymbol{A}^{(1)} \times_2 \boldsymbol{A}^{(2)}$$

Ilmenau University of Technology
Communications Research Laboratory

Data Model

Example: $R = 2,\ M_1 = 4,\ M_2 = 3\ (4 \times 3\ \text{URA}),\ N = 5$





Ilmenau University of Technology
Communications Research Laboratory

Outline

- Introduction
  - ⇒ Tensor-based algorithms vs. matrix-based algorithms
  - ⇒ Application example: *R*-D harmonic retrieval
- **Subspace estimation**
  - ⇒ HOSVD-based enhanced subspace estimate
  - ⇒ Perturbation analysis of matrix-based techniques
  - ⇒ Extension to the tensor case
- Analytical performance evaluation for Tensor-ESPRIT-type algorithms
- Simulation results
- Conclusions

Ilmenau University of Technology
Communications Research Laboratory

R-D Tensor-ESPRIT-type methods

Matrix case | **measurements** | **Tensor case**

$\boldsymbol{X} \in \mathbb{C}^{(M_1 \cdot M_2) \times N}$ $\quad$ $\boldsymbol{\mathcal{X}} \in \mathbb{C}^{M_1 \times M_2 \times N}$

1) Signal subspace estimation

[RHD08] M. Haardt, F. Roemer, and G. Del Galdo, "Higher-order SVD based subspace estimation to improve the parameter estimation accuracy in multi-dimensional harmonic retrieval problems," *IEEE Transactions on Signal Processing*, vol. 56, pp. 3198 - 3213, July 2008.

Ilmenau University of Technology
Communications Research Laboratory

Signal Subspace Estimation

Matrix case

$$\boldsymbol{X} = \underbrace{\boldsymbol{A} \cdot \boldsymbol{S}}_{\text{signal part}} + \underbrace{\boldsymbol{N}}_{\text{noise part}} \qquad \Rightarrow \boldsymbol{X} \approx \boldsymbol{U}_s \cdot \boldsymbol{\Sigma}_s \cdot \boldsymbol{V}_s^H$$

signal part → **rank *d***

$\boldsymbol{U}_s$ → **Basis for the signal subspace** $(M \times d)$, $M = M_1 \cdot M_2$

$\Rightarrow \boldsymbol{A} \approx \boldsymbol{U}_s \cdot \boldsymbol{T}_{\mathrm{M}}$ $\Rightarrow$ *$\boldsymbol{A}$ and $\boldsymbol{U}_s$ span the same column space*

Tensor case

$$\boldsymbol{\mathcal{X}} = \underbrace{\boldsymbol{\mathcal{A}} \times_3 \boldsymbol{S}^{\mathrm{T}}}_{\text{signal part}} + \underbrace{\boldsymbol{\mathcal{N}}}_{\text{noise part}} \qquad \Rightarrow \boldsymbol{\mathcal{X}} \approx \underbrace{\boldsymbol{\mathcal{S}}^{[s]} \times_1 \boldsymbol{U}_1^{[s]} \times_2 \boldsymbol{U}_2^{[s]}}_{\boldsymbol{\mathcal{U}}^{[s]} \in \mathbb{C}^{M_1 \times M_2 \times d}} \times_3 \boldsymbol{U}_3^{[s]}$$

signal part → **rank *d***

$\boldsymbol{\mathcal{U}}^{[s]}$ → **Basis for the signal subspace**

$\Rightarrow \boldsymbol{\mathcal{A}} \approx \boldsymbol{\mathcal{U}}^{[s]} \times_3 \boldsymbol{T}_{\mathrm{T}}$ $\Rightarrow$ *spaces spanned by the 1-mode vectors **and** the 2-mode vectors are equal*

Ilmenau University of Technology
Communications Research Laboratory

R-D Tensor-ESPRIT-type methods

Matrix case | **measurements** | **Tensor case**

$\boldsymbol{X} \in \mathbb{C}^{(M_1 \cdot M_2) \times N}$ $\quad$ $\boldsymbol{\mathcal{X}} \in \mathbb{C}^{M_1 \times M_2 \times N}$

1) Signal subspace estimation

$\boldsymbol{U}_{\mathrm{s}} \in \mathbb{C}^{(M_1 \cdot M_2) \times d}$ $\quad$ $\boldsymbol{\mathcal{U}}^{[\mathrm{s}]} \in \mathbb{C}^{M_1 \times M_2 \times d}$

⇒ we have observed that $\left[\boldsymbol{\mathcal{U}}^{[\mathrm{s}]}\right]_{(3)}^{\mathrm{T}} \in \mathbb{C}^{(M_1 \cdot M_2) \times d}$ represents an **improved** signal subspace **estimate**, provided that $d < \max(M_1, M_2)$

whereas $d_{\max} = \min\left((M_1 - 1) \cdot M_2, M_1 \cdot (M_2 - 1)\right)$

⇒ otherwise it can be shown that: $\left[\boldsymbol{\mathcal{U}}^{[\mathrm{s}]}\right]_{(3)}^{\mathrm{T}} = \boldsymbol{U}_{\mathrm{s}}$

Perturbation analysis for the matrix case

- Unperturbed subspaces

$$\boldsymbol{X}_0 = \boldsymbol{U} \cdot \boldsymbol{\Sigma} \cdot \boldsymbol{V}^{\mathrm{H}} = \begin{bmatrix}\boldsymbol{U}_{\mathrm{s}} & \boldsymbol{U}_{\mathrm{n}}\end{bmatrix} \cdot \begin{bmatrix}\boldsymbol{\Sigma}_{\mathrm{s}} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0}\end{bmatrix} \cdot \begin{bmatrix}\boldsymbol{V}_{\mathrm{s}}^{\mathrm{H}} \\ \boldsymbol{V}_{\mathrm{n}}^{\mathrm{H}}\end{bmatrix}$$

- In the presence of noise

$$\boldsymbol{X} = \boldsymbol{X}_0 + \boldsymbol{N} = \hat{\boldsymbol{U}} \cdot \hat{\boldsymbol{\Sigma}} \cdot \hat{\boldsymbol{V}}^{\mathrm{H}} \qquad \hat{\boldsymbol{U}}_{\mathrm{s}} = \boldsymbol{U}_{\mathrm{s}} + \Delta \boldsymbol{U}_{\mathrm{s}}$$

- First order perturbation analysis

$$\Delta \boldsymbol{U}_{\mathrm{s}} = \boldsymbol{U}_{\mathrm{n}} \cdot \boldsymbol{P} + \boldsymbol{U}_{\mathrm{s}} \cdot \boldsymbol{R}$$

$$\boldsymbol{P} \approx \boldsymbol{U}_{\mathrm{n}}^{\mathrm{H}} \boldsymbol{N} \boldsymbol{V}_{\mathrm{s}} \boldsymbol{\Sigma}_{\mathrm{s}}^{-1}$$

$$\boldsymbol{R} \approx \boldsymbol{D} \odot \left(\boldsymbol{U}_{\mathrm{s}}^{\mathrm{H}} \boldsymbol{N} \boldsymbol{V}_{\mathrm{s}} \boldsymbol{\Sigma}_{\mathrm{s}} + \boldsymbol{\Sigma}_{\mathrm{s}} \boldsymbol{V}_{\mathrm{s}}^{\mathrm{H}} \boldsymbol{N}^{\mathrm{H}} \boldsymbol{U}_{\mathrm{s}}\right)$$

$$[\boldsymbol{D}]_{g,f} = \begin{cases} \frac{1}{\sigma_f^2 - \sigma_g^2} & \text{for } g \neq f; \\ 0 & \text{for } g = f \end{cases} \qquad \boldsymbol{\Sigma}_{\mathrm{s}} = \mathrm{diag}\{[\sigma_1, \ldots, \sigma_d]\}$$

models the perturbation of the signal subspace [LLV93]

models the perturbation of the individual vectors within the signal subspace → no impact on the performance of ESPRIT [LLM08]

Perturbation analysis for the matrix case

[LLV93] F. Li, H. Liu, and R. J. Vaccaro, "Performance analysis for DOA estimation algorithms: Unification, simplifications, and observations", *IEEE Transactions on Aerospace and Electronic Systems*, vol. 29, no. 4, pp. 1170–1184, Oct. 1993.

[LLM08] J. Liu, X. Liu, and X. Ma, "First-order perturbation analysis of singular vectors in singular value decomposition", *IEEE Transactions on Signal Processing*, vol. 56, no. 7, pp. 3044–3049, July 2008.

$$\boldsymbol{X} = \boldsymbol{X}_0 + \boldsymbol{N} = \hat{\boldsymbol{U}} \cdot \hat{\boldsymbol{\Sigma}} \cdot \hat{\boldsymbol{V}}^{\mathrm{H}} \qquad \hat{\boldsymbol{U}}_{\mathrm{s}} = \boldsymbol{U}_{\mathrm{s}} + \Delta \boldsymbol{U}_{\mathrm{s}}$$

- First order perturbation analysis

$$\Delta \boldsymbol{U}_{\mathrm{s}} = \boldsymbol{U}_{\mathrm{n}} \cdot \boldsymbol{P} + \boldsymbol{U}_{\mathrm{s}} \cdot \boldsymbol{R}$$

$$\boldsymbol{P} \approx \boldsymbol{U}_{\mathrm{n}}^{\mathrm{H}} \boldsymbol{N} \boldsymbol{V}_{\mathrm{s}} \boldsymbol{\Sigma}_{\mathrm{s}}^{-1}$$

$$\boldsymbol{R} \approx \boldsymbol{D} \odot \left(\boldsymbol{U}_{\mathrm{s}}^{\mathrm{H}} \boldsymbol{N} \boldsymbol{V}_{\mathrm{s}} \boldsymbol{\Sigma}_{\mathrm{s}} + \boldsymbol{\Sigma}_{\mathrm{s}} \boldsymbol{V}_{\mathrm{s}}^{\mathrm{H}} \boldsymbol{N}^{\mathrm{H}} \boldsymbol{U}_{\mathrm{s}} \right)$$

$$[\boldsymbol{D}]_{g,f} = \begin{cases} \frac{1}{\sigma_f^2 - \sigma_g^2} & \text{for } g \neq f; \\ 0 & \text{for } g = f \end{cases} \qquad \boldsymbol{\Sigma}_{\mathrm{s}} = \mathrm{diag}\{[\sigma_1, \ldots, \sigma_d]\}$$

models the perturbation of the signal subspace [LLV93]

models the perturbation of the individual vectors within the signal subspace → no impact on the performance of ESPRIT [LLM08]

Extension to the tensor case (2-D)

- Signal subspace estimates:

SVD $\boldsymbol{X} = [\boldsymbol{\mathcal{X}}]_{(3)}^{\mathrm{T}} \approx \hat{\boldsymbol{U}}_{\mathrm{s}} \cdot \hat{\boldsymbol{\Sigma}}_{\mathrm{s}} \cdot \hat{\boldsymbol{V}}_{\mathrm{s}}^{\mathrm{H}}$ $\qquad \hat{\boldsymbol{\Sigma}}_3^{[\mathrm{s}]} = \left(\left[\hat{\boldsymbol{\mathcal{S}}}^{[\mathrm{s}]}\right]_{(3)} \cdot \left[\hat{\boldsymbol{\mathcal{S}}}^{[\mathrm{s}]}\right]_{(3)}^{\mathrm{H}} \right)^{\frac{1}{2}} = \hat{\boldsymbol{\Sigma}}_{\mathrm{s}}$

HOSVD $\boldsymbol{\mathcal{X}} \approx \hat{\boldsymbol{\mathcal{S}}}^{[\mathrm{s}]} \times_1 \hat{\boldsymbol{U}}_1^{[\mathrm{s}]} \times_2 \hat{\boldsymbol{U}}_2^{[\mathrm{s}]} \times_3 \hat{\boldsymbol{U}}_3^{[\mathrm{s}]}$

$$\Rightarrow \hat{\boldsymbol{\mathcal{U}}}^{[\mathrm{s}]} = \hat{\boldsymbol{\mathcal{S}}}^{[\mathrm{s}]} \times_1 \hat{\boldsymbol{U}}_1^{[\mathrm{s}]} \times_2 \hat{\boldsymbol{U}}_2^{[\mathrm{s}]} \times_3 \left(\hat{\boldsymbol{\Sigma}}_3^{[\mathrm{s}]} \right)^{-1}$$ different from [HRD08]

$\Rightarrow \left[\hat{\boldsymbol{\mathcal{U}}}^{[\mathrm{s}]}\right]_{(3)}^{\mathrm{T}}$ is the improved signal subspace estimate, replaces $\hat{\boldsymbol{U}}_{\mathrm{s}}$

improvement only if $d < \max(M_1, M_2)$

- A **link** between the **SVD**-based and the **HOSVD**-based subspace estimate

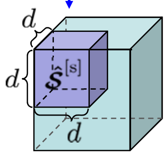
$$\left[\hat{\boldsymbol{\mathcal{U}}}^{[\mathrm{s}]}\right]_{(3)}^{\mathrm{T}} = \left(\hat{\boldsymbol{T}}_1 \otimes \hat{\boldsymbol{T}}_2 \right) \hat{\boldsymbol{U}}_{\mathrm{s}} \qquad \hat{\boldsymbol{T}}_i = \hat{\boldsymbol{U}}_i^{[\mathrm{s}]} \cdot \hat{\boldsymbol{U}}_i^{[\mathrm{s}]\mathrm{H}}$$

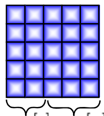
⇒ "**projection** onto the Kronecker structure"
⇒ **do not need** the **core tensor** or a perturbation analysis for it!

Relation between matrix subspace and tensor subspace

$$\mathcal{X} = \hat{\mathcal{S}} \times_1 \hat{U}_1 \times_2 \hat{U}_2 \times_3 \hat{U}_3$$ full HOSVD

$$\hat{U}_r = \left[\hat{U}_r^{[s]},\ \hat{U}_r^{[n]}\right]$$





$\hat{U}_r^{[s]}$ $\hat{U}_r^{[n]}$

$$\hat{\mathcal{S}}^{[s]} = \mathcal{X} \times_1 \hat{U}_1^{[s]^{H}} \times_2 \hat{U}_2^{[s]^{H}} \times_3 \hat{U}_3^{[s]^{H}} \quad (1)$$

truncated core tensor

$$\hat{\mathcal{U}}^{[s]} = \hat{\mathcal{S}}^{[s]} \times_1 \hat{U}_1^{[s]} \times_2 \hat{U}_2^{[s]} \times_3 \left(\hat{\Sigma}_3^{[s]}\right)^{-1}$$ estimated signal subspace tensor

(1)

$$\hat{\mathcal{U}}^{[s]} = \mathcal{X} \times_1 \underbrace{\left(\hat{U}_1^{[s]} \cdot \hat{U}_1^{[s]^{H}}\right)}_{\hat{T}_1} \times_2 \underbrace{\left(\hat{U}_2^{[s]} \cdot \hat{U}_2^{[s]^{H}}\right)}_{\hat{T}_2} \times_3 \left(\hat{\Sigma}_3^{[s]^{-1}} \cdot \hat{U}_3^{[s]^{H}}\right)$$

$$\left[\hat{\mathcal{U}}^{[s]}\right]_{(3)}^{T} = \left(\hat{T}_1 \otimes \hat{T}_2\right) \cdot [\mathcal{X}]_{(3)}^{T} \cdot \hat{U}_3^{[s]^{*}} \cdot \hat{\Sigma}_3^{[s]^{-1}} \quad (2)$$

Relation between matrix subspace and tensor subspace

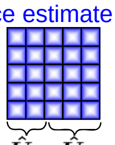
$$\left.\begin{array}{r} [\mathcal{X}]_{(3)} = \hat{U}_3 \cdot \hat{\Sigma}_3 \cdot \hat{V}_3^{H} \\ X = \hat{U} \cdot \hat{\Sigma} \cdot \hat{V}^{H} \end{array}\right\} X = [\mathcal{X}]_{(3)}^{T}$$

$$\hat{V}_s = \hat{U}_3^{[s]^{*}} \qquad \hat{\Sigma}_s = \hat{\Sigma}_3^{[s]} \quad (3)$$

truncated SVDs

matrix-based subspace estimate

$$\hat{U} = \left[\hat{U}_s,\ \hat{U}_n\right]$$



$\hat{U}_s$ $\hat{U}_n$

$$\hat{U}_s = X \cdot \hat{V}_s \cdot \hat{\Sigma}_s^{-1} \quad (4)$$

$$(2)\quad \left[\hat{\mathcal{U}}^{[s]}\right]_{(3)}^{T} = \left(\hat{T}_1 \otimes \hat{T}_2\right) \cdot [\mathcal{X}]_{(3)}^{T} \cdot \underbrace{\hat{U}_3^{[s]^{*}} \cdot \hat{\Sigma}_3^{[s]^{-1}}}$$

(3)

$$\underbrace{X \cdot \hat{V}_s \cdot \hat{\Sigma}_s^{-1}}$$

(4)

$$\hat{U}_s$$

relation between matrix- and tensor-based subspace estimate

$$\left[\hat{\mathcal{U}}^{[s]}\right]_{(3)}^{T} = \left(\hat{T}_1 \otimes \hat{T}_2\right) \hat{U}_s$$

Extension to the tensor case (2-D)

- **Link** between the **SVD**-based and the **HOSVD**-based subspace estimate

$$\left[\hat{\mathcal{U}}^{[s]}\right]^{T}_{(3)} = \left(\hat{T}_1 \otimes \hat{T}_2\right)\hat{U}_s \qquad \hat{T}_i = \hat{U}_i^{[s]} \cdot \hat{U}_i^{[s]^H}$$

$\Rightarrow$ we **never** need to explicitly **compute** the **core tensor**, only the dominant left singular vectors of all three unfoldings

- major impact on **subspace tracking**
- facilitates the performance analysis

$\Rightarrow$ performance analysis for matrix subspace can be reused

- we also need a **first-order perturbation expansion** for the **projectors** T_1 and T_2

Extension to the tensor case (2-D)

- Perturbation for the projection matrices

$$\hat{T}_i = \left(U_i^{[s]} + \Delta U_i^{[s]}\right) \cdot \left(U_i^{[s]^H} + \Delta U_i^{[s]^H}\right)$$
$$\approx T_i + U_i^{[s]} \cdot \Delta U_i^{[s]^H} + \Delta U_i^{[s]} \cdot U_i^{[s]^H} + \cancel{\Delta U_i^{[s]} \cdot \Delta U_i^{[s]^H}}$$

- For $\Delta U_i^{[s]}$: we can apply the perturbation theory to $[\mathcal{X}]_{(i)}$

$$\Delta U_1^{[s]} \approx U_1^{[n]} \cdot \left(U_1^{[n]}\right)^H \cdot [\mathcal{N}]_{(1)} \cdot V_1^{[s]} \cdot \left(\Sigma_1^{[s]}\right)^{-1}$$
$$\Delta U_2^{[s]} \approx U_2^{[n]} \cdot \left(U_2^{[n]}\right)^H \cdot [\mathcal{N}]_{(2)} \cdot V_2^{[s]} \cdot \left(\Sigma_2^{[s]}\right)^{-1}$$

using only [LLV93], for the projectors, the [LLM08] term cancels!

- Insert into previous relation, neglect all higher-order terms

$$\left[\hat{\mathcal{U}}^{[s]}\right]^{T}_{(3)} = U_s + \left[\Delta\mathcal{U}^{[s]}\right]^{T}_{(3)}$$
$$\left[\Delta\mathcal{U}^{[s]}\right]^{T}_{(3)} \approx (T_1 \otimes T_2)\Delta U_s + \left((\Delta U_1^{[s]} \cdot U_1^{[s]^H}) \otimes T_2\right) \cdot U_s + \left(T_1 \otimes (\Delta U_2^{[s]} \cdot U_2^{[s]^H})\right) \cdot U_s$$

Outline

- Introduction
  - ⇒ Tensor-based algorithms vs. matrix-based algorithms
  - ⇒ Application example: *R*-D harmonic retrieval
- Subspace estimation
  - ⇒ HOSVD-based enhanced subspace estimate
  - ⇒ Perturbation analysis of matrix-based techniques
  - ⇒ Extension to the tensor case
- **Analytical performance evaluation for Tensor-ESPRIT-type algorithms**
- Simulation results
- Conclusions

R-D Tensor-ESPRIT-type methods

Matrix case | **measurements** | **Tensor case**

$\boldsymbol{X} \in \mathbb{C}^{(M_1 \cdot M_2) \times N}$ | $\boldsymbol{\mathcal{X}} \in \mathbb{C}^{M_1 \times M_2 \times N}$

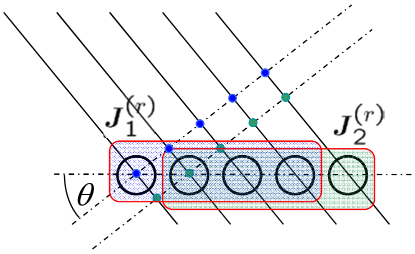
1) Signal subspace estimation

$\boldsymbol{U}_{\mathrm{s}} \in \mathbb{C}^{(M_1 \cdot M_2) \times d}$ | $\boldsymbol{\mathcal{U}}^{[\mathrm{s}]} \in \mathbb{C}^{M_1 \times M_2 \times d}$

2a) Shift invariance equations

Shift Invariance Equations

Shift invariance



$$\boldsymbol{J}_1^{(r)} \cdot \boldsymbol{a}^{(r)}(\mu_i) \cdot e^{j\mu_i^{(r)}} = \boldsymbol{J}_2^{(r)} \cdot \boldsymbol{a}^{(r)}(\mu_i)$$

$$\boldsymbol{J}_1^{(r)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad \boldsymbol{J}_2^{(r)} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

⇓ $d \geq 1$

$$\boldsymbol{J}_1^{(r)} \cdot \boldsymbol{A}^{(r)} \cdot \boldsymbol{\Phi}^{(r)} = \boldsymbol{J}_2^{(r)} \cdot \boldsymbol{A}^{(r)}$$

$$\boldsymbol{\Phi}^{(r)} = \text{diag}\left\{\left[e^{j\mu_1^{(r)}}, e^{j\mu_2^{(r)}}, \ldots, e^{j\mu_d^{(r)}}\right]\right\}$$

⇓ 2-D

Matrix case

$$\begin{aligned} \tilde{\boldsymbol{J}}_1^{(1)} \cdot \boldsymbol{A} \cdot \boldsymbol{\Phi}^{(1)} &= \tilde{\boldsymbol{J}}_2^{(1)} \cdot \boldsymbol{A} \\ \tilde{\boldsymbol{J}}_1^{(2)} \cdot \boldsymbol{A} \cdot \boldsymbol{\Phi}^{(2)} &= \tilde{\boldsymbol{J}}_2^{(2)} \cdot \boldsymbol{A} \end{aligned}$$

⇐

Tensor case

$$\begin{aligned} \mathcal{A} \times_1 \boldsymbol{J}_1^{(1)} \times_3 \boldsymbol{\Phi}^{(1)} &= \mathcal{A} \times_1 \boldsymbol{J}_2^{(1)} \\ \mathcal{A} \times_2 \boldsymbol{J}_1^{(2)} \times_3 \boldsymbol{\Phi}^{(2)} &= \mathcal{A} \times_2 \boldsymbol{J}_2^{(2)} \end{aligned}$$

Shift Invariance Equations

$$\begin{aligned} \tilde{\boldsymbol{J}}_1^{(1)} \cdot \boldsymbol{A} \cdot \boldsymbol{\Phi}^{(1)} &= \tilde{\boldsymbol{J}}_2^{(1)} \cdot \boldsymbol{A} \\ \tilde{\boldsymbol{J}}_1^{(2)} \cdot \boldsymbol{A} \cdot \boldsymbol{\Phi}^{(2)} &= \tilde{\boldsymbol{J}}_2^{(2)} \cdot \boldsymbol{A} \end{aligned} \Rightarrow \boxed{\boldsymbol{A} \approx \boldsymbol{U}_s \cdot \boldsymbol{T}_{\text{M}}} \Rightarrow \begin{aligned} \tilde{\boldsymbol{J}}_1^{(1)} \cdot \boldsymbol{U}_s \cdot \boldsymbol{\Psi}^{(1)} &\approx \tilde{\boldsymbol{J}}_2^{(1)} \cdot \boldsymbol{U}_s \\ \tilde{\boldsymbol{J}}_1^{(2)} \cdot \boldsymbol{U}_s \cdot \boldsymbol{\Psi}^{(2)} &\approx \tilde{\boldsymbol{J}}_2^{(2)} \cdot \boldsymbol{U}_s \end{aligned}$$

$$\boldsymbol{\Psi}^{(r)} = \boldsymbol{T}_{\text{M}} \cdot \boldsymbol{\Phi}^{(r)} \cdot \boldsymbol{T}_{\text{M}}^{-1}$$

$$\begin{aligned} \mathcal{A} \times_1 \boldsymbol{J}_1^{(1)} \times_3 \boldsymbol{\Phi}^{(1)} &= \mathcal{A} \times_1 \boldsymbol{J}_2^{(1)} \\ \mathcal{A} \times_2 \boldsymbol{J}_1^{(2)} \times_3 \boldsymbol{\Phi}^{(2)} &= \mathcal{A} \times_2 \boldsymbol{J}_2^{(2)} \end{aligned}$$

$$\begin{aligned} \mathcal{U}^{[s]} \times_1 \boldsymbol{J}_1^{(1)} \times_3 \boldsymbol{\Psi}^{(1)} &\approx \mathcal{U}^{[s]} \times_1 \boldsymbol{J}_2^{(1)} \\ \mathcal{U}^{[s]} \times_2 \boldsymbol{J}_1^{(2)} \times_3 \boldsymbol{\Psi}^{(2)} &\approx \mathcal{U}^{[s]} \times_2 \boldsymbol{J}_2^{(2)} \end{aligned}$$

$$\boldsymbol{\Psi}^{(r)} = \boldsymbol{T}_{\text{T}}^{-1} \cdot \boldsymbol{\Phi}^{(r)} \cdot \boldsymbol{T}_{\text{T}}$$

$$\boxed{\mathcal{A} \approx \mathcal{U}^{[s]} \times_3 \boldsymbol{T}_{\text{T}}}$$

⇒ shift invariance equations in terms of the estimated signal subspace

⇒ can be solved for $\boldsymbol{\Psi}^{(r)}$

spatial frequencies

$$\mu_i^{(r)} = \arg\left(\lambda_i^{(r)}\right)$$

$$\lambda_i^{(r)} = \text{EV}\left\{\boldsymbol{\Psi}^{(r)}\right\}$$

$$i = 1, 2, \ldots, d$$

R-D Tensor-ESPRIT-type methods

Matrix case | **measurements** | **Tensor case**

$\boldsymbol{X} \in \mathbb{C}^{(M_1 \cdot M_2) \times N}$ | $\boldsymbol{\mathcal{X}} \in \mathbb{C}^{M_1 \times M_2 \times N}$

1) Signal subspace estimation

$\boldsymbol{U}_{\mathrm{s}} \in \mathbb{C}^{(M_1 \cdot M_2) \times d}$ | $\boldsymbol{\mathcal{U}}^{[\mathrm{s}]} \in \mathbb{C}^{M_1 \times M_2 \times d}$

2a) Shift invariance equations

$\tilde{\boldsymbol{J}}_1^{(r)} \cdot \boldsymbol{U}_s \cdot \boldsymbol{\Psi}^{(r)} \approx \tilde{\boldsymbol{J}}_2^{(r)} \cdot \boldsymbol{U}_s$ | $\boldsymbol{\mathcal{U}}^{[s]} \times_r \boldsymbol{J}_1^{(r)} \times_3 \boldsymbol{\Psi}^{(r)} \approx \boldsymbol{\mathcal{U}}^{[s]} \times_1 \boldsymbol{J}_2^{(r)}$

2b) Least Squares method

$\hat{\boldsymbol{\Psi}}^{(r)} = \left[\tilde{\boldsymbol{J}}_1^{(r)} \cdot \boldsymbol{U}_s\right]^{+} \cdot \tilde{\boldsymbol{J}}_2^{(r)} \cdot \boldsymbol{U}_s$ | $\boldsymbol{\Psi}^{(r)\mathrm{T}} = \left(\tilde{\boldsymbol{J}}_1^{(r)} \cdot \left[\boldsymbol{\mathcal{U}}^{[s]}\right]_{(3)}^{\mathrm{T}}\right)^{+} \cdot \tilde{\boldsymbol{J}}_2^{(r)} \cdot \left[\boldsymbol{\mathcal{U}}^{[s]}\right]_{(3)}^{\mathrm{T}}$

3) Joint diagonalization / Simultaneous Schur decomposition (SSD)

$\lambda_i^{(r)} = \mathrm{EV}\left\{\hat{\boldsymbol{\Psi}}\right\} \qquad \mu_i^{(r)} = \arg\left\{\lambda_i^{(r)}\right\}$

Ilmenau University of Technology
Communications Research Laboratory

Performance Analysis for ESPRIT

- **First order expansion** for the estimation error in **standard ESPRIT** [LLV93]

$$\Delta\mu_k = \mathrm{Im}\left\{\boldsymbol{p}_k^{\mathrm{T}} \cdot (\boldsymbol{J}_1 \cdot \boldsymbol{U}_{\mathrm{s}})^{+} \cdot [\boldsymbol{J}_2/\lambda_k - \boldsymbol{J}_1] \cdot \Delta\boldsymbol{U}_{\mathrm{s}} \cdot \boldsymbol{q}_k\right\}$$

$$\boldsymbol{\Psi} = \boldsymbol{Q} \cdot \boldsymbol{\Lambda} \cdot \underbrace{\boldsymbol{Q}^{-1}}_{\boldsymbol{P}} \qquad \boldsymbol{P} = \begin{bmatrix} \boldsymbol{p}_1^{\mathrm{T}} \\ \vdots \\ \boldsymbol{p}_d^{\mathrm{T}} \end{bmatrix} \qquad \begin{aligned} \boldsymbol{Q} &= [\,\boldsymbol{q}_1 \;\; \ldots \;\; \boldsymbol{q}_d\,] \\ \boldsymbol{\Lambda} &= \mathrm{diag}\{[\lambda_1, \ldots, \lambda_d]\} \\ \lambda_k &= \mathrm{e}^{\jmath\mu_k} \\ k &= 1, 2, \ldots, d \end{aligned}$$

- Extension ***R*-D standard ESPRIT**

$\Rightarrow$ Shift invariance equations are solved independently!

$$\Delta\mu_k^{(r)} = \mathrm{Im}\left\{\boldsymbol{p}_k^{\mathrm{T}} \cdot \left(\tilde{\boldsymbol{J}}_1^{(r)} \cdot \boldsymbol{U}_{\mathrm{s}}\right)^{+} \cdot \left[\tilde{\boldsymbol{J}}_2^{(r)}/\lambda_k^{(r)} - \tilde{\boldsymbol{J}}_1^{(r)}\right] \cdot \Delta\boldsymbol{U}_{\mathrm{s}} \cdot \boldsymbol{q}_k\right\} \quad (1)$$

$$\boldsymbol{\Psi}^{(r)} = \boldsymbol{Q} \cdot \boldsymbol{\Lambda}^{(r)} \cdot \boldsymbol{Q}^{-1} \qquad \boldsymbol{\Lambda}^{(r)} = \mathrm{diag}\left\{[\lambda_1^{(r)}, \ldots, \lambda_d^{(r)}]\right\} \qquad \lambda_k = \mathrm{e}^{\jmath\mu_k^{(r)}}$$

- Extension to ***R*-D standard Tensor-ESPRIT**

$$\Delta\mu_k^{(r)} = \mathrm{Im}\left\{\boldsymbol{p}_k^{\mathrm{T}} \cdot \left(\tilde{\boldsymbol{J}}_1^{(r)} \cdot \boldsymbol{U}_{\mathrm{s}}\right)^{+} \cdot \left[\tilde{\boldsymbol{J}}_2^{(r)}/\lambda_k^{(r)} - \tilde{\boldsymbol{J}}_1^{(r)}\right] \cdot \left[\Delta\boldsymbol{\mathcal{U}}^{[\mathrm{s}]}\right]_{(3)}^{\mathrm{T}} \cdot \boldsymbol{q}_k\right\} \quad (2)$$

Ilmenau University of Technology
Communications Research Laboratory

Forward-Backward Averaging

- To assess **Unitary** (Tensor-) ESPRIT, we need **forward-backward averaging**

$$\boldsymbol{X}_0^{(\mathrm{fba})} = [\boldsymbol{X}_0, \quad \boldsymbol{\Pi}_M \cdot \boldsymbol{X}_0^* \cdot \boldsymbol{\Pi}_N] = \left[\boldsymbol{U}_\mathrm{s}^{(\mathrm{fba})}, \ \boldsymbol{U}_\mathrm{n}^{(\mathrm{fba})}\right] \cdot \begin{bmatrix} \boldsymbol{\Sigma}_\mathrm{s}^{(\mathrm{fba})} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \cdot \left[\boldsymbol{V}_\mathrm{s}^{(\mathrm{fba})}, \ \boldsymbol{V}_\mathrm{n}^{(\mathrm{fba})}\right]^\mathrm{H}$$

$$\boldsymbol{X}^{(\mathrm{fba})} = [\boldsymbol{X}, \quad \boldsymbol{\Pi}_M \cdot \boldsymbol{X}^* \cdot \boldsymbol{\Pi}_N]$$

$$\boldsymbol{\Pi} = \begin{bmatrix} 0 & \cdots & 1 \\ \vdots & \iddots & \vdots \\ 1 & \cdots & 0 \end{bmatrix}$$

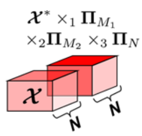
- Using the same reasoning as before we obtain

$$\Delta \boldsymbol{U}_\mathrm{s}^{(\mathrm{fba})} = \boldsymbol{U}_\mathrm{n}^{(\mathrm{fba})} \cdot \boldsymbol{U}_\mathrm{n}^{(\mathrm{fba})^\mathrm{H}} \cdot \boldsymbol{N}^{(\mathrm{fba})} \cdot \boldsymbol{V}_\mathrm{s}^{(\mathrm{fba})} \cdot \boldsymbol{\Sigma}_\mathrm{s}^{(\mathrm{fba})^{-1}}$$

- Similarly, in the tensor case (2-D)

$$\boldsymbol{\mathcal{X}}^{(\mathrm{fba})} = [\boldsymbol{\mathcal{X}} \sqcup_3 \boldsymbol{\mathcal{X}}^* \times_1 \boldsymbol{\Pi}_{M_1} \times_2 \boldsymbol{\Pi}_{M_2} \times_3 \boldsymbol{\Pi}_N]$$

$\boldsymbol{\mathcal{X}}^{(\mathrm{fba})} \triangleq$



$$\left[\Delta \boldsymbol{\mathcal{U}}^{[\mathrm{s}](\mathrm{fba})}\right]_{(3)}^\mathrm{T} = \left(\boldsymbol{T}_1^{(\mathrm{fba})} \otimes \boldsymbol{T}_2^{(\mathrm{fba})}\right) \cdot \Delta \boldsymbol{U}_\mathrm{s}^{(\mathrm{fba})} + \left(\left(\Delta \boldsymbol{U}_1^{[\mathrm{s}](\mathrm{fba})} \cdot \boldsymbol{U}_1^{[\mathrm{s}](\mathrm{fba})^\mathrm{H}}\right) \otimes \boldsymbol{T}_2^{(\mathrm{fba})}\right) \cdot \boldsymbol{U}_\mathrm{s}^{(\mathrm{fba})} + \left(\boldsymbol{T}_1^{(\mathrm{fba})} \otimes \left(\Delta \boldsymbol{U}_2^{[\mathrm{s}](\mathrm{fba})} \cdot \boldsymbol{U}_2^{[\mathrm{s}](\mathrm{fba})^\mathrm{H}}\right)\right) \cdot \boldsymbol{U}_\mathrm{s}^{(\mathrm{fba})}$$

$$\boldsymbol{T}_r^{(\mathrm{fba})} = \boldsymbol{U}_r^{[\mathrm{s}](\mathrm{fba})} \cdot \boldsymbol{U}_r^{[\mathrm{s}](\mathrm{fba})^\mathrm{H}} \quad \text{for } r = 1, 2$$

$$\Delta \boldsymbol{U}_r^{[\mathrm{s}](\mathrm{fba})} = \boldsymbol{U}_r^{[\mathrm{n}](\mathrm{fba})} \cdot \boldsymbol{U}_r^{[\mathrm{n}](\mathrm{fba})^\mathrm{H}} \cdot \left[\boldsymbol{\mathcal{N}}^{(\mathrm{fba})}\right]_{(r)} \cdot \boldsymbol{V}_r^{[\mathrm{s}](\mathrm{fba})} \cdot \boldsymbol{\Sigma}_r^{[\mathrm{s}](\mathrm{fba})^{-1}}$$

Statistical Expectation (2-D)

- Performing **statistical expectation** over white complex (Gaussian) **noise**

$$\mathbb{E}\left\{(\Delta\mu_i^{(r)})^2\right\} = \frac{\sigma_\mathrm{n}^2}{2} \cdot \left\|\boldsymbol{W}_\mathrm{mat}^\mathrm{T} \cdot \boldsymbol{r}_i^{(r)}\right\|_2^2 \quad (3)$$

R-D standard ESPRIT (SE)

$$\mathbb{E}\left\{(\Delta\mu_i^{(r)})^2\right\} = \frac{\sigma_\mathrm{n}^2}{2} \cdot \left\|\boldsymbol{W}_\mathrm{ten}^\mathrm{T} \cdot \boldsymbol{r}_i^{(r)}\right\|_2^2 \quad (5)$$

R-D standard Tensor-ESPRIT (STE)

$$\mathbb{E}\left\{(\Delta\mu_i^{(r)})^2\right\} = \frac{\sigma_\mathrm{n}^2}{2} \cdot \left(\left\|\boldsymbol{W}_\mathrm{mat}^{(\mathrm{fba})^\mathrm{T}} \cdot \boldsymbol{r}_i^{(r)}\right\|_2^2 - \mathrm{Re}\left\{\boldsymbol{r}_i^{(r)^\mathrm{T}} \boldsymbol{W}_\mathrm{mat}^{(\mathrm{fba})} \cdot \boldsymbol{\Pi}_{2MN} \cdot \boldsymbol{W}_\mathrm{mat}^{(\mathrm{fba})^\mathrm{T}} \cdot \boldsymbol{r}_i^{(r)}\right\}\right) \quad (4)$$

R-D Unitary ESPRIT (UE)

$$\mathbb{E}\left\{(\Delta\mu_i^{(r)})^2\right\} = \frac{\sigma_\mathrm{n}^2}{2} \cdot \left(\left\|\boldsymbol{W}_\mathrm{ten}^{(\mathrm{fba})^\mathrm{T}} \cdot \boldsymbol{r}_i^{(r)}\right\|_2^2 - \mathrm{Re}\left\{\boldsymbol{r}_i^{(r)^\mathrm{T}} \boldsymbol{W}_\mathrm{ten}^{(\mathrm{fba})} \cdot \boldsymbol{\Pi}_{2MN} \cdot \boldsymbol{W}_\mathrm{ten}^{(\mathrm{fba})^\mathrm{T}} \cdot \boldsymbol{r}_i^{(r)}\right\}\right) \quad (6)$$

R-D Unitary Tensor-ESPRIT (UTE)

$$\boldsymbol{r}_i^{(r)} = \boldsymbol{q}_i \otimes \left(\left[\left(\tilde{\boldsymbol{J}}_1^{(r)} \boldsymbol{U}_\mathrm{s}\right)^+ \left(\tilde{\boldsymbol{J}}_2^{(r)} / \mathrm{e}^{\jmath \cdot \mu_i^{(r)}} - \tilde{\boldsymbol{J}}_1^{(r)}\right)\right]^\mathrm{T} \cdot \boldsymbol{p}_i\right)$$

σ_n^2: variance of the noise samples
R = 2

$$\boldsymbol{W}_\mathrm{mat} = \left(\boldsymbol{\Sigma}_3^{[\mathrm{s}]^{-1}} \cdot \boldsymbol{U}_3^{[\mathrm{s}]^\mathrm{H}}\right) \otimes \left(\boldsymbol{V}_3^{[\mathrm{n}]} \cdot \boldsymbol{V}_3^{[\mathrm{n}]^\mathrm{H}}\right)^\mathrm{T}$$

$$\boldsymbol{W}_\mathrm{ten} = \left(\boldsymbol{\Sigma}_3^{[\mathrm{s}]^{-1}} \boldsymbol{U}_3^{[\mathrm{s}]^\mathrm{H}}\right) \otimes \left([\boldsymbol{T}_1 \otimes \boldsymbol{T}_2] \boldsymbol{V}_3^{[\mathrm{n}]^*} \boldsymbol{V}_3^{[\mathrm{n}]^\mathrm{T}}\right) + \left(\boldsymbol{U}_\mathrm{s}^\mathrm{T} \otimes \boldsymbol{I}_M\right) \bar{\boldsymbol{T}}_2 \left(\boldsymbol{U}_1^{[\mathrm{s}]^*} \boldsymbol{\Sigma}_1^{[\mathrm{s}]^{-1}} \boldsymbol{V}_1^{[\mathrm{s}]^\mathrm{T}} \otimes \boldsymbol{U}_1^{[\mathrm{n}]} \boldsymbol{U}_1^{[\mathrm{n}]^\mathrm{H}}\right) \cdot \boldsymbol{K}_{M_2 \times (M_1 \cdot N)} + \left(\boldsymbol{U}_\mathrm{s}^\mathrm{T} \otimes \boldsymbol{I}_M\right) \bar{\boldsymbol{T}}_1 \left(\boldsymbol{U}_2^{[\mathrm{s}]^*} \boldsymbol{\Sigma}_2^{[\mathrm{s}]^{-1}} \boldsymbol{V}_2^{[\mathrm{s}]^\mathrm{T}} \otimes \boldsymbol{U}_2^{[\mathrm{n}]} \boldsymbol{U}_2^{[\mathrm{n}]^\mathrm{H}}\right)$$

$\boldsymbol{K}_{p \times q} \cdot \mathrm{vec}\{\boldsymbol{X}\} = \mathrm{vec}\left\{\boldsymbol{X}^\mathrm{T}\right\}, \ \forall \boldsymbol{X} \in \mathbb{C}^{p \times q}$ (commutation matrix)

$$\bar{\boldsymbol{T}}_1 = \begin{bmatrix} \boldsymbol{I}_{M_2} \otimes \boldsymbol{t}_{1,1} \otimes \boldsymbol{I}_{M_2} \\ \vdots \\ \boldsymbol{I}_{M_2} \otimes \boldsymbol{t}_{1,M_1} \otimes \boldsymbol{I}_{M_2} \end{bmatrix} \qquad \bar{\boldsymbol{T}}_2 = \boldsymbol{I}_{M_1} \otimes \begin{bmatrix} \boldsymbol{I}_{M_1} \otimes \boldsymbol{t}_{2,1} \\ \vdots \\ \boldsymbol{I}_{M_1} \otimes \boldsymbol{t}_{2,M_2} \end{bmatrix} \qquad \boldsymbol{t}_{r,m} = [\boldsymbol{T}_r]_{:,m}$$

Special case: single source (2-D)

- For $d = 1$ (**single source**) and an M_1 x M_2 uniform rectangular array (URA)

$$\mathrm{MSE_{SE}} = \mathrm{MSE_{STE}} = \mathrm{MSE_{UE}} = \mathrm{MSE_{UTE}}$$

$$\approx \frac{\sigma_\mathrm{n}^2}{\hat{P}_\mathrm{T} \cdot N} \cdot \left(\frac{1}{(M_1-1)^2 \cdot M_2} + \frac{1}{M_1 \cdot (M_2-1)^2} \right) \quad (7)$$

$$\hat{P}_\mathrm{T} = \|\boldsymbol{S}\|_\mathrm{F}^2 / N$$

- **Deterministic Cramér-Rao Bound**

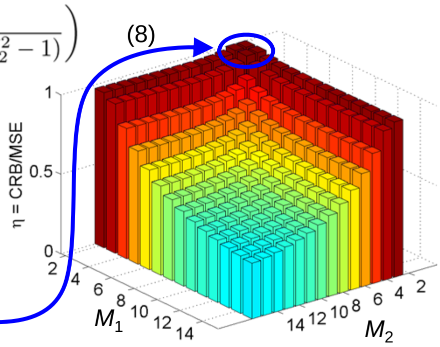
$$\mathrm{CRB} = \frac{\sigma_\mathrm{n}^2}{\hat{P}_\mathrm{T} \cdot N} \cdot \left(\frac{6}{M \cdot (M_1^2 - 1)} + \frac{6}{M \cdot (M_2^2 - 1)} \right) \quad (8)$$

- **Asymptotic efficiency**

$$\eta = \lim_{\sigma_n \to 0} \frac{\mathrm{CRB}}{\mathrm{MSE}}$$

$\Rightarrow$ depends only on M_1 and M_2!

$\Rightarrow M_1 \in \{2,3\}, M_2 \in \{2,3\} \Rightarrow \eta = 1$



Outline

- Introduction
  - $\Rightarrow$ Tensor-based algorithms vs. matrix-based algorithms
  - $\Rightarrow$ Application example: *R*-D harmonic retrieval
- Subspace estimation
  - $\Rightarrow$ HOSVD-based enhanced subspace estimate
  - $\Rightarrow$ Perturbation analysis of matrix-based techniques
  - $\Rightarrow$ Extension to the tensor case
- Analytical performance evaluation for Tensor-ESPRIT-type algorithms
- **Simulation results**
- Conclusions

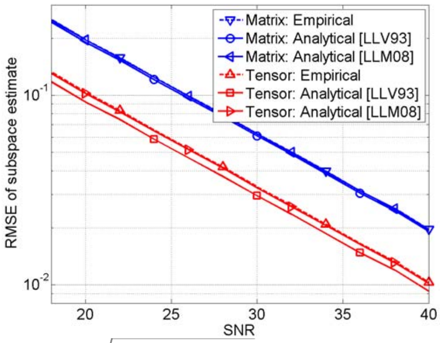
Simulations: 4 uncorrelated sources,
2-D Standard ESPRIT vs. 2-D Standard Tensor-ESPRIT

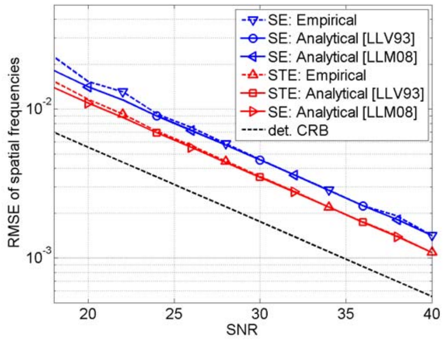
$d = 4$ $\quad (\mu_1, \nu_1) = (-1.5, 1.3)$

$M = 8 \times 8,\ N = 5$ $\quad (\mu_2, \nu_2) = (0.5, -0.2)$

$(\mu_3, \nu_3) = (1.0, 0.7)$

$\rho = 0.0$ $\quad (\mu_4, \nu_4) = (-0.3, -1.5)$





$$\mathrm{RMSE} = \sqrt{\mathbb{E}\left\{\sum_{i=1}^{d} \|\Delta\hat{\boldsymbol{u}}_i\|_2^2\right\}}$$

$$\Delta\hat{\boldsymbol{u}}_i = \hat{\boldsymbol{u}}_i \frac{\hat{\boldsymbol{u}}_i^{\mathrm{H}}\boldsymbol{u}_i}{|\hat{\boldsymbol{u}}_i^{\mathrm{H}}\boldsymbol{u}_i|} - \boldsymbol{u}_i \quad \text{[LLM08]}$$

$$\mathrm{RMSE} = \sqrt{\mathbb{E}\left\{\sum_{i=1}^{d} (\mu_i - \hat{\mu}_i)^2 + (\nu_i - \hat{\nu}_i)^2\right\}}$$

Ilmenau University of Technology
Communications Research Laboratory

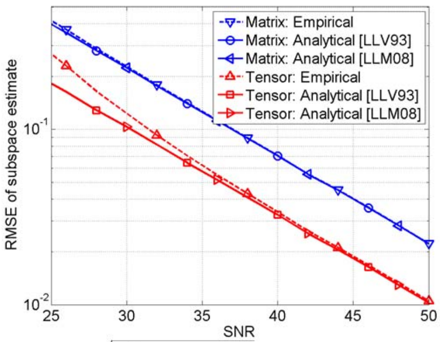
Simulations: 3 correlated sources,
2-D Standard ESPRIT vs. 2-D Standard Tensor-ESPRIT

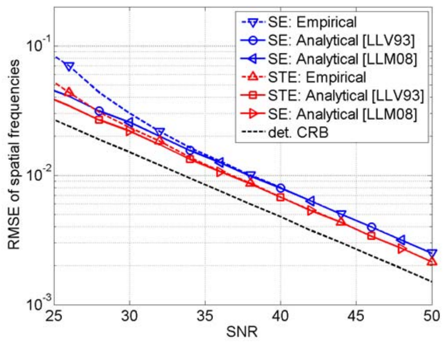
$d = 3$ $\quad (\mu_1, \nu_1) = (0.7, -0.1)$

$M = 8 \times 8,\ N = 20$ $\quad (\mu_2, \nu_2) = (0.9, -0.3)$

$(\mu_3, \nu_3) = (1.1, -0.5)$

$\rho = 0.97$





$$\mathrm{RMSE} = \sqrt{\mathbb{E}\left\{\sum_{i=1}^{d} \|\Delta\hat{\boldsymbol{u}}_i\|_2^2\right\}}$$

$$\Delta\hat{\boldsymbol{u}}_i = \hat{\boldsymbol{u}}_i \frac{\hat{\boldsymbol{u}}_i^{\mathrm{H}}\boldsymbol{u}_i}{|\hat{\boldsymbol{u}}_i^{\mathrm{H}}\boldsymbol{u}_i|} - \boldsymbol{u}_i \quad \text{[LLM08]}$$

$$\mathrm{RMSE} = \sqrt{\mathbb{E}\left\{\sum_{i=1}^{d} (\mu_i - \hat{\mu}_i)^2 + (\nu_i - \hat{\nu}_i)^2\right\}}$$

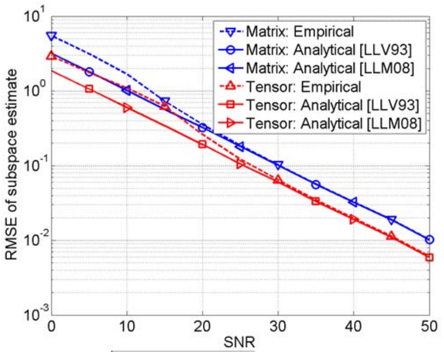
Ilmenau University of Technology
Communications Research Laboratory

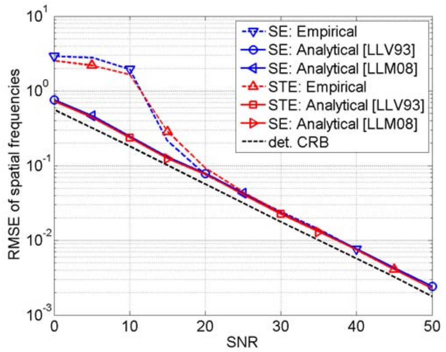
Simulations: 2 close sources, 2-D Standard ESPRIT vs. 2-D Standard Tensor-ESPRIT

$d = 2$ $\quad (\mu_1, \nu_1) = (1.0, 1.0)$

$M = 5 \times 5,\ N = 20$ $\quad (\mu_2, \nu_2) = (0.95, 0.95)$

$\rho = 0.0$



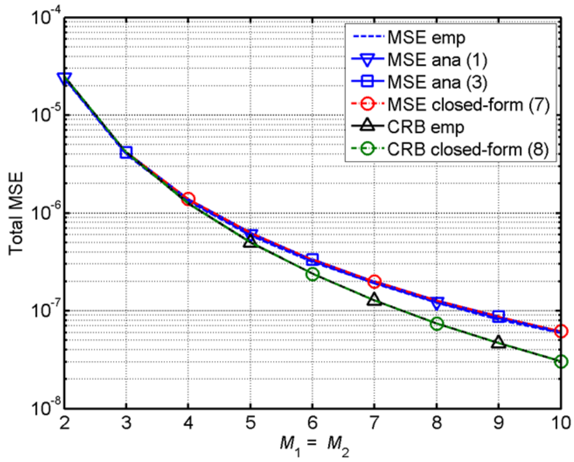


$$\mathrm{RMSE} = \sqrt{\mathbb{E}\left\{\sum_{i=1}^{d} \|\Delta\hat{\boldsymbol{u}}_i\|_2^2\right\}}$$

$$\Delta\hat{\boldsymbol{u}}_i = \hat{\boldsymbol{u}}_i \frac{\hat{\boldsymbol{u}}_i^{\mathrm{H}}\boldsymbol{u}_i}{|\hat{\boldsymbol{u}}_i^{\mathrm{H}}\boldsymbol{u}_i|} - \boldsymbol{u}_i$$

$$\mathrm{RMSE} = \sqrt{\mathbb{E}\left\{\sum_{i=1}^{d} (\mu_i - \hat{\mu}_i)^2 + (\nu_i - \hat{\nu}_i)^2\right\}}$$

Simulations: Single source (2-D)



$\mathrm{SNR} = 40$ dB

$N = 10$

$\mu^{(1)} = 0.3,\ \mu^{(2)} = -0.1$

$R = 2$

$$\mathrm{MSE} = \sum_{i=1}^{d}\sum_{r=1}^{R}\left(\Delta\mu_i^{(r)}\right)^2$$

(1): function of noise realization

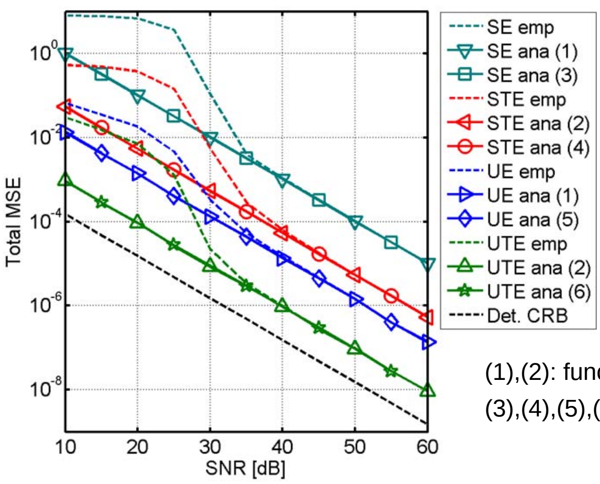
(3): statistical expectation

(7),(8): compact closed-form expressions for $d = 1$

$\Rightarrow$ **no improvement** from using tensors for a **single source**

$\Rightarrow$ asymptotically **efficient** for $M_1 \in \{2,3\}, M_2 \in \{2,3\}$

Simulations: Two correlated sources (2-D)



$M_1 = 5,\ M_2 = 6$

$N = 20$

$\mu_1^{(1)} = 1.0,\ \mu_1^{(2)} = -0.5$

$\mu_2^{(1)} = -0.5,\ \mu_2^{(2)} = 1.0$

$|\rho| = 0.9999$

$R = 2$

$$\mathrm{MSE} = \sum_{i=1}^{d}\sum_{r=1}^{R}\left(\Delta\mu_i^{(r)}\right)^2$$

(1),(2): function of noise realization

(3),(4),(5),(6): statistical expectation

$\Rightarrow$ significant **improvement** from using tensors for **correlated** sources

Ilmenau University of Technology
Communications Research Laboratory

Outline

- Introduction
  - $\Rightarrow$ Tensor-based algorithms vs. matrix-based algorithms
  - $\Rightarrow$ Application example: *R*-D harmonic retrieval
- Subspace estimation
  - $\Rightarrow$ HOSVD-based enhanced subspace estimate
  - $\Rightarrow$ Perturbation analysis of matrix-based techniques
  - $\Rightarrow$ Extension to the tensor case
- Analytical performance evaluation for Tensor-ESPRIT-type algorithms
- Simulation results
- **Conclusions**

Ilmenau University of Technology
Communications Research Laboratory

Conclusions (1)

- **Tensor-based** signal processing has many key **advantages**
  - ⇒ **uniqueness** of trilinear (multilinear) decomposition
  - ⇒ improved **identifiability**
  - ⇒ **focus of this talk**: enhanced subspace estimate achieved through **multilinear rank reduction** via the Higher-Order SVD (HOSVD)
    - can be used to improve any multidimensional subspace-based estimation technique
- Established the fundamental **link between** the **SVD**- and the **HOSVD**-based subspace estimates
  - ⇒ **projection** of the matrix-based estimate onto the **Kronecker** structure of the estimated *r*-mode subspaces ($r = 1, 2, \ldots, R$)
  - ⇒ **no** need to calculate the **core tensor** explicitly
- Analytical **perturbation** expansion
  - ⇒ allows to **quantify** the improvement in the subspace estimate
  - ⇒ can, for example, be used to obtain **analytical** MSE expressions for **Tensor-ESPRIT**-type algorithms

Conclusions (2)

- Closed-form expressions for the statistical **expectation** with respect to white (Gaussian) **noise**
- These results allow us to reliably assess the **performance** of *R*-D Standard **Tensor-ESPRIT** and *R*-D Unitary **Tensor-ESPRIT**
  - ⇒ by computing **analytically**
    - to what extend and
    - under which conditions

    Tensor-ESPRIT-type algorithms outperform matrix-based algorithms
    - **no** improvement for a **single source** and for $d \geq \max(M_1, M_2)$
    - particularly **strong improvement** for **correlated sources** and **small** number of **snapshots**
- Enables us to compute the **asymptotic efficiency** *analytically*
  - ⇒ only depending on the array size in case of a single source

References

[HRD08] M. Haardt, F. Roemer, and G. Del Galdo, “Higher-order SVD based subspace estimation to improve the parameter estimation accuracy in multi-dimensional harmonic retrieval problems,” *IEEE Trans. Signal Processing*, vol. 56, pp. 3198 - 3213, July 2008

[RBH10] F. Roemer, H. Becker, and M. Haardt, “Analytical performance analysis for multi-dimensional Tensor-ESPRIT-type parameter estimation algorithms,” submitted to the *Proc. IEEE Int. Conf. Acoust., Speech, and Signal Processing (ICASSP 2010)*, Dallas, TX, Mar. 2010.

[RBHW09] F. Roemer, H. Becker, M. Haardt, and M. Weis, “Analytical performance evaluation for HOSVD-based parameter estimation schemes,” in *Proc. IEEE Int. Workshop on Computational Advances in Multi-Sensor Adaptive Processing (CAMSAP 2009)*, Aruba, Dutch Antilles, Dec. 2009.

[RH07] F. Roemer and M. Haardt, “Tensor-structure structured least squares (TS-SLS) to improve the performance of multi-dimensional ESPRIT-type algorithms,” in *Proc. IEEE Int. Conf. Acoust., Speech, and Signal Processing (ICASSP)*, vol. II, Honolulu, HI, pp. 893 - 896, Apr. 2007.

[RHD06] F. Roemer, M. Haardt, and G. Del Galdo, “Higher order SVD based subspace estimation to improve multi-dimensional parameter estimation algorithms,” in *Proc. 40th Asilomar Conf. on Signals, Systems, and Computers*, Pacific Grove, CA, pp. 961-965, Nov. 2006, invited paper.