Symmetric tensor decomposition

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Introduction

We present an algorithm for decomposing a symmetric tensor, of dimension n and order d, as a sum of rank-1 symmetric tensors, extending the algorithm of Sylvester devised in 1886 for binary forms.

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$$v = \sum_{i_1<\ldots<i_d} A_{i_1,\ldots,i_d} e_{i_1}\ldots e_{i_d} \in \wedge^d E$$

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- The goal is to find such a decomposition with the minimal r.

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- Another important application field is Data Analysis, for instance, Independent Component Analysis, originally introduced for symmetric tensors whose rank did not exceed dimension.

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- Among these popular methods, we refer to “PARAFAC“ techniques, extensively applied to ill-posed problems.

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- We give necessary and sufficient condition for the existence of a decomposition of rank r, based on rank conditions of Hankel operators or commutation properties.

- This algorithm is not restricted to strictly sub-generic ranks and fully exploits the symmetries.

Binary case

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Theorem (Sylvester, 1886)

A binary quantic $p(x_1,x_2)=\sum_{i=0}^{d}\binom{d}{i}c_i\,x_1^i\,x_2^{d-i}$ *can be written as a sum of* d^{th} *powers of* r *distinct linear forms in* $\mathbb{C}$ *as:*

$$p(x_1,x_2)=\sum_{j=1}^{r}\lambda_j\,(\alpha_j\,x_1+\beta_j\,x_2)^d, \tag{1}$$

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$$\begin{bmatrix} c_0 & c_1 & \cdots & c_r \\ \vdots & & & \vdots \\ c_{d-r} & \cdots & c_{d-1} & c_d \end{bmatrix}\mathbf{q}=\mathbf{0}. \tag{2}$$

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and (**ii**) *the polynomial $q(x_1,x_2)=\sum_{l=0}^{r} q_l\, x_1^l\, x_2^{r-l}$ admits r distinct roots, i.e. can be written as $q(x_1,x_2)=\prod_{j=1}^{r}(\beta_j^*\,x_1-\alpha_j^*\,x_2)$.*

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The Sylvester's theorem yields the following algorithm:

Input : Given a binary polynomial $p(x_1, x_2)$ of degree d with coefficients $a_i = \binom{d}{i} c_i$, $0 \leq i \leq d$, define the Hankel matrix $H[r]$ of dimensions $d - r + 1 \times r + 1$ with entries $H[r]_{ij} = c_{i+j-2}$:

$$H[r] = \begin{bmatrix} c_0 & c_1 & \cdots & c_r \\ \vdots & & & \vdots \\ c_{d-r} & \cdots & c_{d-1} & c_d \end{bmatrix}. \tag{3}$$

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Output : A decomposition of p as $p(x_1, x_2) = \sum_{j=1}^{r} \lambda_j \, \mathbf{k_j}(\mathbf{x})^{\mathbf{d}}$ with minimal r.

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$$\begin{bmatrix} \alpha_1^d & \ldots & \alpha_r^d \\ \alpha_1^{d-1}\beta_1 & \ldots & \alpha_r^{d-1}\beta_r \\ \alpha_1^{d-2}\beta_1^2 & \ldots & \alpha_r^{d-1}\beta_r^2 \\ \vdots & \vdots & \vdots \\ \beta_1^d & \ldots & \beta_r^d \end{bmatrix} \lambda = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_d \end{bmatrix}$$

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6. The decomposition is $p(x_1, x_2) = \sum_{j=1}^{r} \lambda_j \, \mathbf{k_j}(\mathbf{x})^{\mathbf{d}}$, where $\mathbf{k_j}(\mathbf{x}) = (\alpha_{\mathbf{j}} \, \mathbf{x_1} + \beta_{\mathbf{j}} \, \mathbf{x_2})$.

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A symmetric tensor $[a_{j_0,\dots,j_{n-1}}]$ of order d and dimension n can be associated to a homogeneous polynomial $f(\mathbf{x}) \in \mathbf{S_d}$:

$$f(\mathbf{x}) = \sum_{\mathbf{j_0+j_1+\cdots+j_{n-1}=d}} \mathbf{a_{j_0,j_1,\dots,j_{n-1}} x_0^{j_0} x_1^{j_1} \cdots x_{n-1}^{j_{n-1}}}. \tag{4}$$

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Our goal is to compute a decomposition of f as a sum of d^{th} powers of linear forms, i.e.

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where $\lambda_i \neq 0$, $\mathbf{k_i} \neq \mathbf{0}$, and r is the smallest possible.

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where $\lambda_i \neq 0$, $\mathbf{k_i} \neq \mathbf{0}$, and r is the smallest possible.
This minimal r is called the rank of f.

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Consider the following map from the projective space $\mathbb{P}^{n-1}$ to the projective space of symmetric tensors:

$$\begin{array}{rcccl} \nu & : & \mathbb{P}(S_1) & \rightarrow & \mathbb{P}(S_d) \\ & & \mathbf{k}(\mathbf{x}) & \mapsto & \mathbf{k}(\mathbf{x})^{\mathbf{d}}. \end{array}$$

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A tensor of rank 1 is a point of the Veronese.
A tensor of rank r is in the linear space spanned by r points of the Veronese variety.
The closure of the r-dimensional linear space spanned by r points of the Veronese is called the r-1 secant variety.

Reformulations

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Let $f, g \in S_d$, where $f = \sum_{|\alpha|=d} f_\alpha x_0^{\alpha_0} \cdots x_{n-1}^{\alpha_{n-1}}$ and $g = \sum_{|\alpha|=d} g_\alpha x_0^{\alpha_0} \cdots x_{n-1}^{\alpha_{n-1}}$.

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Using this non-degenerate inner product, we can associate an element of S_d with an element S_d^*, through the following map:

$$\begin{aligned} \tau : S_d &\to S_d^* \\ f &\mapsto f^*, \end{aligned}$$

where the linear form f^* is defined as $f^* : g \mapsto \langle f, g \rangle$.

Reformulations

Proposition

Let $f = (a_0 x_0 + \ldots + a_{n-1} x_{n-1})^d \in S_d$, *then*

$$f^*(g) = \langle f, g \rangle = g(a_0, \ldots, a_{n-1}) = \mathbf{ev}_A(g)$$

with $A = (a_0, \ldots, a_{n-1}) \in \mathbb{K}^{n-1}$, $g \in S_d$ *and* $\mathbf{ev}_A$ *is the evaluation in A.*

Reformulations

The decomposition problem can be reformulated as follows:

Given $f^* \in S_d^*$, find the minimal number of non-zero points $\mathbf{k_1}, \ldots, \mathbf{k_r} \in \mathbb{K}^\mathbf{n}$ and non-zero scalars $\lambda_1, \ldots, \lambda_r \in \mathbb{K} - \{0\}$ such that

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$$f^* = \sum_{i=1}^{r} \lambda_i \, \mathbf{ev}_{\mathbf{k_i}}.$$

Note that by a generic change of variables, we can assume that all the coordinates $\mathbf{k_{i,0}}$ are equal to 1.

Hankel operators and quotient algebras

Definition

Let $R = \mathbb{K}[x_1, \ldots, x_{n-1}]$ be the polynomial ring in $n-1$ variables and $\Lambda \in R^*$ be a linear form. We define the Hankel operator H_Λ from R to R^* as

$$\begin{aligned} H_\Lambda \quad : \quad & R \longrightarrow R^* \\ & p \longmapsto p \star \Lambda \end{aligned}$$

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We denote by $\mathbb{H}_\Lambda$ the matrix of H_Λ in the basis $\{x^\alpha\}$ and $\{\mathbf{d}^\alpha\}$ (where $\{\mathbf{d}^\alpha\}$ is the dual basis of the monomial basis $\{x^\alpha\}$). Thus

$$\mathbb{H}_\Lambda = (\Lambda(x^{\alpha+\beta}))_{\alpha,\beta}.$$

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Let $\mathcal{A}_\Lambda := R/I_\Lambda$ be the quotient algebra of polynomials modulo the ideal I_Λ.

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$$\begin{aligned} M_a \quad : \quad & \mathcal{A}_\Lambda \rightarrow \mathcal{A}_\Lambda \\ & b \mapsto ba. \end{aligned}$$

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- Let M_a^t be the be its transposed operator:

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Note that, by definition, we have

$$H_{a\star\Lambda} = M_a^t \circ H_\Lambda$$

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If $\mathrm{rank}(H_\Lambda) = r < \infty$, *then*

Hankel operators and quotient algebras

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$$\Lambda = \sum_{i=1}^{d} \mathbf{ev}_{\zeta_i} \circ p_i(\partial) \tag{6}$$

Moreover the multiplicity of ζ_i *is the dimension of the vector space spanned by the inverse system generated by* $\mathbf{ev}_{\zeta_i} \circ p_i(\partial)$.

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- *the common eigenvectors of the operators* $(M_{x_i}^t)_{1 \leq i \leq n}$ *are (up to scalar)* $\mathbf{ev}_{\zeta_i}$.

Hankel operators and quotient algebras

Theorem

Let $\Lambda \in R^*$. $\Lambda = \sum_{i=1}^{r} \lambda_i \mathbf{ev}_{\zeta_i}$ with $\lambda_i \neq 0$ and ζ_i distinct points of $\mathbb{K}^n$, iff $\mathrm{rank} H_\Lambda = r$ and I_Λ is a radical ideal.

Truncated Hankel operators

The problem of decomposition can be reformulated as follows :

Given $f^* \in R_d^*$ find the smallest r such that there exists $\Lambda \in R^*$ which extends f^* with H_Λ of rank r and I_Λ a radical ideal.

Truncated Hankel operators

Definition

Given $f^* \in R_d^*$ a linear form and B a set of monomials of degree at most d, we define the Hankel operator $\Lambda_{f^*}(\mathbf{h})$ by:

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We denote by $\mathcal{H}^B_{\Lambda_{f^*}}(\mathbf{h})$ its matrix:

$$\mathcal{H}^B_{\Lambda_{f^*}}(\mathbf{h}) = (h_{\alpha+\beta})_{\alpha,\beta \in B}.$$

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Definition

If $\mathcal{H}^B_{\Lambda_{f^*}}(\mathbf{h})$ is invertible in $\mathbb{K}(\mathbf{h})$ (that is the rational polynomial functions in **h**), then we define the multiplication operators

$$\mathcal{M}_i^B(\mathbf{h}) := (\mathcal{H}^B_{\Lambda_{f^*}}(\mathbf{h}))^{-1} \mathcal{H}^B_{x_i \star \Lambda_{f^*}}(\mathbf{h}).$$

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Definition

Let B be a subset of monomials in R. We say that B is connected to 1 if:
- $1 \in B$,
- $\forall m \neq 1 \in B$ there exists $i \in [1, n]$ and $m' \in B$ such that $m = x_i m'$.

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Then $\Lambda_{f^*}(\mathbf{h})$ *admits an extension* $\tilde{\Lambda} \in R^*$ *such that* $H_{\tilde{\Lambda}}$ *is of rank* r *with* B *a basis of* $A_{\tilde{\Lambda}}$ *iff*

$$\mathcal{M}_i^B(\mathbf{h}) \circ \mathcal{M}_j^B(\mathbf{h}) - \mathcal{M}_j^B(\mathbf{h}) \circ \mathcal{M}_i^B(\mathbf{h}) = 0 \quad (1 \leq i < j \leq n) \tag{7}$$

and $\det(\mathcal{H}_{\Lambda_{f^*}}^B)(\mathbf{h}) \neq 0$. *Moreover, such a* $\tilde{\Lambda}$ *is unique.*

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  until the eigenvalues are simple.
- Solve the linear system in $(c_j)_{j=1,\ldots,k}$: $\Lambda = \sum_{j=1}^{r} c_j \mathbf{ev}_{\zeta_j}$ where $\zeta_j \in \mathbb{K}^n$ are the eigenvectors found in step 4.

Example

Assume that a tensor of dimension three and order 5 corresponds to the following homogeneous polynomial:

$f = -1549440\, x_0 x_1 x_2^3 + 2417040\, x_0 x_1^2 x_2^2 + 166320\, x_0^2 x_1 x_2^2 - 829440\, x_0 x_1^3 x_2 - 5760\, x_0^3 x_1 x_2 - 222480\, x_0^2 x_1^2 x_2 + 38\, x_0^5 - 497664\, x_1^5 - 1107804\, x_2^5 - 120\, x_0^4 x_1 + 180\, x_0^4 x_2 + 12720\, x_0^3 x_1^2 + 8220\, x_0^3 x_2^2 - 34560\, x_0^2 x_1^3 - 59160\, x_0^2 x_2^3 + 831840\, x_0 x_1^4 + 442590\, x_0 x_2^4 - 5591520\, x_1^4 x_2 + 7983360\, x_1^3 x_2^2 - 9653040\, x_1^2 x_2^3 + 5116680\, x_1 x_2^4.$

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The minimum decomposition of the polynomial as a sum of powers of linear forms is

$$(x_0 + 2\,x_1 + 3\,x_2)^5 + (x_0 - 2\,x_1 + 3\,x_2)^5 + \frac{1}{3}(x_0 - 12\,x_1 - 3\,x_2)^5 + \frac{1}{5}(x_0 + 12\,x_1 - 13\,x_2)^5,$$

that is, the corresponding tensor is of rank 4.

Example

The whole matrix is 21×21. We show only the 10×10 principal minor.

	1	x_1	x_2	x_1^2	$x_1 x_2$	x_2^2	x_1^3	$x_1^2 x_2$	$x_1 x_2^2$	x_2^3
1	38	−24	36	1272	−288	822	−3456	−7416	5544	−5916
x_1	−24	1272	−288	−3456	−7416	5544	166368	−41472	80568	−77472
x_2	36	−288	822	−7416	5544	−5916	−41472	80568	−77472	88518
x_1^2	1272	−3456	−7416	166368	−41472	80568	−497664	−1118304	798336	−965304
$x_1 x_2$	−288	−7416	5544	−41472	80568	−77472	−1118304	798336	−965304	1023336
x_2^2	822	5544	−5916	80568	−77472	88518	798336	−965304	1023336	−1107804
x_1^3	−3456	166368	−41472	−497664	−1118304	798336	$h_{6,0,0}$	$h_{5,1,0}$	$h_{4,2,0}$	$h_{3,3,0}$
$x_1^2 x_2$	−7416	−41472	80568	−1118304	798336	−965304	$h_{5,1,0}$	$h_{4,2,0}$	$h_{3,3,0}$	$h_{2,4,0}$
$x_1 x_2^2$	5544	80568	−77472	798336	−965304	1023336	$h_{4,2,0}$	$h_{3,3,0}$	$h_{2,4,0}$	$h_{1,5,0}$
x_2^3	−5916	−77472	88518	−965304	1023336	−1107804	$h_{3,3,0}$	$h_{2,4,0}$	$h_{1,5,0}$	$h_{0,6,0}$

Notice that we do not know the elements in some positions of the matrix.

Example

In our example the 4×4 principal minor is of full rank, so there is no need for re-arranging the matrix. The matrix ${\mathbb{H}^B}_\Lambda$ is

$$\mathbb{H}_\Lambda^B = \begin{bmatrix} 38 & -24 & 36 & 1272 \\ -24 & 1272 & -288 & -3456 \\ 36 & -288 & 822 & -7416 \\ 1272 & -3456 & -7416 & 166368 \end{bmatrix}$$

with monomial basis B equal to $\{1, x_1, x_2, x_1^2\}$.

Example

The shifted matrix $\mathbb{H}^B_{x_1 \star \Lambda}$ is

$$\mathbb{H}^B_{x_1 \Lambda} = \begin{bmatrix} -24 & 1272 & -288 & -3456 \\ 1272 & -3456 & -7416 & 166368 \\ -288 & -7416 & 5544 & -41472 \\ -3456 & 166368 & -41472 & -497664 \end{bmatrix}$$

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We check the commutations relations.

Example

We solve the generalized eigenvalue/eigenvector problem $(\mathbb{H}_{x_i \star \Lambda} - \lambda \mathbb{H}_{\Lambda})X = 0$ for $i = 1, 2$.

Example

We get the common eigenvectors in the basis $B = \{1, x_1, x_2, x_1^2\}$

$$\begin{bmatrix} 1 \\ -12 \\ -3 \\ 144 \end{bmatrix}, \begin{bmatrix} 1 \\ 12 \\ -13 \\ 144 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

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Thus, we can deduce the roots and write the following decomposition:

$$f = c_1(x_0 + 2x_1 + 3x_2)^5 + c_2(x_0 - 2x_1 + 3x_2)^5 + c_3(x_0 - 12x_1 - 3x_2)^5 + c_4(x_0 + 12x_1 - 13x_2)^5$$

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It remains to compute c_i's. We get that: $c_1 = 1$, $c_2 = 1$, $c_3 = 1/3$ and $c_4 = 1/5$.

Example

Consider a tensor of dimension three and order 4, that corresponds to the following homogeneous polynomial

$$f = 79\, x_0 x_1^3 + 56\, x_0^2 x_2^2 + 49\, x_1^2 x_2^2 + 4\, x_0 x_1 x_2^2 + 57\, x_0^3 x_1 ,$$

the rank of which is 6.

Example

	1	x_1	x_2	x_1^2	$x_1 x_2$	x_2^2	x_1^3	$x_1^2 x_2$	$x_1 x_2^2$	x_2^3	x_1^4	$x_1^3 x_2$	$x_1^2 x_2^2$	$x_1 x_2^3$	x_2^4
1	0	$\frac{57}{4}$	0	0	0	$\frac{28}{3}$	$\frac{79}{4}$	0	$\frac{1}{3}$	0	0	0	$\frac{49}{6}$	0	0
x_1	$\frac{57}{4}$	0	0	$\frac{79}{4}$	0	$\frac{1}{3}$	0	0	$\frac{49}{6}$	0	h_{500}	h_{410}	h_{320}	h_{230}	h_{140}
x_2	0	0	$\frac{28}{3}$	0	$\frac{1}{3}$	0	0	$\frac{49}{6}$	0	0	h_{410}	h_{320}	h_{230}	h_{140}	h_{050}
x_1^2	0	$\frac{79}{4}$	0	0	0	$\frac{49}{6}$	h_{500}	h_{410}	h_{320}	h_{230}	h_{600}	h_{510}	h_{420}	h_{330}	h_{240}
$x_1 x_2$	0	0	$\frac{1}{3}$	0	$\frac{49}{6}$	0	h_{410}	h_{320}	h_{230}	h_{140}	h_{510}	h_{420}	h_{330}	h_{240}	h_{150}
x_2^2	$\frac{28}{3}$	$\frac{1}{3}$	0	$\frac{49}{6}$	0	0	h_{320}	h_{230}	h_{140}	h_{050}	h_{420}	h_{330}	h_{240}	h_{150}	h_{060}
x_1^3	$\frac{79}{4}$	0	0	h_{500}	h_{410}	h_{320}	h_{600}	h_{510}	h_{420}	h_{330}	h_{700}	h_{610}	h_{520}	h_{430}	h_{340}
$x_1^2 x_2$	0	0	$\frac{49}{6}$	h_{410}	h_{320}	h_{230}	h_{510}	h_{420}	h_{330}	h_{240}	h_{610}	h_{520}	h_{430}	h_{340}	h_{250}
$x_1 x_2^2$	$\frac{1}{3}$	$\frac{49}{6}$	0	h_{320}	h_{230}	h_{140}	h_{420}	h_{330}	h_{240}	h_{150}	h_{520}	h_{430}	h_{340}	h_{250}	h_{160}
x_2^3	0	0	0	h_{230}	h_{140}	h_{050}	h_{330}	h_{240}	h_{150}	h_{060}	h_{430}	h_{340}	h_{250}	h_{160}	h_{070}
x_1^4	0	h_{500}	h_{410}	h_{600}	h_{510}	h_{420}	h_{700}	h_{610}	h_{520}	h_{430}	h_{800}	h_{710}	h_{620}	h_{530}	h_{440}
$x_1^3 x_2$	0	h_{410}	h_{320}	h_{510}	h_{420}	h_{330}	h_{610}	h_{520}	h_{430}	h_{340}	h_{710}	h_{620}	h_{530}	h_{440}	h_{350}
$x_1^2 x_2^2$	$\frac{49}{6}$	h_{320}	h_{230}	h_{420}	h_{330}	h_{240}	h_{520}	h_{430}	h_{340}	h_{250}	h_{620}	h_{530}	h_{440}	h_{350}	h_{260}
$x_1 x_2^3$	0	h_{230}	h_{140}	h_{330}	h_{240}	h_{150}	h_{430}	h_{340}	h_{250}	h_{160}	h_{530}	h_{440}	h_{350}	h_{260}	h_{170}
x_2^4	0	h_{140}	h_{050}	h_{240}	h_{150}	h_{060}	h_{340}	h_{250}	h_{160}	h_{070}	h_{440}	h_{350}	h_{260}	h_{170}	h_{080}

Example

In our example the 6×6 principal minor is of full rank. The matrix $\mathbb{H}_\Lambda$ is

$$\mathbb{H}_\Lambda = \begin{bmatrix} 0 & \frac{57}{4} & 0 & 0 & 0 & \frac{28}{3} \\ \frac{57}{4} & 0 & 0 & \frac{79}{4} & 0 & \frac{1}{3} \\ 0 & 0 & \frac{28}{3} & 0 & \frac{1}{3} & 0 \\ 0 & \frac{79}{4} & 0 & 0 & 0 & \frac{49}{6} \\ 0 & 0 & \frac{1}{3} & 0 & \frac{49}{6} & 0 \\ \frac{28}{3} & \frac{1}{3} & 0 & \frac{49}{6} & 0 & 0 \end{bmatrix}$$

The columns (and the rows) of the matrix correspond to the monomials $\{1, x_1, x_2, x_1^2, x_1 x_2, x_2^2\}$.

Example

The shifted matrix $\mathbb{H}_{x_1\star\Lambda}$ is

$$\mathbb{H}_{x_1\star\Lambda} = \begin{bmatrix} \frac{57}{4} & 0 & 0 & \frac{79}{4} & 0 & \frac{1}{3} \\ 0 & \frac{79}{4} & 0 & 0 & 0 & \frac{49}{6} \\ 0 & 0 & \frac{1}{3} & 0 & \frac{49}{6} & 0 \\ \frac{79}{4} & 0 & 0 & h_{500} & h_{410} & h_{320} \\ 0 & 0 & \frac{49}{6} & h_{410} & h_{320} & h_{230} \\ \frac{1}{3} & \frac{49}{6} & 0 & h_{320} & h_{230} & h_{140} \end{bmatrix}$$

Since not all the entries of $\mathbb{H}_{x_1\Lambda}$ are known, we need to compute them in order to proceed further.

Example

Since we have only two variables, there is only one matrix equation,

$$\mathbb{M}_{x_i}\mathbb{M}_{x_j} - \mathbb{M}_{x_j}\mathbb{M}_{x_i} = \mathbb{H}_{\Lambda}^{-1}\mathbb{H}_{x_1\star\Lambda}\mathbb{H}_{\Lambda}^{-1}\mathbb{H}_{x_2\star\Lambda} - \mathbb{H}_{\Lambda}^{-1}\mathbb{H}_{x_2\star\Lambda}\mathbb{H}_{\Lambda}^{-1}\mathbb{H}_{x_1\Lambda} = \mathbb{O}.$$

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Many of the resulting equations are trivial. After discarding them, we have 6 unknonws $\{h_{500}, h_{410}, h_{320}, h_{230}, h_{140}, h_{050}\}$ and 15 equations.

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A solution of the system is the following

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We subsitute these values to $\mathbb{H}_{x_1\Lambda}$ and we continue the algorithm as in the previous example.