Approximate Robust Optimal Control of Periodic Systems with Invariants and High-Index Differential Algebraic Systems

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Abstract: In this paper we present solution approaches for uncertain periodic optimal control problems with invariants that arise e.g., after index reduction of high-index differential algebraic systems. There are two difficulties to be addressed: first, we encounter a redundancy in the periodic boundary constraints which is due to the presence of invariants. And second, we have to deal with the presence of uncertainties. To address the first problem we discuss both a projection and a null-space based reformulation approaches that avoid the redundancies in the constraints. Concerning the uncertainties, we discuss an approximate robust optimal control formulation based on Lyapunov differential equations. Here, the invariants and periodic boundary constraints have to be taken into account, too. We illustrate our techniques by optimizing an open-loop controlled inverted pendulum which is described by index three differential algebraic equations and is affected by unknown forces.

1. INTRODUCTION

We are interested in the periodic optimal control problem (OCP) of the form:

$$\underset{y(\cdot),u(\cdot),T}{\text{minimize}} \ \bar{J}(y(\cdot),u(\cdot),T) \tag{1}$$

s.t.:

$$\dot{y}(t) = f(y(t),u(t),w(t)), \qquad \forall t \in [0,T] \tag{2}$$

$$0 \geq h(y(t),u(t)), \tag{3}$$

$$y(0) = y(T), \tag{4}$$

where $y:\mathbb{R}\to\mathbb{R}^{n_y}$ and $u:\mathbb{R}\to\mathbb{R}^{n_u}$ denote the time dependent state and control functions, respectively. The end time $T\in\mathbb{R}_{++}$ can also be considered as a decision variable. The function f describes the dynamics of the system, while the functions h_i (with $i\in\{1,\ldots,n_h\}$) represent the inequality path constraints of the system. These functions are assumed to be sufficiently differentiable in their arguments. In this paper periodic boundary conditions (4) are considered.

Additionally, we assume that inside of the dynamics $f(y,u,w)$ an invariant $c(y(t)) = 0$ is intrinsically included. This invariant can be a result of a transformation from a high-index differential algebraic system into a differential system or a conserved quantity which is included into the differential dynamics, e.g., energy conservation. In order to maintain the invariant through the dynamics we need to fix its value for at least one point on the time interval $[0,T]$, e.g., at $t = T$ by adding the final time constraints:

$$c(y(T)) \; = \; 0. \tag{5}$$

The first difficulty in the treatment of the periodic optimal control problem with invariants (1) - (5) is a redundancy in the boundary conditions (4) - (5). In the present paper two approaches for the transformation of the redundant constraints (4) - (5) are proposed: the *projection* method and the *null-space* approach.

The second difficulty is due to the presence of the time varying uncertain functions $w(t)\in\mathbb{R}^{n_w}$ in the dynamics f. The function w is only known to be contained in a common uncertainty set $w\in W$ defined by

$$W \;:=\; \left\{ w \,\middle|\, \int_0^\infty w(\tau)^\top w(\tau)\,\mathrm{d}\tau \leq \gamma^2 \right\}. \tag{6}$$

In order to solve the uncertain optimal control problem (1) - (5) in the present paper an approximate robust optimal control approach based on the solution of Lyapunov differential equations is employed.

The structure of the paper is as follows. In Section 2 we introduce two approaches for treating the invariants in periodic optimal control problems, i.e. the projection and the null-space approaches. In Section 3 we discuss the approximate robust optimal control algorithm based on the solution of Lyapunov differential equations which can be be applied to optimal control problems with dynamic invariants. Section 4 gives an illustrative example in which the methodology is applied. For the numerical implementation we exploit the open source software `ACADO Toolkit` by Houska et al. [2011]. The paper concludes in Section 5 with a summary and an outlook on how the presented techniques and results might become relevant for the realization and control of more complex mechanical systems in the near future.

2. TREATING INVARIANTS IN PERIODIC OPTIMAL CONTROL

In this section we introduce two alternative approaches for the treatment of invariants in periodic optimal control problems. We consider the *nominal* optimal control problem (1) - (3) for

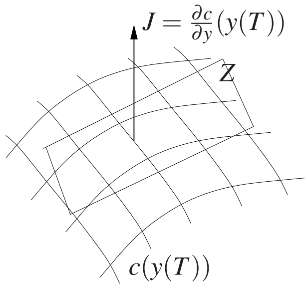

Fig. 1. Illustration of the decomposition into two subspaces for $n_y = 3$ and $n_c = 1$.

the fixed uncertainty function $w(t) = 0$ with the boundary conditions (4) - (5). For convenience we rewrite these conditions here in the form

Nominal redundant boundary conditions:

$$y(0) = y(T), \tag{7}$$

$$c(y(T)) = 0. \tag{8}$$

The constraints (7) - (8) are partly redundant because the invariant $c(y(T))$ will also be zero at the begin of the time interval. This creates problems for the numerical solution. Thus, we need to impose periodicity only in the directions that are tangential to the invariant constraint. One way to achieve this is to define an orthogonal projection operator ϕ that projects any state violating the invariant back to the invariant manifold, and to impose periodicity only after having applied this operator to the terminal state. Then, the invariant constraint can be left away as it is anyway satisfied. The proposed decomposition into two subspaces is illustrated in Figure 1.

Assumption 1. In the following we assume that the Jacobian matrix $J = \frac{dc}{dy}(y(T)) \in \mathbb{R}^{n_c \times n_y}$ has full rank.

The nonlinear projection operator can then be approximated by the first step of a Newton-type procedure to satisfy the constraint as follows:

Nominal projected boundary conditions:

$$y(0) - \phi(y(T)) = 0, \tag{9}$$

$$\phi(y) = y - J^\top \left(J J^\top\right)^{-1} c(y). \tag{10}$$

Introducing the (non-unique) null-space matrix $Z \in \mathbb{R}^{n_y \times (n_y - n_c)}$ of the derivative matrix J that satisfies:

$$J\,Z = 0 \qquad \text{and} \qquad Z^\top Z = \mathbb{I}$$

the redundant boundary conditions (7) - (8) can also be reformulated as:

Nominal Null-space boundary conditions:

$$Z^\top\,(y(0) - y(T)) = 0, \tag{11}$$

$$c(y(T)) = 0. \tag{12}$$

The nominal redundant boundary conditions (7) - (8) imply the nominal projected boundary conditions (9) - (10) and the nominal null-space boundary conditions (11) - (12). The implication in the other direction is in general only valid in case the algebraic constraints c are linear in $y(T)$.

In the following we denote the redundant optimal control problem (1) - (5) with $w(t) = 0$ as a nominal redundant OCP. The nominal OCP with projected boundary conditions now reads as:

Nominal projected OCP (POCP)

$$\underset{y(\cdot),u(\cdot),R,T}{\text{minimize}} \;\; \bar{J}(y(\cdot),u(\cdot),T) \tag{13}$$

s.t.:

$$(2)-(3) \qquad \text{with } w(t) = 0,$$

$$0 = y(0) - y(T) + J^\top R\, c(y(T)), \tag{14}$$

$$J\,J^\top R = \mathbb{I}. \tag{15}$$

Here, in order to compute the matrix $(JJ^T)^{-1} \in \mathbb{R}^{n_c \times n_c}$ the slack variables $R \in \mathbb{R}^{n_c \times n_c}$, defined by $J\,J^\top R = \mathbb{I}$, are introduced.

Employing the alternative null-space approach for reformulation of the redundant boundary conditions the nominal OCP reads as:

Nominal Null-space OCP (NOCP)

$$\underset{y(\cdot),u(\cdot),Z,T}{\text{minimize}} \;\; \bar{J}(y(\cdot),u(\cdot),T) \tag{16}$$

s.t.:

$$(2)-(3) \qquad \text{with } w(t) = 0,$$

$$Z^\top\,(y(0) - y(T)) = 0, \qquad J\,Z = 0, \tag{17}$$

$$c(y(T)) = 0, \qquad Z^\top Z = \mathbb{I}. \tag{18}$$

In this formulation the $n_y \times (n_y - n_c)$ entries of the null-space matrix Z become new decision variables. In the next proposition we give some relations between the solutions of nominal POCP/NOCP and the nominal redundant OCP.

Proposition 1. If Assumption 1 holds, any solution to the problem POCP or NOCP yields a lower bound on the original problem (1) - (5).

Proof: Every feasible point for nominal redundant OCP is also feasible for POCP and NOCP. Thus, any optimal solution for the problems POCP or NOCP yields a lower bound for the nominal redundant OCP. □

The following lemma provides conditions under which the linear independence constraint qualification (LICQ) for POCP and NOCP are satisfied.

Lemma 1. Let us start with a point $(y(t), u(t))$ which is feasible for the POCP or NOCP. If Assumption 1 holds, and matrix $(\mathbb{I} - V\,\bar{A})$ has full rank, where $V = \mathbb{I} - J^\top \left(J\,J^\top\right)^{-1} J$ and $\bar{A} = \frac{\partial y(T)}{\partial y(0)}$, then, the LICQ holds at the point (y, u) for the POCP or NOCP, respectively.

Proof:

We start with discretizing the nominal projected optimal control problem (13) - (15) using e.g., the single shooting method. Given a piecewise control function u the discrete counterpart for this optimal control problem reads as:

$$\underset{y_0 \in \mathbb{R}^{n_y}, u \in \mathbb{R}^{N n_u}}{\text{minimize}} \;\; \bar{J}[y_0, u] \tag{19}$$

s.t.:

$$y_0 - \xi(y_0, u) + J^\top \left(J\,J^\top\right)^{-1} c(\xi(y_0, u)) = 0, \tag{20}$$

where $\xi(y_0, u) = y(T)$ is a solution of the differential equation

$$\dot{y}(t) = f(y(t), u(t), 0)$$

parametrized by $y(0) = y_0$ and the discrete control u. The LICQ for the discrete minimization problem (19) - (20) is equivalent to the condition that the matrix $\left(\mathbb{I} - V\bar{A}, -V\bar{B}\right)$, with $\bar{B} = \frac{\partial \xi}{\partial u}$

has full row-rank. This condition is satisfied if the left block, i.e., the matrix $(\mathbb{I} - V\,\bar{A})$ has full rank.

For the NOCP problem (16) - (18) the proof is similar. The discrete counterpart for this optimal control problem reads as:

$$\underset{y_0 \in \mathbb{R}^{n_y}, u \in \mathbb{R}^{Nn_u}}{\text{minimize}} \quad \bar{J}[y_0, u] \tag{21}$$

s.t.:

$$Z^\top (y_0 - \xi(y_0, u)) = 0, \tag{22}$$

$$c(\xi(y_0, u)) = 0. \tag{23}$$

Then, the LICQ for this discrete minimization problem is equivalent to the condition that the matrix

$$\begin{pmatrix} Z^\top - Z^\top \bar{A}, & -Z^\top \bar{B} \\ J\bar{A}, & J\bar{B} \end{pmatrix} \in \mathbb{R}^{n_y \times (n_y + Nn_u)}$$

has full row-rank. This is satisfied if the left block

$$\begin{pmatrix} Z^\top - Z^\top \bar{A} \\ J\bar{A} \end{pmatrix} = \begin{pmatrix} Z^\top \\ J \end{pmatrix} \left(\mathbb{I} - V\,\bar{A}\right)$$

has full rank. Since the matrix $\begin{pmatrix} Z^\top \\ J \end{pmatrix}$ builds a basis and thus has full rank, this is true if the matrix $\left(\mathbb{I} - V\,\bar{A}\right)$ has full rank. The matrix V, specified as above, represents a projection on the null-space Z orthogonal to J. The assumptions of the Lemma include that the matrix $\left(\mathbb{I} - V\,\bar{A}\right)$ has full rank. This concludes the proof. □

Corollary 1. If Assumption 1 holds at the given point $(y(t), u(t))$ and the underlying linearized differential system is asymptotically stable on the null-space Z orthogonal to J then the LICQ is satisfied.

Proof: Due to its definition, the asymptotic stability of the linearized differential system on the null-space Z orthogonal to J implies that the eigenvalues of the projection of $\bar{A}$ on the null-space Z, which is defined by $V\bar{A}$, are less than one. Thus, the matrix $(\mathbb{I} - V\,\bar{A})$ has full rank. Application of Lemma 1 guarantees that the LICQ is satisfied. □

3. APPROXIMATE ROBUST OPTIMAL CONTROL FOR EQUATIONS WITH INVARIANTS

In order to address optimal control problems with uncertainties an approximate robust optimal control strategy based on Lyapunov differential equations is employed. In this section, we formulate a robust counterpart for the optimal control problems (13) - (15) and (16) - (18) taking in account the fact that the uncertainties w are contained in the bounded uncertainty set W defined by (6).

A similar worst-case formulation for uncertain optimization problem was developed in Ben-Tal and Nemirovski [1998], Ben-Tal et al. [2005]. In order to transfer these ideas to the periodic optimal control case we assume that for given periodic controls $u(t)$ there is a unique solution $\xi[t, u(\cdot), w]$ of the infinite periodic differential system with invariants

$$\begin{aligned} \dot{y}(\tau) \quad &= f(y(\tau), u(\tau), 0),\ \forall \tau \in [-\infty, T] \\ c(y(T)) \ &= \ 0. \end{aligned} \tag{24}$$

Then, the robust counterpart for the optimal control problem (1) - (5) can be formulated as

$$\begin{aligned} &\underset{y(\cdot), u(\cdot), T}{\text{minimize}} \ \bar{J}(y(\cdot), u(\cdot), T) \\ &\text{s.t.:} \ \max_{w(\cdot) \in W} h_i(\xi[t, u(\cdot), w], u(t)) \leq 0, \quad \forall t \in [0, T], \end{aligned} \tag{25}$$

where the constraints have to be satisfied for all indices $i \in [1, \ldots, n_h]$.

Due to the fact that there are no suitable numerical algorithms available in order to solve min-max robust optimal control problem (25), we employ some heuristics that allows us to approximately solve this optimal control problem. There exist several possibilities for the approximation of problem (25). In the present paper linearization techniques are applied (see Diehl et al. [2006], Houska and Diehl [2009]), although some approaches propose to use higher order terms for the approximation (see Nagy and Braatz [2004, 2007]).

In order to formulate the approximated robust counterpart for (1) - (5) we linearize both the differential dynamics and the path constraints h around a reference solution $y(t)$ for a given control $u(t)$ and fixed uncertainties $w(t)$. Then, the path constraints in (25) can be replaced by

$$\tilde{h}_i(y(t), u(t)) = h_i + \max_{w(\cdot) \in W} (C_i, H(t)w)_{\mathbb{R}^{n_y}} \tag{26}$$

with the linear operator $H(t) : w(t) \mapsto \delta y(t)$ defined by

$$\begin{aligned} &\delta\dot{y}(t) = A(t)\delta y(t) + B(t)w(t), \qquad \forall t \in [-\infty, T] \\ &\delta y(0) \, = \, \delta y(T), \qquad J\,\delta y(T) \, = \, 0, \end{aligned}$$

where we use the shorthands

$$\begin{aligned} A(t) \ &:= \ \frac{\partial f(y, u, 0)}{\partial y}, \ B(t) \ := \ \frac{\partial f(y, u, 0)}{\partial w}, \\ C(t) \ &:= \ \frac{\partial h(y, u)}{\partial y}. \end{aligned}$$

Introducing the adjoint operator $H^\star$ as

$$(C, H(t)w)_{\mathbb{R}^{n_y}} = (H^*(t)C, w)_{\mathbb{R}^{n_w}}$$

it was shown in Houska [2007] that the approximated path constraints $\tilde{h}_i$ in (26) can be calculated as:

$$\tilde{h}_i(y(t), u(t)) = h_i + \gamma \sqrt{C_i^\top P(t) C_i},$$

where the matrix valued function $P(t) \ := \ H(t)^* H(t)$ is a solution of the Lyapunov differential equation

$$\dot{P}(t) = A(t)P(t) + P(t)A(t)^\top + B(t)B(t)^\top,\ \forall t \in [0, T] \tag{27}$$

with periodic boundary conditions, $P(0) = P(T)$.

Depending on which approach for treating invariants in the nominal periodic system is employed, i.e. the projection approach or the null-space approach, the boundary conditions for the Lyapunov system (27) take different formulations, which are equivalent to each other. If the projection approach was applied in the nominal case, the boundary conditions for the Lyapunov system (27) take the form:

Lyapunov projected boundary conditions:

$$P(0) - VP(T)V = 0, \tag{28}$$

$$V = V^\top = \frac{\partial \phi}{\partial y}(y(T)) = I - J^\top \left(JJ^\top\right)^{-1} J. \tag{29}$$

In case the null-space approach was used for the reformulation of the nominal redundant boundary conditions, the corresponding Lyapunov periodic boundary conditions read as:

Lyapunov Null-space boundary conditions:

$$Z^\top \, (P(0) - P(T))\, Z \, = \, 0, \qquad Z^\top \, P(0)\, J^\top \, = \, 0, \tag{30}$$

$$J\, P(0)\, J^\top \, = \, 0, \qquad J\, Z \, = \, 0, \qquad Z^\top Z \, = \, \mathbb{I}. \tag{31}$$

In the following lemma the equivalence between these two types of Lyapunov boundary conditions is proved.

Lemma 2. The sets (28) - (29) and (30) - (31) of the boundary conditions for the Lyapunov differential equation (27) are equivalent.

Proof:

In order to prove the equivalence we apply a basis transformation, which is specified by the matrix $(Z\,\bar{Z})^\top$, to $P(0) - VP(T)V^\top = 0$. We define $Z \in \mathbb{R}^{n_y \times (n_y - n_c)}$ as the null-space of the derivative matrix J, i.e. $J\,Z = 0 \in \mathbb{R}^{n_c \times (n_y - n_c)}$, and $\bar{Z} \in \mathbb{R}^{n_y \times n_c}$ - an orthogonal complement to Z. $\bar{Z}$ can be chosen to be the row-space of J, i.e. $\bar{Z} = J^\top$.

$$\begin{pmatrix} Z^\top \\ J \end{pmatrix} \left(P(0) - VP(T)V^\top\right) (Z\,J^\top) =$$
$$\left(\begin{array}{c|c} Z^\top\,(P(0) - P(T))\,Z & Z^\top\,P(0)\,J^\top \\ \hline J\,P(0)\,Z & J\,P(0)\,J^\top \end{array}\right) = 0. \qquad (32)$$

Exploiting the symmetry property in (32) the periodicity conditions $P(0) - VP(T)V^\top = 0$ are equivalent to

$$\begin{aligned} Z^\top\,(P(0) - P(T))\,Z &= 0, \\ Z^\top\,P(0)\,J^\top = 0, \qquad J\,P(0)\,J^\top &= 0. \end{aligned} \qquad (33)$$

□

In the following lemma we establish the necessary and sufficient conditions to a) guarantee the existence of the solution of the Lyapunov differential equation (27) with the boundary conditions (28) - (29) or (30) - (31) and b) to provide the stability of the underlying differential system with invariants. This lemma is an extension of the Lyapunov lemma, cf. (Bolzern and Colaneri [1988]), applied to the specific boundary conditions.

Lemma 3. The periodic Lyapunov differential equation system (27) - (29) and (27), (30) - (31) has a unique and positive semi-definite solution $P(t) \succeq 0$, whose projection on the null-space Z of J is positive definite, i.e. $Z^\top P(t) Z \succ 0$, for all $t \in \mathbb{R}$, if and only if:

a). The projection $Z^\top X Z$ of the monodromy matrix $X := Y(T, 0)$ on the null-space Z is asymptotically stable.

b). The projection $Z^\top Q(T) Z$ of the reachability Grammian matrix $Q(T)$ is positive definite, i.e. $Z^\top Q(T) Z \succ 0$. The reachability Grammian matrix $Q(t) \in \mathbb{R}^{n_y \times n_y}$ is defined as:

$$Q(t) := \int_0^t Y(t,\tau) B(\tau) B(\tau)^\top Y(t,\tau)^\top d\tau.$$

The fundamental solution $Y : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{n_y \times n_y}$ is obtained as

$$\frac{\partial Y(t,\tau)}{\partial t} = A(t) Y(t,\tau) \quad \text{with} \quad Y(\tau,\tau) = 1, \qquad (34)$$

for all $t, \tau \in \mathbb{R}$. Note that due to the invariants holds: $J\,Y(zT, \tau) = 0$, for all $z \in \mathbb{Z}$.

Proof:

The function $P : \mathbb{R} \to \mathbb{R}^{n_y \times n_y}$ given by

$$P(t) := Y(t,0)\,P_0\,Y(t,0)^\top + Q(t) \qquad (35)$$

for all $t \in \mathbb{R}$ is the unique solution of the Lyapunov differential equation (27) with the initial condition $P(0) = P_0 \in \mathbb{R}^{n_y \times n_y}$. The boundary conditions (28) have the form

$$V\,P(T)\,V = V\,X\,P_0\,X^\top\,V + V\,Q(T)\,V = P_0. \qquad (36)$$

Applying a basis transformation $(Z\,J^\top)$ to the equation (36) and employing the fact that $V\,(Z\,J^\top) = Z\,J^\top$, we obtain the following relation

$$\bar{P}_0 = \bar{X}\,\bar{P}_0\,\bar{X} + \bar{Q}(T), \qquad (37)$$

where

$$(\bar{P}_0, \bar{X}, \bar{Q}(T)) = \begin{pmatrix} Z^\top \\ J \end{pmatrix} (P_0, X, Q(T))(Z\,J^\top).$$

Note that $J\,X = 0$ and $J\,Q(T) = 0$ due to the invariants. The equation in (37) is linear in $\bar{P}_0$, thus, using standard linear algebra arguments (see Bolzern and Colaneri [1988]) the positive definite solution $Z^\top\,P_0\,Z$ exists if and only if:

a). The matrix $Z^\top\,Q(T)\,Z$ is positive definite.

b). The eigenvalues of the projected monodromy matrix $Z^\top\,X\,Z$ are all contained in the open unit disc, i.e. the projection $Z^\top\,X\,Z$ is asymptotically stable.

Moreover, if $Z^\top P_0 Z$ is positive definite, it immediately follows from (35) and (37), that the projected matrix $Z^\top P(t) Z$ is positive definite for all $t > 0$, since $Q(t) \succeq 0$ for all $t \in [0, T]$ due to its construction.

Because of the property (34) of the fundamental solution $Y(t, \tau)$ the projections of the reachability matrix $Q(T)$ satisfy

$$J\,Q(T)\,Z = 0, \qquad J\,Q(T)\,J^\top = 0,$$

which directly implies

$$J\,P_0\,Z = 0, \qquad J\,P_0\,J^\top = 0$$

to be a solution of (37).

□

In summary, if the projection approach is employed in order to treat the invariants in the boundary conditions, the robust counterpart for the uncertain OCP (1) - (5) can be formulated as:

Robust projected OCP

$$\underset{y(\cdot),P(\cdot),u(\cdot),R,V,T}{\text{minimize}} \quad \bar{J}(y(\cdot), u(\cdot), T) \qquad (38)$$

s.t.:

$$\dot{y}(t) = f(y(t), u(t), 0), \qquad \forall t \in [0, T] \qquad (39)$$

$$\dot{P}(t) = A(t)P(t) + P(t)A(t)^\top + B(t)B(t)^\top, \qquad (40)$$

$$0 \geq h_i(y(t), u(t)) + \gamma\sqrt{C^i(t)P(t)C^i(t)^\top}, \qquad (41)$$

$$y(0) - \phi(y(T)) = 0, \qquad (42)$$

$$P(0) - VP(T)V^\top = 0, \qquad (43)$$

$$V = I - J^\top R J, \qquad J J^\top R = I. \qquad (44)$$

In case if the null-space approach was applied the robust counterpart for the optimal control problem (1) - (5) takes the form:

Robust Null-space OCP

$$\underset{y(\cdot),P(\cdot),u(\cdot),Z,T}{\text{minimize}} \quad \bar{J}(y(\cdot), u(\cdot), T) \qquad (45)$$

s.t.:

$$(39) - (41),$$

$$Z^\top\,(y(0) - y(T)) = 0, \qquad J\,Z = 0, \qquad (46)$$

$$c(y(T)) = 0, \qquad Z^\top\,Z = \mathbb{I}, \qquad (47)$$

$$Z^\top\,(P(0) - P(T))\,Z = 0, \qquad (48)$$

$$J\,P(0)\,Z = 0, \qquad J\,P(0)\,J^\top = 0, \qquad (49)$$

where the null-space matrix $Z \in \mathbb{R}^{n_y \times (n_y - n_c)}$ replaces R in (38) - (44) and is a new decision variable.

4. ILLUSTRATIVE EXAMPLE

The system considered for the illustrative example is an inverted pendulum mounted on a joint that can be moved in the *vertical* direction only. See Figure 2 for a sketch of the system. In has been observed in many experiments that the pendulum can be stabilized in an open-loop manner by imposing an oscillatory motion of the joint of the appropriate frequency and amplitude. In Houska [2011] this system was approximated by an elastic link and stability optimization was performed. In the example which we propose in this paper, the elastic link is replaced by an infinitely rigid link.

4.1 System model

An orthonormal fixed reference frame $\{e_x, e_z\}$ is defined, where e_z spans the vertical direction. The position of the mass of the pendulum m is given by $P_m = xe_x + ze_z$, and the position of the joint is given by $P_J = ve_z$. The pendulum arm is modeled as a rigid link that constrains the pendulum mass to evolve on the one-dimensional manifold defined by:

$$C = \frac{1}{2}\left(x^2 + (z-v)^2 - l^2\right) = 0, \tag{50}$$

with l being the length of the arm. The kinetic and potential energy functions of the system read:

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{z}^2\right), \quad V = m\,g\,z.$$

Defining the generalized coordinates $q = [x \;\; z]^T$, and using the Lagrange function $\mathscr{L} = T - V - \lambda\, C$, the pendulum dynamics can be computed using the Lagrange equation $\frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{q}} - \frac{\partial \mathscr{L}}{\partial q} = F$, resulting in the index-three differential algebraic system:

$$m\,\ddot{q} + \begin{bmatrix} 0 \\ m\,g \end{bmatrix} + \lambda \begin{bmatrix} x \\ z-v \end{bmatrix} = F, \quad C = 0, \tag{51}$$

where F is the vector of generalized forces acting on the system. For any $t_0 \in \mathbb{R}$, using $\ddot{C}(t) = 0, \dot{C}(t_0) = 0, C(t_0) = 0$, equation (51) can be reformulated as the index-one differential algebraic system (together with the consistency conditions):

$$\begin{bmatrix} m\mathbb{I}_2 & C_q^T \\ C_q & 0 \end{bmatrix} \begin{bmatrix} \ddot{q} \\ \lambda \end{bmatrix} = \begin{bmatrix} F - V_q \\ -\dot{C}_q\,\dot{q} - \dot{C}_v\,\dot{v} - \dot{C}_{\dot{v}}\,u \end{bmatrix},$$

$$C(t_0) = 0, \qquad \dot{C}(t_0) = \left(C_q\,\dot{q} + C_v\,\dot{v}\right)_{t=t_0} = 0,$$

where $\dot{C}_q = \frac{\partial \dot{C}}{\partial q}$, $\dot{C}_v = \frac{\partial \dot{C}}{\partial v}$, and $\dot{C}_{\dot{v}} = \frac{\partial \dot{C}}{\partial \dot{v}}$. The control function u in the model is chosen to be the acceleration $\ddot{v}$ of the joint. This system can equivalently be rewritten as:

$$\begin{bmatrix} m & 0 & x \\ 0 & m & z-v \\ x & z-v & 0 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{z} \\ \lambda \end{bmatrix} = \begin{bmatrix} F_x \\ F_z - m\,g \\ -\dot{v}^2 + 2\,\dot{v}\,\dot{z} - \dot{x}^2 - \dot{z}^2 - u\,(v-z) \end{bmatrix},$$

$$C(t_0) = \frac{1}{2}\left(x^2 + (z-v)^2 - l^2\right)_{t=t_0} = 0, \tag{52}$$

$$\dot{C}(t_0) = \left(\dot{x}\,x + \dot{v}\,(v-z) - \dot{z}\,(v-z)\right)_{t=t_0} = 0,$$

where $F = [F_x \;\; F_z]^T$. The generalized forces F are the viscous terms:

$$F = -\mu\dot{q}.$$

Because λ appears linearly in (52) it can be eliminated, so that the system (52) can be reformulated as a differential system.

Table 1. Model parameters

Parameter	Value
m	1 [kg]
l	1 [m]
μ	2 $[\mathrm{N/m \cdot s^{-1}}]$
$x_{\max}$	0.2 [m]
κ	0.01
g	9.81

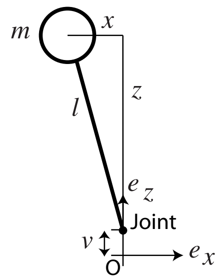

Fig. 2. Sketch of the pendulum

Defining $y = [x \;\; z \;\; \dot{x} \;\; \dot{z} \;\; v \;\; \dot{v}]^T$, and $c = \left[C \;\; \dot{C}\right]^T$, the equations of motion (52) take the form:

$$\dot{y} = f(y, u, w), \qquad c(0) = 0. \tag{53}$$

The right-hand side function f becomes

$$f(y, u, w) = (\dot{x},\, \dot{z},\, \ddot{x},\, \ddot{z},\, \dot{v},\, u)^\top,$$

with $\ddot{x} = -(\lambda\,x + \mu\,\dot{x})/m + w$ and $\ddot{z} = -(\lambda\,(z-v) + \mu\,\dot{z})/m - g$. Numerical values for the physical constraints involved in the problem are given in Table 1. The function w is assumed to be a time-dependent uncertainty which acts on the mass in the horizontal direction.

4.2 Optimization problem

The proposed optimization problem seeks at finding an open-loop stable periodic orbit, a control input u and the optimal period time $T_p \in \mathbb{R}_{++}$, such that the pendulum stays in its inverted position. Moreover, we want to robustify the path constraints, defined by

$$-x_{\max} \leq x \leq x_{\max}, \qquad \text{for all t} \in [0, T]$$

with respect to uncertainties w. For this purpose we choose an objective function of the Lagrange type, defined by

$$L(y, u, T_p, \gamma) \;=\; -\gamma + \frac{\kappa}{T_p}\int_0^{T_p} \|u(t)\|^2 dt,$$

where γ is a robustification parameter, which is included into the objective function. κ is a regularization parameter. The period time T_p is a decision variable.

In order to construct an approximate robust optimal control formulation for the inverted pendulum example, we have first to introduce the matrix valued function $P(t)$ as a solution of the corresponding Lyapunov differential equation and second to add the robustness margins to the inequality path constraints. To reduce the number of additional differential states we use the symmetry property of the matrix $P(t)$ and assume that the states v and $\dot{v}$ are not affected by uncertainties. Thus, altogether we have 16 differential states. Summarizing, the approximate robust and stable counterpart for the optimal control of the inverted pendulum, if the projection approach is employed in order to reformulate the redundancy in the boundary constraints, can be defined as:

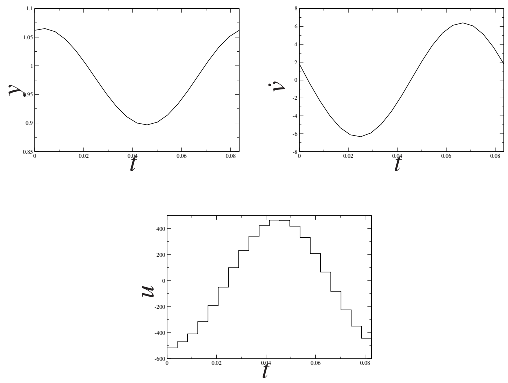

Fig. 3. Left: optimal trajectory of the mass point; right: optimal velocity of the joint; bottom: optimal control input

$$
\begin{aligned}
\min_{y,u,T_p,V,J,\gamma}\quad & -\gamma + \frac{1}{T_p}\int_0^{T_p}\left(\kappa\,\|u(t)\|^2 + \operatorname{tr} P(t)\right)dt \\
\text{s.t.}\quad & \dot{y} = f(y,u,w), \qquad c(0) = 0, \\
& \dot{P} = f_y P + P f_y^T + f_w f_w^T, \\
& y(0) - \phi(y(T)) = 0, \\
& P(0) - V(J)P(T)V^\top(J) = 0, \\
& V(J) = \mathbb{I} - J^\top (JJ^\top)^{-1} J, \qquad J = \frac{\partial c}{\partial y(T)}, \\
& \gamma\sqrt{P_{11}} - x_{\max} \le x \le x_{\max} - \gamma\sqrt{P_{11}},
\end{aligned} \tag{54}
$$

where $f_y = \frac{\partial f}{\partial y}$ and $f_w = \frac{\partial f}{\partial w}$. In order to numerically solve the robustified optimal control problem we use the open source software `ACADO Toolkit` by Houska et al. [2011], which implements multiple shooting with an SQP method. The control is discretized by piecewise constant functions. In order to integrate the differential system and to compute the sensitivities, that are needed for the optimization procedure, we use a special type of integrator, the so called `LYAPINT` integrator, which was developed in Sternberg et al. [2011] and represents a new feature within the `ACADO Toolkit`. The `LYAPINT` integrator efficiently exploits a particular structure of the Lyapunov differential equation and thus reduces the overall computation time. One SQP iteration takes 5.3 seconds.

For the proposed inverted pendulum example it was possible to find an open-loop stable and robust solution. The maximal confidence level is $\gamma = 0.278$. The optimal cycle duration is $T = 82.67$ ms. The spectral radius of the projected monodromy matrix is $\rho(Z^\top X Z) = 0.89259 < 1$. The optimized robust and open-loop stable trajectories for the y-component of the mass, the velocity $\dot{v}$ of the joint and the optimal control input u are shown in Figure 3. Alternatively, we can also apply the null-space approach in order to reformulate the redundant boundary conditions. The numerical results are similar, but the drawback of this approach is that the null-space matrix Z is not unique. The locally optimal solution for the robust projected OCP (54) turns out also to be a local solution for the robust OCP with redundant boundary conditions. Nevertheless, in general if a local solution for the robust projected (Null-space) OCP is not feasible for the redundant OCP, one possibility to find a local solution for the redundant OCP is to add some penalization terms into the objective function and start optimization from the given solution as an initial guess.

5. CONCLUSION

In the present paper, we have discussed a strategy to numerically deal with uncertain periodic optimal control problems with invariants as they arise in the context of high-index differential algebraic systems. In the first part of the paper, we have concentrated on a projection and a null-space method which avoid the redundancies in the periodic boundary conditions with respect to the invariant manifold. Lemma 1 has shown that both methods lead to a set of numerically well-behaved constraints in the sense that the linear independence constraint qualification can be obtained under mild regularity assumptions.

The second part of the paper has focused on the uncertainty which might be present in the dynamic system. Here, the flow of the uncertainty through the dynamic system can approximately be computed by propagating a Lyapunov differential equation. Note that this Lyapunov differential equation has been used to impose both approximate robustness with respect to constraints as well as the nominal open-loop stability of the periodic system. Here, the conditions for the asymptotic stability of periodic orbits have been discussed in Lemma 3 assuming that the invariant manifold is given. Finally, we have shown how the approach can be used to find open-loop stable orbits of an inverted pendulum.

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