NMPC based on Huber Penalty Functions to Handle Large Deviations of Quadrature States

Sébastien Gros and Moritz Diehl

***Abstract*— Nonlinear Model Predictive Control for mechanical applications is often used to perform the tracking of time-varying reference trajectories, and is typically implemented using penalty functions based on L_2 norms. Controllers for mechanical systems, however, are often required to handle large deviations from the reference trajectory. In such cases, it has been observed that NMPC schemes based on L_2 norms can have undesirably aggressive behaviors. Heuristics can be developed to tackle these issues, but they require intricate and non-systematic tuning procedures. This paper proposes an NMPC scheme based on Huber penalty functions to handle large deviation of quadrature state from its reference, offering an intuitive and easy-to-tune alternative. The behavior of the proposed NMPC scheme is analysed, and the conditions for its nominal stability are established. The control scheme is illustrated on a simulated crane.**

***Keywords* : nonlinear model predictive control, Huber penalty function, large deviation from the reference, mechanical systems.**

I. Introduction

Nonlinear model predictive control (NMPC) is an effective way of tackling problems with constraints and nonlinear dynamics. NMPC re-calculates at every sampling instant a control policy that minimizes a penalty function defined over a horizon window in the future. Thought the properties of NMPC have been studied for the general class of $\mathcal{K}$ penalty functions [8], [4], [6], [11], [10], in practice L_2-norms are preferred because they are straightforward to implement, can be efficiently treated using Gauss-Newton hessian approximations, and yield controllers having an intuitive behavior.

NMPC has been extensively used in the process industry [1], where it is often assumed that the error between the system state and its fixed reference is relatively small. However, NMPC is more and more used for mechanical applications. Controllers for mechanical applications are often required to track infeasible trajectories, handle large reference jumps, or perform obstacle avoidance, potentially resulting in large deviations form their reference.

In such situations, it has been observed that NMPC based on L_2-norms can become very aggressive, i.e. it yields a significant activation of the inputs bounds and state constraints, and taps strongly into the system nonlinearities. The latter often requires an expensive line search to ensure the convergence of the underlying Newton-type scheme. More crucially, in an L_2-norms NMPC scheme, the penalty associated to a state deviating strongly from its reference completely dominates the cost function, so that competing penalties are almost disregarded. This is especially a problem when such competing penalties must weigh in the cost function regardless of the deviation from the reference (this is e.g. the case for penalties associated to the alleviation of structural fatigue [13]).

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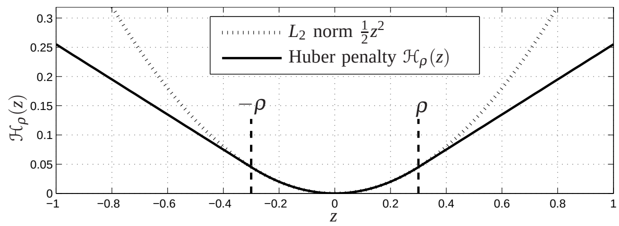

Fig. 1. Huber penalty function $\mathcal{H}_\rho(z)$ for $\rho = 0.3$.

Heuristics such as smoothing and saturation of the regulation error, or a temporary reduction of the L_2-norms weighting matrices can be used to tackled such issues [7]. However these heuristics can be difficult to set up, and can result in intricate and non-intuitive closed-loop behaviors. As an alternative, this paper proposes a systematic way of dealing with large tracking errors of quadrature states, i.e. states that do not enter the system dynamics (see Section II), using the Huber penalty function [3] $\mathcal{H}_\rho : \mathbb{R} \to \mathbb{R}_+$ given by (see Fig. 1):

$$\mathcal{H}_\rho(z) = \begin{cases} \frac{1}{2}z^2, & |z| \leq \rho \\ \rho\left(|z| - \frac{1}{2}\rho\right), & |z| > \rho \end{cases}, \; z \in \mathbb{R}, \tag{1}$$

The Huber penalty function $\mathcal{H}_\rho(z)$ is equivalent to a L_2 norm within the region $[-\rho, \rho]$, and to a L_1 norm outside. As a result, for small tracking errors the NMPC based on the Huber penalty function is equivalent to a classical L_2-norm NMPC, but for large errors the gradient of the cost function based on the Huber penalty function is smaller than for the L_2-norm cost function, hence reducing the incentive for strong control actions. The NMPC based on the Huber penalty function can therefore be tuned to have a soft response to large tracking errors, without sacrificing its responsiveness for small deviation from the reference. The zone where the NMPC behaves as a classical L_2-norm NMPC can be directly adjusted via parameter ρ, allowing a straightforward tuning of the proposed scheme.

The paper is organized as follows. Section II details the proposed NMPC scheme. Section III-A proposes an analysis of the behavior of the proposed scheme for large tracking errors of the quadrature state, Section III-B establishes its nominal stability, Section IV presents an illustrative example.

II. NMPC BASED ON THE HUBER PENALTY FUNCTION

For sake of brevity, only the case of a scalar quadrature state $q \in \mathbb{R}$ is considered here. The following form of discrete system will be studied:

$$x_{i+1} = f(x_i, u_i), \qquad q_{i+1} = q_i + \mathcal{I}(x_i), \tag{2}$$

where $f : \mathbb{R}^n \times \mathbb{R}^{n_u} \to \mathbb{R}^n$ and $\mathcal{I} : \mathbb{R}^n \to \mathbb{R}$ are smooth, nonlinear functions representing the system dynamics, $[x, q] \in \mathbb{R}^{n+1}$ is the $n+1$-dimensional system state vector, $u \in \mathbb{R}^{n_u}$ the n_u-dimensional input vector. In the following, the index i is reserved for the current time instants, while the index k is used for the predicted times.

A classical form of L_2-norm NMPC scheme with terminal equality constraints for system (2) reads:

$$\begin{aligned}
\mathcal{P}_2(x_i, q_i, N) = \min_{s,u,I} \quad & \frac{1}{2}\sum_{k=0}^{N} I_k^2 + \underbrace{\frac{1}{2}\sum_{k=0}^{N} s_k^T Q s_k + \frac{1}{2}\sum_{k=0}^{N-1} u_k^T R u_k}_{=\Phi(s,u)} \\
s.t. \quad & s_{k+1} - f(s_k, u_k) = 0, \quad s_0 - x_i = 0, \\
& I_{k+1} - I_k - \mathcal{I}(s_k) = 0, \quad I_0 - q_i = 0, \\
& s_N = 0, \quad I_N = 0, \\
& h(s_k, u_k) \le 0, \quad k = 0, \ldots, N-1. \qquad (3)
\end{aligned}$$

where h stands for the set of state and input constraints, Q and R are user-defined weighting matrices, I and s are the predicted quadrature state q and system state x respectively. Consider the following alternative NMPC scheme where the quadratic penalty function $\sum_{k=0}^{N} I_k^2$ in the NMPC scheme (3) is replaced by the Huber penalty function $\sum_{k=0}^{N} \mathcal{H}_\rho(I_k)$, and the terminal constraint $I_N = 0$ is removed:

$$\begin{aligned}
\mathcal{P}_H(x_i, q_i, N) = \min_{s,u,I} \quad & \sum_{k=0}^{N} \mathcal{H}_\rho(I_k) + \Phi(s, u), \\
s.t. \quad & s_{k+1} - f(s_k, u_k) = 0, \quad s_0 - x_i = 0, \\
& I_{k+1} - I_k - \mathcal{I}(s_k) = 0, \quad I_0 - q_i = 0, \\
& s_N = 0, \\
& h(s_k, u_k) \le 0, \quad k = 0, \ldots, N-1. \qquad (4)
\end{aligned}$$

Using the standard, smooth reformulation of the Huber penalty function (see [3] p. 299 for details), the resulting NMPC scheme reads :

$$\begin{aligned}
\mathcal{P}_H(x_i, q_i, N) = \min_{s,u,I,\mu,\nu} \quad & \sum_{k=0}^{N} \left(\rho \nu_k + \frac{1}{2}\mu_k^2 \right) + \Phi(s, u) \\
s.t. \quad & s_{k+1} - f(s_k, u_k) = 0, \quad s_0 - x_i = 0 \\
& I_{k+1} - I_k - \mathcal{I}(s_k) = 0, \quad I_0 - q_i = 0 \\
& s_N = 0, \\
& h(s_k, u_k) \le 0, \quad k = 0, \ldots, N-1 \\
& \nu_k \ge 0, \quad k = 0, \ldots, N \\
& -\mu_k - \nu_k \le I_k \le \mu_k + \nu_k, \\
& 0 \le \mu_k \le \rho. \qquad (5)
\end{aligned}$$

In addition, the inequality constraints:

$$\mu_N - \mu_k \le 0, \qquad \nu_N - \nu_k \le 0, \qquad k = 0, \ldots, N-1 \tag{6}$$

are introduced as a replacement of the terminal constraint $I_N = 0$. It should be observed that (5)-(6) implements a Huber penalty on I_k if and only if the constraints (6) are not (strictly) active. Otherwise, an extra penalty is added for $|I_N|$ not being the lowest value of the trajectory.

Constraints (6) can be understood as a relaxation of the terminal equality constraint $I_N = 0$, hence avoiding the aggressive control actions required to satisfy $I_N = 0$ for large initial conditions $|q_i|$. The effect of the constraints (6) on the stability of the NMPC scheme (5) will be developed in Section III-B.

III. ANALYSIS OF THE HUBER NMPC SCHEME

This section establishes some fundamental properties of the NMPC scheme (5)-(6). First some notations are introduced. In the following, the equality constraints in (5) are lumped together in:

$$g_s = \begin{bmatrix} s_0 - x \\ s_1 - f(s_0, u_0) \\ \cdots \\ s_N - f(s_{N-1}, u_{N-1}) \\ s_N \end{bmatrix}, \; g_I = \begin{bmatrix} I_0 - q_i \\ I_1 - I_0 - \mathcal{I}(s_0) \\ \cdots \\ I_N - I_{N-1} - \mathcal{I}(s_{N-s}) \end{bmatrix},$$

and the inequality constraints in:

$$h_I = \begin{bmatrix} I_k - \mu_k - \nu_k \\ -I_k - \mu_k - \nu_k \end{bmatrix}, \quad h_S = \begin{bmatrix} \mu_k - \rho \\ -\nu_k \end{bmatrix},$$
$$h_L = \begin{bmatrix} \mu_N - \mu_k \\ \nu_N - \nu_k \end{bmatrix}. \tag{7}$$

We define the vector of decision variables w as $w = [I_0 \ldots I_N \; \nu_0 \ldots \nu_N \; \mu_0 \ldots \mu_N \; s_0 \ldots s_N \; u_0 \ldots u_{N-1}]^T \in \mathbb{R}^d$, with $d = N(n + n_u + 3) + n + 3$ and $\overline{w}$ as the solution of (5)-(6). In the following, the open sets $\vartheta_+ = \{ w \in \mathbb{R}^d | I_k > \rho, \forall k \}$, $\vartheta_- = \{ w \in \mathbb{R}^d | I_k < -\rho, \forall k \}$ and $\vartheta = \vartheta_+ \cup \vartheta_-$ will be used, as well as the notations $\mathbf{1}_N = [1 \; 1 \ldots]^T \in \mathbb{R}^N$, and $\mathbf{0}_N$, $\mathbf{0}_{N\times M}$, $\mathbf{I}_N$ for a $N \times N$ matrix of zeros, $N \times M$ matrix of zeros and $N \times N$ identity matrix, respectively. The subscripts will be sometimes dropped when the dimensions are clear form the context. The cost function J will be defined as:

$$J(w) = \sum_{k=0}^{N} \left(\frac{1}{2}\mu_k^2 + \rho \nu_k \right) + \Phi(s, u).$$

The cost and constraints sensitivities with respect to w read:

$$\nabla J = \begin{bmatrix} \mathbf{0}_{N+1} \\ \rho \mathbf{1}_{N+1} \\ \mu \\ Qs \\ Ru \end{bmatrix}, \quad \nabla h_L = \begin{bmatrix} \mathbf{0}_{N+1\times N} & \mathbf{0}_{N+1\times N} \\ \mathbf{0}_{N+1\times N} & \mathcal{M} \\ \mathcal{M} & \mathbf{0}_{N+1\times N} \\ \mathbf{0} & \mathbf{0} \\ \ldots & \ldots \end{bmatrix} \tag{8}$$

$$\nabla h_I = \begin{bmatrix} \mathbf{I}_{N+1} & -\mathbf{I}_{N+1} \\ -\mathbf{I}_{N+1} & -\mathbf{I}_{N+1} \\ -\mathbf{I}_{N+1} & -\mathbf{I}_{N+1} \\ \mathbf{0} & \mathbf{0} \\ \ldots & \ldots \end{bmatrix}, \nabla h_S = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I}_{N+1} \\ \mathbf{I}_{N+1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \ldots & \ldots \end{bmatrix},$$

where $\mathcal{M} = \begin{bmatrix} -\mathbf{I}_N \\ \mathbf{1}_N^T \end{bmatrix}$. Subsections III-A and III-B establish some fundamental properties of the NMPC controller (5)-(6).

A. Insensitivity of the solution for large deviations

In this subsection, it will be established that the control policy delivered by the NMPC scheme (5)-(6) becomes insensitive to q_i for large values of $|q_i|$. This statement is further discussed at the end of the following Lemma.

Lemma 1: for any given initial conditions x_i, the solution $\bar{s}$, $\bar{u}$ to the Huber NMPC scheme (5)-(6) is insensitive to q_i if $w \in \vartheta_+$ or $w \in \vartheta_-$.

Proof: For $\bar{w} \in \vartheta_+$ or $\bar{w} \in \vartheta_-$, the constraint $-\bar{\nu} \leq 0$ is not active, and $\mathrm{sign}(\bar{I}_k) = \mathrm{sign}(q_i), \forall k$. Then the set of possibly (but not necessarily) active constraints is:

$$h_\vartheta = \begin{bmatrix} \gamma\bar{I}_k - \bar{\mu}_k - \bar{\nu}_k \\ \bar{\mu}_k - \rho \\ \bar{\mu}_N - \bar{\mu}_k \\ \bar{\nu}_N - \bar{\nu}_k \\ h(\bar{s}_j, \bar{u}_j) \end{bmatrix}, \quad k = 0,\dots,N, \; j = 0,\dots,N-1.$$

where $\gamma = \mathrm{sign}(q_i)$. Define h_A the subset of active constraints in h_ϑ, and the Lagrange function:

$$\mathcal{L} = J + \lambda_s^T g_s + \lambda_I^T g_I + \xi_A^T h_A.$$

A solution $\bar{w}$ of (5)-(6) satisfies the KKT conditions:

$$\nabla\mathcal{L} = 0, \quad g_s = 0, \quad g_I = 0, \quad h_A = 0, \tag{9}$$

for some Lagrange multipliers $\bar{\lambda}_s$, $\bar{\lambda}_I$ and $\bar{\xi}_A \geq 0$, where ∇ is the derivative operator with respect to the decision variables w. The sensitivity of (9) with respect to q_i is then given by:

$$H\frac{d\bar{w}}{dq_i} + \nabla_{wq_i}\mathcal{L} + \nabla g_I \frac{d\bar{\lambda}_I}{dq_i} + \nabla_w g_s \frac{d\bar{\lambda}_s}{dq_i} + \nabla h_A \frac{d\bar{\xi}_A}{dq_i} = 0, \tag{10}$$

$$\nabla g_I^T \frac{d\bar{w}}{dq_i} + \nabla_{q_i} g_I = 0, \tag{11}$$

$$\nabla g_s^T \frac{d\bar{w}}{dq_i} = 0, \tag{12}$$

$$\nabla h_A^T \frac{d\bar{w}}{dq_i} = 0, \tag{13}$$

where $H = \nabla_{ww}\mathcal{L}$ is the Hessian of the Lagrange function $\mathcal{L}$. Defining:

$$\delta^T = \begin{bmatrix} \mathbf{1}_{N+1}^T & \gamma\mathbf{1}_{N+1}^T & 0 & \dots & 0 \end{bmatrix},$$

it will be established in the following that $\frac{d\bar{w}}{dq_i} = \delta$ is a solution of (10)-(13). The Lagrange function depends linearly on I, q_i and ν, hence it can be verified that $\nabla_{wq_i}\mathcal{L} = 0$ and:

$$H = \begin{bmatrix} \mathbf{0}_{2N+2} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{N+1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \nabla_{ss}\mathcal{L} & \nabla_{su}\mathcal{L} \\ \mathbf{0} & \mathbf{0} & \nabla_{us}\mathcal{L} & \nabla_{uu}\mathcal{L} \end{bmatrix}.$$

It follows that $H\delta = 0$ and (10) holds with $\frac{d\bar{\lambda}_I}{dq_i} = 0$, $\frac{d\bar{\lambda}_s}{dq_i} = 0$ and $\frac{d\bar{\xi}_A}{dq_i} = 0$. Observing that g_I is independent of ν and

$$\nabla g_I = \begin{bmatrix} \nabla_I g_I \\ \mathbf{0}_{2N+2} \\ \nabla_s g_I \\ \mathbf{0}_{N+1} \end{bmatrix}, \quad \nabla_I g_I = \begin{bmatrix} 1 & -1 & 0 & \dots \\ 0 & 1 & -1 & \dots \\ 0 & 0 & 1 & \dots \\ & & \dots & \end{bmatrix},$$

it can be verified that:

$$\left(\delta^T \nabla g_I\right)^T = \left(\mathbf{1}_{N+1}^T \nabla_I g_I\right)^T = \begin{bmatrix} 1 \\ 0 \\ \dots \end{bmatrix}, \quad \nabla_{q_i} g_I = \begin{bmatrix} -1 \\ 0 \\ \dots \end{bmatrix}$$

and it follows that (11) holds. Moreover, g_s is independent of I and ν, therefore $\nabla g_s^T \delta = 0$ and (12) holds.

The sensitivity of h_ϑ is given by:

$$\nabla h_\vartheta = \begin{bmatrix} \gamma\mathbf{I}_{N+1} & \mathbf{0}_{N+1} & \mathbf{0}_{N+1\times N} & \mathbf{0}_{N+1\times N} & \mathbf{0} \\ -\mathbf{I}_{N+1} & \mathbf{0}_{N+1} & \mathbf{0}_{N+1\times N} & \mathcal{M} & \mathbf{0} \\ -\mathbf{I}_{N+1} & \mathbf{I}_{N+1} & \mathcal{M} & \mathbf{0}_{N+1\times N} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \nabla_s h \\ \dots & \dots & \dots & \dots & \nabla_u h \end{bmatrix}$$

Since $\mathbf{1}_{N+1}^T\mathcal{M} = 0$, it can be observed that $\delta^T\nabla h_\vartheta = 0$, so that (13) holds on any active set $A \subseteq \{0,\dots,N-1\}$, then $\frac{d\bar{w}}{dq_i} = \delta$, and

$$\frac{d\bar{I}}{dq_i} = \mathbf{1}_{N+1}, \quad \frac{d\bar{u}}{dq_i} = 0, \quad \frac{d\bar{s}}{dq_i} = 0$$

□

Discussion: lemma 1 entails that there is a *limit* to how far the deviation of q_i from its reference can impact the control policy of the NMPC, i.e. the domination of the penalty associated to q_i over the competing penalties (lumped together in $\boldsymbol{\Phi}$) is limited. In contrast, in a classical L_2-norm NMPC scheme the domination of the penalty associated to q_i is unlimited.

B. Nominal stability

In this section, the nominal stability of the NMPC scheme (5)-(6) is investigated. In the following, the notation $\bar{I}(x_i,q_i,N)$, $\bar{u}(x_i,q_i,N)$ and $\bar{s}(x_i,q_i,N)$ will be used for the solution of (5)-(6) corresponding to the initial values x_i, q_i, and horizon N, yielding the optimal cost function $\mathcal{P}_{\mathrm{H}}(x_i,q_i,N)$.

The following Lemma establishes that under some conditions, $\mathcal{P}_{\mathrm{H}}(x_i,q_i,N)$ is a Lyapunov function of system (2) controlled by the NMPC scheme (5)-(6). First, three key assumptions are introduced:

1) $\mathcal{J}(0) = 0$ and $f(0,0) = 0$,
2) the inequality constraints $h(s,u) \leq 0$ are not active at $s = 0, u = 0$,
3) the Quadratic Programm (QP):

$$\begin{aligned} \min_{\eta,\Xi} \quad & \frac{1}{2}\eta^T\eta + \Xi \\ s.t. \quad & \eta^T \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \nabla_{I,s,u} g_I & \nabla_{I,s,u} g_s \end{bmatrix} = 0, \quad \eta^T \begin{bmatrix} \mathcal{M} \\ \mathbf{0} \end{bmatrix} \leq 0 \\ & \eta^T \begin{bmatrix} \mathbf{1}_{N+1} \\ \mathbf{0} \end{bmatrix} \leq \Xi, \quad \eta^T \begin{bmatrix} -\mathbf{I}_{N+1} \\ \mathbf{I}_{N+1} \\ \mathbf{0} \end{bmatrix} \leq 0, \end{aligned} \tag{14}$$

solved at $u = 0$, $s = 0$ admits a solution $\eta \in \mathbb{R}^{(n+n_u+2)N+n+2}$, $\Xi \in \mathbb{R}$ with $\Xi < 0$.

Assumption 3 is discussed at the end of this section. In the following, the partition $\eta^T = \begin{bmatrix} \eta_S^T & \eta_I^T & \eta_s^T & \eta_u^T \end{bmatrix}$ will be used, with $\eta_S \in \mathbb{R}^{N+1}$, $\eta_I \in \mathbb{R}^{N+1}$, $\eta_s \in \mathbb{R}^{n(N+1)}$ and $\eta_u \in \mathbb{R}^{n_u N}$.

Lemma 2: Let Ω be the set of feasible initial conditions (x_0, q_0) for problem (5)-(6), then under assumptions 1-3, the optimal cost function $\mathcal{P}_H$ is a Lyapunov function for the nominal closed-loop system:

$$x_{i+1} = f(x_i, \bar{u}_0), \qquad q_{i+1} = q_i + \mathcal{I}(x_i)$$

in the set Ω, where $\bar{u}_0$ is the first element of the sequence $\bar{u}(x_i, q_i, N)$.

Proof: first an upper bound for $\mathcal{P}_H(x_{i+1}, q_{i+1}, N) - \mathcal{P}_H(x_i, q_i, N)$ is computed. In the absence of perturbation and model error, the initial values at time $i+1$ match the predicted trajectories, i.e.:

$$x_{i+1} = \bar{s}_1(x_i, q_i, N), \qquad q_{i+1} = \bar{I}_1(x_i, q_i, N).$$

Then consider the shifted trajectories (where the arguments (x_i, q_i, N) are omitted):

$$s_S = \begin{bmatrix} \bar{s}_1 \\ \dots \\ \bar{s}_N \\ \mathbf{0}_{n\times 1} \end{bmatrix}, \quad I_S = \begin{bmatrix} \bar{I}_1 \\ \dots \\ \bar{I}_N \\ \bar{I}_N \end{bmatrix}, \quad u_S = \begin{bmatrix} \bar{u}_1 \\ \dots \\ \bar{u}_{N-1} \\ \mathbf{0}_{n_u\times 1} \end{bmatrix}, \tag{15}$$

$$\mu_S = \begin{bmatrix} \bar{\mu}_1 \\ \dots \\ \bar{\mu}_N \\ \bar{\mu}_N \end{bmatrix}, \quad \nu_S = \begin{bmatrix} \bar{\nu}_1 \\ \dots \\ \bar{\nu}_N \\ \bar{\nu}_N \end{bmatrix},$$

which are *feasible* for problem (5)-(6) with initial values x_{i+1}, q_{i+1}. They yield the cost function $J_i \geq \mathcal{P}_H(x_{i+1}, q_{i+1}, N)$ given by:

$$J_i = \mathcal{P}_H(x_i, q_i, N) - \Phi(x_i, \bar{u}_0) - \left(\frac{1}{2}\bar{\mu}_0^2 + \rho\bar{\nu}_0\right) + \left(\frac{1}{2}\bar{\mu}_N^2 + \rho\bar{\nu}_N\right).$$

It follows that:

$$\mathcal{P}_H(x_{i+1}, q_{i+1}, N) - \mathcal{P}_H(x_i, q_i, N) \leq \Delta,$$

with

$$\Delta = -\Phi(x_i, \bar{u}_0) - \frac{1}{2}\left(\bar{\mu}_0^2 - \bar{\mu}_N^2\right) - \rho\left(\bar{\nu}_0 - \bar{\nu}_N\right).$$

Because of the inequality constraints (6), $\bar{\mu}_0 \geq \bar{\mu}_N$, $\bar{\nu}_0 \geq \bar{\nu}_N$ hold, and since $\Phi \geq 0$, it follows that $\Delta \leq 0$.

Next it is established that $\Delta = 0 \Rightarrow x_i = 0, q_i = 0$. Clearly:

$$\Delta = 0 \quad \Rightarrow \quad x_i = 0,\ \bar{\mu}_0 = \bar{\mu}_N,\ \bar{\nu}_0 = \bar{\nu}_N \tag{16}$$

holds. Then the trajectory $u = 0$, $s = 0$, $I = q_i$, $\mu_k = \bar{\mu}_0$, $\nu_k = \bar{\nu}_0$ is feasible, hence the optimal cost function $\mathcal{P}_H(x_i, q_i, N)$ is upper bounded by:

$$\mathcal{P}_H(x_i, q_i, N) \leq \Phi(0,0) + \sum_{k=0}^{N} \frac{1}{2}\bar{\mu}_0^2 + \rho\bar{\nu}_0 = (N+1)\left(\frac{1}{2}\bar{\mu}_0^2 + \rho\bar{\nu}_0\right). \tag{17}$$

Using the inequality constraints (6), $\bar{\mu}_k \geq \bar{\mu}_N$, $\bar{\nu}_k \geq \bar{\nu}_N$, (16), $\bar{\mu}_0 = \bar{\mu}_N$, $\bar{\nu}_0 = \bar{\nu}_N$, and (17):

$$(N+1)\left(\frac{1}{2}\bar{\mu}_0^2 + \rho\bar{\nu}_0\right) \geq \mathcal{P}_H(x_i, q_i, N) =$$

$$\Phi(\bar{s}, \bar{u}) + \sum_{k=0}^{N} \frac{1}{2}\bar{\mu}_k^2 + \rho\bar{\nu}_k \geq \Phi(\bar{s}, \bar{u}) + \sum_{k=0}^{N} \frac{1}{2}\bar{\mu}_N^2 + \rho\bar{\nu}_N =$$

$$\Phi(\bar{s}, \bar{u}) + \sum_{k=0}^{N} \frac{1}{2}\bar{\mu}_0^2 + \rho\bar{\nu}_0 = \Phi(\bar{s}, \bar{u}) + (N+1)\left(\frac{1}{2}\bar{\mu}_0^2 + \rho\bar{\nu}_0\right),$$

and it follows that $\Phi(\bar{s}, \bar{u})$=0, hence if $\Delta = 0$ then the optimal trajectory is $\bar{u} = 0$, $\bar{s} = 0$, $\bar{I} = q_i$, $\bar{\mu}_k = \bar{\mu}_N$, and $\bar{\nu}_k = \bar{\nu}_N$. Then the gradient ∇J_i and the (not necessarily active) inequality constraints read:

$$\nabla J_i = \begin{bmatrix} \mathbf{0}_{N+1} \\ \rho\mathbf{1}_{N+1} \\ \bar{\mu}_0\mathbf{1}_{N+1} \\ \mathbf{0}_{2N+1} \end{bmatrix}, \quad h_{\Delta=0} = \begin{bmatrix} \gamma\bar{I} - \bar{\mu} - \bar{\nu} \\ C(\bar{\mu} - \rho) \\ (C-1)\bar{\nu} \\ \bar{\mu}_N - \bar{\mu}_{0,\dots,N-1} \\ \bar{\nu}_N - \bar{\nu}_{0,\dots,N-1} \end{bmatrix},$$

where

$$C = \begin{cases} 0 & \text{if} \quad |q_i| \leq \rho \quad \text{(i.e. for } \nu = 0) \\ 1 & \text{if} \quad |q_i| > \rho \quad \text{(i.e. for } \mu = \rho) \end{cases}, \quad \gamma = \operatorname{sign}(q_i)$$

Since $\bar{s} = 0$, $\bar{u} = 0$, assumption 2 guarantees that the inequality constraints $h(s,u)$ are not active, and constraints $h(s,u)$ are therefore omitted in $h_{\Delta=0}$. The sensitivity of $h_{\Delta=0}$ is given by:

$$\nabla h_{\Delta=0} = \begin{bmatrix} \gamma\mathbf{I}_{N+1} & \mathbf{0}_{N+1} & \mathbf{0}_{N+1} & \mathbf{0}_{N+1\times N} & \mathbf{0}_{N+1\times N} \\ -\mathbf{I}_{N+1} & \mathbf{0}_{N+1} & (C-1)\mathbf{I}_{N+1} & \mathbf{0}_{N+1\times N} & \mathcal{M} \\ -\mathbf{I}_{N+1} & C\mathbf{I}_{N+1} & \mathbf{0}_{N+1} & \mathcal{M} & \mathbf{0}_{N+1\times N} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix}.$$

Assumption 3 guarantees that that the direction:

$$\delta w^T = \begin{bmatrix} \gamma\eta_I^T & C\eta_S^T & (1-C)\eta_S^T & \eta_s^T & \eta_u^T \end{bmatrix},$$

yields $\delta w^T \nabla g_I = 0$, $\delta w^T \nabla g_s = 0$, and the inequalities $\eta_S^T \mathcal{M} \leq 0$, and $\eta_I^T \mathbf{I}_{N+1} - \eta_S^T \mathbf{I}_{N+1} \leq 0$. Since $C(C-1) = 0$, it can be verified that

$$\delta w^T h_{\Delta=0} \leq 0.$$

As a result, direction δw is feasible. Moreover,

$$\delta w^T \nabla J_i = \{C\rho + (1-C)\bar{\mu}_0\}\, \eta_S^T \mathbf{1}_{N+1}.$$

Using $\rho, \bar{\mu}_0 > 0$ and since assumption 3 guarantees that $\eta_S^T \mathbf{1}_{N+1} \leq \Xi < 0$, it follows that $\delta w^T \nabla J_i < 0$, unless $C, \bar{\mu}_0 = 0$ therefore $\bar{u} = 0$, $\bar{s} = 0$, $\bar{I} = q_i = 0$ is the only possible optimal trajectory for $\Delta = 0$. As a result, function $\mathcal{P}_H(x_i, q_i, N)$ has the following properties on Ω:

$$\mathcal{P}_H(x_{i+1}, q_{i+1}, N) - \mathcal{P}_H(x_i, q_i, N) \leq \Delta(x_i, q_i, \bar{u}_0),$$

with

$$\Delta(x_i, q_i, \bar{u}_0) \leq 0, \quad \text{and} \quad \Delta(x_i, q_i, \bar{u}_0) = 0 \;\Rightarrow\; x_i = 0, q_i = 0.$$

$\square$

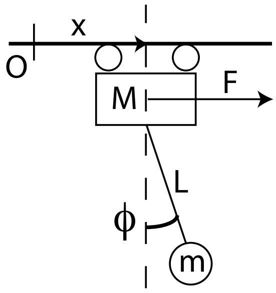

Fig. 2. Schematic of the crane.

TABLE I

MODEL & CONTROL PARAMETERS

Parameter	Value	Unit
m	1	(kg)
M	1	(kg)
L	1	(m)
g	9.81	(ms^{-2})
N	20	$(-)$
ρ	$5\cdot 10^{-2}$	(m)

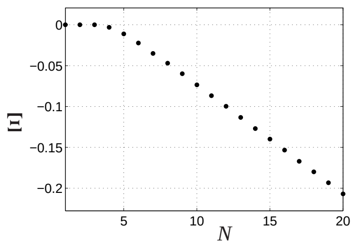

Fig. 3. Illustration of assumption 3 (see Section III-B). The decision variable Ξ obtained for various prediction horizons N by solving QP (14).

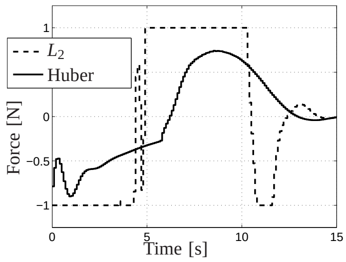

Fig. 4. Control input F for the L_2-norm NMPC (3) (dashed line), and for the NMPC (5)-(6) based on the Huber penalty function (solid line). The L_2-norm NMPC results in strong activations of the input bounds.

Discussion of Lemma 2 :

- assumption 3 essentially demands that for a trajectory $u=0, s=0, q=q_0$, there is a feasible perturbation $\delta u, \delta s, \delta q$ that reduces the cost function. It should be observed that QP (14) needs to be solved off-line only once for $u=0$ and $s=0$, using $\eta=0$ as initial guess. A solution with $\Xi<0$ is a certificate that assumption 3 is fulfilled.
- it should be observed that the inequality constraints (6) play a key role in the stability result established in Lemma 2, since they are needed to ensure that a) $\mathcal{P}_{\mathrm{H}}(x_i, q_i, N)$ is non-increasing, and that b) $x_i=0$, $q_i=0$ is the only point where $\mathcal{P}_{\mathrm{H}}(x_i, q_i, N)$ is non-decreasing.

IV. Illustrative example

This section proposes to compare the NMPC scheme (5)-(6) and the L_2-norm NMPC scheme (3) on a simulated crane (see Fig. 2). The state vector is $\left[x, \phi, \dot{x}, \dot{\phi}\right]$ where x is the cart position and ϕ the angle of the crane from the vertical axis. The continuous dynamics are given by:

$$f_{\mathrm{c}}(\phi, \dot{x}, \dot{\phi}, F) = \left[\begin{array}{c} \dot{x} \\ \dot{\phi} \\ \frac{F - m(L\sin(\phi)\dot{\phi}^2 - g\cos(\phi)\sin(\phi))}{M - m\cos(\phi)^2} \\ -\frac{g(m+M)\sin(\phi) + \cos(\phi)(Lm\sin(\phi)\dot{\phi}^2 - F)}{L(m\sin(\phi)^2 + M)} \end{array}\right],$$

and discretized using multiple-shooting [2] at a sampling time $T_{\mathrm{S}} = 0.1\,[\mathrm{s}]$. The quadrature state q is the position x, which does not appear in the dynamics f_{c}. The model and control parameters are summarized in Table I.

The fixed reference was chosen as $\left[x, \phi, \dot{x}, \dot{\phi}\right] = 0$. A prediction horizon $N = 50$ samples was used, corresponding to a time horizon of $T_{\mathrm{h}} = NT_{\mathrm{S}} = 5\,[\mathrm{s}]$. The function Φ was chosen as:

$$\Phi = \frac{1}{2}\left(10^{-2}F^2 + 10^{-1}\dot{\phi}^2\right),$$

The Huber penalty function parameter ρ was chosen as $\rho = 0.25\,[\mathrm{cm}]$. Trajectories were simulated, using the initial condition $\left[x, \phi, \dot{x}, \dot{\phi}\right] = [10, 0, 0, 0]$. It can be seen that the control input F undergoes significant saturation. Moreover, because of the strong penalty associated to the large deviation of x from its reference, the L_2 NMPC disregards the penalty on $\dot{\phi}$ such that ϕ undergoes strong oscillations (dashed line in Fig. 5). Because the impact of the penalty associated x on the control policy of the Huber NMPC is limited (see Sec. III-A), such oscillation does not occur when using the Huber NMPC (solid line in Fig. 5). The adjunction of a constraint on $\dot{\phi}$ has little benefit for the reduction of the oscillations of the angle ϕ (see Fig. 7).

Assumption 3 was verified for various prediction horizons N. It can be seen in Fig. 3 that $\Xi = 0$ for very short horizons and decreases linearly for longer horizons. Lemma 2 is illustrated in Fig. 6. It can be seen that function $\mathcal{P}_{\mathrm{H}}(x_i, q_i, N)$ is monotonically decreasing.

The simulations were performed using ACADO toolkit [9], [5], and implemented in code generation. The computational times obtained for both the L_2-norm NMPC scheme and the Huber NMPC scheme are reported in Fig. 8. It can be observed that the implementation of the Huber norm increases significantly the computational burden of the NMPC scheme, it does not jeopardize the real-time feasibility of the proposed example.

V. Conclusion & Future work

This paper has proposed a NMPC scheme based on the Huber penalty function to address the shortcoming of more classical L_2-norm based NMPC scheme when dealing with large deviations from the reference. The study was limited to the handling of a single quadrature state. The proposed scheme behaves as a standard NMPC scheme when the

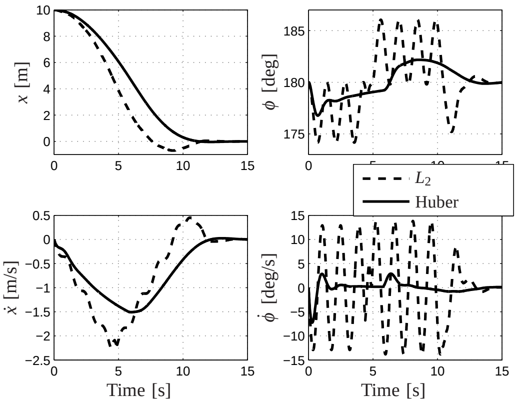

Fig. 5. State trajectories for the L_2-norm NMPC (3) (dashed line), and for the NMPC (5)-(6) based on the Huber penalty function (solid line). The L_2-norm NMPC results in strong oscillation of ϕ.

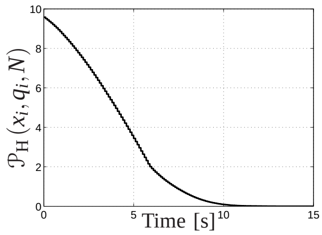

Fig. 6. Illustration of Lemma 2. Function $\mathcal{P}_{\mathrm{H}}(x_i, q_i, N)$ is a Lyapunov function for the trajectories displayed in Fig. 5 (plain lines)

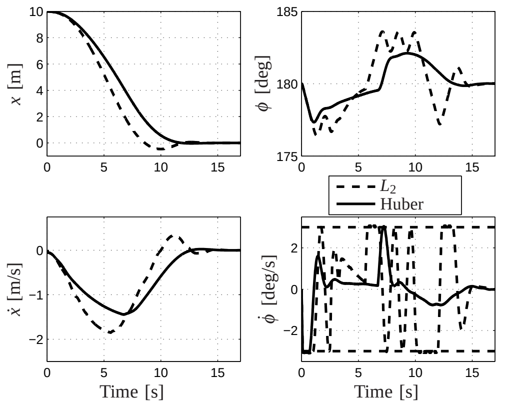

Fig. 7. State trajectories for the initial conditions $[x, \phi, \dot{x}, \dot{\phi}] = [10, 0, 0, 0]$, for the L_2-norm NMPC (3) (dashed line), and for the NMPC (5)-(6) based on the Huber penalty function (solid line), the addition of a constraint on $\dot{\phi}$ is not generally a means to reduce the oscillations of $\dot{\phi}$.

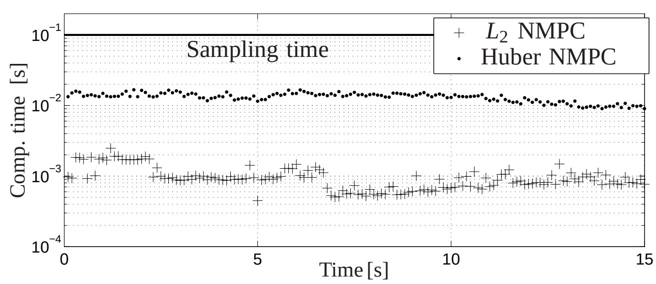

Fig. 8. Computational time for the L_2-norm vs. the Huber-norm NMPC.

system is close to its reference, but yields significantly less aggressive control action when far from its reference. The tuning is intuitive, and based on a single parameter. Some properties of the proposed scheme were established as well as its nominal stability. The proposed scheme was illustrated using a simulated overhanging crane.

Future work will explore the case of multiple quadrature states, and the more general case of non-quadrature states. An extension of the nominal stability result to the case of terminal sets, and the case of no terminal constraint will be considered expanding on the work presented in [8].

The smooth reformulation of the Huber penalty in (5) introduces a significant extra computational burden in the NMPC scheme. This computational burden can arguably be alleviated by using a first-order approach to solve the underlying QP problems [12], where the Lipschitz continuity of the Hessian of the NMPC problem is not required, and the Huber penalty function (1) can be used directly.

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