DSP-CIS

Part-IV : Filter Banks & Time-Frequency Transforms

Chapter-12 : Filter Bank Design

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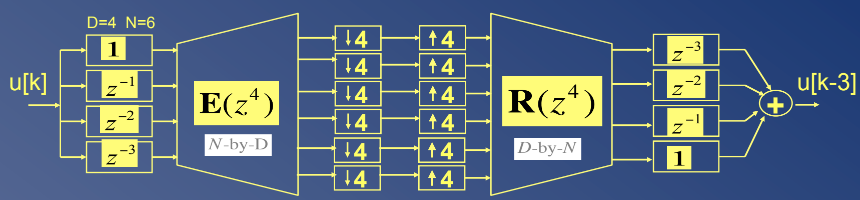
Part-IV : Filter Banks & Time-Frequency Transforms

Filter Bank Design Problem Statement

Perfect Reconstruction (PR) condition (D=N and D<N)
(based on polyphase representation of analysis/synthesis bank))



$$\mathbf{R}(z).\mathbf{E}(z) = z^{-\delta} I_D$$

Beautifully Simple!!

Will use this for Perfect Reconstruction Filter Bank (PR-FB) Design

Filter Bank Design Problem Statement

Two design targets :

- ✪ **Filter specifications**, e.g. stopband attenuation, passband ripple, transition band, etc. (for each (analysis) filter!)
- ✪ **Perfect reconstruction** (PR) property.

Challenge will be in addressing two design targets at once
(e.g. 'PR only' (without filter specs) is easy, see ex. Chapter-11)

PS: Can also do 'Near-Perfect Reconstruction Filter Bank Design', i.e. optimize filter specifications and at the same time minimize aliasing/ distortion (=numerical optimization). Not covered here…

General PR-FB Design: Maximum Decimation (D=N)

$$\mathbf{R}(z).\mathbf{E}(z) = z^{-\delta} I_N$$

(= N-by-N matrices)

- Design Procedure:
  1. Design all analysis filters (see Part-II).
  2. This determines E(z) (=polyphase matrix).
  3. Assuming E(z) can be inverted (?), synthesis filters are

$$\mathbf{R}(z) = z^{-\delta}.\mathbf{E}^{-1}(z)$$

(delta to make synthesis causal, see ex. p.7)

- Will consider only **FIR** analysis filters, leading to simple polyphase decompositions (see Chapter-2)
- However, FIR E(z) then generally leads to **IIR** R(z), where stability is a concern…

General PR-FB Design: Maximum Decimation (D=N)

PS: Inversion of matrix transfer functions ?…

- The inverse of a scalar (i.e. 1-by-1 matrix) FIR transfer function is **always** IIR (except for contrived examples)

$$\mathbf{E}(z) = (2 - z^{-1}) \Rightarrow \mathbf{R}(z) = \mathbf{E}^{-1}(z) = \frac{1}{(2 - z^{-1})}$$

- …but the inverse of an N-by-N (N>1) FIR transfer function **can** be FIR

$$\mathbf{E}(z) = \begin{bmatrix} z^{-2} + \frac{1}{2} & z^{-1} \\ 2z^{-1} & 2 \end{bmatrix} \Rightarrow \mathbf{R}(z) = \mathbf{E}^{-1}(z) = \begin{bmatrix} 2 & -z^{-1} \\ -2z^{-1} & z^{-2} + \frac{1}{2} \end{bmatrix}$$

$$\det(\mathbf{E}(z)) = 1$$

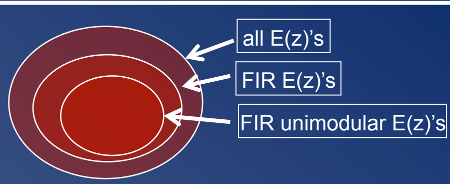
PS: Compare this to inversion of integers and integer matrices

$\mathbf{E} = 2 \rightarrow \mathbf{R} = \mathbf{E}^{-1} = \frac{1}{2}$ …but… $\mathbf{E} = \begin{bmatrix} 5 & 6 \\ 4 & 5 \end{bmatrix} \rightarrow \mathbf{R} = \mathbf{E}^{-1} = \begin{bmatrix} 5 & -6 \\ -4 & 5 \end{bmatrix}$, $\det(\mathbf{E}) = 1$

General PR-FB Design: Maximum Decimation (D=N)

Question:

Can we build **FIR** E(z)'s (N-by-N) that have an **FIR** inverse?



Answer:

YES, `**unimodular**' E(z)'s, i.e. matrices with determinant=constant*z^d

e.g.

$$\mathbf{E}(z)=\mathbf{E}_L.\begin{bmatrix} I_{N-1} & 0 \\ 0 & z^{-1}\end{bmatrix}.\mathbf{E}_{L-1}.\begin{bmatrix} I_{N-1} & 0 \\ 0 & z^{-1}\end{bmatrix}...\mathbf{E}_1.\begin{bmatrix} I_{N-1} & 0 \\ 0 & z^{-1}\end{bmatrix}.\mathbf{E}_0$$

$$\mathbf{R}(z)=\mathbf{E}_0^{-1}.\begin{bmatrix} z^{-1}.I_{N-1} & 0 \\ 0 & 1\end{bmatrix}.\mathbf{E}_1^{-1}...\begin{bmatrix} z^{-1}.I_{N-1} & 0 \\ 0 & 1\end{bmatrix}.\mathbf{E}_{L-1}^{-1}.\begin{bmatrix} z^{-1}.I_{N-1} & 0 \\ 0 & 1\end{bmatrix}.\mathbf{E}_L^{-1}$$

$$\Rightarrow \mathbf{R}(z).\mathbf{E}(z)=z^{-L}.I_N$$

where the E_l's are constant (= not a function of z) invertible matrices

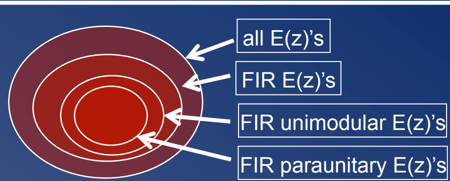
Design Procedure:

Optimize E_l's to meet filter specs (ripple, etc.) for all analysis filters (at once)

= not-so-easy but DOABLE !

General PR-FB Design: Maximum Decimation (D=N)

An interesting special case of this is obtained when the E_l's are orthogonal (=real) matrices or unitary (=complex) matrices



$$\mathbf{E}_l^H=\mathbf{E}_l^{-1},\quad l=0...L$$

$$\mathbf{E}(z)=\mathbf{E}_L.\begin{bmatrix} I_{N-1} & 0 \\ 0 & z^{-1}\end{bmatrix}.\mathbf{E}_{L-1}.\begin{bmatrix} I_{N-1} & 0 \\ 0 & z^{-1}\end{bmatrix}...\mathbf{E}_1.\begin{bmatrix} I_{N-1} & 0 \\ 0 & z^{-1}\end{bmatrix}.\mathbf{E}_0$$

$$\mathbf{R}(z)=\mathbf{E}_0^{H}.\begin{bmatrix} z^{-1}.I_{N-1} & 0 \\ 0 & 1\end{bmatrix}.\mathbf{E}_1^{H}...\begin{bmatrix} z^{-1}.I_{N-1} & 0 \\ 0 & 1\end{bmatrix}.\mathbf{E}_{L-1}^{H}.\begin{bmatrix} z^{-1}.I_{N-1} & 0 \\ 0 & 1\end{bmatrix}.\mathbf{E}_L^{H}$$

$$\Rightarrow \mathbf{R}(z).\mathbf{E}(z)=z^{-L}.I_N$$

E(z) and R(z) are then 'paraunitary' (definition omitted) and the analysis and synthesis FB are said to be 'paraunitary' FBs

PS: Before proceeding compare formulas with lossless lattice realizations in Chapter-5…

General PR-FB Design: Maximum Decimation (D=N)

Paraunitary PR-FBs have great properties: (proofs omitted)

- If polyphase matrix E(z) is paranunitary, then analysis filters are power complementary (=form lossless 1-input/N-output system)

$$\sum_{n=0}^{N-1} \left| H_n(e^{j\omega}) \right|^2 = 1 \quad \text{(or other constant)}$$

(see Chapter 5)

- Synthesis filters are obtained from analysis filters by conjugating the analysis filter coefficients + reversing the order

$$f_n[k] = h_n^*[L-k], \qquad 0 \le n \le N-1$$

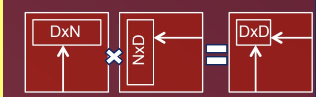
- Hence magnitude response of synthesis filter F_n is the same as magnitude response of corresponding analysis filter H_n

$$\left| F_n(e^{j\omega}) \right| = \left| H_n(e^{j\omega}) \right|, \qquad 0 \le n \le N-1$$

…and so synthesis filters are also power complementary

General PR-FB Design: Oversampled FBs (D<N)

$$\mathbf{R}(z).\mathbf{E}(z) = z^{-\delta} I_D$$



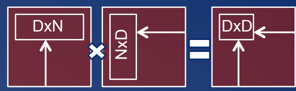
- Design Procedure:
  1. Design all analysis filters (see Part-II).
  2. This determines E(z) (=polyphase matrix).
  3. Find R(z) such that PR condition is satisfied (how? read on…)

= easy if step-3 is doable…

- Will consider only **FIR** analysis filters, leading to simple polyphase decompositions (see Chapter-2)
- It will turn out that when D<N an **FIR** R(z) can always be found (except in contrived cases)…

General PR-FB Design: Oversampled FBs (D<N)

$$\mathbf{R}(z).\mathbf{E}(z) = z^{-\delta} I_D$$



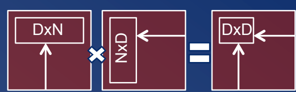
- Given E(z) how can R(z) be computed?
  - Assume every entry in E(z) is L_E-th order FIR (i.e. L_E +1 coefficients)
  - Assume every entry in R(z) is L_R-th order FIR (i.e. L_R +1 coefficients)
  - Hence number of unknown coefficients in R(z) is **D.N.(L_R +1)**
  - Every entry in R(z).E(z) is (L_E+L_R)-th order FIR (i.e. L_E+L_R+1 coefficients) (cfr. polynomial multiplication / linear convolution)
  - Hence PR condition is equivalent to **D.D.(L_E+L_R+1)** linear equations in the unknown coefficients (*)
  - Can be solved (except in contrived cases) if $D.N.(L_R+1) \geq D.D.(L_E+L_R+1)$

$$\rightarrow \quad L_R \geq \frac{D}{(N-D)} L_E - 1$$

(*) Try to write down these equations!

General PR-FB Design: Oversampled FBs (D<N)

$$\mathbf{R}(z).\mathbf{E}(z) = z^{-\delta} I_D$$



- Given E(z) how can R(z) be computed?
  - (continued) …
  - Can be solved (except in contrived cases) if $D.N.(L_R+1) \geq D.D.(L_E+L_R+1)$

$$\rightarrow \quad L_R \geq \frac{D}{(N-D)} L_E - 1$$

  - If D<N, then L_R can be made sufficiently large so that the (underdetermined) set of equations can be solved, i.e. an R(z) can be found (!).
  - Note that if D=N, then L_R in general has to be infinitely large, i.e. R(z) in general has to be IIR (see p.5)

DFT-Modulated FBs

- All design procedures so far involve monitoring of characteristics (passband ripple, stopband suppression,…) of <u>all</u> (analysis) filters, which may be tedious.

- Design complexity may be reduced through usage of `uniform' and `<u>modulated</u>' filter banks.
  - DFT-modulated FBs (read on..)
  - Cosine-modulated FBs (not covered, but interesting design!)

- Will consider
  - Maximally decimated DFT-modulated FBs
  - Oversampled DFT-modulated FBs

Maximally Decimated DFT-Modulated FBs (D=N)

Uniform versus non-uniform (analysis) filter bank:

- **<u>N-channel uniform FB</u>:** $H_n(z) = H_0(z.e^{-j2\pi n/N}) \quad n = 0,...,N-1$

  i.e. frequency responses are <u>uniformly shifted</u> over the unit circle

  Ho(z)= `prototype' filter (=one and only filter that has to be designed)

  Time domain equivalent is: $h_n[k] = h_0[k].e^{j2\pi k.n/N}$

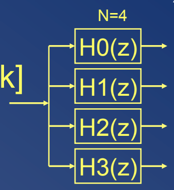
- Non-uniform = everything that is not uniform

  e.g. for speech & audio applications (cfr. human hearing)

Maximally Decimated DFT-Modulated FBs (D=N)

Uniform filter banks can be realized cheaply based on polyphase decompositions + DFT(FFT) (hence name `DFT-modulated FB)

1. Analysis FB



If

$$H_0(z), H_1(z), \ldots, H_{N-1}(z) \quad \text{with} \quad H_n(z) = H_0(z.e^{-j2\pi n/N})$$

$$H_0(z) = \sum_{\bar{n}=0}^{N-1} z^{-\bar{n}}.E_{\bar{n}}(z^N)$$

(N-fold polyphase decomposition)

then

$$H_n(z) = H_0(z.e^{-j2\pi n/N}) = \sum_{\bar{n}=0}^{N-1} z^{-\bar{n}}.e^{j2\pi n\bar{n}/N}.E_{\bar{n}}(z^N \overbrace{e^{-j2\pi nN/N}}^{1})$$

$$= \sum_{\bar{n}=0}^{N-1} z^{-\bar{n}}.W^{-n\bar{n}}.E_{\bar{n}}(z^N) \,, \quad \text{with} \quad W = e^{-j2\pi/N}$$

i.e. →

Maximally Decimated DFT-Modulated FBs (D=N)

i.e. →

$$\begin{bmatrix} H_0(z) \\ H_1(z) \\ H_2(z) \\ \vdots \\ H_{N-1}(z) \end{bmatrix} .U(z) = \overbrace{\begin{bmatrix} W^0 & W^0 & W^0 & \ldots & W^0 \\ W^0 & W^{-1} & W^{-2} & \ldots & W^{-(N-1)} \\ W^0 & W^{-2} & W^{-4} & \ldots & W^{-2(N-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ W^0 & W^{-(N-1)} & W^{-2(N-1)} & \ldots & W^{-(N-1)^2} \end{bmatrix}}^{N.\mathbf{F}^{-1}} . \begin{bmatrix} E_0(z^N) \\ z^{-1}.E_1(z^N) \\ z^{-2}.E_2(z^N) \\ \vdots \\ z^{-N+1}.E_{N-1}(z^N) \end{bmatrix} .U(z)$$

$$W = e^{-j2\pi/N}$$

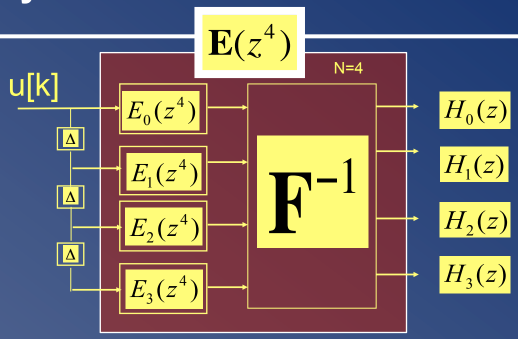
where F is NxN DFT-matrix

This means that filtering with the H_n's can be implemented by first filtering with the polyphase components and then applying an inverse DFT

PS: To simplify formulas the factor N in $N.F^{-1}$ will be left out from now on (i.e. absorbed in the polyphase components)

Maximally Decimated DFT-Modulated FBs (D=N)

i.e.

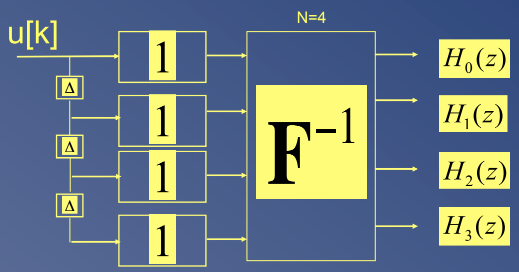


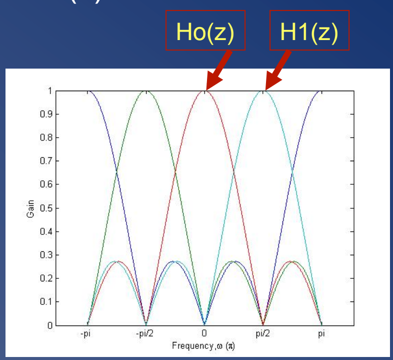
Conclusion: economy in…

- Implementation complexity (for FIR filters):
  N filters for the price of 1, plus inverse DFT (=FFT) !
- Design complexity:
  Design `prototype' Ho(z), then other Hn(z)'s are automatically `co-designed' (same passband ripple, etc…) !

Maximally Decimated DFT-Modulated FBs (D=N)

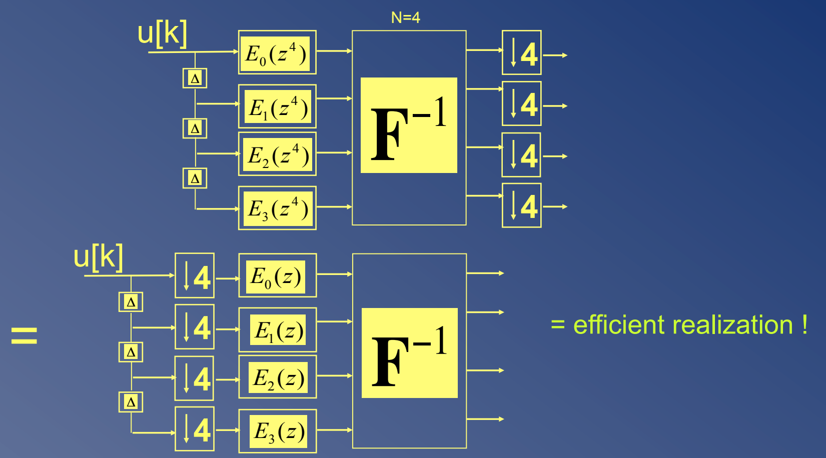
- Special case: DFT-filter bank, if all $E_n(z)=1$





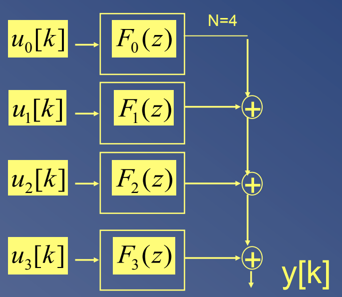
Maximally Decimated DFT-Modulated FBs (D=N)

- DFT-modulated analysis FB + maximal decimation



Maximally Decimated DFT-Modulated FBs (D=N)

2. Synthesis FB



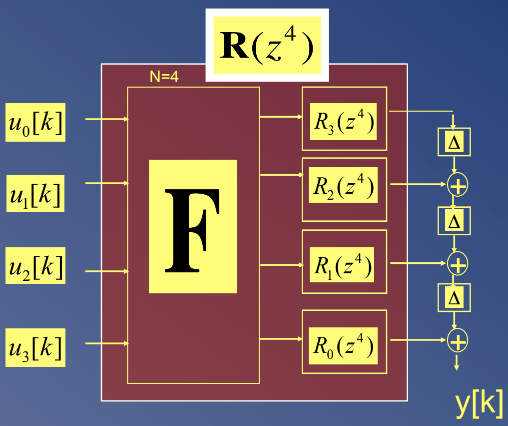
phase shift added for convenience

$F_0(z),F_1(z),\ldots,F_{N-1}(z)$ with $F_n(z)=e^{j2\pi n/N}F_0(z.e^{-j2\pi n/N})$

$$F_0(z)=\sum_{\bar{n}=0}^{N-1} z^{-\bar{n}}.R_{\bar{n}}(z^N) \quad \Rightarrow \quad F_n(z)= \;\ldots = \sum_{\bar{n}=0}^{N-1} z^{-\bar{n}}.W^{n(N-1-\bar{n})}.R_{\bar{n}}(z^N)$$

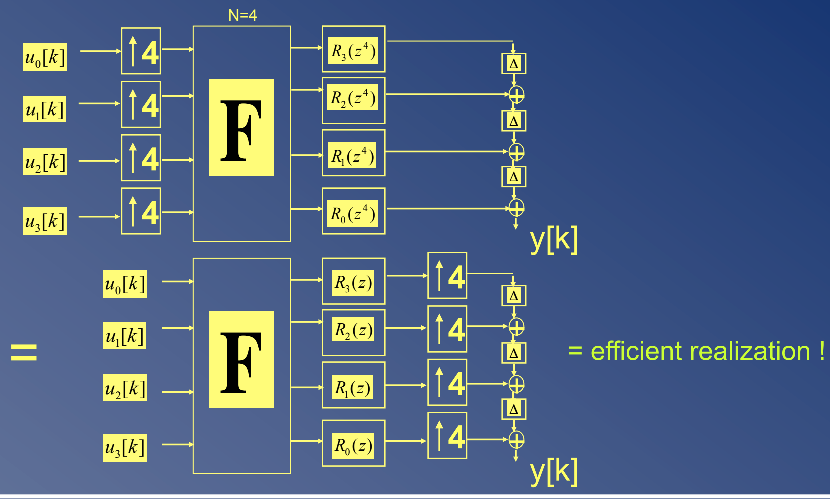
Maximally Decimated DFT-Modulated FBs (D=N)

Similarly simple derivation then leads to…



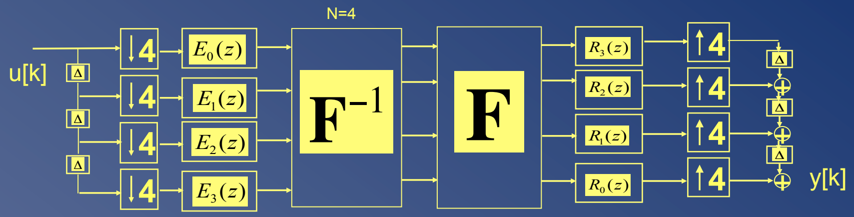
Maximally Decimated DFT-Modulated FBs (D=N)

- Expansion + DFT-modulated synthesis FB :



Maximally Decimated DFT-Modulated FBs (D=N)

How to achieve **Perfect Reconstruction** (PR)
with maximally decimated DFT-modulated FBs?



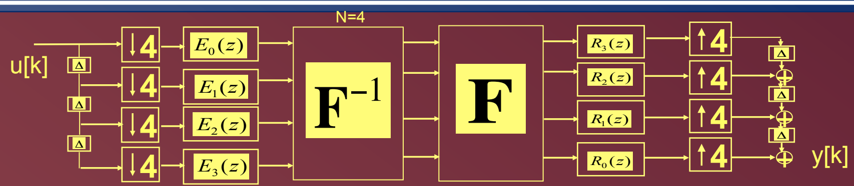
$$\mathbf{R}(z).\mathbf{E}(z) = z^{-\delta} I_N$$

$$\rightarrow \quad \mathbf{E}(z) = \mathbf{F}^{-1}.diag\left[E_n(z)\right] \quad \Rightarrow \quad \mathbf{R}(z) = z^{-\delta}.\mathbf{E}^{-1}(z) = z^{-\delta}.diag\left[E_n^{-1}(z)\right].\mathbf{F}$$

$$\rightarrow \quad R_n(z) = z^{-\delta}.E^{-1}_{N-1-n}(z)$$

Polyphase components of synthesis bank prototype filter are obtained by inverting polyphase components of analysis bank prototype filter

Maximally Decimated DFT-Modulated FBs (D=N)



- Design Procedure:
  1. Design prototype analysis filter Ho(z) (see Part-II).
  2. This determines $E_n(z)$ (=polyphase components).
  3. Assuming all $E_n(z)$'s can be inverted (?), choose synthesis filters

$$R_n(z) = z^{-\delta}.E^{-1}_{N-1-n}(z)$$

- Will consider only **FIR** prototype analysis filter, leading to simple polyphase decomposition (Chapter-2).
- However, FIR $E_n(z)$'s generally again lead to IIR $R_n(z)$'s, where stability is a concern…

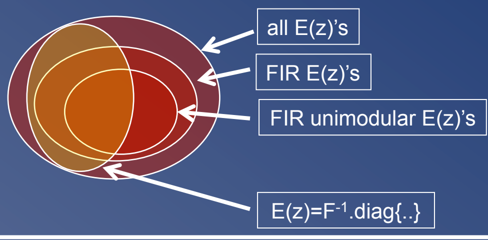
Maximally Decimated DFT-Modulated FBs (D=N)

This does not leave much design freedom…

- **FIR & Unimodular E(Z)?** ..such that Rn(z) are also FIR

  Only obtained when each $E_n(z)$ is 'unimodular', i.e. $E_n(z)=\text{constant}.z^d$

  Simple example is $E_n(z)=w_n \Rightarrow R_{N-1-n}(z)=w_n^{-1}$, where w_n's are constants, which leads to `windowed' IDFT/DFT bank, a.k.a. `short-time Fourier transform' (see Chapter-14)



Maximally Decimated DFT-Modulated FBs (D=N)

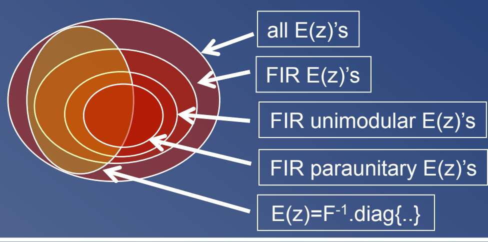
This does not leave much design freedom…

- **FIR & Paraunitary E(Z)?**

  ..such that Rn(z) are FIR + power complementary FB's.

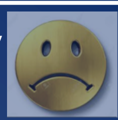
  Only obtained when each $E_n(z)$ is paraunitary (i.e. all-pass) (and FIR), i.e. $En(z)=\pm 1.z^d$.

  i.e. only trivial modifications of IDFT/DFT filter bank !



Maximally Decimated DFT-Modulated FBs (D=N)

- **Bad news:** Not much design freedom for maximally decimated DFT-modulated FB's…



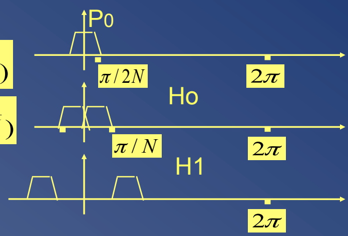
- **Good news:** More design freedom with...
  - Cosine-modulated FB's (not covered, but interesting design!)

    $P_0(z)$ is prototype lowpass filter, cutoff at $\pm\pi/2N$ for N filters

    Then...

    $$H_0(z)=\alpha_0.P_0(z.e^{-j(0.5)\frac{\pi}{N}})+\alpha_0^*.P_0(z.e^{j(+0.5)\frac{\pi}{N}})$$

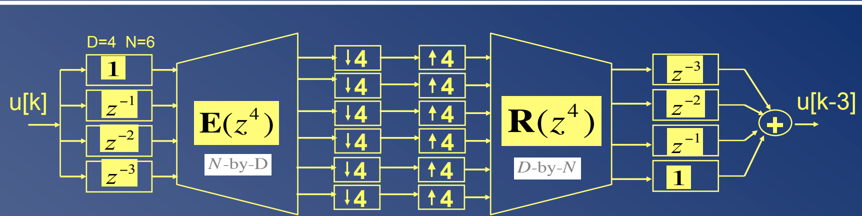
    $$H_1(z)=\alpha_1.P_0(z.e^{-j(1+0.5)\frac{\pi}{N}})+\alpha_1^*.P_0(z.e^{j(1+0.5)\frac{\pi}{N}})$$



    etc..

    PS: Real-valued filter coefficients here!
  - Oversampled DFT-modulated FB's (read on..)

Oversampled DFT-Modulated FBs (D<N)



- In maximally decimated DFT-modulated FB, we had

  $\mathbf{E}(z)=\mathbf{F}^{-1}.diag[E_n(z)]$ $\mathbf{R}(z)=diag[R_{N-1-n}(z)].\mathbf{F}$ (N-by-N matrices)
- In oversampled DFT-modulated FB, will have

  $$\underbrace{\mathbf{E}(z)}_{\text{N-by-D}}=\underbrace{\mathbf{F}^{-1}}_{\text{N-by-N}}.\underbrace{\mathbf{B}(z)}_{\text{N-by-D}}\qquad \underbrace{\mathbf{R}(z)}_{\text{D-by-N}}=\underbrace{\mathbf{C}(z)}_{\text{D-by-N}}\underbrace{.\mathbf{F}}_{\text{N-by-N}}$$

  with B(z) (tall-thin) and C(z) (short-fat) structured/sparse matrices
- Will give 2 examples in next slides, other cases are similar..

Oversampled DFT-Modulated FBs (D<N)

Example-1: #channels N = 8 — analysis FB Ho(z),H1(z),…,H7(z)
decimation D = 4
prototype analysis filter Ho(z)
will consider N'-fold polyphase expansion, with

Should not try to understand this…

$$N' = \frac{N.D}{\gcd(N,D)} = \frac{8.4}{4} = 8$$

$$H_0(z) = \sum_{k=-\infty}^{\infty} h_0[k].z^{-k} = \sum_{n'=0}^{7} z^{-n'}.E_{n'}(z^8) \quad , \quad E_{n'}(z) = \sum_{k=-\infty}^{\infty} h_0[8.k+n'].z^{-k}$$

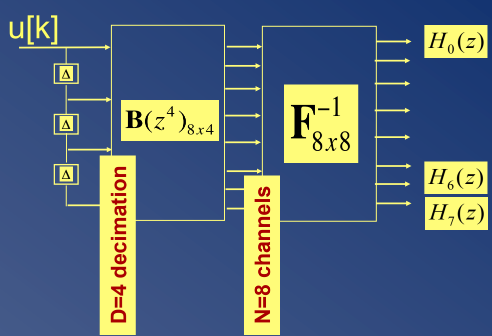
PS: A scale factor *N* will again absorbed in polyphase components (see p.16)

Oversampled DFT-Modulated FBs (D<N)

Example-1: Define B(z)…

$$\mathbf{B}(z^4) = \begin{bmatrix} E_0(z^8) & 0 & 0 & 0 \\ 0 & E_1(z^8) & 0 & 0 \\ 0 & 0 & E_2(z^8) & 0 \\ 0 & 0 & 0 & E_3(z^8) \\ z^{-4}.E_4(z^8) & 0 & 0 & 0 \\ 0 & z^{-4}.E_5(z^8) & 0 & 0 \\ 0 & 0 & z^{-4}.E_6(z^8) & 0 \\ 0 & 0 & 0 & z^{-4}.E_7(z^8) \end{bmatrix}$$

and construct FB as…



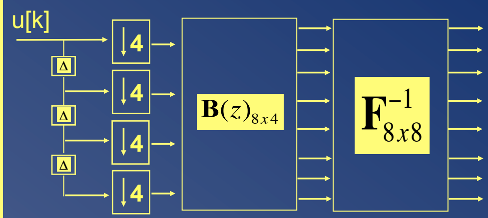
Proof is simple…

$$\mathbf{F}^{-1}.\mathbf{B}(z^4).\begin{bmatrix} 1 \\ z^{-1} \\ z^{-2} \\ z^{-3} \end{bmatrix}.U(z) = \mathbf{F}^{-1}.\begin{bmatrix} E_0(z^8) & 0 & 0 & 0 \\ 0 & E_1(z^8) & 0 & 0 \\ 0 & 0 & E_2(z^8) & 0 \\ 0 & 0 & 0 & E_3(z^8) \\ z^{-4}.E_4(z^8) & 0 & 0 & 0 \\ 0 & z^{-4}.E_5(z^8) & 0 & 0 \\ 0 & 0 & z^{-4}.E_6(z^8) & 0 \\ 0 & 0 & 0 & z^{-4}.E_7(z^8) \end{bmatrix}.\begin{bmatrix} 1 \\ z^{-1} \\ z^{-2} \\ z^{-3} \end{bmatrix}.U(z) = \mathbf{F}^{-1}.\begin{bmatrix} E_0(z^8) \\ z^{-1}.E_1(z^8) \\ z^{-2}.E_2(z^8) \\ z^{-3}.E_3(z^8) \\ z^{-4}.E_4(z^8) \\ z^{-5}.E_5(z^8) \\ z^{-6}.E_6(z^8) \\ z^{-7}.E_7(z^8) \end{bmatrix}.U(z) = \begin{bmatrix} H_0(z) \\ H_1(z) \\ H_2(z) \\ H_3(z) \\ H_4(z) \\ H_5(z) \\ H_6(z) \\ H_7(z) \end{bmatrix}.U(z)$$

Oversampled DFT-Modulated FBs (D<N)

Example-1: With 4-fold decimation, this is…

$$\mathbf{B}(z)=\begin{bmatrix} E_0(z^2) & 0 & 0 & 0 \\ 0 & E_1(z^2) & 0 & 0 \\ 0 & 0 & E_2(z^2) & 0 \\ 0 & 0 & 0 & E_3(z^2) \\ z^{-1}.E_4(z^2) & 0 & 0 & 0 \\ 0 & z^{-1}.E_5(z^2) & 0 & 0 \\ 0 & 0 & z^{-1}.E_6(z^2) & 0 \\ 0 & 0 & 0 & z^{-1}.E_7(z^2) \end{bmatrix}$$

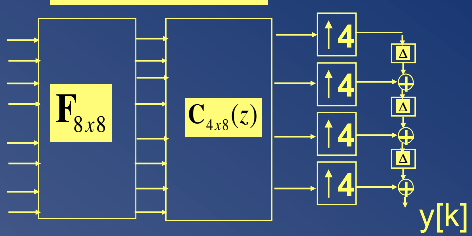


$$\mathbf{E}(z)=\mathbf{F}^{-1}.\mathbf{B}(z)$$

Oversampled DFT-Modulated FBs (D<N)

Example-1: Synthesis FB is similarly derived…

$$\mathbf{R}(z)=\mathbf{C}(z).\mathbf{F}$$

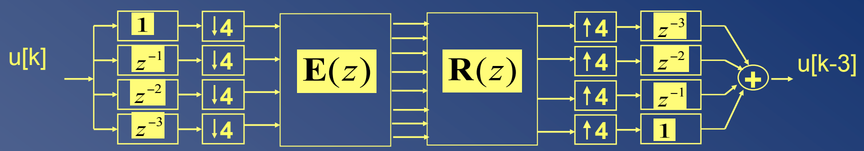


where…

$$\mathbf{C}(z)=\begin{bmatrix} z^{-1}.R_7(z^2) & 0 & 0 & 0 & R_3(z^2) & 0 & 0 & 0 \\ 0 & z^{-1}.R_6(z^2) & 0 & 0 & 0 & R_2(z^2) & 0 & 0 \\ 0 & 0 & z^{-1}.R_5(z^2) & 0 & 0 & 0 & R_1(z^2) & 0 \\ 0 & 0 & 0 & z^{-1}.R_4(z^2) & 0 & 0 & 0 & R_0(z^2) \end{bmatrix}$$

Oversampled DFT-Modulated FBs (D<N)

Example-1: Perfect Reconstruction?



$\mathbf{E}(z) = \mathbf{F}^{-1}.\mathbf{B}(z)$ $\mathbf{R}(z) = \mathbf{C}(z).\mathbf{F}$

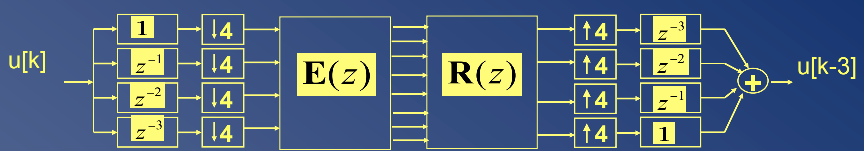
where…

$$z^{-\delta}I = \mathbf{R}(z).\mathbf{E}(z) = \mathbf{C}(z).\mathbf{F}.\mathbf{F}^{-1}.\mathbf{B}(z) = \mathbf{C}(z).\mathbf{B}(z)$$

$$= \begin{bmatrix} z^{-1}.R_7(z^2) & 0 & 0 & 0 & R_3(z^2) & 0 & 0 & 0 \\ 0 & z^{-1}.R_6(z^2) & 0 & 0 & 0 & R_2(z^2) & 0 & 0 \\ 0 & 0 & z^{-1}.R_5(z^2) & 0 & 0 & 0 & R_1(z^2) & 0 \\ 0 & 0 & 0 & z^{-1}.R_4(z^2) & 0 & 0 & 0 & R_0(z^2) \end{bmatrix} \begin{bmatrix} E_0(z^2) & 0 & 0 & 0 \\ 0 & E_1(z^2) & 0 & 0 \\ 0 & 0 & E_2(z^2) & 0 \\ 0 & 0 & 0 & E_3(z^2) \\ z^{-1}.E_4(z^2) & 0 & 0 & 0 \\ 0 & z^{-1}.E_5(z^2) & 0 & 0 \\ 0 & 0 & z^{-1}.E_6(z^2) & 0 \\ 0 & 0 & 0 & z^{-1}.E_7(z^2) \end{bmatrix}$$

Oversampled DFT-Modulated FBs (D<N)

Example-1: Perfect Reconstruction?



$\mathbf{E}(z) = \mathbf{F}^{-1}.\mathbf{B}(z)$ $\mathbf{R}(z) = \mathbf{C}(z).\mathbf{F}$

hence…

$$z^{-1}I = \mathbf{R}(z).\mathbf{E}(z)$$

$$\Leftrightarrow$$

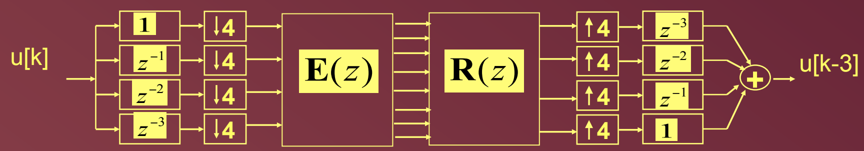
$$R_7(z^2).E_0(z^2) + R_3(z^2)E_4(z^2) = 1$$

$$R_6(z^2).E_1(z^2) + R_2(z^2)E_5(z^2) = 1$$

etc.

Oversampled DFT-Modulated FBs (D<N)

Example-1: Perfect Reconstruction?



Design Procedure :

1. Design FIR prototype analysis filter Ho(z)
2. This determines En(z) (=polyphase components)
3. Compute pairs of FIR Rn(z)'s (L_R+1 coefficients each) from pairs of FIR En(z)'s (L_E+1 coefficients each)

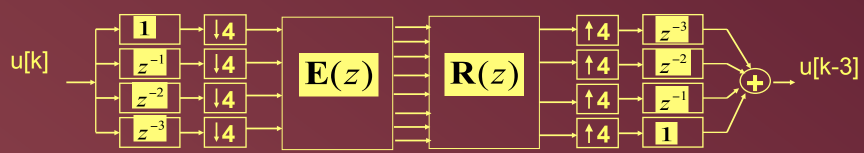
$$R_7(z^2).E_0(z^2)+R_3(z^2)E_4(z^2)\overset{(*)}{=}1$$
$$R_6(z^2).E_1(z^2)+R_2(z^2)E_5(z^2)=1$$
etc.

(*) L_R+L_E+1 equations in $2(L_R+1)$ unknowns can be solved if $L_E-1 \leq L_R$ (except in contrived cases)

= EASY !

Oversampled DFT-Modulated FBs (D<N)

Example-1: Perfect Reconstruction?



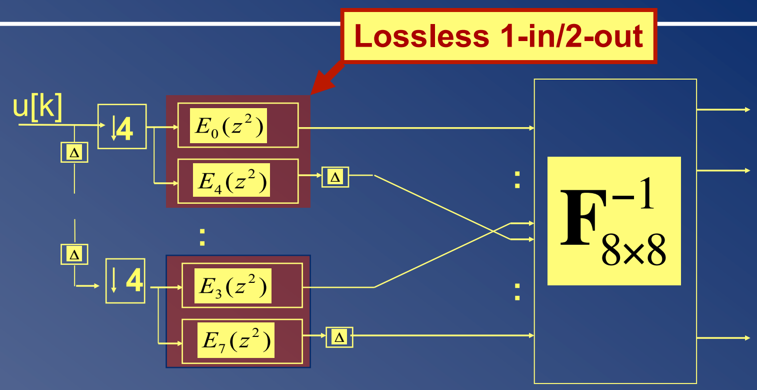
Design Procedure :

PS: If in addition $E_n(z)$ and $E_{n+4}(z)$ (n=0,1,2,3) are designed to be power complementary (i.e. form a lossless 1 input/2 output system) then the analysis and synthesis FB are **paraunitary**, i.e with power complementary analysis filters and power complementary synthesis filters (proof omitted)

Oversampled DFT-Modulated FBs (D<N)

..that is

p.30 =



- Design Procedure: Optimize parameters (=angles) of 4 (=D) FIR lossless lattices (defining polyphase components of Ho(z)) such that Ho(z) satisfies specifications.

= not-so-easy but DOABLE !

Oversampled DFT-Modulated FBs (D<N)

Example-2 (non-integer oversampling) :

#channels N = 6 analysis filters Ho(z),H1(z),…,H5(z)

decimation D = 4

prototype analysis filter Ho(z)

will consider N'-fold polyphase expansion, with

Should not try to understand this…

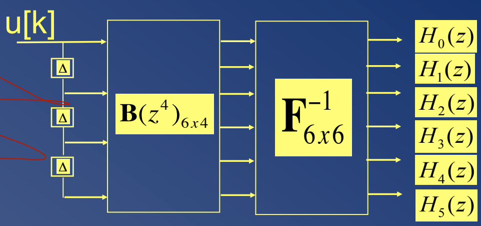
$$N' = \frac{N.D}{\gcd(N,D)} = \frac{6.4}{2} = 12$$

$$H_0(z) = \sum_{k=-\infty}^{\infty} h_0[k].z^{-k} = \sum_{n'=0}^{11} z^{-n'}.E_{n'}(z^{12}) \;, \qquad E_{n'}(z) = \sum_{k=-\infty}^{\infty} h_0[12.k+n'].z^{-k}$$

Oversampled DFT-Modulated FBs (D<N)

Example-2: Define B(z)… and construct FB as…

$$\mathbf{B}(z^4)=\begin{bmatrix} E_0(z^{12}) & 0 & z^{-4}.E_6(z^{12}) & 0 \\ 0 & E_1(z^{12}) & 0 & z^{-4}.E_7(z^{12}) \\ z^{-8}.E_8(z^{12}) & 0 & E_2(z^{12}) & 0 \\ 0 & z^{-8}.E_9(z^{12}) & 0 & E_3(z^{12}) \\ z^{-4}.E_4(z^{12}) & 0 & z^{-8}.E_{10}(z^{12}) & 0 \\ 0 & z^{-4}.E_5(z^{12}) & 0 & z^{-8}.E_{11}(z^{12}) \end{bmatrix}$$



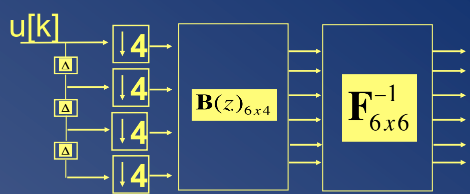
Proof is simple…

$$\mathbf{F}^{-1}.\mathbf{B}(z^4).\begin{bmatrix}1\\ z^{-1}\\ z^{-2}\\ z^{-3}\end{bmatrix}.U(z)=\ldots=\mathbf{F}^{-1}.\underbrace{\begin{bmatrix} E_0(z^{12})+z^{-6}.E_6(z^{12})\\ z^{-1}E_1(z^{12})+z^{-7}.E_7(z^{12})\\ z^{-2}E_2(z^{12})+z^{-8}.E_8(z^{12})\\ z^{-3}E_3(z^{12})+z^{-9}.E_9(z^{12})\\ z^{-4}E_4(z^{12})+z^{-10}.E_{10}(z^{12})\\ z^{-5}E_5(z^{12})+z^{-11}.E_{11}(z^{12})\end{bmatrix}}_{=\text{6-fold polyphase decomposition}}.U(z)=\begin{bmatrix}H_0(z)\\ H_1(z)\\ H_2(z)\\ H_3(z)\\ H_4(z)\\ H_5(z)\end{bmatrix}.U(z)$$

Oversampled DFT-Modulated FBs (D<N)

Example-2: With 4-fold decimation, this is…

$$\mathbf{B}(z)=\begin{bmatrix} E_0(z^{3}) & 0 & z^{-1}.E_6(z^{3}) & 0 \\ 0 & E_1(z^{3}) & 0 & z^{-1}.E_7(z^{3}) \\ z^{-2}.E_8(z^{3}) & 0 & E_2(z^{3}) & 0 \\ 0 & z^{-2}.E_9(z^{3}) & 0 & E_3(z^{3}) \\ z^{-1}.E_4(z^{3}) & 0 & z^{-2}.E_{10}(z^{3}) & 0 \\ 0 & z^{-1}.E_5(z^{3}) & 0 & z^{-2}.E_{11}(z^{3}) \end{bmatrix}$$



$$\mathbf{E}(z)=\mathbf{F}^{-1}.\mathbf{B}(z)$$

- Synthesis FB R(z)=C(z).F similarly derived
- PR conditions similarly derived, leading to undetermined sets of equations to compute synthesis prototype from analysis prototype (try it)

  = EASY !

- Paraunitary (power complementary) analysis and synthesis filter bank also possible (details omitted)