DSP-CIS

Part-IV : Filter Banks & Time-Frequency Transforms

Chapter-11 : Filter Bank Preliminaries

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Part-IV : Filter Banks & Time-Frequency Transforms

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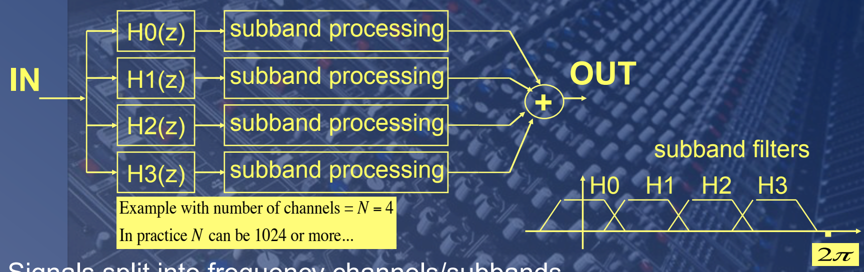
Filter Bank Set-Up

What we have in mind is this… :

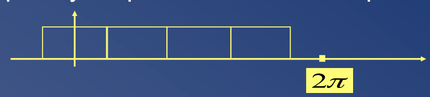


- Signals split into frequency channels/subbands
- Per-channel/subband processing
- Reconstruction : synthesis of processed signal
- Applications : see below (audio coding etc.)
- In practice, this is implemented as a **multi-rate** structure for higher efficiency (see next slides)

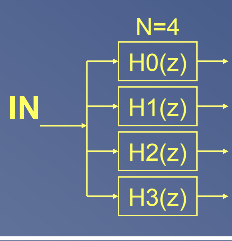
Filter Bank Set-Up

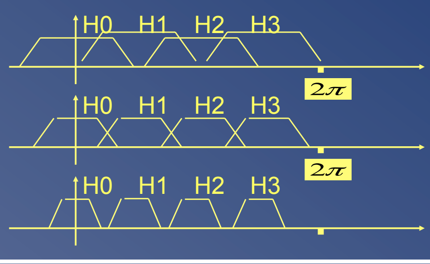
Step-1: Analysis filter bank

- Collection of N filters (`analysis filters', `decimation filters') with a common input signal
- **Ideal** (but non-practical) frequency responses = ideal bandpass filters



- **Typical** frequency responses (overlapping, non-overlapping,…)





Filter Bank Set-Up

Step-2: Decimators (downsamplers)

- To increase efficiency, subband sampling rate is reduced by factor D

  (= Nyquist sampling theorem (for passband signals))

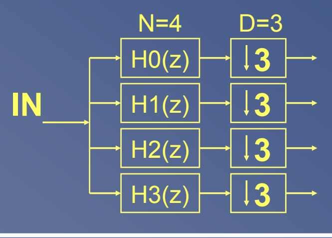
- **Maximally decimated** filter banks (=critically downsampled): **D=N**

  \# subband samples = # fullband samples

  this sounds like maximum efficiency, but aliasing (see below)!

- **Oversampled** filter banks (=non-critically downsampled): **D<N**

  \# subband samples > # fullband samples



PS: analysis filters Hn(z) are now also **decimation/anti-aliasing filters** to avoid aliasing in subband signals after decimation (see Chapter-2)

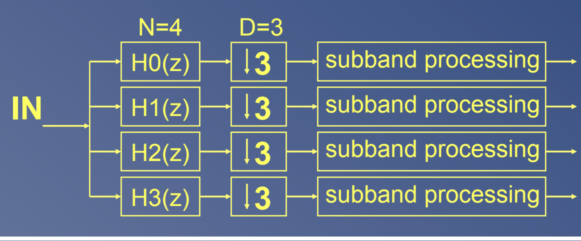
Filter Bank Set-Up

Step-3: Subband processing

- Example :

  coding (=compression) + (transmission or storage) + decoding

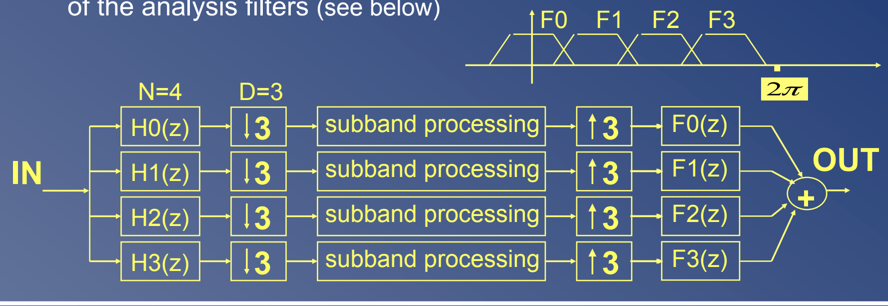
- Filter bank design mostly assumes subband processing has `unit transfer function' (output signals=input signals), i.e. mostly ***ignores*** presence of subband processing



Filter Bank Set-Up

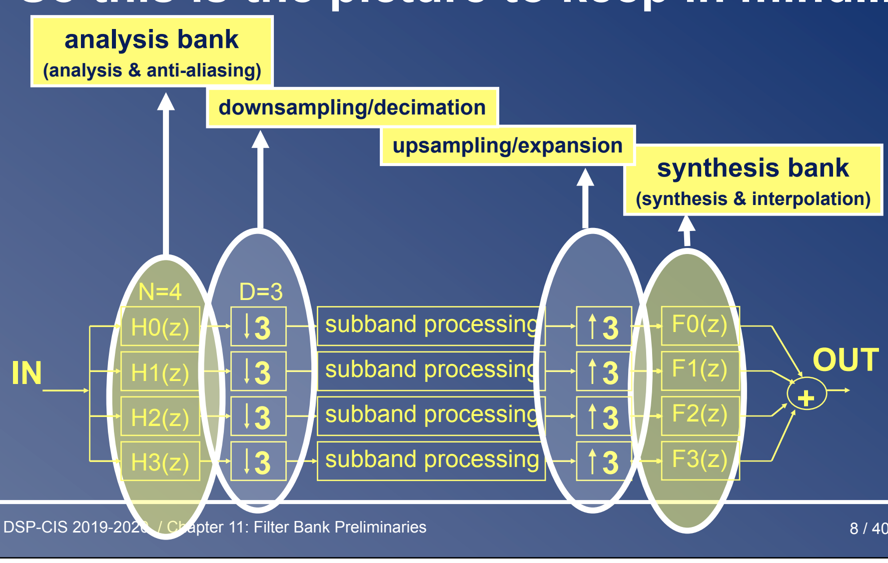
Step-4&5: Expanders (upsamplers) & synthesis filter bank

- Restore original fullband sampling rate by D-fold upsampling
- Upsampling has to be followed by interpolation filtering (to 'fill the zeroes' & remove spectral images, see Chapter-2)
- Collection of N filters (`synthesis', `interpolation') with summed output
- Frequency responses : preferably `matched' to frequency responses of the analysis filters (see below)



Filter Bank Set-Up

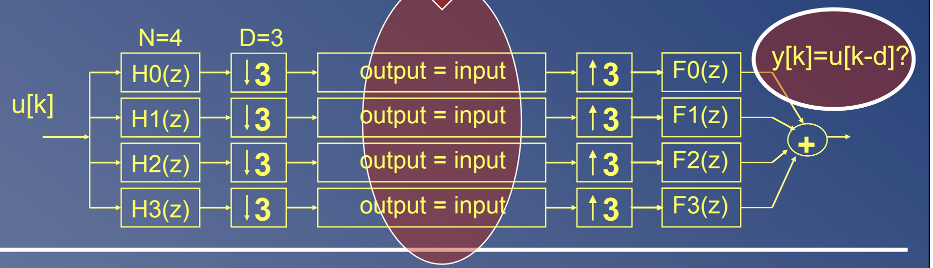
So this is the picture to keep in mind...



Filter Bank Set-Up

A crucial concept concept will be Perfect Reconstruction (PR)

- Assume subband processing does not modify subband signals (e.g. lossless coding/decoding)
- The overall aim would then be to have PR, i.e. that the output signal is equal to the input signal up to at most a delay: y[k]=u[k-d]
- But: downsampling introduces aliasing, so achieving PR will be non-trivial



Filter Bank Applications

- **Subband coding :**

Coding = Fullband signal split into subbands & downsampled
(=analysis filters + decimators)
subband signals separately encoded
(e.g. subband with smaller energy content encoded with fewer bits)

Decoding = reconstruction of subband signals, then fullband
signal synthesis (=expanders + synthesis filters)

Example : **Image coding** (e.g. wavelet filter banks)

Example : **Audio coding**
e.g. digital compact cassette (DCC), MiniDisc, MPEG, ...
Filter bandwidths and bit allocations chosen to further exploit perceptual properties of human hearing (perceptual coding, masking, etc.)

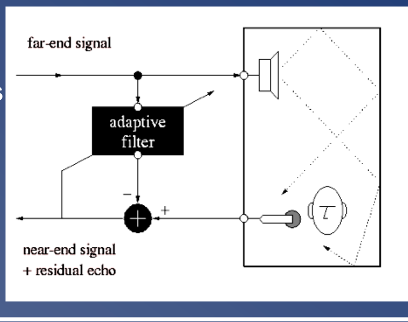
Filter Bank Applications

- **Subband adaptive filtering** :
  - Example : Acoustic echo cancellation
    Adaptive filter models (time-varying) acoustic echo path and produces a copy of the echo, which is then subtracted from microphone signal.

  = Difficult problem !

  ✪ long acoustic impulse responses

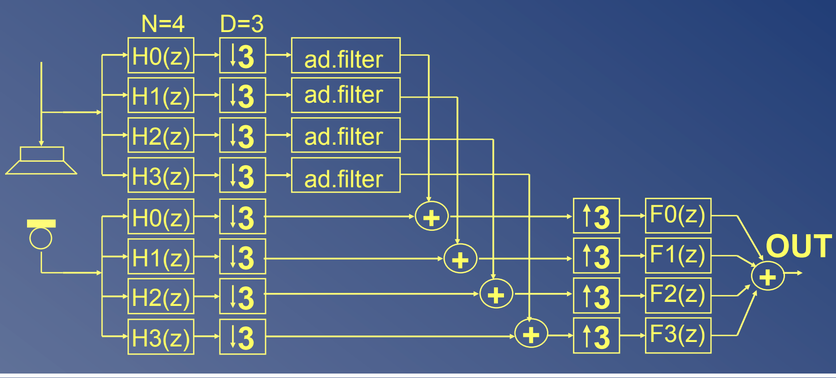
  ✪ time-varying



Filter Bank Applications

- Subband filtering = N (simpler) subband modeling problems instead of one (more complicated) fullband modeling problem

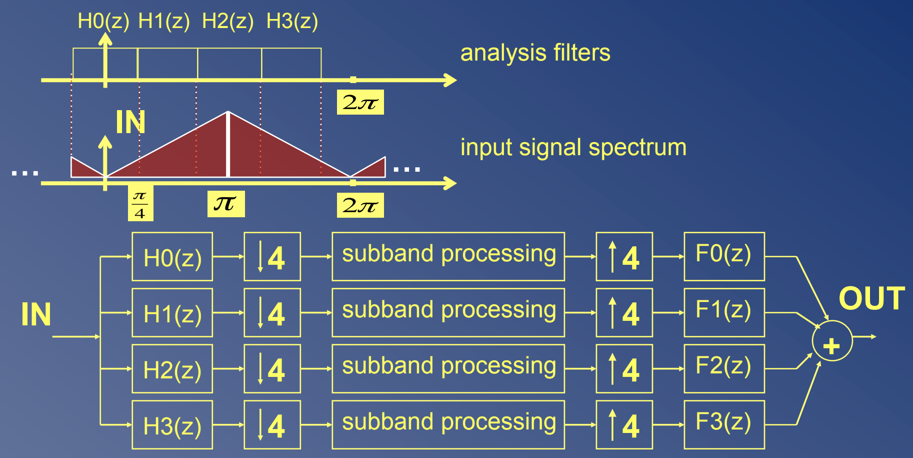
- Perfect Reconstruction (PR) guarantees distortion-free desired near-end speech signal



Ideal Filter Bank Operation

$D = N = 4 \ \ (*)$

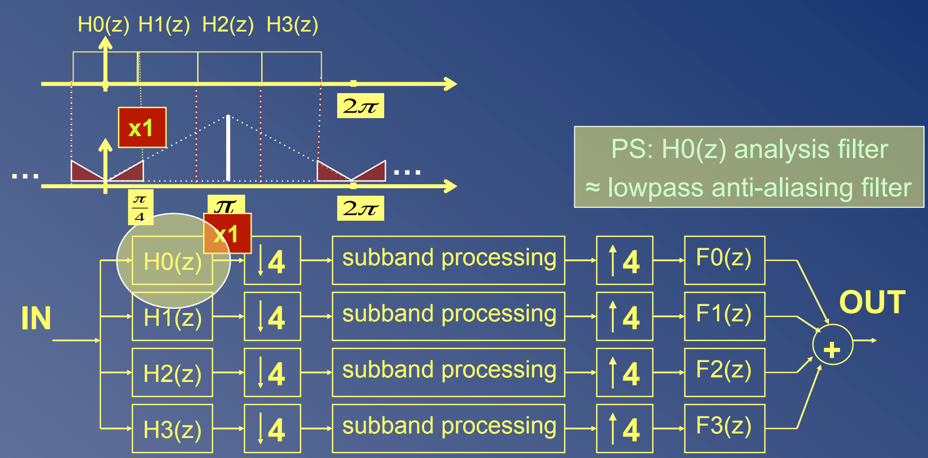
- With ideal analysis/synthesis filters, filter bank operates as follows (1)



(*) Similar figures for other D=N & for oversampled (D<N) case

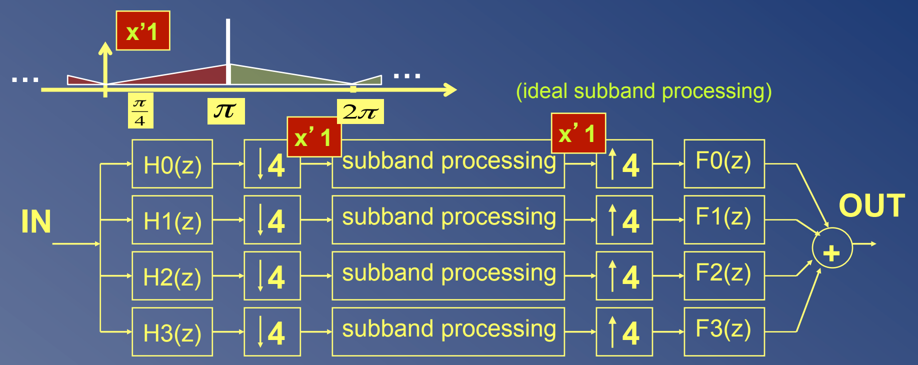
Ideal Filter Bank Operation

- With ideal analysis/synthesis filters, filter bank operates as follows (2)



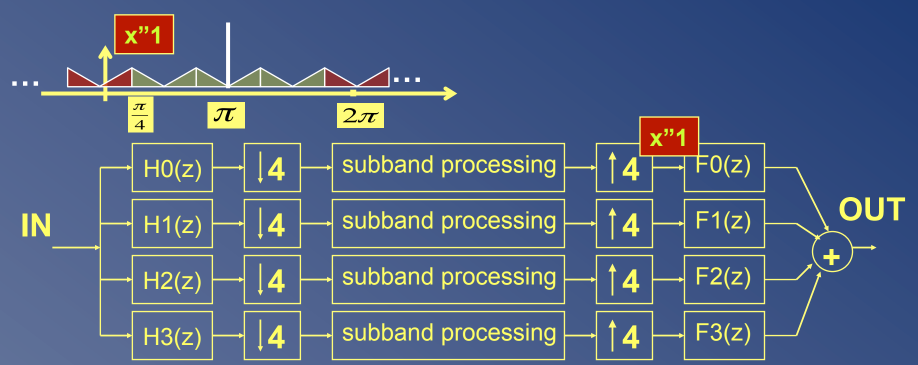
Ideal Filter Bank Operation

- With ideal analysis/synthesis filters, filter bank operates as follows (3)



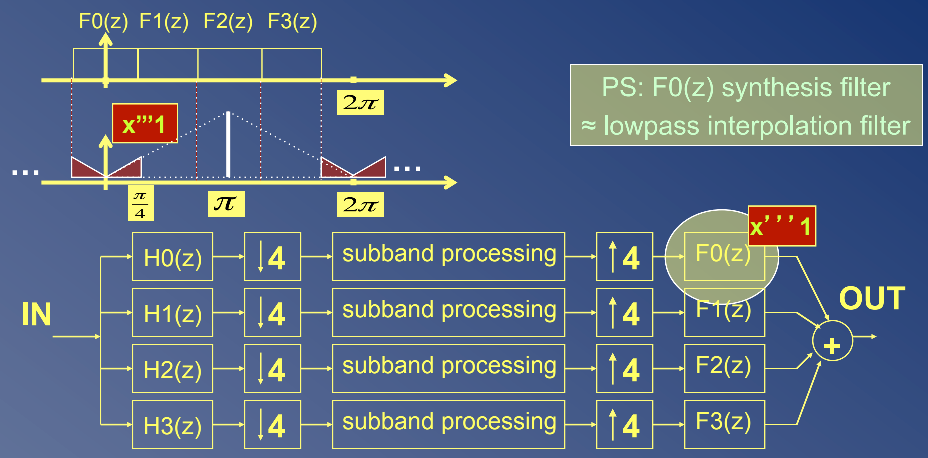
Ideal Filter Bank Operation

- With ideal analysis/synthesis filters, filter bank operates as follows (4)



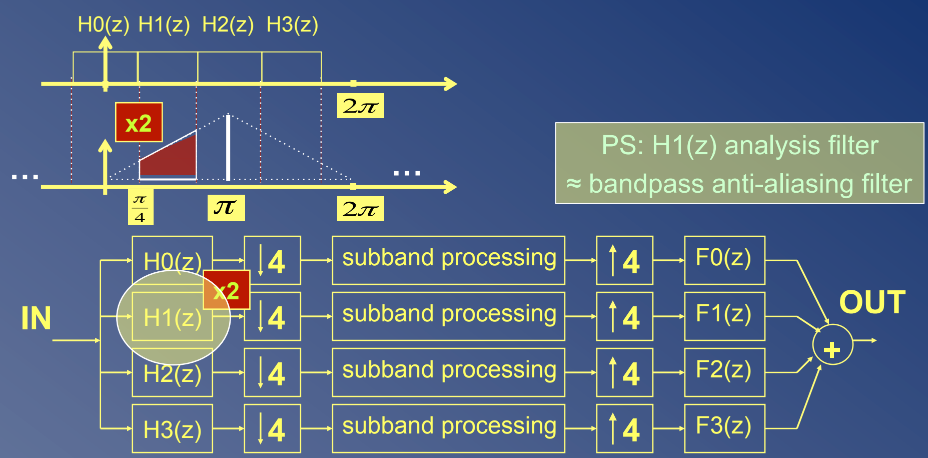
Ideal Filter Bank Operation

- With ideal analysis/synthesis filters, filter bank operates as follows (5)



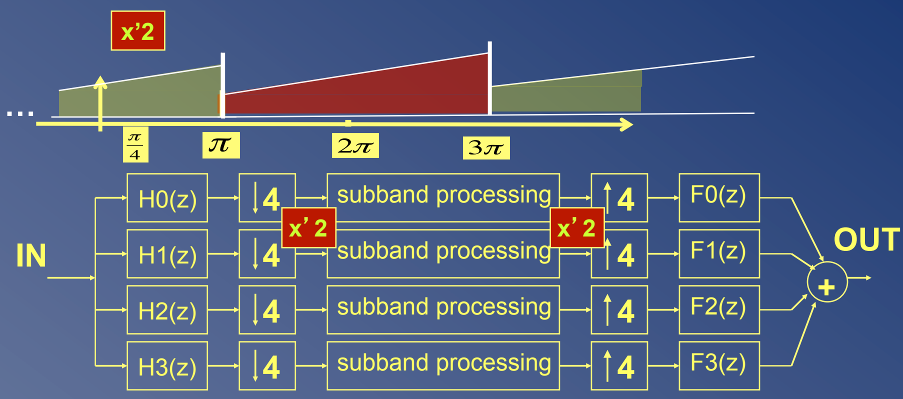
Ideal Filter Bank Operation

- With ideal analysis/synthesis filters, FB operates as follows (6)



Ideal Filter Bank Operation

- With ideal analysis/synthesis filters, filter bank operates as follows (7)



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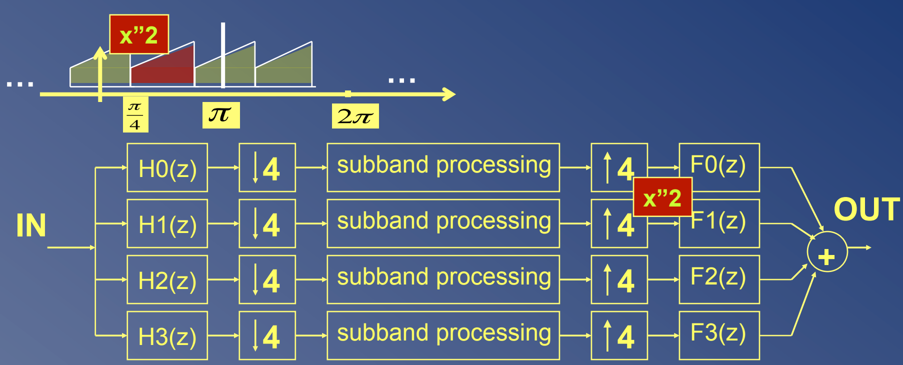
Ideal Filter Bank Operation

- With ideal analysis/synthesis filters, filter bank operates as follows (8)



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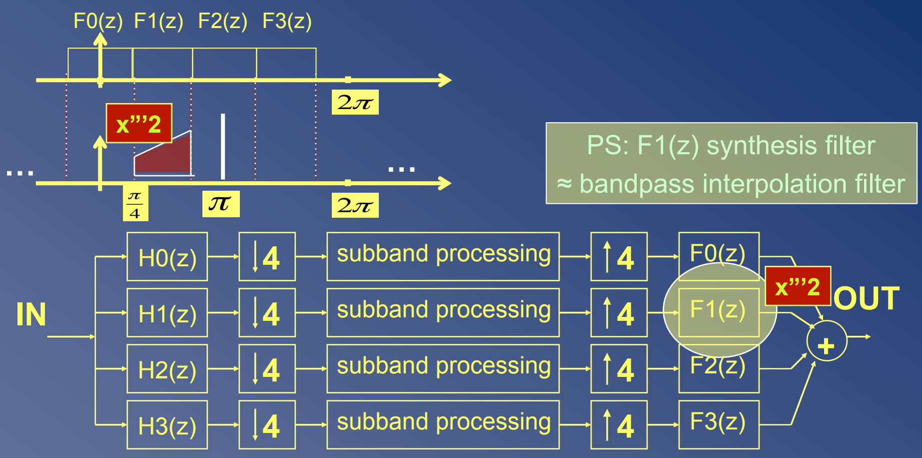
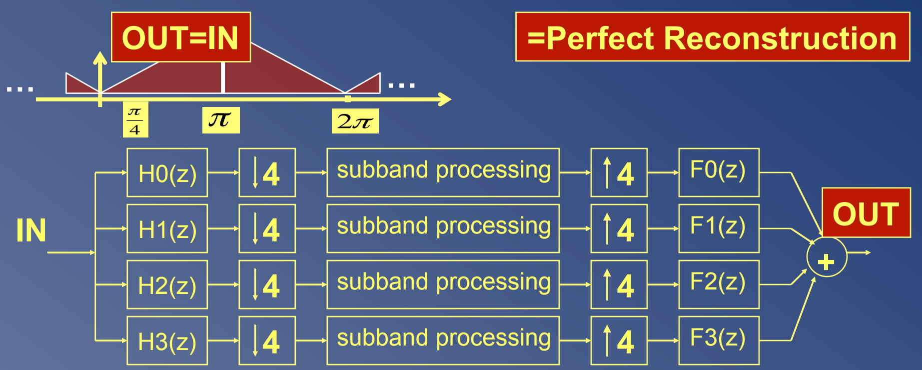
Ideal Filter Bank Operation

- With ideal analysis/synthesis filters, filter bank operates as follows (9)



Ideal Filter Bank Operation

- With ideal analysis/synthesis filters, filter bank operates as follows (10)



Now try this with non-ideal filters…?

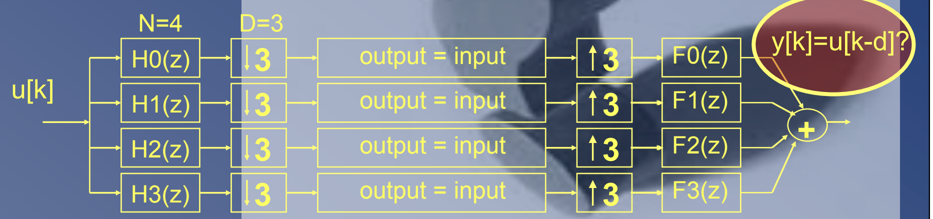
Non-Ideal Filter Bank Operation

Question :

Can y[k]=u[k-d] be achieved with <u>non-ideal</u> filters i.e. in the presence of aliasing ?

Answer :

YES !! **<u>Perfect Reconstruction Filter Banks</u>** (PR-FB) with synthesis bank designed to <u>remove</u> aliasing effects !

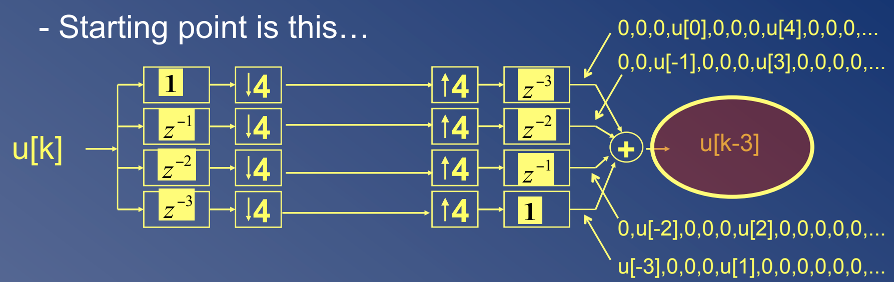


Non-Ideal Filter Bank Operation

$D = N = 4 \quad (*)$

A very simple PR-FB is constructed as follows

- Starting point is this…



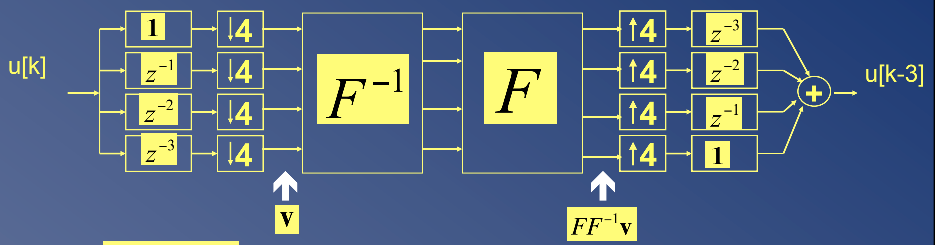
As y[k]=u[k-d] this can be viewed as a (1st) (maximally decimated) PR-FB (with lots of aliasing in the subbands!)

All analysis/synthesis filters are seen to be pure delays, hence are not frequency selective (i.e. far from ideal case with ideal bandpass filters, not yet very interesting….)

(*) Similar figures for other D=N

Non-Ideal Filter Bank Operation

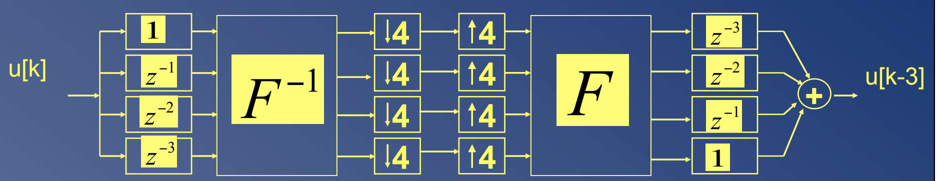
- Now insert DFT-matrix (discrete Fourier transform) and its inverse (I-DFT)...



as $F.F^{-1} = I$ this clearly does not change the input-output relation (hence PR property preserved)

Non-Ideal Filter Bank Operation

- …and reverse order of decimators/expanders and DFT-matrices (not done in an efficient implementation!) :



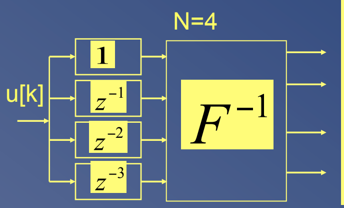
=analysis filter bank =synthesis filter bank

This is the `**DFT/IDFT filter bank**'

It is a first (or 2nd) example of a (maximally decimated) PR-FB!

Non-Ideal Filter Bank Operation

What do analysis filters look like? (N-channel case)



$$\begin{bmatrix} H_0(z) \\ H_1(z) \\ H_2(z) \\ \vdots \\ H_{N-1}(z) \end{bmatrix} = \frac{1}{N} \begin{bmatrix} W^0 & W^0 & W^0 & \dots & W^0 \\ W^0 & W^{-1} & W^{-2} & \dots & W^{-(N-1)} \\ W^0 & W^{-2} & W^{-4} & \dots & W^{-2(N-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ W^0 & W^{-(N-1)} & W^{-2(N-1)} & \dots & W^{-(N-1)^2} \end{bmatrix} . \begin{bmatrix} 1 \\ z^{-1} \\ z^{-2} \\ \vdots \\ z^{-N+1} \end{bmatrix}$$

$$W = e^{-j2\pi/N}$$

This is seen/known to represent a collection of filters Ho(z),H1(z),..., each of which is a frequency shifted version of Ho(z) :

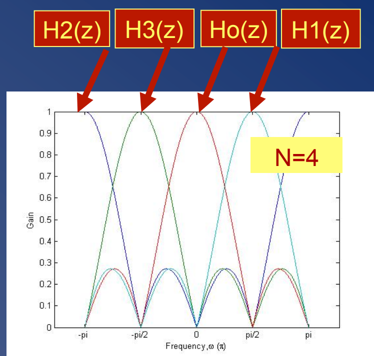
$$H_n(e^{j\omega}) = H_0(e^{j(\omega - n.(2\pi/N))}) \qquad H_0(z) = \frac{1}{N}.(1 + z^{-1} + z^{-2} + \dots + z^{-N+1})$$

i.e. the H_n are obtained by uniformly shifting the `prototype' Ho over the frequency axis.

Non-Ideal Filter Bank Operation

The prototype filter Ho(z) is a not-so-great **lowpass filter** with significant sidelobes. Ho(z) and Hi(z)'s are thus far from ideal lowpass/bandpass filters.

Synthesis filters are shown to be equal to analysis filters (up to a scaling)



Hence (maximal) decimation introduces significant ALIASING in the decimated subband signals

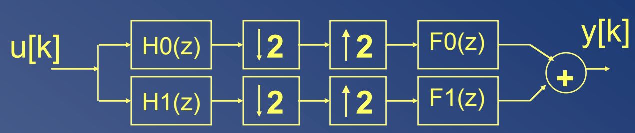
Still, we know this is a PR-FB (see construction previous slides), which means the synthesis filters can apparently restore the aliasing distortion.

This is remarkable, it means PR can be achieved even with non-ideal filters!

Now comes the hard part…(?)

- 2-channel case:
  Simple (maximally decimated, D=N) example to start with…
- N-channel case:
  Polyphase decomposition based approach

$D = N = 2$



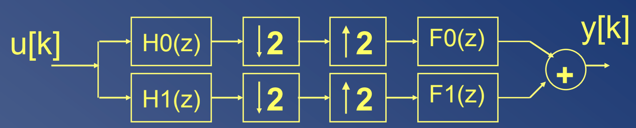
It is proved that... (try it!)

$$Y(z) = \underbrace{\frac{1}{2}.\{H_0(z).F_0(z) + H_1(z)F_1(z)\}}_{T(z)}.U(z) + \underbrace{\frac{1}{2}.\{H_0(-z).F_0(z) + H_1(-z)F_1(z)\}}_{A(z)}.U(-z)$$

- U(-z) represents aliased signals (*), hence
  A(z) is referred to as `alias transfer function'
- T(z) referred to as `distortion function' (amplitude & phase distortion)
  Note that T(z) is also the transfer function obtained after removing the up- and downsampling (up to a scaling) (!)

(*) $U(-z)\big|_{z=e^{j\omega}} = U(-e^{j\omega}) = U(e^{j(\omega+\pi)})$

Perfect Reconstruction : 2-Channel Case



- Requirement for `alias-free' filter bank :

  $$A(z) = 0$$

  If A(z)=0, then Y(z)=T(z).U(z)
  hence the complete filter bank behaves as a **LTI system**
  (despite/without up- & downsampling)!

- Requirement for `perfect reconstruction' filter bank
  (= alias-free + distortion-free):

  $$A(z) = 0 \quad + \quad T(z) = z^{-\delta}$$

Perfect Reconstruction : 2-Channel Case

- A solution is as follows: (ignore details) [Smith&Barnwell 1984] [Mintzer 1985]

  i) $F_0(z) = H_1(-z), \qquad F_1(z) = -H_0(-z)$

  so that (alias cancellation) $A(z) = \ldots = 0$ ☺

  ii) `power symmetric' Ho(z) (real coefficients case)

  $$\left|H_0(e^{j(\frac{\pi}{2}+\omega)})\right|^2 + \left|H_0(e^{j(\frac{\pi}{2}-\omega)})\right|^2 = 1$$

  iii) $h_1[k] = (-1)^k . h_0[L-k]$ ☺

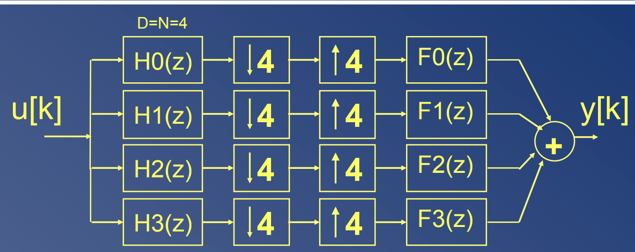
  so that (distortion function) $T(z) = \ldots = 1$ *ignore the details!*

This is a so-called `*paraunitary*' perfect reconstruction bank (see below), based on a *lossless system* Ho,H1 : $\left|H_0(e^{j\omega})\right|^2 + \left|H_1(e^{j\omega})\right|^2 = 1$

This is already pretty complicated…

Perfect Reconstruction : N-Channel Case

$D = N$



It is proved that... (try it!)

$$Y(z) = \underbrace{\frac{1}{N}.\{\sum_{n=0}^{N-1} H_n(z).F_n(z)\}}_{T(z)}.U(z) + \frac{1}{N}.\sum_{n=1}^{N-1}\underbrace{\{\sum_{\bar{n}=0}^{N-1} H_{\bar{n}}(z.W^n).F_{\bar{n}}(z)\}}_{A_n(z)}.U(z.W^n)$$

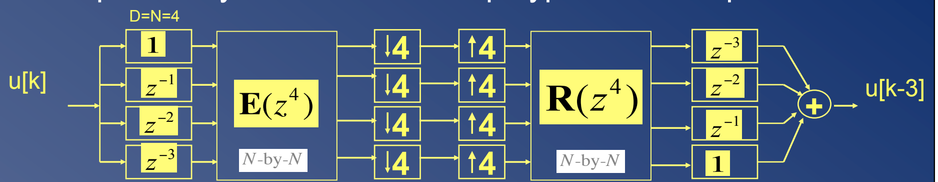
- 2nd term represents aliased signals, hence all `alias transfer functions' $A_n(z)$ should ideally be **zero** (for all n=1..N-1)
- T(z) is referred to as `distortion function' (amplitude & phase distortion). For perfect reconstruction, T(z) should be a **pure delay**

Sigh !!…Too Complicated!!...

Perfect Reconstruction Theory

$D = N$

A simpler analysis results from a polyphase description :



$$\begin{bmatrix} H_0(z) \\ \vdots \\ H_{N-1}(z) \end{bmatrix} = \overbrace{\begin{bmatrix} E_{0|0}(z^N) & \dots & E_{0|N-1}(z^N) \\ \vdots & & \vdots \\ E_{N-1|0}(z^N) & \dots & E_{N-1|N-1}(z^N) \end{bmatrix}}^{\mathbf{E}(z^N)} . \begin{bmatrix} 1 \\ \vdots \\ z^{-(N-1)} \end{bmatrix}$$

n-th row of E(z) has N-fold (=D-fold) polyphase components of $H_n(z)$

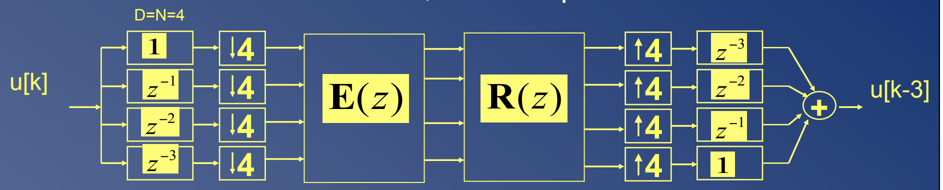
$$\begin{bmatrix} F_0(z) \\ \vdots \\ F_{N-1}(z) \end{bmatrix}^T = \begin{bmatrix} z^{-(N-1)} \\ \vdots \\ 1 \end{bmatrix}^T . \overbrace{\begin{bmatrix} R_{0|0}(z^N) & \dots & R_{N-1|0}(z^N) \\ \vdots & & \vdots \\ R_{0|N-1}(z^N) & \dots & R_{N-1|N-1}(z^N) \end{bmatrix}}^{\mathbf{R}(z^N)}$$

n-th column of R(z) has N-fold polyphase components of $F_n(z)$

Do not continue until you understand how formulae correspond to block scheme!

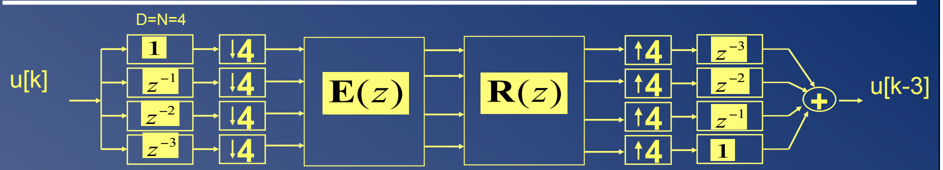
Perfect Reconstruction Theory

- With the `noble identities’, this is equivalent to:



Necessary & sufficient conditions for
i) alias cancellation
ii) perfect reconstruction
are then derived, based on the product $\mathbf{R}(z).\mathbf{E}(z)$

Perfect Reconstruction Theory



i) Necessary & sufficient condition for alias-free FB is…:

$$\mathbf{R}(z).\mathbf{E}(z) = \text{`pseudo-circulant'}$$

a pseudo-circulant matrix is a circulant matrix with the additional feature that elements below the main diagonal are multiplied by 1/z, i.e.

D=N=4

$$\mathbf{R}(z).\mathbf{E}(z) = \begin{bmatrix} p_0(z) & p_1(z) & p_2(z) & p_3(z) \\ z^{-1}.p_3(z) & p_0(z) & p_1(z) & p_2(z) \\ z^{-1}.p_2(z) & z^{-1}.p_3(z) & p_0(z) & p_1(z) \\ z^{-1}.p_1(z) & z^{-1}.p_2(z) & z^{-1}.p_3(z) & p_0(z) \end{bmatrix}$$

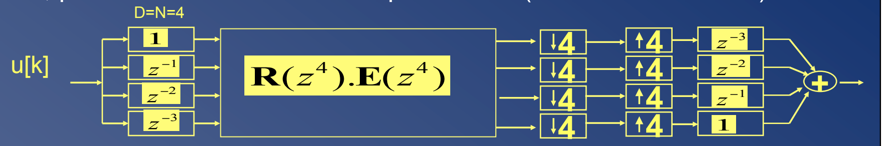
& then 1^{st} row of R(z).E(z) are polyphase cmpnts of `distortion function’ T(z)

Read on ➔

Perfect Reconstruction Theory

Read on ➔ This can be verified as follows:

First, previous block scheme is equivalent to (cfr. Noble identities)

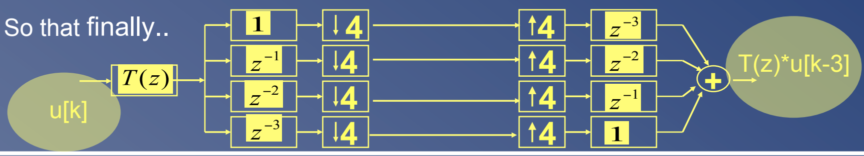


Then (iff R.E is pseudo-circ.)…

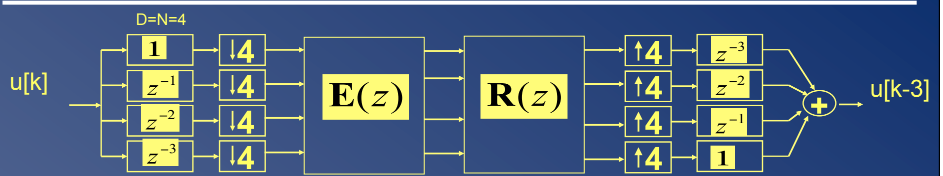
D=N=4

$$\mathbf{R}(z^4).\mathbf{E}(z^4).\begin{bmatrix}1\\ z^{-1}\\ z^{-2}\\ z^{-3}\end{bmatrix}.U(z)=\ldots=\begin{bmatrix}1\\ z^{-1}\\ z^{-2}\\ z^{-3}\end{bmatrix}.\underbrace{(p_0(z^4)+z^{-1}p_1(z^4)+z^{-2}p_2(z^4)+z^{-3}p_3(z^4))}_{T(z)}.U(z)$$

So that finally..



Perfect Reconstruction Theory



ii) Necessary & sufficient condition for **PR** is then…

(i.e. where T(z)=pure delay, hence $p_r(z)$=pure delay=$z^{-\delta}$, and all other $p_n(z)$=0)

$$\mathbf{R}(z).\mathbf{E}(z)=\begin{bmatrix}0 & z^{-\delta}I_{N-r}\\ z^{-\delta-1}.I_r & 0\end{bmatrix},\quad 0\le r\le N-1$$

I_n is nxn identity matrix, r is arbitrary

Example (r=0) :

$$\mathbf{R}(z).\mathbf{E}(z)=z^{-\delta}I_N$$

for conciseness, will use this from now on !

➔ PR-FB design in chapter-12

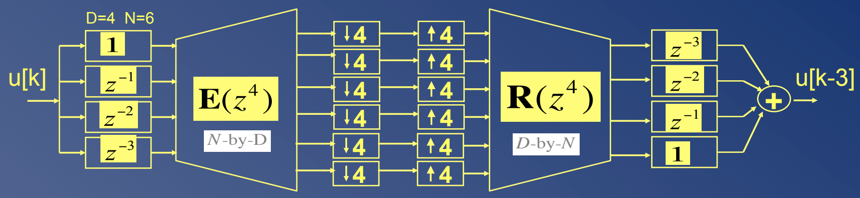
Beautifully simple!! (compared to page 33)

Perfect Reconstruction Theory

$D < N$

A similar PR condition can be derived for oversampled FBs

The polyphase description (compare to p.34) is then…



$$\begin{bmatrix} H_0(z) \\ \vdots \\ H_{N-1}(z) \end{bmatrix} = \underbrace{\begin{bmatrix} E_{0|0}(z^D) & \dots & E_{0|D-1}(z^D) \\ \vdots & & \vdots \\ E_{N-1|0}(z^D) & \dots & E_{N-1|D-1}(z^D) \end{bmatrix}}_{\mathbf{E}(z^D)} \cdot \begin{bmatrix} 1 \\ \vdots \\ z^{-(D-1)} \end{bmatrix}$$

n-th row of E(z) has D-fold polyphase components of Hn(z)

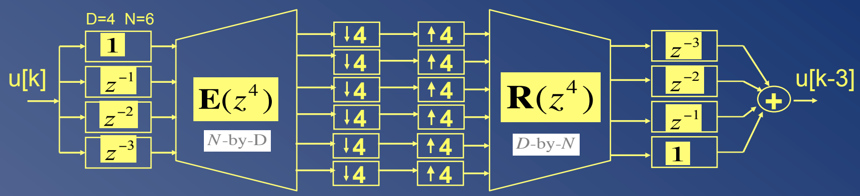
$$\begin{bmatrix} F_0(z) \\ \vdots \\ F_{N-1}(z) \end{bmatrix}^T = \begin{bmatrix} z^{-(D-1)} \\ \vdots \\ 1 \end{bmatrix}^T \cdot \underbrace{\begin{bmatrix} R_{0|0}(z^D) & \dots & R_{N-1|0}(z^D) \\ \vdots & & \vdots \\ R_{0|D-1}(z^D) & \dots & R_{N-1|D-1}(z^D) \end{bmatrix}}_{\mathbf{R}(z^D)}$$

n-th column of R(z) has D-fold polyphase components of Fn(z)

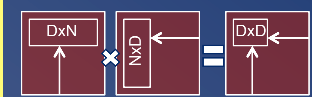
Note that **E** is an N-by-D ('tall-thin') matrix, **R** is a D-by-N ('short-fat') matrix !

Perfect Reconstruction Theory

Simplified (r=0 on p.38) condition for **PR** is then…



$$\mathbf{R}(z).\mathbf{E}(z) = z^{-\delta} I_D$$



In the D=N case (p.38), the PR condition has a product of square matrices. PR-FB design (Chapter 11) will then involve matrix inversion, which is mostly problematic.

In the D<N case, the PR condition has a product of a 'short-fat' matrix and a 'tall-thin' matrix. This will lead to additional PR-FB design flexibility (see Chapter 12).

Again beautifully simple!! (compared to page 33)