Linear Algebra and Complementarity Problems

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Back to the Roots

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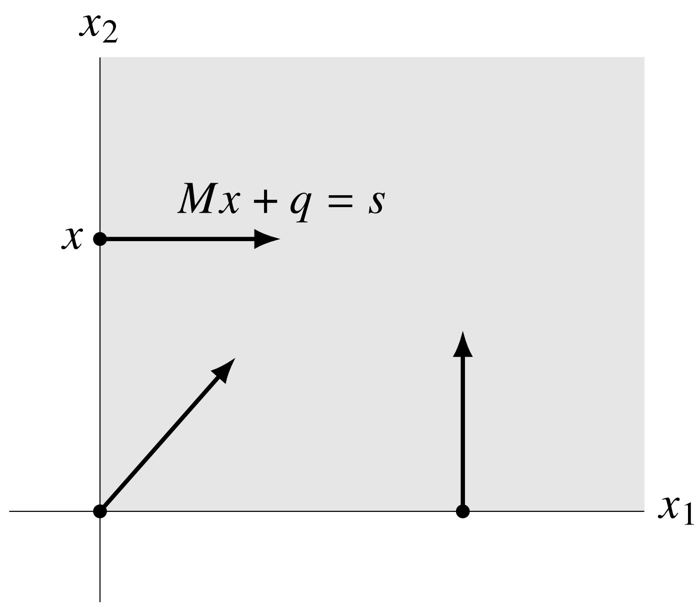
Linear complementarity problem (LCP)

$$s = Mx + q, \qquad 0 \leq x \perp s \geq 0$$

- variables are n-vectors x, s
- $x \perp s$ is the *complementarity* condition: $x_k s_k = 0$ for $k = 1, \ldots, n$



recognized and studied as a fundamental problem in optimization since 1960s

[Cottle and Dantzig, 1968]

Applications

Early applications (1960s)

- extends linear and quadratic programming
- Nash equilibrium in bimatrix games

Equilibrium models (economics, game theory, traffic networks, ...)

Engineering applications

- piecewise-linear circuits and systems
- hybrid systems
- contact mechanics

Linear program as LCP

Primal: minimize $c^T x$
subject to $Ax \leq b$
$x \geq 0$

Dual: maximize $-b^T y$
subject to $-A^T y \leq c$
$y \geq 0$

Optimality conditions

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 & A^T \\ -A & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ b \end{bmatrix}, \qquad 0 \leq \begin{bmatrix} x \\ y \end{bmatrix} \perp \begin{bmatrix} u \\ v \end{bmatrix} \geq 0$$

an LCP with skew-symmetric coefficient matrix

Quadratic program as LCP

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}x^T P x + q^T x \\ \text{subject to} & Ax \leq b \\ & x \geq 0 \end{array}$$

(P symmetric positive semidefinite)

Optimality conditions

$$\left[\begin{array}{c} u \\ v \end{array}\right] = \left[\begin{array}{cc} P & A^T \\ -A & 0 \end{array}\right] \left[\begin{array}{c} x \\ y \end{array}\right] + \left[\begin{array}{c} q \\ b \end{array}\right], \qquad 0 \leq \left[\begin{array}{c} x \\ y \end{array}\right] \perp \left[\begin{array}{c} u \\ v \end{array}\right] \geq 0$$

an LCP with (nonsymmetric) positive semidefinite coefficient matrix

Algorithms

Monotone LCP: M positive semidefinite (not necessarily symmetric)

- interior-point methods
- first-order splitting methods

Copositive LCP: M is a copositive matrix

$$x^T M x \geq 0 \quad \text{for all } x \geq 0$$

- Lemke's pivoting algorithm (1965): a continuation (homotopy) algorithm
- also useful as a constructive proof of an important existence theorem

Local methods: various first and second order methods

Complete solution: algorithm by Bart De Moor and Joos Vandewalle (1987)

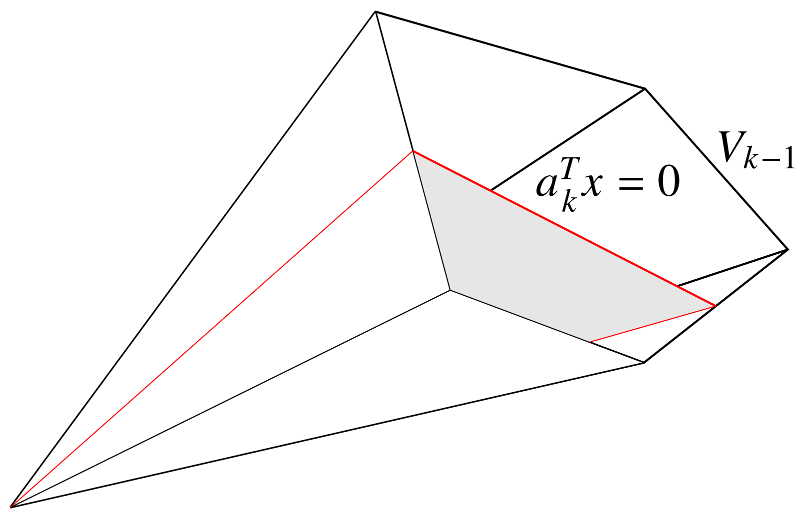
Double description method [Motzkin, Raiffa, Thompson, Thrall, 1953]

finds all extreme points of a polyhedron defined by linear inequalities

Example: polyhedral cone described by

$$x \geq 0, \qquad a_1^T x = 0, \qquad a_2^T x = 0, \qquad \ldots, \qquad a_m^T x = 0$$

- define $V_k = \{x \mid x \geq 0,\ a_1^T x = 0, \ldots, a_k^T x = 0\}$
- algorithm constructs extreme rays of V_k from extreme rays of V_{k-1}
- extreme rays of V_k are convex combinations of adjacent extreme rays of V_{k-1} on opposite sides of hyperplane $a_k^T x = 0$



Double description method to find all solutions of LCP

[De Moor and Vandewalle, 1987] [De Moor, Vandenberghe, Vandewalle, 1992]

Generalized LCP

$$x \geq 0, \qquad Ax = 0, \qquad \prod_{i \in I_k} x_i = 0, \quad k = 1, \ldots, l$$

variable x is n-vector; $I_1, \ldots, I_l$ are subsets of $\{1, 2, \ldots, n\}$

- solution set is a union of polyhedral cones
- at each step of DDM, discard extreme solutions that are not complementary
- this actually improves the efficiency
- further generalizations in Ph.D. of Bart De Schutter
  [De Schutter and De Moor, 1995]

Example in piecewise-linear circuit analysis

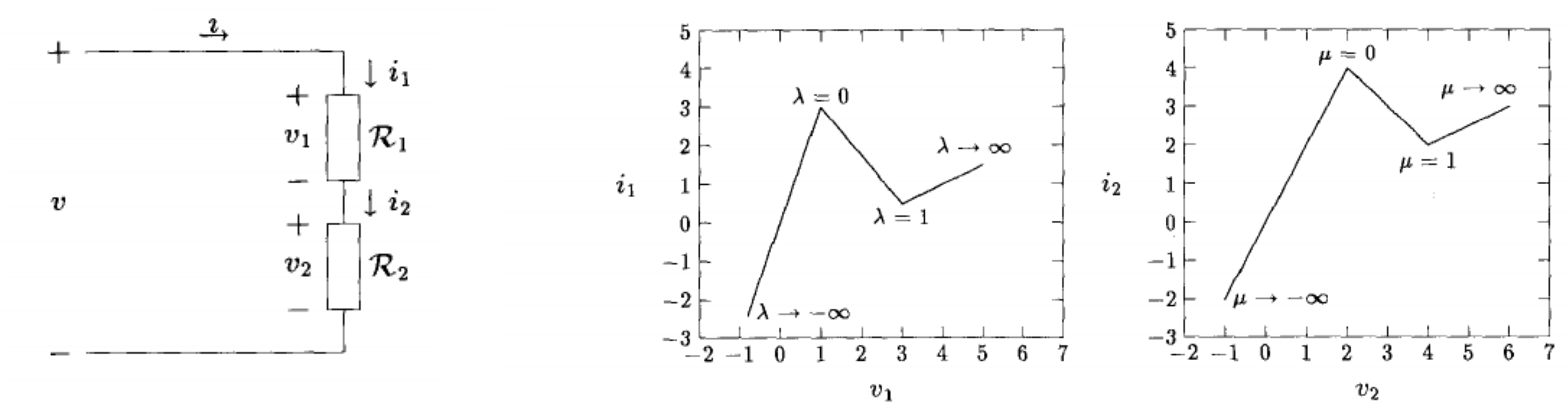


Fig. 1. Series connection of two piecewise linear resistors and their respective current-voltage characteristics.

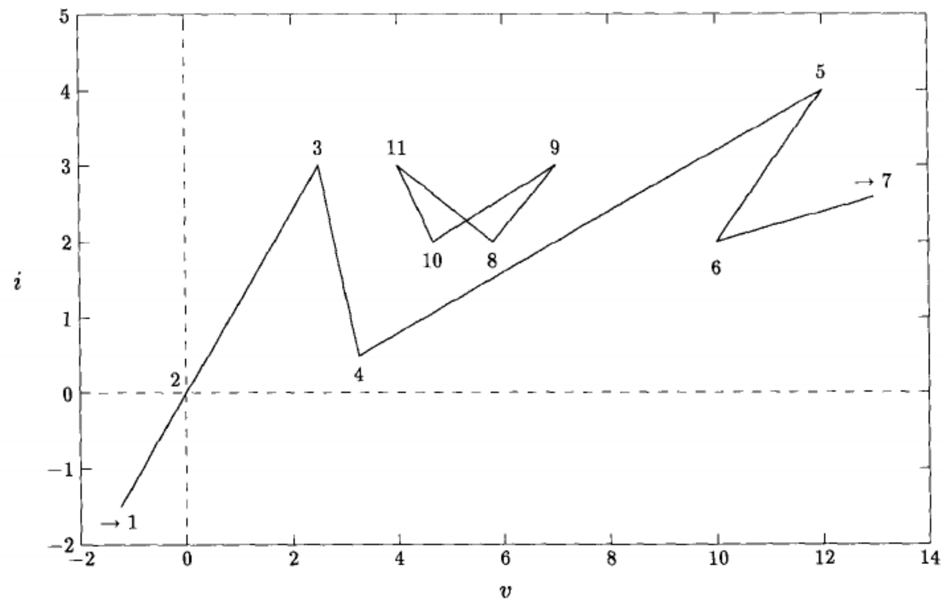


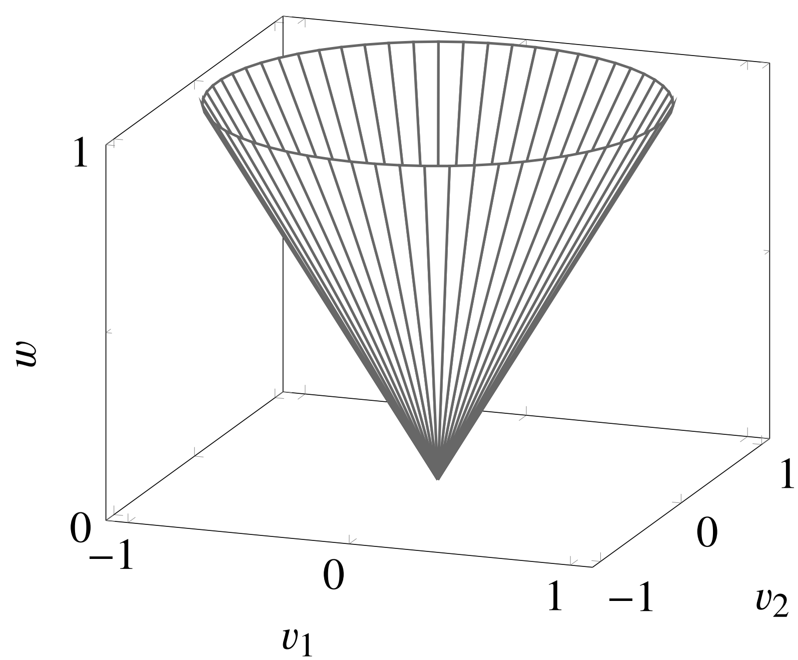
Fig. 2. Resulting current-voltage characteristic of the series connection circuit in Figure 1.

Conic LCP

$$s = Mx + q, \qquad 0 \preceq x \perp s \succeq_* 0$$

- $x \succeq 0$ means $x \in K$, where K is a convex cone
- $s \succeq_* 0$ means $s \in K^*$, where K^* is the dual cone
- important example: $K = K^*$ is product of second order cones

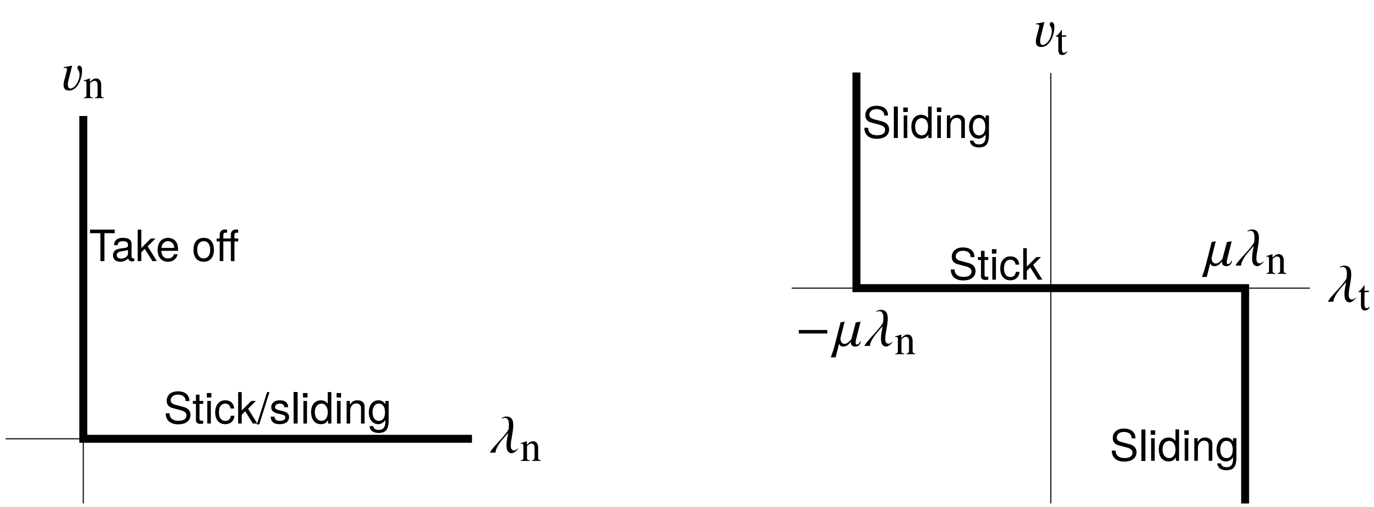
$$\mathcal{Q} = \{(v, w) \in \mathbf{R}^m \times \mathbf{R} \mid \|v\| \leq w\}$$



Conic LCP in mechanics: contact with Coulomb friction

States per contact (friction coefficient μ)		
Stick	$\|\lambda_t\| \leq \mu\lambda_n$	$(v_t, v_n) = 0$
Sliding	$\|\lambda_t\| = \mu\lambda_n$	$v_t \propto -\lambda_t,\ v_n = 0$
Take off	$(\lambda_t, \lambda_n) = 0$	$v_n \geq 0$

- $v_n \in \mathbf{R}$ is normal velocity, $v_t \in \mathbf{R}^2$ is tangential velocity
- $\lambda_n \in \mathbf{R}$ is normal force, $\lambda_t \in \mathbf{R}^2$ is tangential force



Frictional contact in two dimensions

Conic LCP in mechanics

- frictional contact can be modeled via conic complementarity (2nd order cone)
- leads to dynamical systems with inequality and complementarity constraints
- simulation requires solution of sequences of LCPs
- applications in computer animation and robotics
- important motivation for J.J. Moreau's pioneering work in convex analysis