From (second-order) matrices to (higher-order) tensors

Lieven De Lathauwer

Back to the roots
KU Leuven, Belgium, 8 July 2022

KU LEUVEN
STADIUS Center for Dynamical Systems, Signal Processing and Data Analytics
EOS THE EXCELLENCE OF SCIENCE
erc European Research Council
Established by the European Commission

Looking back: bachelor and master in engineering:

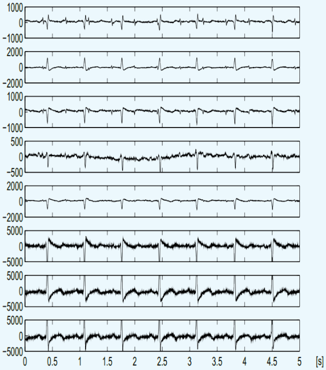
In the beginning, there was, ...

- Linear Algebra (Joos)
- Singular Value Decomposition (SVD)
- Applications (biomedical)

⋮

- Systems Theory
- System Identification
- Control (Bart)

⋮



Final year

- Master's thesis: Fetal Heart Rate Estimation (Dirk Callaerts)
- Student teaching assistant: linear algebra

Looking back: Ph.D. in engineering

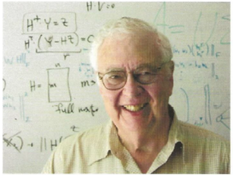
- Multilinear algebra and higher-order tensors
- SVD
- SISTA/STADIUS
- Joos and Bart
- Higher-order statistics, signal processing
- Numerical linear algebra (power method)
- Gene Golub, Stanford (Prof. SVD)

Gene Golub Around the World Commemoration

February 29, 2008

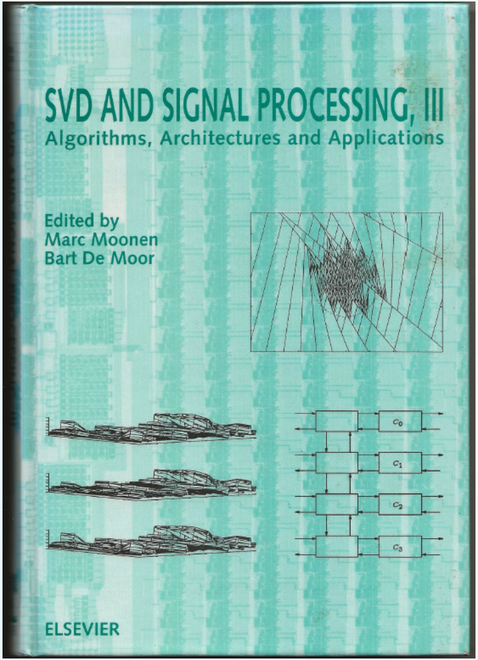
K.U.Leuven,
Dept. Electrical Engineering
Leuven, Belgium

B. De Moor, P. Van Dooren,
S. Van Huffel




ESPRIT

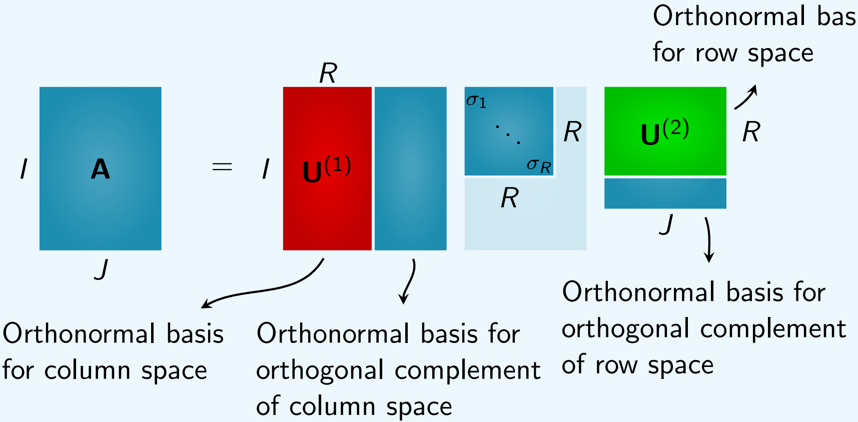

ymmetric
eralization to best rank-(P, Q, R) approxima-


SVD AND SIGNAL PROCESSING, III
Algorithms, Architectures and Applications
Edited by
Marc Moonen
Bart De Moor
ELSEVIER

Matrix Singular Value Decomposition

$$\mathbf{A} = \mathbf{U}^{(1)} \cdot \mathbf{S} \cdot \mathbf{U}^{(2)^\mathsf{T}}$$

$$\mathbf{A}^\mathsf{T} = \mathbf{U}^{(2)} \cdot \mathbf{S}^\mathsf{T} \cdot \mathbf{U}^{(1)^\mathsf{T}}$$



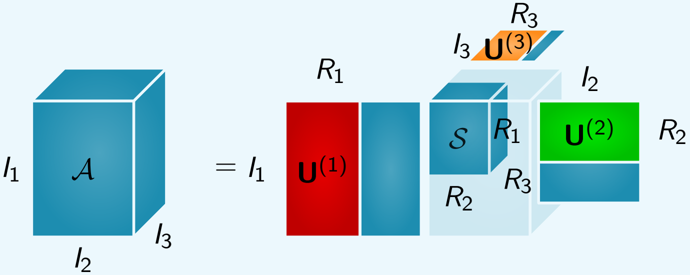
Multilinear Singular Value Decomposition

Definition:

$$\mathcal{A} = \mathcal{S} \bullet_1 \mathbf{U}^{(1)} \bullet_2 \mathbf{U}^{(2)} \bullet_3 \cdots \bullet_N \mathbf{U}^{(N)}$$

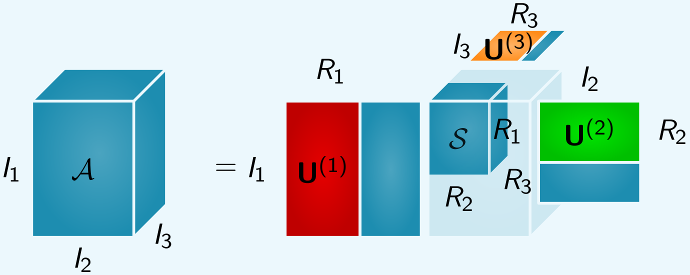
in which $\mathcal{S}$ is all-orthogonal and ordered
$\mathbf{U}^{(1)}, \mathbf{U}^{(2)}, \ldots, \mathbf{U}^{(N)}$ are orthogonal

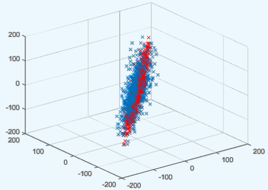
Surprising fact: $\mathcal{S}$ is not diagonal

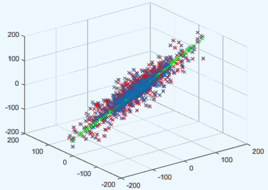


[Tucker 1964; De Lathauwer, De Moor, et al. 2000a]

From linear to multilinear PCA: dominant subspaces and compression

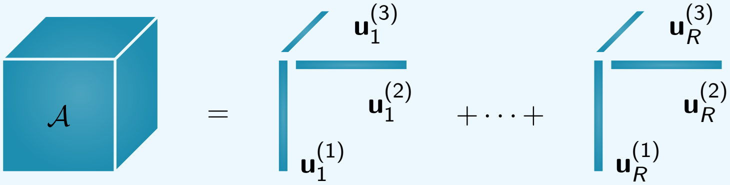






Canonical polyadic decomposition

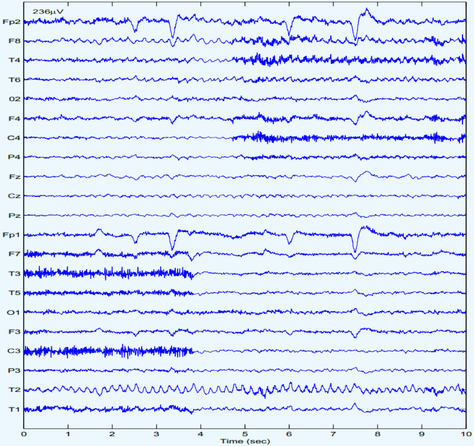
Definition: decomposition in minimal number of rank-1 terms [Harshman 1970; Carrol and Chang 1970]



Surprising fact: unique under mild conditions on number of terms and differences between terms

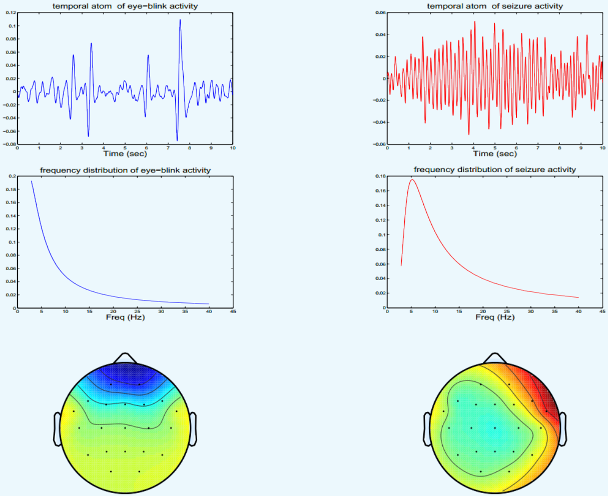
Additional constraints such as orthogonality, triangularity, ... are not required, but may be imposed.

Application: detection epileptic seizure in EEG



Tensorization: biorthogonal wavelet

Components: eye blink and epileptic activity



[De Vos, Vergult, et al. 2007; Acar, Aykut-Bingol, et al. 2007]

Canonical polyadic decomposition

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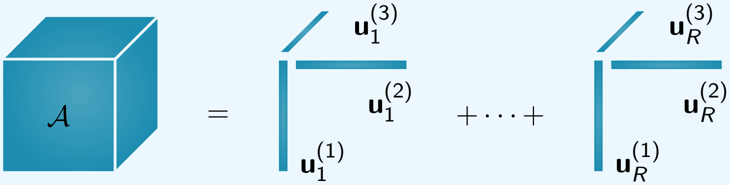
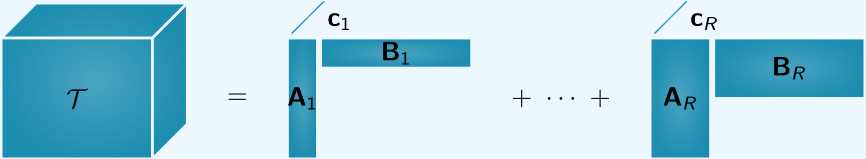


Surprising fact: unique under mild conditions on number of terms and differences between terms

Additional constraints such as orthogonality, triangularity, ... are not required, but may be imposed.

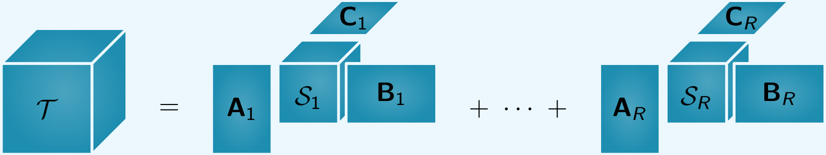
Decomposition in multilinear rank-$(L, L, 1)$ terms



Surprising fact: unique under mild conditions

[De Lathauwer 2008; De Lathauwer 2011; Sørensen and De Lathauwer 2015; Sørensen, Domanov, et al. 2015; Domanov and De Lathauwer 2020]

Decomposition in ML rank-(R_1, R_2, R_3) terms



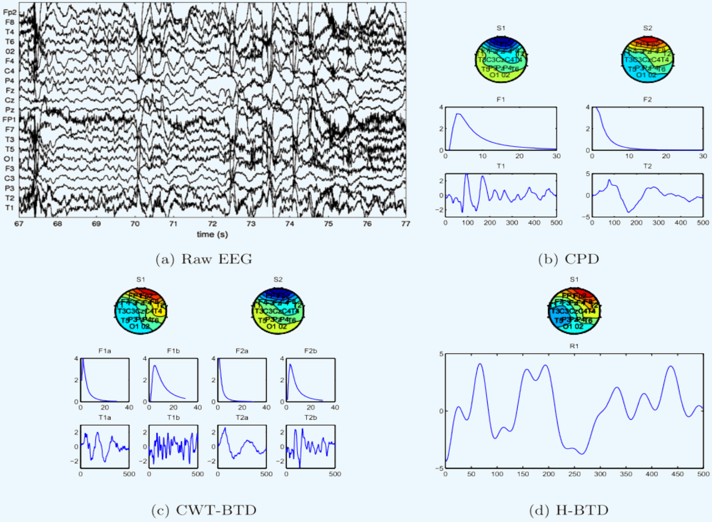
Unique under mild conditions

Rank $\leftrightarrow$ multilinear rank

Atoms (rank-1) $\leftrightarrow$ molecules (low ML rank)

[De Lathauwer 2008]

Application: detection epileptic seizure in EEG



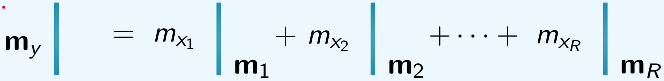
(a) Raw EEG (b) CPD

(c) CWT-BTD (d) H-BTD

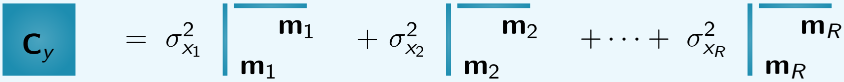
[Hunyadi, Camps, et al. 2014]

ICA overview

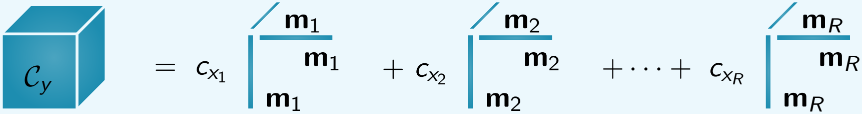
First order:



Second order:



Higher order:



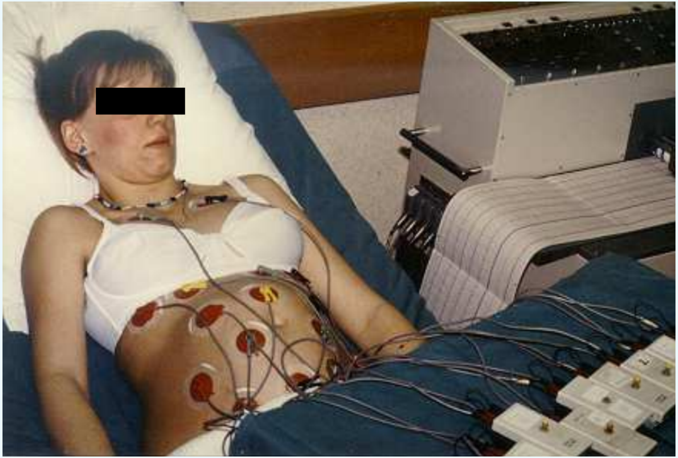
Algorithmic approaches:

higher order only

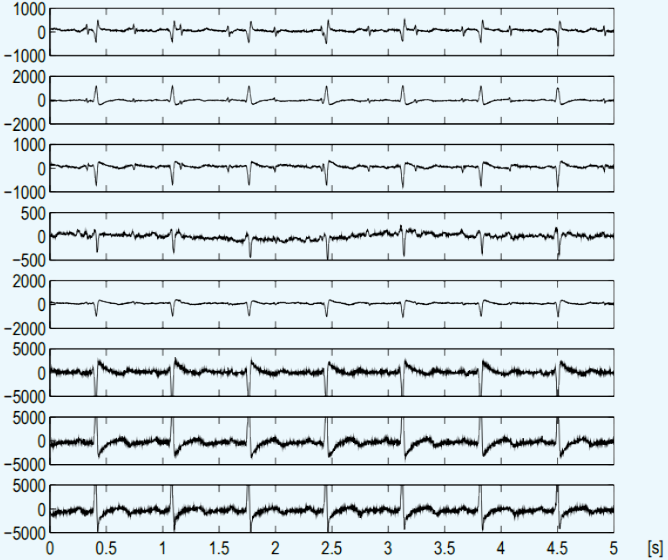
all orders jointly

first second order (prewhitening), then higher order

Application: Fetal ElectroCardiogram Extraction (FECG)

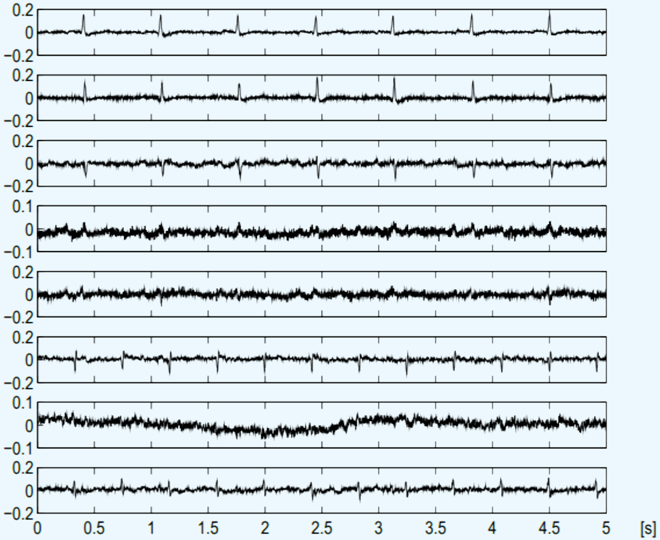


Abdominal and thoracic recordings



ICA results for FECG extraction

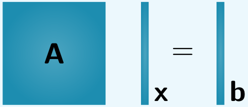
Independent components:



[De Lathauwer, De Moor, et al. 2000b]

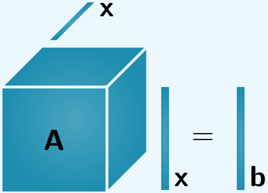
Linear and polynomial equations

Linear



$\sum_j a_{ij} x_j = b_i$

Quadratic



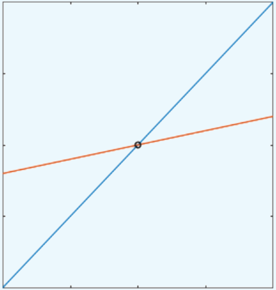
$\sum_{jk} a_{ijk} x_j x_k = b_i$

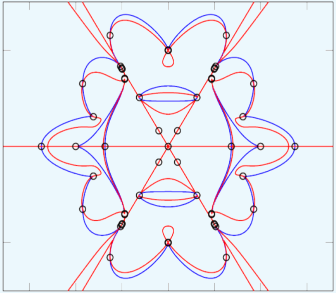
Polynomial

Tensor higher-order

$\sum_{j_1 j_2 \ldots j_D} a_{i j_1 j_2 \ldots j_D} x_{j_1} \ldots x_{j_D} = b_i$

From linear to multilinear/polynomial





Linear equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 - b_1 = 0 \\ a_{21}x_1 + a_{22}x_2 - b_2 = 0 \end{cases}$$

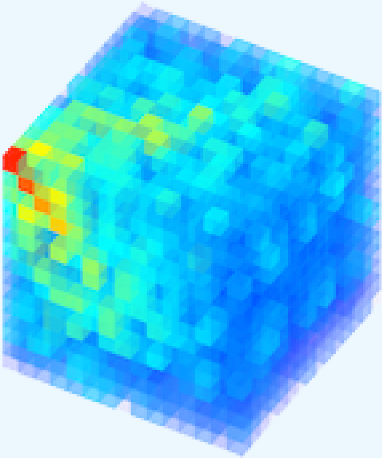
Polynomial equations

$$\begin{cases} p_1(x_1, x_2) = 0 \\ p_2(x_1, x_2) = 0 \end{cases}$$

[Sorber, Barel, et al. 2014]

Tensorlab

- Tensorlab 1: Tensor decompositions
- Complex Optimization Toolbox
  `www.esat.kuleuven.be/sista/cot`
- Tensorlab 2: Structured Data Fusion
- Tensorlab 3: Large-scale algorithms, tensorization
- Extensive documentation and demos

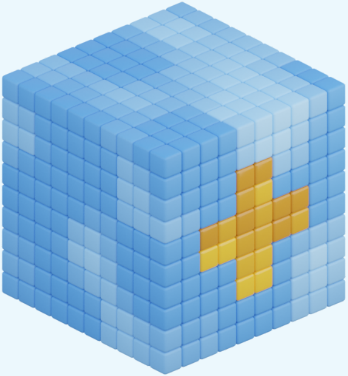


`www.tensorlab.net`

Announcement

Release Tensorlab+

- code supporting 34 papers
- 150+ experiments
- reproducing 180+ figures and table
- 32 tutorials and demos



`www.tensorlab.net`
`www.tensorlabplus.net`

Congratulations, Bart!

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