Back to the roots: a spectrum of what was realized

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Outline

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Eigenvalues and vectors: For matrix $A \in \mathbb{R}^{n \times n}$:

$$Ax = x\lambda \,, x \in \mathbb{C}^n, \lambda \in \mathbb{C}, x \neq 0$$

Characteristic equation - fundamental theorem of algebra:

$$p(\lambda) = \det(\lambda I_n - A) = \lambda^n + \alpha_1 \lambda^{n-1} + \ldots + \alpha_{n-1}\lambda + \alpha_n = 0$$

Since Galois, for $n \geq 5$: no solution in radicals !

Numerical linear algebra = iterative algorithms + finite precision machines

Cayley-Hamilton:

$$A^n + \alpha_1 A^{n-1} + \ldots + \alpha_{n-1} A + \alpha_n I_n = 0$$

Eigenvalue decomposition - Jordan Canonical Form (JCF):

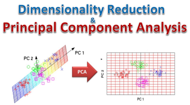
$$A = XJX^{-1}$$

Eigen-objects: Operator (object) = object × scalar

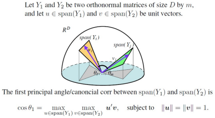
Continuous spectrum:

$d(e^{(\alpha \pm j\beta t)})/dt = (e^{(\alpha \pm j\beta t)})(\alpha \pm j\beta t)$, $(d\ ./dt + \int .\ dt)e^{\alpha t} = e^{\alpha t}(\frac{\alpha^2+1}{\alpha}\ )$

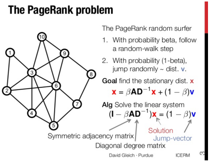
Discrete spectrum: e.g. standing waves



PCA



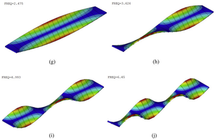
Can. Corr./Principal Angles



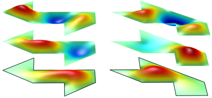
Graph spectral analysis

$$\frac{\partial^2 y}{\partial t^2} = \frac{1}{v_w^2} \frac{\partial^2 y}{dt^2}$$

Wave equation

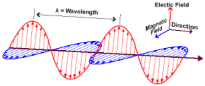


Modal shapes



Hear the shape of a drum?

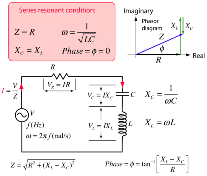
Answer: No !



Maxwell's laws

1. $\nabla \cdot \mathbf{D} = \rho_V$
2. $\nabla \cdot \mathbf{B} = 0$
3. $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$
4. $\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}$

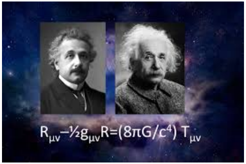
Maxwell's field equations



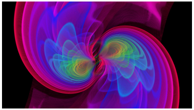
RLC circuits-resonances

$$H(t)|\psi(t)\rangle = i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle$$

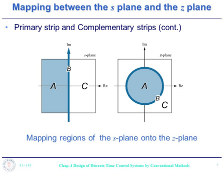
Schrödinger equation



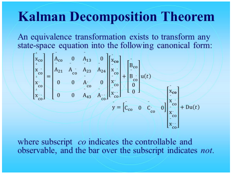
Matter curves spacetime moves matter



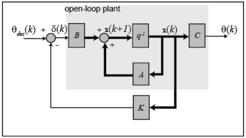
Gravitational waves



Stability



Controllability/observability



Pole placement

Observers	Kalman Filter Riccati Hamil. EVP	H_∞-filter Riccati Sympl. EVP
Control	LQR Riccati Hamil. EVP	H_∞-control Riccati Sympl. EVP

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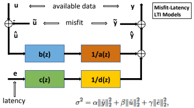
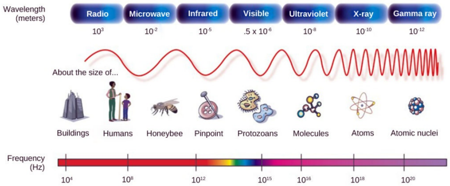
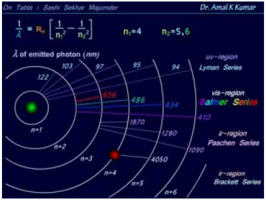
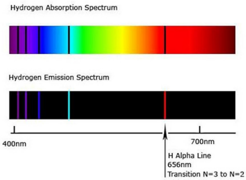


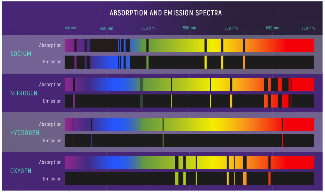
LS LTI System ID = EVP !

If you want to find the secrets of the universe, think in terms of energy, frequency and vibration. Nikola Tesla





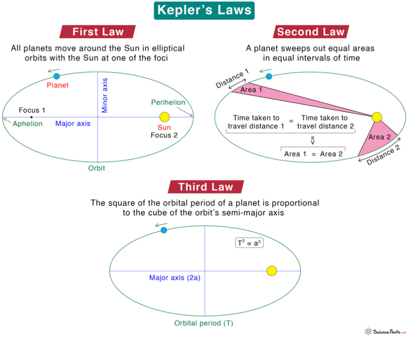




Outline

From Kepler to Newton: realization from data to (internal) state

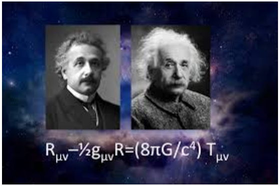
Kepler (1571-1630)



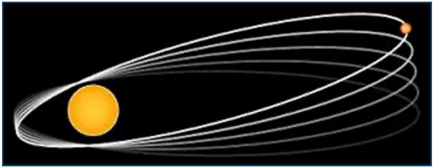
Newton (1642-1726)

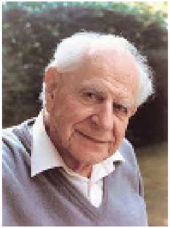
$$\mathbf{F} = m\,\mathbf{a}$$
$$F = G\,m\,M/r^2$$

Einstein and Popper



Einstein (1879-1955)





Popper (1902-1994)

Popper′s demarcation criterion :

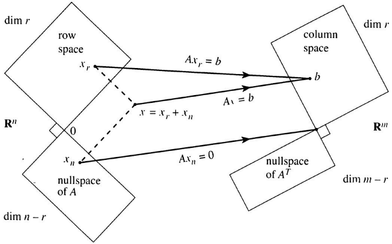
A model/theory is scientific when refutable

Models forbid more than they allow

In this talk, eigen-(multi-)spectra to...:

- characterize (multi-)shift invariant subspaces
- realize regular linear time-invariant state space models from 1D model-compliant data
- realize mD shift-invariant state space models from mD model-compliant data
- realize 1D state space models from data that are not model-compliant via mD realization

Fundamental Theorem of linear algebra: 4 matrix subspaces and their dimension/rank (SVD)



Fundamental Theorem of algebra: n roots

$$p(\lambda) = \det(\lambda I_n - A) = \lambda^n + \alpha_1 \lambda^{n-1} + \ldots + \alpha_{n-1}\lambda + \alpha_n = 0$$

The Theorems meet in Cayley - Hamilton

$$A^n + \alpha_1 A^{n-1} + \ldots + \alpha_{n-1} A + \alpha_n I_n = 0$$

Outline

Taylor / McLaurin series expansion

$$\begin{aligned} f(z) &= f(0) + \frac{f'(0)}{1!} z + \frac{f''(0)}{2!} z^2 + \ldots + \frac{f^{(k)}}{k!} z^k + \ldots \\ &= \gamma_0 + \gamma_1 z + \gamma_2 z^2 + \ldots + \gamma_k z^k + \ldots \end{aligned}$$

When rational in z^{-1} ?

$$\begin{aligned} f(z^{-1}) &= \frac{\beta_0 z^n + \beta_1 z^{n-1} + \ldots + \beta_{n-1} z + \beta_n}{\alpha_0 z^n + \alpha_1 z^{n-1} + \ldots + \alpha_{n-1} z + \alpha_n} \\ &= \frac{\sum_{i=0}^n \beta_i z^{n-i}}{\sum_{i=0}^n \alpha_i z^{n-i}} \\ &= \gamma_0 + \gamma_1 z^{-1} + \gamma_2 z^{-2} + \ldots = \sum_{i=0}^{\infty} \gamma_i z^{-i} \end{aligned}$$

$$\Longrightarrow \sum_{i=0}^{n} \beta_i z^{n-i} = \left(\sum_{i=0}^{n} \alpha_i z^{n-i}\right)\left(\sum_{i=0}^{\infty} \gamma_i z^{-i}\right)$$

Kronecker (1823-1891)

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Example: $n = 2$:

$$\beta_0 z^2 + \beta_1 z + \beta_2 = (\alpha_0 z^2 + \alpha_1 z + \alpha_2)(\gamma_0 + \gamma_1 z^{-1} + \gamma_2 z^{-2} + \gamma_3 z^{-3} + \ldots)$$

Equate likewise powers of z:

$$\begin{array}{rrcl}
z^2: & \beta_0 & = & \alpha_0\gamma_0 \\
z^1: & \beta_1 & = & \alpha_0\gamma_1 + \alpha_1\gamma_0 \\
z^0: & \beta_2 & = & \alpha_0\gamma_2 + \alpha_1\gamma_1 + \alpha_2\gamma_0 \\
z^{-1}: & 0 & = & \alpha_0\gamma_3 + \alpha_1\gamma_2 + \alpha_2\gamma_1 \\
z^{-2}: & 0 & = & \alpha_0\gamma_4 + \alpha_1\gamma_3 + \alpha_2\gamma_2 \\
\vdots & \vdots & & \vdots \\
z^{-k}: & 0 & = & \alpha_0\gamma_{k+2} + \alpha_1\gamma_{k+1} + \alpha_2\gamma_k
\end{array}$$

Coefficients $\gamma_k, k \geq 2$ satisfy 3-term linear recurrence

$$\alpha_0\gamma_{k+2} + \alpha_1\gamma_{k+1} + \alpha_2\gamma_k = 0, k \geq 2,$$

with initial conditions $\gamma_0, \gamma_1, \gamma_2$ from set of linear equations.

Kronecker and Hankel

$$\begin{pmatrix} \alpha_2 & \alpha_1 & \alpha_0 & 0 & 0 & 0 & \cdots \\ 0 & \alpha_2 & \alpha_1 & \alpha_0 & 0 & \ddots & \ddots \\ 0 & 0 & \alpha_2 & \alpha_1 & \alpha_0 & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} \gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 & \cdots \\ \gamma_3 & \gamma_4 & \gamma_5 & \iddots & \vdots \\ \gamma_4 & \gamma_5 & \gamma_6 & \iddots & \vdots \\ \gamma_5 & \gamma_6 & \gamma_7 & \iddots & \vdots \\ \iddots & \iddots & \iddots & \iddots & \vdots \end{pmatrix} = 0$$

Rational function series expansion $\Longleftrightarrow$ Hankel matrix rank deficient

Banded Toeplitz $\times$ rank deficient Hankel $= 0$
Rank Hankel $=$ degree of rational function
Recurrence relation coefficients $=$ denominator

Hankel (1839-1873)

Outline

State $x_k \in \mathbb{R}^n$, output $y_k \in \mathbb{R}^l$:

$$x_{k+1} = Ax_k \qquad X(z) = (zI_n - A)^{-1}x_0 = (I_n + Az^{-1} + Az^{-2} + \ldots)x_0$$

$$y_k = Cx_k = CA^k x_0 \qquad Y(z) = C(zI_n - A)^{-1}x_0 = Cx_0 + (CAx_0)z^{-1} + (CA^2x_0)z^{-2} + \ldots = y_0 + y_1 z^{-1} + y_2 z^{-2} + \ldots$$

Resolvent is rational:

$$(zI_n - A)^{-1} = \operatorname{adj}(A)/\det(zI_n - A)$$

Kronecker/Hankel: Factorize (SVD) to go from data to state space model

$$\begin{pmatrix} y_1 & y_2 & y_3 & y_4 & \ldots \\ y_2 & y_3 & y_4 & y_5 & \vdots \\ y_3 & y_4 & y_5 & y_6 & \vdots \\ \iddots & \iddots & \iddots & \iddots & \vdots \end{pmatrix} = \begin{pmatrix} C \\ CA \\ CA^2 \\ CA^3 \\ \vdots \end{pmatrix} \begin{pmatrix} x_0 & Ax_0 & A^2x_0 & \ldots \end{pmatrix}$$

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With characteristic equation and Cayley-Hamilton

$$p(\lambda) = \det(\lambda I_n - A) = \lambda^n + \alpha_1 \lambda^{n-1} + \ldots + \alpha_{n-1}\lambda + \alpha_n = 0$$

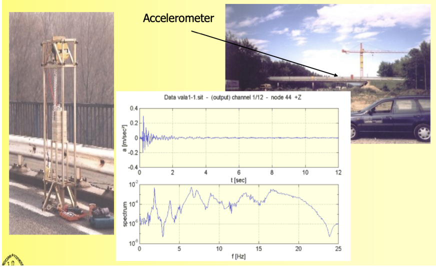
$$\begin{pmatrix} \alpha_n & \alpha_{n-1} & \alpha_{n-2} & \ldots & \alpha_1 & \alpha_0 & 0 & 0 & \ldots \\ 0 & \alpha_n & \alpha_{n-1} & \ddots & \alpha_2 & \alpha_1 & \alpha_0 & 0 & \ddots \\ 0 & 0 & \alpha_n & \ddots & \ddots & \alpha_2 & \alpha_1 & \alpha_0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} C \\ CA \\ CA^2 \\ CA^3 \\ \vdots \end{pmatrix} = T \,.\, \Gamma = 0$$

Left null space (Γ) = banded Toeplitz T.
Right null space (T) = shift-invariant:

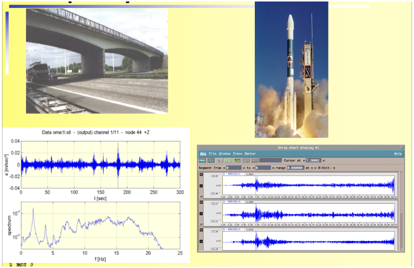
$$\boxed{\underline{\Gamma}\, A = \overline{\Gamma} \iff \text{rank}(\ \underline{\Gamma}\ \ \overline{\Gamma}\ ) = n = \text{rank}(\underline{\Gamma})\ (\text{PRC}) \implies A = \underline{\Gamma}^{\dagger}\, \overline{\Gamma}}$$

Outline

Application: Impulse response and stochastic realization

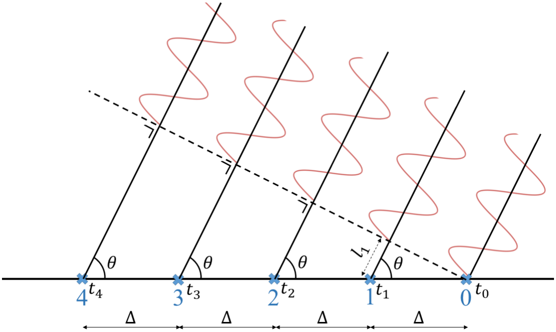


Impulse response



Stochastic realization

Application: Direction of Arrival: Uniform linear array, narrow band sources, far field



$$\begin{aligned} y_i(t) &= \sin(\omega t + \frac{\omega(i\Delta)\cos\theta}{c}) = \sin(\omega t + \varphi_i) \\ &= \sin(\omega t)\cos\varphi_i + \cos(\omega t)\sin\varphi_i = \begin{pmatrix} \cos\varphi_i & \sin\varphi_i \end{pmatrix} \begin{pmatrix} \sin\omega t \\ \cos\omega t \end{pmatrix} \end{aligned}$$

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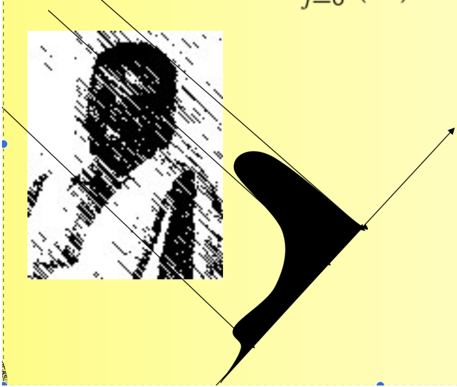
Application: Shape from moments

Calculate moments of 'pdf' and show that

$$\int_{-T}^{T} p_f(t,\theta) t^k dt = \sum_{j=0}^{k} \binom{k}{j} \cos^{k-j}(\theta) \sin^j(\theta) \mu_{k-j,j}$$



$$\mu_{pq} \rightarrow \tau_k$$

$$\tau_k = \sum_{j=1}^{n} a_j z_j^k$$

→ Realization theory !

Application: Cepstrum realization

Power cepstrum = power spectrum of log of power spectrum

$$\log \Phi(z) = \sum_{-\infty}^{+\infty} c_k . z^{-k}$$

Cepstral coefficients $c_k = c_{-k}, \forall k;$

$$c_0 = 2 \log \rho$$

$$k\, c_k = \sum \alpha_i^k - \sum \beta_i^k$$

i-th cepstral coefficient
= sum of i-th powers
of poles and zeros

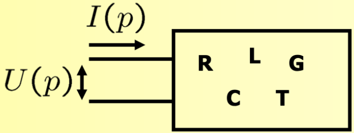
→ Realization theory !

Application: Electrical circuit power spectrum by R, L, C, T, G

A transfer function $Z(p)$ is **realizable** as a passive electrical circuit

$\Leftrightarrow$ there exists an interconnection of a finite number of R's, L's, C's, T's and G's such that

$$Z(p) = \frac{U(p)}{I(p)}$$



$\Leftrightarrow Z(p)$ is positive real

$\Leftrightarrow p \in \mathbb{C}_+ \Rightarrow Z(p) \in \mathbb{C}_+$

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mD shift invariant systems $(m = 2)$

$x_{k,l} \in \mathbb{R}^n$, $y_{k,l} \in \mathbb{R}$:

$$\begin{aligned} x_{k+1,l} &= A_1 \, x_{k,l} \\ x_{k,l+1} &= A_2 \, x_{k,l} \qquad A_1 A_2 = A_2 A_1 \\ y_{k,l} &= C \, x_{k,l} \end{aligned}$$

$$Y = \left(\begin{array}{c|cc|ccc|cc} y_{00} & y_{10} & y_{01} & y_{20} & y_{11} & y_{02} & y_{30} & \cdots \\ \hline y_{10} & y_{20} & y_{11} & y_{30} & y_{21} & y_{12} & y_{40} & \cdots \\ y_{01} & y_{11} & y_{02} & y_{21} & y_{12} & y_{03} & y_{31} & \cdots \\ \hline y_{20} & y_{30} & y_{21} & y_{40} & y_{31} & y_{22} & y_{50} & \cdots \\ y_{11} & y_{21} & y_{12} & y_{31} & y_{22} & y_{13} & y_{41} & \cdots \\ y_{02} & y_{12} & y_{13} & y_{22} & y_{13} & y_{04} & y_{32} & \cdots \\ \hline y_{30} & y_{40} & y_{31} & y_{50} & \cdots & \cdots & \cdots & \cdots \\ y_{21} & y_{31} & y_{22} & y_{41} & \cdots & \cdots & \cdots & \cdots \\ y_{12} & y_{22} & y_{13} & y_{32} & \cdots & \cdots & \cdots & \cdots \\ y_{03} & y_{13} & y_{04} & y_{23} & \cdots & \cdots & \cdots & \cdots \\ \hline y_{40} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array}\right) \qquad \mathbf{rank}(Y) = n$$

$$= \Gamma \;\Delta = \left(\begin{array}{c} C \\ \hline CA_1 \\ CA_2 \\ \hline CA_1^2 \\ CA_1A_2 \\ CA_2^2 \\ \hline \vdots \end{array}\right) \left(\begin{array}{c|cc|cc} x_0 & A_1x_0 & A_2x_0 & A_1^2x_0 & \ldots \end{array}\right)$$

The column space of Γ is a **multi-shift-invariant subspace**:

$$\underline{\Gamma}\, A_1 = S_1\Gamma = \begin{pmatrix} C \\ \hline CA_1 \\ CA_2 \\ \hline CA_1^2 \\ CA_1A_2 \\ CA_2^2 \\ \hline \vdots \\ \hline CA_1^{p-2} \\ CA_1^{p-3}A_2 \\ \vdots \\ CA_2^{p-2} \end{pmatrix} A_1 = \begin{pmatrix} CA_1 \\ \hline CA_1^2 \\ CA_1A_2 \\ \hline CA_1^3 \\ CA_1^2A_2 \\ CA_1A_2^2 \\ \hline \vdots \\ \hline CA_1^{p-1} \\ CA_1^{p-2}A_2 \\ \vdots \\ CA_1A_2^{p-2} \end{pmatrix} \quad \text{and} \quad \underline{\Gamma}\, A_2 = S_2\Gamma$$

- Selector matrix S_1 selects the block rows $(2,4,5,7,8,9,\ldots)$.
- Selector matrix S_2 selects the block rows $(3,5,6,8,9,10,\ldots)$.
- Find A_1, A_2 by solving set of linear equations (PRC: $\mathrm{rank}(\underline{\Gamma}) = \mathrm{n}$)

$$\boxed{\mathrm{A}_1 = \underline{\Gamma}^{\dagger} S_1\Gamma \ \text{ and } \ A_2 = \underline{\Gamma}^{\dagger} S_2\Gamma \ .}$$

- A multi-shift invariant subspace is determined by the eigenvalues of its shifts A_1 and A_2

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Outline

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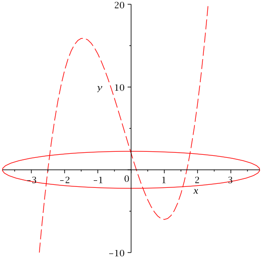
- All mD generalizations of DOA, shape-from-moments, power spectra, etc.
- Bilinear system identification
- *Rooting multivariable polynomials*
- *Multi-parameter eigenvalue problems*
- *Global optimum of prediction-error-methods*
- ...

Application: Two polynomials in two variables

- Consider

$$\begin{cases} p(x,y) &= x^2 + 3y^2 - 15 = 0 \\ q(x,y) &= y - 3x^3 - 2x^2 + 13x - 2 = 0 \end{cases}$$



- Fix a monomial order, e.g., $1 < x < y < x^2 < xy < y^2 < x^3 < x^2y < \ldots$

- Construct quasi-Toeplitz Macaulay matrix M:

$$\begin{array}{c} \\ p(x,y) \\ q(x,y) \\ x \cdot p(x,y) \\ y \cdot p(x,y) \end{array} \begin{array}{c} \begin{array}{cccccccccc} 1 & x & y & x^2 & xy & y^2 & x^3 & x^2y & xy^2 & y^3 \end{array} \\ \begin{bmatrix} -15 & & & 1 & & 3 & & & & \\ -2 & 13 & 1 & -2 & & & -3 & & & \\ & -15 & & & & & 1 & & 3 & \\ & & -15 & & & & & 1 & & 3 \end{bmatrix} \end{array} \begin{pmatrix} 1 \\ x \\ y \\ x^2 \\ xy \\ \vdots \\ xy^2 \\ y^3 \end{pmatrix} = 0$$

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$$\begin{cases} p(x,y) & = & x^2 + 3y^2 - 15 = 0 \\ q(x,y) & = & y - 3x^3 - 2x^2 + 13x - 2 = 0 \end{cases}$$

Continue to enlarge M:

it #	form	1	x	y	x^2	xy	y^2	x^3	x^2y	xy^2	y^3	x^4	x^3y	x^2y^2	xy^3	y^4	x^5	x^4y	x^3y^2	x^2y^3	xy^4	y^5	$\rightarrow$
$d=3$	p	-15			1		3																
	xp		-15					1		3													
	yp			-15					1		3												
	q	-2	13	1	-2			-3															
$d=4$	x^2p				-15							1		3									
	xyp					-15							1		3								
	y^2p						-15							1		3							
	xq		-2		13	1		-2				-3											
	yq			-2		13	1		-2				-3										
$d=5$	x^3p							-15									1		3				
	x^2yp								-15									1		3			
	xy^2p									-15									1		3		
	y^3p										-15									1		3	
	x^2q				-2			13	1				-2				-3						
	xyq					-2			13	1				-2				-3					
	y^2q						-2			13	1				-2				-3				
	$\downarrow$							$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$

- $\#$ rows grows faster than $\#$ cols $\Rightarrow$ overdetermined system
- If solution exists: rank deficient by construction!

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nD realization in the null space

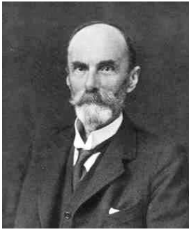
- Macaulay matrix M:

$$M = \begin{bmatrix} \times & \times & \times & \times & 0 & 0 & 0 \\ 0 & \times & \times & \times & \times & 0 & 0 \\ 0 & 0 & \times & \times & \times & \times & 0 \\ 0 & 0 & 0 & \times & \times & \times & \times \end{bmatrix}$$

- Solutions generate vectors in kernel of M:

$$MK = 0$$

- Number of solutions s follows from rank decisions 'mind-the-gap':



Francis Sowerby Macaulay

Vandermonde nullspace K built from s solutions (x_i, y_i):

$$\left[\begin{array}{c|c|c|c} 1 & 1 & \ldots & 1 \\ \hline x_1 & x_2 & \ldots & x_s \\ y_1 & y_2 & \ldots & y_s \\ \hline x_1^2 & x_2^2 & \ldots & x_s^2 \\ x_1 y_1 & x_2 y_2 & \ldots & x_s y_s \\ y_1^2 & y_2^2 & \ldots & y_s^2 \\ \hline x_1^3 & x_2^3 & \ldots & x_s^3 \\ x_1^2 y_1 & x_2^2 y_2 & \ldots & x_s^2 y_s \\ x_1 y_1^2 & x_2 y_2^2 & \ldots & x_s y_s^2 \\ y_1^3 & y_2^3 & \ldots & y_s^3 \\ \hline x_1^4 & x_2^4 & \ldots & x_4^4 \\ x_1^3 y_1 & x_2^3 y_2 & \ldots & x_s^3 y_s \\ x_1^2 y_1^2 & x_2^2 y_2^2 & \ldots & x_s^2 y_s^2 \\ x_1 y_1^3 & x_2 y_2^3 & \ldots & x_s y_s^3 \\ y_1^4 & y_2^4 & \ldots & y_s^4 \\ \hline \vdots & \vdots & \vdots & \vdots \end{array}\right]$$

Setting up an eigenvalue problem in x

- Choose s linear independent rows in K

$$S_1 K$$

- This corresponds to finding linear dependent columns in M

1	1	$\ldots$	1
x_1	x_2	$\ldots$	x_s
y_1	y_2	$\ldots$	y_s
x_1^2	x_2^2	$\ldots$	x_s^2
$x_1 y_1$	$x_2 y_2$	$\ldots$	$x_s y_s$
y_1^2	y_2^2	$\ldots$	y_s^2
x_1^3	x_2^3	$\ldots$	x_s^3
$x_1^2 y_1$	$x_2^2 y_2$	$\ldots$	$x_s^2 y_s$
$x_1 y_1^2$	$x_2 y_2^2$	$\ldots$	$x_s y_s^2$
y_1^3	y_2^3	$\ldots$	y_s^3
x_1^4	x_2^4	$\ldots$	x_4^4
$x_1^3 y_1$	$x_2^3 y_2$	$\ldots$	$x_s^3 y_s$
$x_1^2 y_1^2$	$x_2^2 y_2^2$	$\ldots$	$x_s^2 y_s^2$
$x_1 y_1^3$	$x_2 y_2^3$	$\ldots$	$x_s y_s^3$
y_1^4	y_2^4	$\ldots$	y_s^4
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Shifting the selected rows gives (shown for 3 columns)

$$
\left[\begin{array}{c|c|c}
1 & 1 & 1 \\ \hline
x_1 & x_2 & x_3 \\
y_1 & y_2 & y_3 \\ \hline
x_1^2 & x_2^2 & x_3^2 \\
x_1 y_1 & x_2 y_2 & x_3 y_3 \\
y_1^2 & y_2^2 & y_3^2 \\ \hline
x_1^3 & x_2^3 & x_3^3 \\
x_1^2 y_1 & x_2^2 y_2 & x_3^2 y_3 \\
x_1 y_1^2 & x_2 y_2^2 & x_3 y_3^2 \\
y_1^3 & y_2^3 & y_3^3 \\ \hline
x_1^4 & x_2^4 & x_4^4 \\
x_1^3 y_1 & x_2^3 y_2 & x_3^3 y_3 \\
x_1^2 y_1^2 & x_2^2 y_2^2 & x_3^2 y_3^2 \\
x_1 y_1^3 & x_2 y_2^3 & x_3 y_3^3 \\
y_1^4 & y_2^4 & y_3^4 \\ \hline
\vdots & \vdots & \vdots
\end{array}\right]
\longrightarrow \text{“shift with } x\text{”} \longrightarrow
\left[\begin{array}{c|c|c}
1 & 1 & 1 \\ \hline
x_1 & x_2 & x_3 \\
y_1 & y_2 & y_3 \\ \hline
x_1^2 & x_2^2 & x_3^2 \\
x_1 y_1 & x_2 y_2 & x_3 y_3 \\
y_1^2 & y_2^2 & y_3^2 \\ \hline
x_1^3 & x_2^3 & x_3^3 \\
x_1^2 y_1 & x_2^2 y_2 & x_3^2 y_3 \\
x_1 y_1^2 & x_2 y_2^2 & x_3 y_3^2 \\
y_1^3 & y_2^3 & y_3^3 \\ \hline
x_1^4 & x_2^4 & x_4^4 \\
x_1^3 y_1 & x_2^3 y_2 & x_3^3 y_3 \\
x_1^2 y_1^2 & x_2^2 y_2^2 & x_3^2 y_3^2 \\
x_1 y_1^3 & x_2 y_2^3 & x_3 y_3^3 \\
y_1^4 & y_2^4 & y_3^4 \\ \hline
\vdots & \vdots & \vdots
\end{array}\right]
$$

simplified:

$$
\left[\begin{array}{c|c|c}
1 & 1 & 1 \\
x_1 & x_2 & x_3 \\
y_1 & y_2 & y_3 \\
x_1 y_1 & x_2 y_2 & x_3 y_3 \\
x_1^3 & x_2^3 & x_3^3 \\
x_1^2 y_1 & x_2^2 y_2 & x_3^2 y_3
\end{array}\right]
\left[\begin{array}{ccc}
x_1 & & \\
 & x_2 & \\
 & & x_3
\end{array}\right]
=
\left[\begin{array}{c|c|c}
x_1 & x_2 & x_3 \\
x_1^2 & x_2^2 & x_3^2 \\
x_1 y_1 & x_2 y_2 & x_3 y_3 \\
x_1^2 y_1 & x_2^2 y_2 & x_3^2 y_3 \\
x_1^4 & x_2^4 & x_4^4 \\
x_1^3 y_1 & x_2^3 y_2 & x_3^3 y_3
\end{array}\right]
$$

Finding the x-roots

Let $D_x = \mathrm{diag}(x_1, x_2, \ldots, x_s)$, then

$$S_1 K D_x = S_x K,$$

where S_1 and S_x select rows from K w.r.t. shift property We have

$$S_1 K D_x = S_x K$$

Generalized Vandermonde K is not known as such, instead a null space basis Z is calculated, which is a linear transformation of K:

$$ZV = K$$

which leads to

$$(S_x Z)V = (S_1 Z)V D_x$$

Here, V is the matrix with eigenvectors, D_x contains the roots x as eigenvalues.

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Setting up an eigenvalue problem in y

It is possible to shift with y as well...

We find

$$S_1 K D_y = S_y K$$

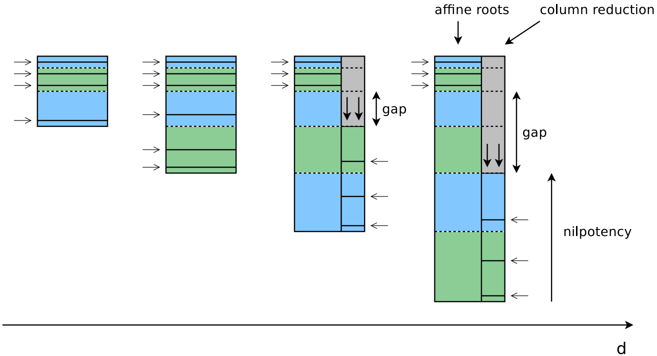
with D_y diagonal matrix of y-components of roots, leading to

$$(S_y Z)V = (S_1 Z) V D_y$$

Some interesting observations:

- same eigenvectors V!
- $(S_x Z)^{-1}(S_1 Z)$ and $(S_y Z)^{-1}(S_1 Z)$ commute
  $\Longrightarrow$ 'commutative algebra'

'Mind the Gap' with roots at infinity !



Rank, nullity and null space: SVD-ize the Macaulay matrix

Basic Algorithm outline

Find a basis for the nullspace of M using an SVD:

$$M = \begin{bmatrix} \times & \times & \times & \times & 0 & 0 & 0 \\ 0 & \times & \times & \times & \times & 0 & 0 \\ 0 & 0 & \times & \times & \times & \times & 0 \\ 0 & 0 & 0 & \times & \times & \times & \times \end{bmatrix} = \begin{bmatrix} X & Y \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} W^T \\ Z^T \end{bmatrix}$$

Hence,

$$MZ = 0$$

Deflate roots at ∞ by detecting 'mind-the-gap' and column compression:

$$ZT = \begin{pmatrix} Z_{11} & 0 \\ Z_{21} & Z_{22} \end{pmatrix}$$

We have

$$S_1 K D = S_{\text{shift}} K$$

with K generalized Vandermonde, not known as such. Instead a basis Z_{11} is computed as

$$Z_{11} V = K$$

which leads to

$$(S_{\text{shift}} Z_{11}) V = (S_1 Z_{11}) V D$$

S_1 selects linear independent rows.
S_{shift} selects rows 'hit' by the shift.

Application: Multiparameter Eigenvalue Problem (MEVP)

Given $A_0, \ldots, A_m \in \mathbb{R}^{p \times q}$ with $p \geq q$, find $\lambda_i \in \mathbb{C}, i = 1, \ldots, m$ and $x \neq 0 \in \mathbb{C}^q$ so that

$$(A_0 + A_1\lambda_1 + \ldots + A_m\lambda_m)\, x = 0$$

Special cases:

- Ordinary EVP: $A_0 \in \mathbb{R}^{n \times n}$, $A_1 = -I_n$, $A_i = 0, i \geq 2$
- 'Generalized' EVP: $A_0, A_1 \in \mathbb{R}^{n \times n}$, $A_i = 0, i \geq 2$

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Basic idea to solve an MEVP (illustrated for $m = 2$)

$$(A_0 + A_1\lambda_1 + A_2\lambda_2)\, x = 0$$

$$\begin{array}{l} \times 1 \\ \times\lambda_1 \\ \times\lambda_2 \\ \times\lambda_1^2 \\ \vdots \end{array} \begin{pmatrix} A_0 & A_1 & A_2 & 0 & 0 & 0 & 0 & \ldots \\ 0 & A_0 & 0 & A_1 & A_2 & 0 & 0 & \ldots \\ 0 & 0 & A_0 & 0 & A_1 & A_2 & 0 & \ldots \\ 0 & 0 & 0 & A_0 & 0 & 0 & A_1 & \ldots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} x \\ \hline x\lambda_1 \\ x\lambda_2 \\ \hline x\lambda_1^2 \\ x\lambda_1\lambda_2 \\ x\lambda_2^2 \\ \hline x\lambda_1^3 \\ \vdots \end{pmatrix} = 0$$

Block 'quasi'-Toeplitz structure + 'generalized' Vandermonde structure

Outline

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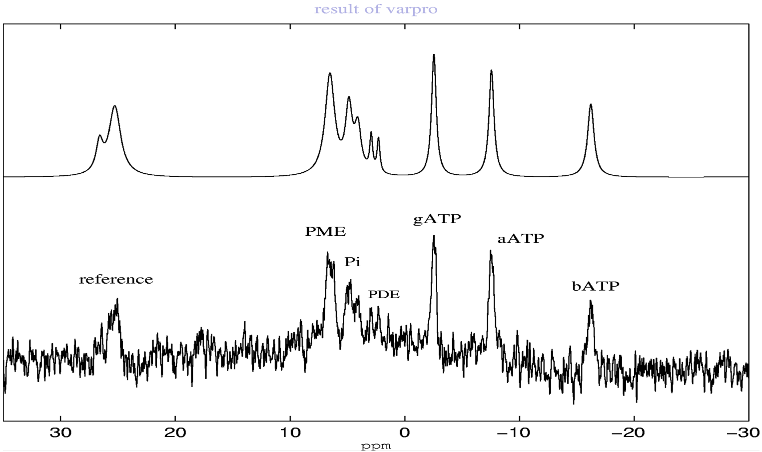
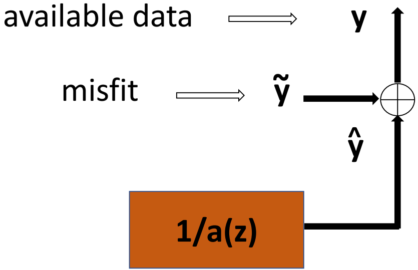
Errors using inadequate data are much less than those using no data at all.
Charles Babbage.



Data not model-compliant

Misfit case: Least squares realization (n_a)



Misfit:
LS realization

$$T_a \hat{y} = 0$$

$$\sigma^2 = \|\tilde{y}\|_2^2$$

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Misfit case: Least squares realization (ref: Kailath 80 !)

Data : $y \in \mathbb{R}^N$. **Model:** Data = model-compliant data + misfit:

$$y = \hat{y} + \tilde{y}$$

Model-compliancy (Popper: models forbid more than allow) :
Image model:

$$\hat{y} = \Gamma\, \hat{x}_0$$

Kernel model

$$\begin{aligned} \hat{Y}\, a &= T^a_{N-n}\, \hat{y} \\ &= \begin{pmatrix} \alpha_n & \alpha_{n-1} & \dots & \alpha_0 & 0 & \dots & 0 \\ 0 & \alpha_n & \alpha_{n-1} & \ddots & \alpha_0 & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} \hat{y}_0 \\ \hat{y}_1 \\ \vdots \\ \hat{y}_{N-1} \end{pmatrix} = 0 \end{aligned}$$

Least squares minimization:

$$\min \|\tilde{y}\|_2^2 \ \text{subject to model} - \text{compliancy}$$

Multi-parameter EVP

$$\boxed{\begin{pmatrix} T^a_{N-n} y & T^a_{N-n}(T^a_{N-n})^T (T^a_{N-2n})^T \end{pmatrix} \begin{pmatrix} -1 \\ g \end{pmatrix} = 0}$$

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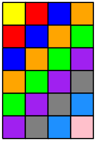
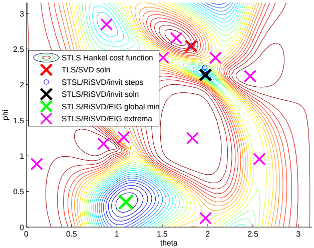
$$\begin{array}{ll} \min_{v} & \tau^2 = v^T M^T D_v^{-1} M v \\ \text{s.t.} & v^T v = 1. \end{array}$$





method	TLS/SVD	STLS inv. it.	STLS eig
v_1	.8003	.4922	**.8372**
v_2	-.5479	-.7757	**.3053**
v_3	.2434	.3948	**.4535**
τ^2	4.8438	3.0518	**2.3822**
global solution?	no	no	**yes**

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Outline

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- Volterra equation of the 2nd kind, affine EVP $Ax = x\lambda + b$
- PdE separation of variables (diffusion and wave equations,...)
- All multivariate optimization problems are MEVPs
- All Prediction Error Methods solve a MEVP (heuristically)
- H_2 model reduction is a MEVP
- Real roots only = realization theory + nonnegative definiteness of Hankel
- etc... etc...

Outline

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What do we have in common ?


Lennart Ljung


Thomas Kailath


Kailath and Obama

LINKÖPING UNIVERSITY

KTH VETENSKAP OCH KONST

European Research Council
Established by the European Commission

European Innovation Council

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What do we have in common ?

- Topics: System identification, power spectrum estimation, DOA, subspace fitting, array signal processing
- Research grants and awards
- Lots of (old) common friends and colleagues: Boyd, Swindlehurst, Moonen, Viberg, van der Veen, Wahlberg, Slock, ...
- 35 years
- Wine


Björn Otte

Kära Björn,

tack för vetenskapen och vänskapen!

(Dear Bjorn, thanks for the science and the friendship !)