KATHOLIEKE UNIVERSITEIT LEUVEN
FACULTEIT DER TOEGEPASTE WETENSCHAPPEN
DEPARTEMENT ELEKTROTECHNIEK
Kardinaal Mercierlaan 94, 3001 Leuven (Heverlee)

MAX-ALGEBRAIC SYSTEM THEORY FOR DISCRETE EVENT SYSTEMS

Promotor:
Prof. dr. ir. B. De Moor

Proefschrift voorgedragen tot
het behalen van het doctoraat
in de toegepaste wetenschappen

door

Bart DE SCHUTTER

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Jury:
Prof. W. Dutré, vice-decaan, voorzitter
Prof. B. De Moor, promotor
Prof. R. Boel (Univ. Gent)
Dr. S. Gaubert (INRIA, Le Chesnay, France)
Prof. G.J. Olsder (TU Delft, Nederland)
Prof. H. Van Brussel
Prof. J. Vandewalle

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Preface

There are some people that I would like to thank for their contribution to this thesis and to the research that will be presented in it.

First of all I want to express my gratitude and my appreciation to my advisor Prof. Bart De Moor, who has introduced me to the area of discrete event systems and the max-plus-algebra. I especially want to thank him for his contagious enthusiasm and creativity, for the pleasant and fruitful cooperation, for the support and the guidance he has given me during this research project, and for the fact that he has encouraged me to continue my research on this topic.

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Furthermore, I also want to thank the two other members of the reading committee: Prof. G.J. Olsder and Prof. H. Van Brussel for their willingness to go through this manuscript and for their constructive criticism and suggestions. I also want to thank Prof. Olsder for the interest he has shown in my research from the very first beginning.

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Abstract

Discrete event systems (DESs) are systems in which the state changes only at discrete points in time in response to the occurrence of particular events. Typical examples of DESs are: flexible manufacturing systems, telecommunication networks, parallel processing systems and traffic control systems. One of the frameworks that can be used to model and to analyze certain types of DESs is the max-plus algebra, which has maximization and addition as basic operations. In this thesis we develop tools for solving some fundamental problems in the max-algebraic system theory for DESs.

First we introduce a mathematical programming problem: the Extended Linear Complementarity Problem (ELCP). We develop an algorithm to find all the solutions of an ELCP.

We show that the problem of solving a system of multivariate max-algebraic polynomial equalities and inequalities is equivalent to an ELCP. This enables us to solve many other max-algebraic problems such as computing max-algebraic matrix factorizations, performing max-algebraic state space transformations, computing state space realizations of the impulse response of a max-linear time-invariant DES, computing max-algebraic singular value decompositions and QR decompositions, and so on.

We also study the max-algebraic characteristic polynomial and state space transformations for max-linear time-invariant DESs. Next we develop a method to solve the minimal state space realization problem for max-linear time-invariant DESs. First we use our results on the max-algebraic characteristic polynomial to develop a procedure to determine a lower bound for the minimal system order of a max-linear time-invariant DES. Then we show that the ELCP can be used to compute all fixed order partial state space realizations and all minimal state space realizations of the impulse response of a max-linear time-invariant DES.

Finally we prove the existence of max-algebraic analogues of two basic matrix decompositions from linear algebra: the singular value decomposition and the QR decomposition.

Korte Inhoud

Een discrete-gebeurtenissysteem (DGS) is een systeem waarin de toestandsovergangen veroorzaakt worden door gebeurtenissen die op discrete tijdstippen plaatsvinden. Typische voorbeelden van DGS'en zijn: flexibele produktiesystemen, telecommunicatienetwerken en spoorwegnetwerken. Bepaalde types DGS'en kunnen gemodelleerd en geanalyseerd worden met behulp van de zogenaamde max-plus-algebra. De basisbewerkingen van de max-plus-algebra zijn het maximum en de optelling. In dit proefschrift ontwikkelen we een aantal technieken om enkele belangrijke problemen uit de max-algebraïsche systeemtheorie voor DGS'en op te lossen.

Eerst stellen we een wiskundige-programmatieprobleem voor: het Uitgebreide Lineaire Complementariteitsprobleem (ULCP). We ontwikkelen een algoritme om alle oplossingen van een ULCP te bepalen.

Vervolgens tonen we aan dat het oplossen van een stelsel multivariabele max-algebraïsche veeltermvergelijkingen en veeltermongelijkheden equivalent is met het oplossen van een ULCP. Dit stelt ons in staat om een aantal andere max-algebraïsche problemen op te lossen zoals b.v. het berekenen van max-algebraïsche matrixfactorisaties, het bepalen van max-algebraïsche toestandsruimtetransformaties, het berekenen van toestandsruimterealisaties van de impulsresponsie van een max-lineair tijdsinvariant DGS, het berekenen van max-algebraïsche singuliere-waardenontbindingen en QR-ontbindingen, ...

We bestuderen de max-algebraïsche karakteristieke veelterm en toestandsruimtetransformaties voor max-lineaire tijdsinvariante DGS'en. Vervolgens ontwikkelen we een methode om het minimale-toestandsruimterealisatieprobleem voor max-lineaire tijdsinvariante DGS'en op te lossen. Steunend op onze resultaten i.v.m. de max-algebraïsche karakteristieke veelterm ontwikkelen we een techniek om een ondergrens voor de minimale systeemorde van een max-lineair tijdsinvariant DGS te bepalen. Daarna tonen we aan dat het ULCP kan gebruikt worden om alle partiële toestandsruimterealisaties van een bepaalde orde en alle minimale toestandsruimterealisaties van de impulsresponsie van een max-lineair tijdsinvariant DGS te berekenen.

Tenslotte bewijzen we het bestaan van max-algebraïsche equivalenten van twee belangrijke matrixontbindingen uit de lineaire algebra: de singuliere-waardenontbinding en de QR-ontbinding.

Notation

Here we list some of the symbols and acronyms that occur frequently in this thesis and with which the reader might not be familiar. The numbers in the last column refer to the page on which the symbol or concept in question is defined.

List of Symbols

Sets

$\emptyset$	the empty set	
$\#S$	cardinality of the set S	
$S \subseteq T$	S is a subset of T	
$S \subset T$	S is a proper subset of T	
$\mathbb{N}$	set of the nonnegative integers: $\mathbb{N} = \{0, 1, 2, \dots\}$	
$\mathbb{N}_0$	set of the positive integers: $\mathbb{N}_0 = \mathbb{N} \setminus \{0\}$	
$\mathbb{Z}$	set of the integers	
$\mathbb{Q}$	set of the rational numbers	
$\mathbb{R}$	set of the real numbers	
$\mathbb{R}^+$	set of the nonnegative real numbers	
$\mathbb{R}_0$	set of the real numbers except for 0: $\mathbb{R}_0 = \mathbb{R} \setminus \{0\}$	
$\mathbb{R}_0^+$	set of the positive real numbers: $\mathbb{R}_0^+ = \mathbb{R}^+ \setminus \{0\}$	
$\mathbb{C}$	set of the complex numbers	
$[a, b]$	closed interval in $\mathbb{R}$: $[a, b] = \{ x \in \mathbb{R} \mid a \leqslant x \leqslant b \}$	
(a, b)	open interval in $\mathbb{R}$: $(a, b) = \{ x \in \mathbb{R} \mid a < x < b \}$	
$\mathcal{C}_n^k$	set of all the subsets with cardinality k of the set $\{1, 2, \dots, n\}$	33
$\mathcal{P}_n$	set of the permutations of the set $\{1, 2, \dots, n\}$	33
$\mathcal{P}_{n,\mathrm{even}}$	set of the even permutations of the set $\{1, 2, \dots, n\}$	34
$\mathcal{P}_{n,\mathrm{odd}}$	set of the odd permutations of the set $\{1, 2, \dots, n\}$	34

Functions

$f : D \to T$	function with domain of definition D and target T	32
$\mathrm{dom}\, f$	domain of definition of the function f	32

$\log x$	natural logarithm of x	
$O(f)$	any real function g such that $\limsup_{x\to\infty} \frac{\lvert g(x)\rvert}{f(x)}$ is finite	32
$\lfloor x \rfloor$	largest integer less than or equal to x	32
$\sim$	asymptotic equivalence	193

Matrices and Vectors

$\mathbb{R}^{m\times n}$	set of the m by n matrices with real entries	
$\mathbb{R}^n$	set of the real column vectors with n components: $\mathbb{R}^n = \mathbb{R}^{n\times 1}$	
A^T	transpose of the matrix A	
I_n	n by n identity matrix	
$O_{m\times n}$	m by n zero matrix	
a_i	ith component of the vector a	32
a_α	subvector of the vector a consisting of the components indexed by α	32
a_{ij}, $(A)_{ij}$	entry of the matrix A on the ith row and the jth column	32
$A_{i,.}$	ith row of the matrix A	32
$A_{.,j}$	jth column of the matrix A	32
$A_{\alpha\beta}$	submatrix of the matrix A consisting of the rows indexed by α and the columns indexed by β	32
$A_{\alpha,.}$	submatrix of the matrix A consisting of the rows indexed by α	32
$\lVert A\rVert_{\mathrm{F}}$	Frobenius norm of the matrix A	192
$\lVert A\rVert_2$	2-norm of the matrix A	192

Extended Linear Complementarity Problems (ELCPs)

$\mathcal{X}^{\mathrm{cen}}$	a minimal complete set of central generators of the solution set of an ELCP	94
$\mathcal{X}^{\mathrm{ext}}$	a minimal complete set of extreme generators of the solution set of an ELCP	94
$\mathcal{X}^{\mathrm{fin}}$	a minimal complete set of finite points of the solution set of an ELCP	94
Λ	the set of ordered pairs of maximal cross-complementary subsets of $\mathcal{X}^{\mathrm{ext}}$ and $\mathcal{X}^{\mathrm{fin}}$	94

Max-Plus Algebra

$\oplus$	max-algebraic addition	34
$\otimes$	max-algebraic multiplication	34
$⌐$	max-algebraic division	35
ε	zero element for $\otimes$: $\varepsilon = -\infty$	35

$x^{\otimes^r}$	rth max-algebraic power of x	35
E_n	n by n max-algebraic identity matrix	35
$\mathcal{E}_{m\times n}$	m by n max-algebraic zero matrix	35
$A^{\otimes^k}$	kth max-algebraic power of the matrix A	36
$\mathbb{R}_\varepsilon$	$\mathbb{R}\cup\{-\infty\}$	35
$\mathbb{S}$	set of the max-real numbers	44
$\mathbb{T}$	set of the max-complex numbers	196
$\mathbb{S}^\oplus$	set of the max-positive or max-zero numbers	44
$\mathbb{S}^\ominus$	set of the max-negative or max-zero numbers	44
$\mathbb{S}^\vee$	set of the max-signed numbers: $\mathbb{S}^\vee = \mathbb{S}^\oplus \cup \mathbb{S}^\ominus$	44
$\mathbb{S}^\bullet$	set of the balanced numbers	44
$\mathbb{R}_{\max}$	max-plus algebra: $\mathbb{R}_{\max} = (\mathbb{R}_\varepsilon, \max, +)$	35
$\mathbb{S}_{\max}$	symmetrized max-plus algebra: $\mathbb{S}_{\max} = (\mathbb{S}, \oplus, \otimes)$	44
$\mathbb{T}_{\max}$	max-complex max-plus algebra: $\mathbb{T}_{\max} = (\mathbb{T}, \oplus, \otimes)$	197
$\ominus$	max-algebraic minus operator	43
$(\cdot)^\bullet$	balance operator	43
∇	balance relation	43
$x^\oplus$	max-positive part of x	44
$x^\ominus$	max-negative part of x	44
$\lvert x\rvert_\oplus$	max-absolute value of x	43
$\lVert A\rVert_\oplus$	max-algebraic norm of the matrix A	45
$\mathrm{sgn}_\oplus(\sigma)$	max-algebraic signature of the permutation σ	46
$\det_\oplus A$	max-algebraic determinant of the matrix A	46
$\mathrm{rank}_\oplus(A)$	max-algebraic minor rank of the matrix A	47
$\mathrm{rank}_{\oplus,\mathrm{wc}}(A)$	max-algebraic weak column rank of the matrix A	158
$\mathrm{rank}_{\oplus,\mathrm{cc}}(A)$	max-algebraic consecutive column rank of the matrix A	165

Max-algebraic State Space Realizations

$H(G)$	block Hankel matrix that corresponds to the impulse response G	158
$\mathcal{R}_r(G,N)$	set of the rth order state space realizations of the first N Markov parameters of the impulse response G	170
$\mathcal{R}_r(G)$	set of the rth order state space realizations of the impulse response G	173
$\mathcal{R}_r^{\mathrm{nor}}(G,N)$	set of the normalized rth order state space realizations of the first N Markov parameters of the impulse response G	173
$\mathcal{R}_r^{\mathrm{nor}}(G)$	set of the normalized rth order state space realizations of the impulse response G	173

Miscellaneous

$\mathcal{G}(A)$	precedence graph of the matrix A	39

$\dim \mathcal{V}$	dimension of the vector space $\mathcal{V}$	54
$\mathcal{L}(\mathcal{P})$	lineality space of the polyhedron $\mathcal{P}$	54
$\binom{n}{k}$	binomial coefficient: $\binom{n}{k} = \dfrac{n!}{(n-k)!\,k!}$	33

We use $\square$ to indicate the end of a proof or an example, and $\diamond$ to indicate the end of a remark.

Remark: The notation we use for the max-algebraic symbols corresponds to a large extent to that of [3], which is one of the basic references in the field of the max-plus algebra. Nevertheless, there are a few differences that are mainly caused by the fact that we use concepts from both conventional algebra and max-plus algebra in this thesis. The main differences are:

- We have added $\oplus$ as a subscript in symbols such as $\det_\oplus$, $\operatorname{rank}_\oplus$, $\operatorname{sgn}_\oplus$, $\|\cdot\|_\oplus$, ... that are used to indicate max-algebraic concepts in order to avoid confusion with similar symbols for linear algebra concepts.
- We use $a^{\otimes^r}$ instead of a^r to denote the max-algebraic power. Furthermore, we never omit the $\otimes$ sign.
- We do not use e to denote the identity element for $\otimes$ in $\mathbb{R}_\varepsilon$ and $\mathbb{S}$. Instead we use its numerical value (0) to avoid confusion with the number $e = \exp(1)$.
- We use $a^\oplus$ and $a^\ominus$ instead of a^+ and a^- to denote the max-positive and the max-negative part of $a \in \mathbb{S}$. $\diamond$

Acronyms

DES	Discrete Event System	1
ELCP	Extended Linear Complementarity Problem	59
MACP	Max-Algebraic Characteristic Polynomial	140
MIMO	Multiple Input Multiple Output	156
LCP	Linear Complementarity Problem	58
QRD	QR Decomposition	192
SISO	Single Input Single Output	156
SVD	Singular Value Decomposition	193

Contents

Max-Algebraïsche Systeemtheorie voor Discrete-Gebeurtenis-systemen

Nederlandse samenvatting

Hoofdstuk 1: Inleiding

Een discrete-gebeurtenissysteem is een asynchroon systeem met toestandsovergangen die veroorzaakt worden door gebeurtenissen, die op discrete tijdstippen plaatsvinden. Typische voorbeelden van discrete-gebeurtenissystemen zijn: flexibele produktiesystemen, telecommunicatienetwerken, spoorwegnetwerken en computerbesturingssystemen. Een gebeurtenis is dan b.v. het afwerken van een onderdeel op een machine, het aankomen van een boodschap of een trein, of de beëindiging van een opdracht.

Voor het beschrijven en bestuderen van discrete-gebeurtenissystemen is er reeds een breed scala van modellen en modelleringstechnieken beschikbaar, zoals Markovketens, Petrinetten, wachtlijnnetwerken, uitgebreide-toestandsmachines, veralgemeende semi-Markovprocessen, max-plus-algebra, formele talen, perturbatie-analyse, computersimulatie, ... [3, 13, 80, 81, 94, 112, 131, 132, 155]. Elk van deze methodologieën heeft zo zijn voor- en nadelen en het antwoord op de vraag welke methode het meest aangewezen is, hangt af van het te modelleren systeem en van wat we nadien met het resulterende model willen uitrichten. Eén van de belangrijkste overwegingen die hierbij in beschouwing moeten genomen worden, is modelleerkracht tegenover analyseerbaarheid: het blijkt immers dat hoe nauwkeuriger een bepaald model het gedrag van het gegeven systeem beschrijft, hoe moeilijker het is om analytisch uitspraken te doen over de eigenschappen van dat model. In ons onderzoek gebruiken wij max-algebraïsche modellen om het gedrag van discrete-gebeurtenissystemen te beschrijven. Alhoewel we op deze manier slechts een beperkte klasse discrete-gebeurtenissystemen kunnen beschrijven, laten deze modellen ons wel toe om een groot aantal eigenschappen van het gegeven systeem analytisch te bepalen.

De elementen van de max-plus-algebra zijn de reële getallen en $-\infty$. De basisbewerkingen van de max-plus-algebra zijn het maximum (voorgesteld door $\oplus$) en de optelling (voorgesteld door $\otimes$). Het blijkt dat er een merkwaardige analogie bestaat tussen de basisbewerkingen van de max-plus-algebra aan de ene kant en de basisbewerkingen uit de conventionele algebra — de optelling en de vermenigvuldiging — aan de andere kant. Heel wat eigenschappen en stellingen uit de conventionele (lineaire) algebra zijn na het vervangen van $+$ door $\oplus$ en van $\times$ door $\otimes$ immers ook geldig in de max-plus-algebra.

Bovendien kunnen we voor bepaalde types van discrete-gebeurtenissystemen — die kunnen omschreven worden als tijdsinvariante deterministische discrete-gebeurtenissystemen waarin de volgorde van de gebeurtenissen en de lengte van de activiteiten vaststaan of op voorhand bepaald kunnen worden — een toestandsruimtebeschrijving opstellen die lineair is in de max-plus-algebra [3, 20, 33]:

$$x(k+1) = A \otimes x(k) \ \oplus \ B \otimes u(k) \tag{0.1}$$

$$y(k) = C \otimes x(k) \ , \tag{0.2}$$

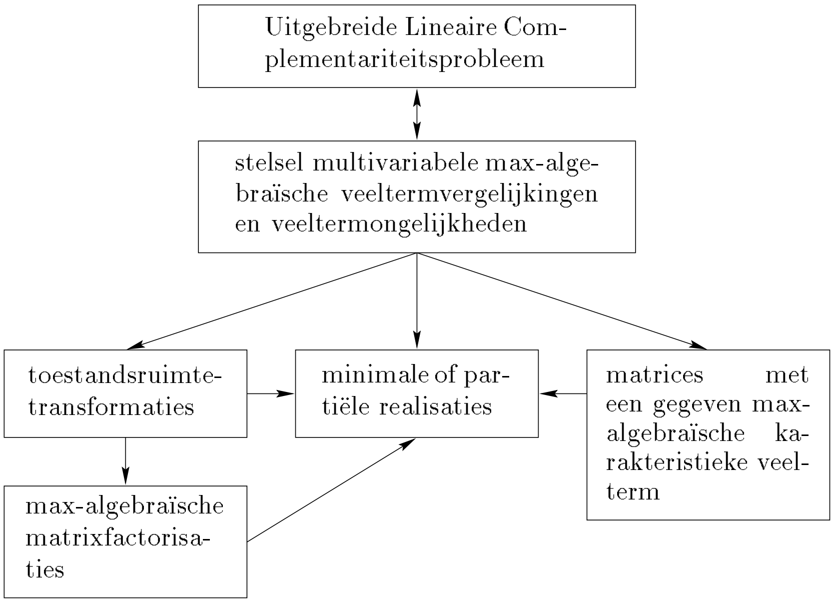
waarbij de vector x de toestand voorstelt (de componenten van $x(k)$ geven de tijdstippen aan waarop de verschillende gebeurtenissen voor de k-de keer plaatsvinden). De vector u bevat de ingangstijden en de vector y bevat de uitgangstijden. Voor een produktiesysteem komen de componenten van $x(k)$ b.v. overeen met de tijdstippen waarop de machines aan het k-de contingent produkten beginnen te werken, de componenten van $u(k)$ geven aan wanneer de grondstoffen voor het $(k-1)$-de contingent produkten in het systeem werden gebracht en de componenten van $y(k)$ geven aan wanneer het k-de contingent afgewerkte produkten het systeem verlaat. Discrete-gebeurtenissystemen die met een model van de vorm (0.1) – (0.2) kunnen beschreven worden, worden max-lineaire tijdsinvariante discrete-gebeurtenissystemen genoemd.

Het eerste doel van ons onderzoek is om uitgaande van de analogie tussen de max-plus-algebra en de conventionele algebra en van de analogie tussen de beschrijving (0.1) – (0.2) en de toestandsruimtebeschrijving voor lineaire tijdsinvariante systemen te onderzoeken welke concepten, technieken en algoritmen uit de lineaire systeemtheorie kunnen overgedragen worden naar de max-algebraïsche systeemtheorie voor discrete-gebeurtenissystemen. Ondanks de sterke gelijkenissen bestaan er echter ook grote verschillen tussen de max-plus-algebra en de conventionele algebra. Daardoor kunnen we de eigenschappen en de methodes van de lineaire systeemtheorie niet zo maar mechanisch overnemen. Dit heeft tot gevolg dat de max-algebraïsche systeemtheorie helemaal nog niet zo sterk ontwikkeld als de lineaire systeemtheorie. In ons onderzoek hebben wij dan ook getracht om methodes en technieken te ontwikkelen voor het oplossen van enkele fundamentele problemen uit de max-algebraïsche systeemtheorie voor discrete-gebeurtenissystemen. Eén van de open problemen in de max-algebraïsche systeemtheorie die we in dit kader bestudeerd hebben is het *minimale-realisatieprobleem* voor max-lineaire tijdsinvariante discrete-gebeurtenissystemen: gegeven de impulsresponsie van een systeem van de vorm (0.1) – (0.2) met onbekende systeemmatrices, zoek A, B en C waarbij de dimensie van A zo klein mogelijk moet zijn. Het minimale-realisatieprobleem kan beschouwd worden als één van de centrale problemen van dit proefschrift. In het kader van ons onderzoek in verband met dit probleem hebben wij ook de max-algebraïsche karakteristieke veelterm, max-algebraïsche toestandsruimtetransformaties en max-algebraïsche matrixfactorisaties bestudeerd. Daarnaast hebben we, met het oog op het ontwikkelen van methodes voor de identificatie van discrete-gebeurtenissystemen, ook onderzoek verricht in verband met max-algebraïsche equivalenten van twee matrixfactorisaties uit de lineaire algebra die een belangrijke rol spelen in een aantal identificatie-algoritmen voor lineaire systemen [138, 139, 140, 142]: de QR-ontbinding en de singulierewaardenontbinding.

Bij dit onderzoek viel ons op dat vele van de optredende max-algebraïsche problemen konden geherformuleerd worden als een wiskundige-programmatieprobleem dat wij het Uitgebreide Lineaire Complementariteitsprobleem (ULCP) genoemd hebben. Dit probleem komt neer op het oplossen van een stelsel lineaire vergelijkingen en ongelijkheden waarin een aantal groepen ongelijkheden voorkomen waarbij er in elke groep ten minste één ongelijkheid voldaan moet zijn met gelijkheid. Wij hebben aangetoond dat het probleem van het oplossen van een stelsel multivariabele max-algebraïsche veeltermvergelijkingen en veeltermongelijkheden geherformuleerd kan worden als een ULCP en omgekeerd. Dit stelt ons in staat om de volgende max-algebraïsche problemen op te lossen:

- het berekenen van max-algebraïsche matrixfactorisaties,
- het berekenen van toestandsruimtetransformaties voor max-lineaire tijdsinvariante discrete-gebeurtenissystemen,
- het berekenen van minimale of partiële realisaties van de impulsresponsie van een max-lineair tijdsinvariant discrete-gebeurtenissysteem,
- het berekenen van matrices met een gegeven max-algebraïsche karakteristieke veelterm,
- het berekenen van de max-algebraïsche QR-ontbinding of de max-algebraïsche singuliere-waardenontbinding van een matrix,
- ...

Alhoewel de problemen uit de lineaire algebra en de lineaire systeemtheorie die met deze max-algebraïsche problemen overeenkomen vaak zeer gemakkelijk op te lossen zijn, is dit niet het geval voor de max-algebraïsche problemen. Voor



Figuur 0.1: De verbanden tussen het ULCP en enkele max-algebraïsche problemen die in dit proefschrift aan bod komen.

bijna al de bovenvermelde max-algebraïsche problemen is de ULCP-benadering op dit moment de enige manier om het probleem op te lossen.

Het verband tussen enkele van de bovenvermelde problemen is weergegeven in Figuur 0.1. We zullen de relaties tussen deze problemen nu wat nader bespreken. We hebben reeds gezegd dat het ULCP en het oplossen van een stelsel multivariabele max-algebraïsche veeltermvergelijkingen en veeltermongelijkheden equivalent zijn. Een gegeven matrix factoriseren als een produkt van twee of meer matrices is een speciaal geval van het oplossen van een stelsel multivariabele max-algebraïsche veeltermvergelijkingen en veeltermongelijkheden en kan dus ook opgelost worden met behulp van een ULCP. We kunnen een toestandsruimtemodel met systeemmatrices A, B en C en begintoestand x_0 transformeren in een equivalent toestandsruimtemodel door een factorisatie te bepalen van de blokmatrices $\begin{bmatrix} A \\ C \end{bmatrix}$ of $\begin{bmatrix} A & B & x_0 \end{bmatrix}$. We zullen ook aantonen dat het berekenen van matrices met een gegeven max-algebraïsche karakteristieke veelterm ook kan gebeuren met behulp van een ULCP.
Voor het bepalen van minimale of partiële realisaties van de impulsresponsie van een max-lineair tijdsinvariant discrete-gebeurtenissysteem zullen we verschillende methodes voorstellen. De eerste methode kan beschouwd worden als een uitgebreide max-algebraïsche matrixfactorisatie. De tweede methode is een twee-stapsmethode waarin eerst de A-matrix bepaald wordt met behulp van

haar max-algebraïsche karakteristieke veelterm, vervolgens worden dan de B- en de C-matrix bepaald via een variant van de uitgebreide matrixfactorisatie die in de eerste methode gebruikt wordt. Tenslotte kunnen we in sommige gevallen ook toestandsruimtetransformaties gebruiken om een minimale realisatie van de impulsresponsie van een max-lineair tijdsinvariant discrete-gebeurtenis-systeem te berekenen, op voorwaarde natuurlijk dat we reeds over een andere (niet-minimale) realisatie van dit systeem beschikken.

Laten we nu wat nader ingaan op de belangrijkste resultaten die in dit proefschrift worden voorgesteld.

Hoofdstuk 2: Achtergrondinformatie

In dit hoofdstuk geven we een inleiding tot de max-plus-algebra en de gesymmetriseerde max-plus-algebra. We behandelen ook enkele basisbegrippen uit de systeemtheorie voor max-lineaire tijdsinvariante discrete-gebeurtenissystemen.

Om de lezer toe te laten de rest van deze samenvatting te kunnen volgen, zullen we nu kort enkele van de belangrijkste definities en begrippen behandelen die in dit hoofdstuk van het proefschrift voorgesteld worden.
De basisbewerkingen van de max-plus-algebra werden reeds vermeld:

$$x \oplus y = \max(x, y)$$

$$x \otimes y = x + y$$

voor $x, y \in \mathbb{R} \cup \{-\infty\}$. We definiëren $\varepsilon = -\infty$ en $\mathbb{R}_\varepsilon = \mathbb{R} \cup \{\varepsilon\}$. De max-algebraïsche macht is als volgt gedefinieerd:

$$a^{\otimes^r} = ra$$

met $a, r \in \mathbb{R}$.
De bewerkingen $\oplus$ en $\otimes$ worden als volgt uitgebreid tot matrices:

$$(A \oplus B)_{ij} = a_{ij} \oplus b_{ij} = \max(a_{ij}, b_{ij})$$

$$(A \otimes B)_{ij} = \bigoplus_k a_{ik} \otimes b_{kj} = \max_k (a_{ik} + b_{kj})$$

voor alle i, j. Bovendien geldt: $A^{\otimes^k} = \underbrace{A \otimes A \otimes \ldots \otimes A}_{k \text{ keer}}$.
De matrix E_n is de $n \times n$ max-algebraïsche eenheidsmatrix. De matrix $\mathcal{E}_{m \times n}$ is de $m \times n$ max-algebraïsche nulmatrix.

Alhoewel er geen symmetrische elementen bestaan voor de $\oplus$-bewerking, kunnen we toch een soort symmetrisatie van de max-plus-algebra doorvoeren. Dit geeft aanleiding tot het invoeren van de $\ominus$-bewerking, die kan beschouwd worden als een max-algebraïsche versie van de $-$-bewerking. Het symmetriseren betekent ook dat we =-tekens moeten vervangen door ∇-tekens (de ∇-relatie kan beschouwd worden als een afgezwakte versie van de gelijkheidsrelatie). Nu kunnen we de volgende verzamelingen invoeren:

- de verzameling van de max-positieve elementen en ε: $\mathbb{S}^{\oplus} = \mathbb{R}_{\varepsilon}$,
- de verzameling van de max-negatieve elementen en ε: $\mathbb{S}^{\ominus} = \{\ominus a \,|\, a \in \mathbb{R}_{\varepsilon}\}$,
- de verzameling van de gebalanceerde elementen: $\mathbb{S}^{\bullet} = \{ a^{\bullet} = a \oplus (\ominus a) \,|\, a \in \mathbb{R}_{\varepsilon}\}$.

We definiëren $\mathbb{S} = \mathbb{S}^{\oplus} \cup \mathbb{S}^{\ominus} \cup \mathbb{S}^{\bullet}$ en $\mathbb{S}^{\vee} = \mathbb{S}^{\oplus} \cup \mathbb{S}^{\ominus}$.

Tenslotte behandelen we nog enkele begrippen uit de systeemtheorie voor max-lineaire tijdsinvariante discrete-gebeurtenissystemen, d.w.z. voor discrete-gebeurtenissystemen die kunnen beschreven worden met behulp van een model van de vorm

$$x(k+1) = A \otimes x(k) \;\oplus\; B \otimes u(k) \tag{0.3}$$

$$y(k) = C \otimes x(k) \tag{0.4}$$

met begintoestand $x(0) = x_0$. In deze samenvatting zullen we een model van de vorm (0.3) – (0.4) karakteriseren door het drietal (A, B, C).
Voor de eenvoud beschouwen we nu even een systeem met één ingang en één uitgang. Als we een max-algebraïsche eenheidsimpuls:

$$e(k) = \begin{cases} 0 & \text{als } k = 0\,, \\ \varepsilon & \text{als } k \neq 0\,, \end{cases}$$

aan de ingang aanleggen en als $x(0) = \mathcal{E}_{n \times 1}$, dan bekomen we

$$x(1) = B, \;\; x(2) = A \otimes B, \; \ldots, \;\; x(k) = A^{\otimes^{k-1}} \otimes B, \; \ldots .$$

Dit betekent dat

$$y(k) = C \otimes A^{\otimes^{k-1}} \otimes B \qquad \text{voor alle } k \in \mathbb{N}_0\,.$$

Definieer nu $G_k = C \otimes A^{\otimes^{k}} \otimes B$. De rij $\{G_k\}_{k=0}^{\infty}$ wordt de *impulsresponsie* van het discrete-gebeurtenissysteem genoemd aangezien ze overeenkomt met de uitgang ten gevolge van een max-algebraïsche eenheidsimpuls aan de ingang.

Hoofdstuk 3: Het Uitgebreide Lineaire Complementariteitsprobleem

Tijdens ons onderzoek viel ons op dat een groot aantal max-algebraïsche problemen eigenlijk te herleiden is tot het oplossen van een stelsel multivariabele max-algebraïsche veeltermvergelijkingen en veeltermongelijkheden. Dit probleem kan op zijn beurt geherformuleerd worden als een wiskundige-programmatieprobleem, dat wij het Uitgebreide Lineaire Complementariteitsprobleem (ULCP) genoemd hebben en dat als volgt gedefinieerd is:

> Gegeven $A \in \mathbb{R}^{p \times n}$, $B \in \mathbb{R}^{q \times n}$, $c \in \mathbb{R}^p$, $d \in \mathbb{R}^q$ en m deelverzamelingen ϕ_1, ϕ_2, ... , ϕ_m van $\{1, 2, \ldots, p\}$, bepaal een vector $x \in \mathbb{R}^n$ zodat
>
> $$\sum_{j=1}^{m} \prod_{i \in \phi_j} (Ax - c)_i = 0 \tag{0.5}$$
>
> met $Ax \geqslant c$ en $Bx = d$.

Dit probleem is een uitbreiding van het Lineaire Complementariteitsprobleem (LCP), dat één van de basisproblemen van het wiskundig programmeren is [30].

Laten we de voorwaarde (0.5) eens even van naderbij bekijken. Aangezien $Ax \geqslant c$ zullen alle factoren die in (0.5) voorkomen niet-negatief zijn. Dit heeft tot gevolg dat alle termen van de sommatie ook niet-negatief zijn. Opdat een som van niet-negatieve termen gelijk aan 0 zou kunnen zijn, moet elk van de termen gelijk zijn aan 0. Dit betekent dat voorwaarde (0.5) equivalent is met de volgende voorwaarde:

$$\prod_{i \in \phi_j} (Ax - c)_i = 0 \qquad \text{voor } j = 1, 2, \ldots, m \ . \tag{0.6}$$

Dit betekent op zijn beurt dat in elke verzameling ϕ_j er minstens één index i moet zitten zodat $(Ax - c)_i = 0$ of nog $(Ax)_i = c_i$, wat wil zeggen dat de i-de ongelijkheid van het stelsel $Ax \geqslant c$ voldaan is met gelijkheid.
Daarom kunnen we de volgende interpretatie geven aan voorwaarde (0.5):

> Elke verzameling ϕ_j komt overeen met een groep ongelijkheden uit het stelsel $Ax \geqslant c$ en in elke groep moet ten minste één ongelijkheid voldaan zijn met gelijkheid.

Hieruit volgt dat de oplossingsverzameling van een ULCP in het algemeen bestaat uit een unie van zijvlakken van een veelvlak.

We hebben reeds gezegd dat het ULCP een uitbreiding is van het LCP. In het proefschrift tonen we aan dat vele andere uitbreidingen van het LCP kunnen beschouwd worden als speciale gevallen van het ULCP. Aangezien de oplossingsverzameling van elke lineaire uitbreiding van het LCP bestaat uit de unie van zijvlakken van een veelvlak en aangezien de unie van een willekeurige verzameling zijvlakken van een willekeurig veelvlak kan beschreven worden met behulp van een ULCP, beweren wij dat elke lineaire uitbreiding van het LCP een speciaal geval is van het ULCP.

Wij ontwikkelen een algoritme dat ons in staat stelt om de *volledige* oplossingsverzameling van een ULCP te beschrijven. In dit algoritme wordt in elke stap een nieuwe vergelijking of ongelijkheid van het stelsel $Ax \geqslant c$ en $Bx = d$

in beschouwing genomen en wordt de doorsnede bepaald van de halfruimte of het hypervlak dat gedefinieerd wordt door de nieuwe vergelijking of ongelijkheid en het veelvlak dat gedefinieerd wordt door de vorige vergelijkingen en ongelijkheden. Dit levert dan uiteindelijk een beschrijving op van de oplossingsverzameling met behulp van eindige punten, generatoren voor de extreme stralen en een basis voor de lineaire deelruimte geassocieerd met de maximale affiene deelruimten van de oplossingsverzameling. Een voorbeeld zal verduidelijken wat we hiermee bedoelen.

Voorbeeld 0.0.1 Beschouw het volgende ULCP:

Bepaal $x, y, z \in \mathbb{R}$ zodat

$$(x+y+1)(-x-y+1)(z+1) \;=\; 0 \tag{0.7}$$

met

$$x+y \geqslant -1 \tag{0.8}$$

$$-x-y \geqslant -1 \tag{0.9}$$

$$z \geqslant -1 \;\; . \tag{0.10}$$

We definiëren de volgende vlakken:

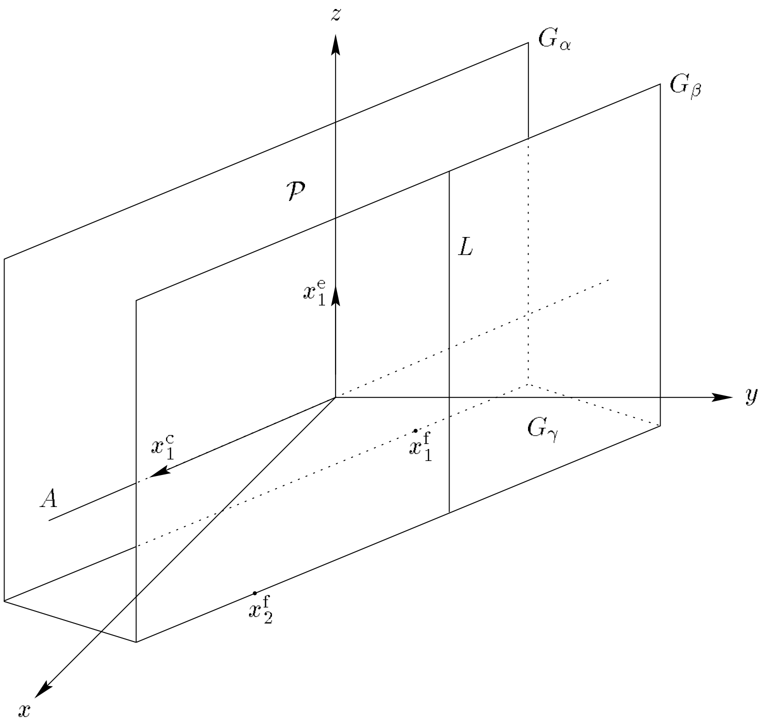
$$\alpha = \left\{ \begin{bmatrix} x & y & z \end{bmatrix}^T \in \mathbb{R}^3 \;\middle|\; x+y=-1 \right\}$$
$$\beta = \left\{ \begin{bmatrix} x & y & z \end{bmatrix}^T \in \mathbb{R}^3 \;\middle|\; -x-y=-1 \right\}$$
$$\gamma = \left\{ \begin{bmatrix} x & y & z \end{bmatrix}^T \in \mathbb{R}^3 \;\middle|\; z=-1 \right\}$$

en de volgende rechte:

$$A = \left\{ \begin{bmatrix} x & y & z \end{bmatrix}^T \in \mathbb{R}^3 \;\middle|\; x+y=0 \text{ en } z=0 \right\} \;.$$

In Figuur 0.2 hebben we het veelvlak $\mathcal{P}$ voorgesteld dat gedefinieerd wordt door het stelsel (0.8) – (0.10). De zijvlakken van het veelvlak hebben we G_α, G_β en G_γ genoemd: $G_\alpha = \mathcal{P} \cap \alpha$, $G_\beta = \mathcal{P} \cap \beta$ en $G_\gamma = \mathcal{P} \cap \gamma$. De doorsnede van G_β en het yz-vlak hebben we L genoemd.
Het is duidelijk dat elke oplossing van het ULCP tot $\mathcal{P}$ moet behoren. Als we nu de voorwaarde (0.7) in rekening brengen, dan zien dat de som van de x- en de y-component van een oplossing 1 of -1 moet zijn of dat de z-component -1 moet zijn. Dit betekent dat de oplossingen van het ULCP op de zijvlakken van $\mathcal{P}$ moeten liggen. Als we nu de volgende punten definiëren:

$$x_1^{\mathrm{f}} = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}, \; x_2^{\mathrm{f}} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \; x_1^{\mathrm{e}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \; x_1^{\mathrm{c}} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \;,$$



Figuur 0.2: De oplossingsverzameling van het ULCP van Voorbeeld 1.

dan kan elk willekeurig punt v_α van G_α geschreven worden als

$$v_\alpha = \lambda x_1^{\mathrm{c}} + \kappa x_1^{\mathrm{e}} + x_1^{\mathrm{f}} \tag{0.11}$$

met $\lambda \in \mathbb{R}$ en $\kappa \geqslant 0$. Elk willekeurig punt v_β van G_β kan geschreven worden als

$$v_\beta = \lambda x_1^{\mathrm{c}} + \kappa x_1^{\mathrm{e}} + x_2^{\mathrm{f}} \tag{0.12}$$

met $\lambda \in \mathbb{R}$ en $\kappa \geqslant 0$. Elk willekeurig punt v_γ van G_γ kan geschreven worden als

$$v_\gamma = \lambda x_1^{\mathrm{c}} + \mu_1 x_1^{\mathrm{f}} + \mu_2 x_2^{\mathrm{f}} \tag{0.13}$$

met $\lambda \in \mathbb{R}$, $\mu_1, \mu_2 \geqslant 0$ en $\mu_1 + \mu_2 = 1$.
Dit betekent dat elke oplossing $v = \begin{bmatrix} x & y & z \end{bmatrix}^T$ van het ULCP kan geschreven worden als een combinatie van de vorm (0.11), (0.12) of (0.13) met $\lambda \in \mathbb{R}$,

$\kappa, \mu_1, \mu_2 \geqslant 0$ en $\mu_1 + \mu_2 = 1$.
De punten x_1^{f} en x_2^{f} zijn eindige punten van de oplossingsverzameling van het ULCP. De vector x_1^{e} is een generator voor de extreme stralen (zoals b.v. de halfrechte L). Aangezien elke rechte die behoort tot $G_\alpha \cup G_\beta \cup G_\gamma$ evenwijdig is met de vectorrechte A, is A de lineaire deelruimte geassocieerd met de maximale affiene deelruimten van de oplossingsverzameling van het ULCP. De verzameling $\{x_1^{\mathrm{c}}\}$ is een basis voor de rechte A. □

In de praktijk kan het door ons ontwikkelde algoritme enkel toegepast worden op niet al te omvangrijke ULCP's: de uitvoeringstijd van het algoritme stijgt immers exponentieel met het aantal onbekenden en polynomiaal met het aantal vergelijkingen en ongelijkheden. Daarom moet er zeker gezocht worden naar efficiëntere algoritmen om één oplossing van een ULCP te vinden. Dit zal echter geen gemakkelijke opgave zijn, aangezien het ULCP een NP-hard probleem is: dit betekent dat het waarschijnlijk zeer moeilijk of zelfs onmogelijk zal zijn om snelle, efficiënte algoritmen te ontwikkelen om het algemene ULCP op te lossen.

Hoofdstuk 4: Toepassingen van het Uitgebreide Lineaire Complementariteitsprobleem in de Max-Plus-Algebra

Beschouw het volgende probleem:

> Gegeven $p_1 + p_2$ positieve natuurlijke getallen $m_1, m_2, \ldots, m_{p_1+p_2}$ en reële getallen a_{ki}, b_k en c_{kij} voor $k = 1, 2, \ldots, p_1 + p_2$, $i = 1, 2, \ldots, m_k$ en $j = 1, 2, \ldots, n$, bepaal $x \in \mathbb{R}^n$ zodat
>
> $$\bigoplus_{i=1}^{m_k} a_{ki} \otimes \bigotimes_{j=1}^{n} x_j^{\otimes^{c_{kij}}} = b_k \qquad \text{voor } k = 1, 2, \ldots, p_1\,, \tag{0.14}$$
>
> $$\bigoplus_{i=1}^{m_k} a_{ki} \otimes \bigotimes_{j=1}^{n} x_j^{\otimes^{c_{kij}}} \leqslant b_k \qquad \text{voor } k = p_1 + 1, p_1 + 2, \ldots, p_1 + p_2\,. \tag{0.15}$$

Dit probleem wordt een stelsel multivariabele max-algebraïsche veeltermvergelijkingen en veeltermongelijkheden genoemd. Merk op dat de exponenten ook negatief of reëel mogen zijn.

Beschouw nu één vergelijking van het type (0.14). Als we een dergelijke vergelijking herschrijven met behulp van de conventionele bewerkingen $+$, $\times$ en max dan bekomen we

$$\max_{i=1,\ldots,m_k} \left(a_{ki} + \sum_{j=1}^{n} c_{kij} x_j \right) = b_k \ .$$

Dit betekent dat

$$a_{ki} + \sum_{j=1}^{n} c_{kij} x_j \leqslant b_k \qquad \text{voor } i =, 1, 2, \ldots, m_k$$

waarbij voor ten minste één index i de waarde van het linkerlid van de ongelijkheid gelijk moet zijn aan b_k.
Dit betekent dat (0.14) beschouwd kan worden als een stelsel van p_1 groepen lineaire ongelijkheden waarbij in elke groep ten minste één ongelijkheid voldaan moet zijn met gelijkheid.
Op een gelijkaardige wijze vinden we dat (0.15) beschouwd kan worden als een stelsel van p_2 groepen lineaire ongelijkheden (zonder extra voorwaarde).
We hebben dus nu een stelsel van lineaire vergelijkingen en ongelijkheden waarin een aantal groepen ongelijkheden voorkomen waarbij er in elke groep ten minste één ongelijkheid voldaan moet zijn met gelijkheid, m.a.w. we hebben een ULCP.
Omgekeerd kan ook aangetoond worden dat elke ULCP kan geherformuleerd worden als een stelsel multivariabele max-algebraïsche veeltermvergelijkingen en veeltermongelijkheden. Zo komt de ULCP van Voorbeeld 0.0.1 overeen met het volgende — zeer eenvoudige — stelsel multivariabele max-algebraïsche veeltermvergelijkingen en veeltermongelijkheden:

$$\text{Bepaal } x, y, z \in \mathbb{R} \text{ zodat } x^{\otimes^{-1}} \otimes y^{\otimes^{-1}} \oplus x \otimes y \oplus z^{\otimes^{-1}} = 1 \, .$$

Hiermee hebben we dus aangetoond dat het ULCP equivalent is met een stelsel multivariabele max-algebraïsche veeltermvergelijkingen en veeltermongelijkheden.
In het proefschrift leiden we ook een methode af om oplossingen van een stelsel multivariabele max-algebraïsche veeltermvergelijkingen en veeltermongelijkheden te vinden waarvan sommige componenten gelijk zijn aan ε. Dit stelt ons in staat om alle oplossingen van een probleem van de vorm (0.14) – (0.15) te vinden. Dit betekent ook dat we het ULCP kunnen gebruiken om een aantal belangrijke problemen in de max-plus-algebra en de max-min-plus-algebra op te lossen, zoals

- het berekenen van max-algebraïsche matrixfactorisaties:

  Gegeven een matrix $T \in \mathbb{R}_\varepsilon^{m \times n}$ en een positief natuurlijk getal l, bepaal $P \in \mathbb{R}_\varepsilon^{m \times l}$ en $Q \in \mathbb{R}_\varepsilon^{l \times n}$ zodat $T = P \otimes Q$.

  Factorisaties van T als een produkt van drie of meer matrices ook kunnen berekend worden met behulp van het ULCP. Bovendien kunnen we ook een bepaalde structuur voor de factoren opleggen: diagonaal, bovendriehoeks-, symmetrisch, ...

- het oplossen van een stelsel max-lineaire vergelijkingen in de gesymmetriseerde max-plus-algebra:

Gegeven $A \in \mathbb{S}^{m \times n}$ en $b \in \mathbb{S}^m$, bepaal $x \in (\mathbb{S}^\vee)^n$ zodat $A \otimes x \nabla b$.

of

Gegeven $A \in \mathbb{S}^{m \times n}$, bepaal $x \in (\mathbb{S}^\vee)^n$ zodat $A \otimes x \nabla b$ en $\|x\|_\oplus = 0$.

- het construeren van matrices met een gegeven max-algebraïsche karakteristieke veelterm (zie Hoofdstuk 5),
- het bepalen van toestandsruimtetransformaties voor max-lineaire tijdsinvariante discrete-gebeurtenissystemen (zie Hoofdstuk 6),
- het berekenen van minimale toestandsruimterealisaties voor max-lineaire tijdsinvariante discrete-gebeurtenissystemen (zie Hoofdstuk 6),
- het berekenen van max-algebraïsche singuliere-waardenontbindingen en max-algebraïsche QR-ontbindingen (zie Hoofdstuk 7),
- het oplossen van gemengde max-min-problemen:

  Gegeven positieve natuurlijke getallen m_k, m_{kl_1} voor $k = 1, 2, \ldots, m$ en $l_1 = 1, 2, \ldots, m_k$ en reële getallen $a_{kl_1l_2}$, b_k en $c_{kl_1l_2j}$ voor $k = 1, 2, \ldots, m$, $l_1 = 1, 2, \ldots, m_k$, $l_2 = 1, 2, \ldots, m_{kl_1}$ en $j = 1, 2, \ldots, n$, bepaal $x \in \mathbb{R}^n$ zodat

$$\bigoplus_{l_1=1}^{m_k}{}' \bigoplus_{l_2=1}^{m_{kl_1}} a_{kl_1l_2} \otimes \bigotimes_{j=1}^{n} x_j{}^{\otimes^{c_{kl_1l_2j}}} = b_k \tag{0.16}$$

  voor $k = 1, 2, \ldots, m$,

  waarbij de $\oplus'$-bewerking als volgt gedefinieerd is:

$$x \oplus' y = \min(x, y)$$

  voor $x, y \in \mathbb{R}_\varepsilon$.
  De ULCP-aanpak kan ook gebruikt worden indien sommige vergelijkingen in (0.16) ongelijkheden zijn in plaats van gelijkheden.

- het oplossen van max-max- en max-min-problemen:

  Gegeven positieve natuurlijke getallen m_k, p_k voor $k = 1, 2, \ldots, m$ en reële getallen a_{ki}, b_{kij}, c_{kl} en d_{klj} voor $k = 1, 2, \ldots, m$, $i = 1, 2, \ldots, m_k$, $j = 1, 2, \ldots, n$ en $l = 1, 2, \ldots, p_k$, bepaal $x \in \mathbb{R}^n$ zodat

$$\bigoplus_{i=1}^{m_k} a_{ki} \otimes \bigotimes_{j=1}^{n} x_j{}^{\otimes^{b_{kij}}} = \bigoplus_{l=1}^{p_k} c_{kl} \otimes \bigotimes_{j=1}^{n} x_j{}^{\otimes^{d_{klj}}} \tag{0.17}$$

  voor $k = 1, 2, \ldots, m$.

  De ULCP-aanpak kan ook gebruikt worden indien sommige vergelijkingen in (0.17) ongelijkheden zijn in plaats van gelijkheden of indien sommige van de $\bigoplus$-sommaties vervangen worden door $\bigoplus'$-sommaties.

Hoofdstuk 5: De Max-Algebraïsche Karakteristieke Veelterm

In dit hoofdstuk kijken we naar nodige en voldoende voorwaarden voor de coëfficiënten van de max-algebraïsche karakteristieke veelterm van een matrix met elementen in $\mathbb{R}_\varepsilon$. Voor matrices met een dimensie die kleiner is dan 5, zijn we erin geslaagd om nodige en voldoende voorwaarden voor de coëfficiënten van de max-algebraïsche karakteristieke veelterm af te leiden. Bovendien beschikken we over analytische uitdrukkingen die ons in staat stellen om, indien de coëfficiënten van een gegeven max-algebraïsche veelterm aan de nodige en voldoende voorwaarden voldoen, een matrix te bepalen waarvan de max-algebraïsche karakteristieke veelterm gelijk is aan de gegeven max-algebraïsche veelterm. Voor het algemene geval (d.w.z. voor matrices met een dimensie groter dan of gelijk aan 5) hebben we enkel nodige voorwaarden voor de coëfficiënten van de max-algebraïsche karakteristieke veelterm kunnen afleiden.
Beschouw nu het volgende probleem:

> Gegeven een max-algebraïsche veelterm van de vorm
>
> $$\lambda^{\otimes^n} \oplus \bigotimes_{k=1}^{n} b_k \otimes \lambda^{\otimes^{n-k}} \quad ,$$
>
> bepaal een matrix $A \in \mathbb{R}_\varepsilon^{n \times n}$ zodat de max-algebraïsche karakteristieke veelterm van A gelijk is aan de gegeven max-algebraïsche veelterm.

In het proefschrift tonen we aan dat dit probleem geherformuleerd kan worden als een stelsel van multivariabele max-algebraïsche veeltermvergelijkingen en veeltermongelijkheden. Dit betekent dat we het ULCP kunnen gebruiken om een matrix met een gegeven max-algebraïsche karakteristieke veelterm te construeren.

Hoofdstuk 6: Toestandsruimtetransformaties en Toestandsruimterealisatie voor Max-Lineaire Tijdsinvariante Discrete-Gebeurtenissystemen

In het eerste deel van dit hoofdstuk onderzoeken we toestandsruimtetransformaties voor max-lineaire tijdsinvariante discrete-gebeurtenissystemen. We vertrekken van een gegeven drietal (A, B, C) en een begintoestand x_0. Nu willen we een ander drietal $(\tilde{A}, \tilde{B}, \tilde{C})$ en een bijbehorende begintoestand $\tilde{x}_0$ bepalen zodat beide drietallen met hun overeenkomstige begintoestanden hetzelfde ingangs-uitgangsgedrag beschrijven.
Net zoals voor lineaire systemen kan men voor max-lineaire tijdsinvariante discrete-gebeurtenissystemen (max-algebraïsche) gelijkvormigheidstransformaties gebruiken om een drietal (A, B, C) met bijbehorende begintoestand x_0 om

te zetten in een equivalent drietal $(\tilde{A}, \tilde{B}, \tilde{C})$ met bijbehorende begintoestand $\tilde{x}_0$:

$$\tilde{A} = T \otimes A \otimes T^{\otimes^{-1}}, \quad \tilde{B} = T \otimes B, \quad \tilde{C} = C \otimes T^{\otimes^{-1}} \quad \text{en} \quad \tilde{x}_0 = T \otimes x_0 \ .$$

Aangezien de klasse van de matrices die inverteerbaar zijn in de max-plus-algebra zeer klein is, is het bereik van deze max-algebraïsche gelijkvormigheids-transformaties beperkt. Wij stellen twee nieuwe transformaties voor die kunnen beschouwd worden als een uitbreiding van max-algebraïsche gelijkvormigheids-transformaties maar die een veel groter bereik hebben. De eerste transformatie, de L-transformatie, vertrekt van een factorisatie van de matrices A en C als

$$A = \hat{A} \otimes L \quad \text{en} \quad C = \hat{C} \otimes L \ .$$

Als we nu de volgende matrices en de volgende vector definiëren:

$$\tilde{A} = L \otimes \hat{A}, \quad \tilde{B} = L \otimes B, \quad \tilde{C} = \hat{C} \quad \text{en} \quad \tilde{x}_0 = L \otimes x_0 \ ,$$

dan beschrijft het drietal $(\tilde{A}, \tilde{B}, \tilde{C})$ met begintoestand $\tilde{x}_0$ hetzelfde ingangs-uitgangsgedrag als het drietal (A, B, C) met begintoestand x_0.
De tweede transformatie, de M-transformatie, is de duale transformatie van de L-transformatie. Zij vertrekt van een factorisatie van de matrices A en B en de vector x_0 als

$$A = M \otimes \hat{A}, \quad B = M \otimes \hat{B} \quad \text{en} \quad x_0 = M \otimes \hat{x}_0 \ .$$

Nu resulteert

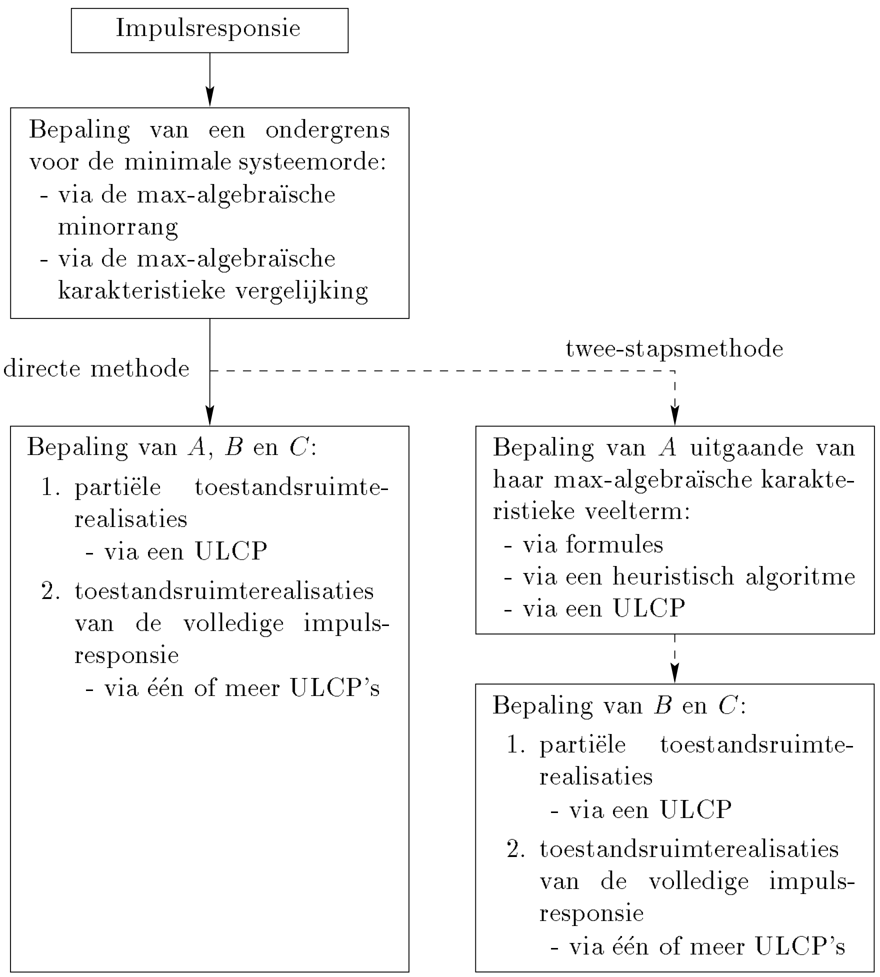
$$\tilde{A} = \hat{A} \otimes M, \quad \tilde{B} = \hat{B}, \quad \tilde{C} = C \otimes M \quad \text{en} \quad \tilde{x}_0 = \hat{x}_0$$

in een equivalent drietal met bijbehorende begintoestand.
Aangezien deze transformaties vertrekken van een max-algebraïsche matrixfactorisatie, kunnen we ze berekenen met behulp van een ULCP.

In het tweede deel van dit hoofdstuk behandelen we het minimale-realisatie-probleem voor max-lineaire tijdsinvariante discrete-gebeurtenissystemen:

> Gegeven de impulsresponsie $\{G_k\}_{k=0}^{\infty}$ van een max-lineair tijdsinvariant discrete-gebeurtenissysteem, bepaal matrices A, B en C met zo klein mogelijke dimensies zodat $C \otimes A^{\otimes^k} \otimes B = G_k$ voor $k = 0, 1, 2, \ldots$.

In Figuur 0.3 hebben we de verschillende stappen van de door ons ontwikkelde methodes om dit probleem op te lossen voorgesteld.
Eerst bepalen we een ondergrens voor de minimale systeemorde — dit is de kleinst mogelijke afmeting van de A-matrix over de verzameling van alle realisaties van de gegeven impulsresponsie. Voor de eenvoud veronderstellen we



Figuur 0.3: Een overzicht van de methodes om het minimale-realisatieprobleem voor max-lineaire tijdsinvariante discrete-gebeurtenissystemen op te lossen.

eerst dat we te maken hebben met een systeem met één ingang en één uitgang. Met de termen van de gegeven impulsresponsie $\{g_k\}_{k=0}^{\infty}$ construeren we de volgende Hankelmatrix:

$$H = \begin{bmatrix} g_0 & g_1 & g_2 & \cdots \\ g_1 & g_2 & g_3 & \cdots \\ g_2 & g_3 & g_4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} .$$

De max-algebraïsche minorrang van H is een ondergrens voor de minimale systeemorde [54, 56]. In het proefschrift stellen we een alternatieve manier voor om een ondergrens voor de minimale systeemorde te bepalen waarbij we gebruik maken van de resultaten van Hoofdstuk 5. We zoeken naar een relatie van de vorm

$$\bigoplus_{i=0}^{r} a_i \otimes H_{.,k+r-i} \nabla \mathcal{E}_{\infty\times 1} \qquad \text{voor } k = 1, 2, 3, \ldots \tag{0.18}$$

zodat de coëfficiënten $a_0, a_1, \ldots, a_r$ overeenkomen met de coëfficiënten van de max-algebraïsche karakteristieke veelterm van een matrix met elementen in $\mathbb{R}_\varepsilon$. Merk op dat hier de nodige en/of voldoende voorwaarden voor de coëfficiënten van de max-algebraïsche karakteristieke veelterm van een matrix met elementen in $\mathbb{R}_\varepsilon$ die werden afgeleid in Hoofdstuk 5, van pas komen. De kleinst mogelijke r waarvoor we een relatie van de vorm (0.18) kunnen vinden die overeenkomt met de max-algebraïsche karakteristieke veelterm van een matrix met elementen in $\mathbb{R}_\varepsilon$, is een ondergrens voor de minimale systeemorde. Deze methode kan gemakkelijk uitgebreid worden naar systemen met meer dan één ingang of uitgang.

In de eerste methode voor het bepalen van de systeemmatrices worden A, B en C tegelijkertijd berekend. Deze methode hebben wij de directe methode genoemd. Ze bestaat uit twee grote stappen. In de eerste stap van de directe methode gaan we op zoek naar *partiële* realisaties van de r-de orde van de gegeven impulsresponsie: voor een bepaalde $N \in \mathbb{N}_0$ bepalen we matrices $A \in \mathbb{R}_\varepsilon^{r\times r}$, $B \in \mathbb{R}_\varepsilon^{r\times m}$ en $C \in \mathbb{R}_\varepsilon^{l\times r}$ zodat

$$C \otimes A^{\otimes^k} \otimes B = G_k \qquad \text{voor } k = 0, 1, \ldots, N-1\,. \tag{0.19}$$

Aangezien dit kan beschouwd worden als een soort van uitgebreide max-algebraïsche matrixfactorisatie kunnen we het ULCP gebruiken om partiële realisaties van de gegeven impulsresponsie te bepalen.

In de tweede stap van de directe methode voor het bepalen van de systeemmatrices kijken we dan hoe de verzameling van de partiële realisaties van de eerste N termen van de impulsresponsie evolueert als N naar ∞ streeft. Hierbij doen zich twee gevallen voor:

- Vanaf een bepaalde N verandert de verzameling van de partiële realisaties niet meer. Dit betekent dat we vanaf dan realisaties van de volledige impulsresponsie hebben.

- De verzameling van de partiële realisaties wordt steeds kleiner en kleiner naarmate N toeneemt, maar de limiet wordt niet bereikt voor een eindige waarde van N. In dit geval kunnen we na het invoeren van een aantal normalisaties toch gemakkelijk bepalen wat de uiteindelijke limietverzameling zal zijn. Na het wegwerken van de normalisaties bekomen we dan de verzameling van alle realisaties van de gegeven impulsresponsie.

Aangezien de ULCP's die in de directe methode optreden vaak zo groot zijn dat ze in de praktijk niet op te lossen zijn met ons algoritme, bespreken we ook een twee-stapsmethode voor het bepalen van de systeemmatrices die in veel kleinere ULCP's resulteert. In plaats van A, B en C tegelijkertijd te bepalen, kunnen we eerst de A-matrix bepalen vertrekkende van de coëfficiënten a_0, a_1, ..., a_r van haar max-algebraïsche karakteristieke veelterm die uit (0.18) volgen. Dit kan gebeuren met behulp van de formules die in Hoofdstuk 5 werden afgeleid, met behulp van het heuristisch algoritme van Appendix B of met behulp van een ULCP. Vervolgens bepalen we B en C via (0.19) waarbij de matrix A nu geen onbekende meer is. Hierbij gebruiken we dan dezelfde procedure als in de directe methode: eerst bepalen we partiële realisaties en vervolgens kijken we dan wat er gebeurt als N naar ∞ streeft. In het proefschrift tonen we echter aan dat niet voor alle mogelijke A-matrices die uit de eerste stap kunnen volgen, er B- en C-matrices kunnen gevonden worden zodat (A, B, C) een realisatie is van de gegeven impulsresponsie. Dit houdt in dat de twee-stapsmethode niet altijd gebruikt kan worden. Daarom hebben we in Figuur 0.3 het pad dat met deze methode overeenkomt met streepjeslijnen aangegeven.

Hoofdstuk 7: De Singuliere-Waardenontbinding en de QR-Ontbinding in de Gesymmetriseerde Max-Plus-Algebra

In dit hoofdstuk bewijzen we het bestaan van een soort max-algebraïsche singuliere-waardenontbinding die als volgt gedefinieerd is:

> Beschouw een matrix $A \in \mathbb{S}^{m \times n}$. Dan bestaan er een max-algebraïsche diagonaal matrix $\Sigma \in \mathbb{R}_\varepsilon^{m \times n}$ en matrices $U \in (\mathbb{S}^\vee)^{m \times m}$ en $V \in (\mathbb{S}^\vee)^{n \times n}$ zodat
>
> $$A \,\nabla\, U \otimes \Sigma \otimes V^T \tag{0.20}$$
>
> $$U^T \otimes U \,\nabla\, E_m \tag{0.21}$$
>
> $$V^T \otimes V \,\nabla\, E_n \tag{0.22}$$
>
> $$\|A\|_\oplus \geqslant \sigma_1 \geqslant \sigma_2 \geqslant \ldots \geqslant \sigma_r \tag{0.23}$$
>
> waarbij $r = \min(m, n)$ en $\sigma_i = (\Sigma)_{ii}$ voor $i = 1, 2, \ldots, r$.
> Elke "ontbinding" van de vorm (0.20) waarbij Σ een max-algebraïsche diagonaal matrix is, waarbij U en V matrices zijn met elementen in $\mathbb{S}^\vee$ en waarbij aan de voorwaarden (0.21)–(0.23) voldaan is, wordt een max-algebraïsche singuliere-waardenontbinding (SWO) van A genoemd.

We tonen aan dat de max-algebraïsche SWO van een matrix kan berekend worden met behulp van een ULCP. We geven ook aan hoe de max-algebraïsche

SWO gebruikt zou kunnen worden in een procedure voor het identificeren van een max-lineair tijdsinvariant discrete-gebeurtenissysteem vertrekkende van de *gemeten* impulsresponsie van het systeem, d.w.z. in een situatie waarbij we niet over de *exacte* impulsresponsie beschikken, maar wel over de impulsresponsie met ruis erop gesuperponeerd. Naar analogie met SWO-gebaseerde methodes voor het bepalen van een (benadering van) de minimale systeemorde op basis van een gemeten ingangs-uitgangsgedrag zouden we de max-algebraïsche SWO kunnen gebruiken voor het schatten van de minimale systeemorde van een max-lineair tijdsinvariant discrete-gebeurtenissysteem op basis van de gemeten impulsresponsie.

Alhoewel wij in [42] reeds het bestaan van de max-algebraïsche SWO hebben bewezen, wordt in dit hoofdstuk een volledig nieuwe bewijstechniek ontwikkeld om het bestaan van de max-algebraïsche SWO aan te tonen. Met behulp van deze bewijstechniek kan op vrij eenvoudige wijze ook het bestaan van een aantal andere matrixontbindingen aangetoond worden die max-algebraïsche equivalenten zijn van matrixontbindingen uit de lineaire algebra. Om dit te illustreren bewijzen wij ook het bestaan van de max-algebraïsche QR-ontbinding, die als volgt gedefinieerd is:

> Beschouw een matrix $A \in \mathbb{S}^{m \times n}$. Dan bestaan er een matrix $Q \in (\mathbb{S}^{\vee})^{m \times m}$ en een max-algebraïsche bovendriehoeksmatrix $R \in (\mathbb{S}^{\vee})^{m \times n}$ zodat
>
> $$A \nabla Q \otimes R \tag{0.24}$$
>
> $$Q^T \otimes Q \nabla E_m \tag{0.25}$$
>
> $$\|R\|_{\oplus} \leqslant \|A\|_{\oplus} \quad . \tag{0.26}$$
>
> Elke "ontbinding" van de vorm (0.24) waarbij R een max-algebraïsche bovendriehoeksmatrix is, waarbij de elementen van Q en R tot $\mathbb{S}^{\vee}$ behoren en waarbij aan de voorwaarden (0.25) – (0.26) voldaan is, wordt een max-algebraïsche QR-ontbinding van A genoemd.

Op analoge wijze kan het bestaan van een max-algebraïsch equivalent van de eigenwaardenontbinding voor symmetrische matrices, de LU-ontbinding, de Schurontbinding, enz. bewezen worden. Al deze matrixontbindingen kunnen ook berekend worden met behulp van een ULCP.

Om het bestaan te bewijzen van de bovenvermelde max-algebraïsche matrixontbindingen steunen we op een verband tussen de gesymmetriseerde max-plus-algebra en een ring van reële functies met optelling en vermenigvuldiging als basisbewerkingen. Door dit verband uit te breiden naar complexe functies kunnen we de verzameling van de max-complexe getallen invoeren en de bewerkingen $\oplus$ en $\otimes$ ook uitbreiden tot max-complexe getallen. Dit resulteert in een uitgebreide max-algebraïsche structuur waarin dan nog meer matrixontbindingen zouden kunnen gedefinieerd worden die max-algebraïsche equivalenten zijn van complexe matrixontbindingen uit de lineaire algebra.

Hoofdstuk 8. Besluiten en Open Problemen

Eén van de voornaamste doelstellingen van ons onderzoek was het verder ontwikkelen van de max-algebraïsche systeemtheorie voor max-lineaire tijdsinvariante discrete-gebeurtenissystemen. Daartoe hebben we een aantal problemen bestudeerd die allemaal op de ene of de andere manier verband houden met het minimale-toestandsruimterealisatieprobleem voor max-lineaire tijdsinvariante discrete-gebeurtenissystemen.

We hebben aangetoond dat vele fundamentele max-algebraïsche problemen die opduiken bij allerlei mogelijke oplossingsmethodes voor het minimale-realisatieprobleem (en andere problemen uit de max-algebraïsche systeemtheorie voor discrete-gebeurtenissystemen) geherformuleerd kunnen worden als een wiskundige-programmatieprobleem: het Uitgebreide Lineaire Complementariteitsprobleem (ULCP). Daarom hebben we het ULCP uitvoerig bestudeerd en een algoritme ontwikkeld om alle oplossingen van een ULCP te bepalen. Dit heeft ons uiteindelijk in staat gesteld om een oplossingsmethode te ontwikkelen voor het minimale-realisatieprobleem, dat als één van de belangrijkste basisproblemen uit de systeemtheorie voor max-lineaire tijdsinvariante discrete-gebeurtenissystemen kan beschouwd worden. Op een gelijkaardige wijze kunnen we ook heel wat andere max-algebraïsche problemen oplossen zoals b.v. het berekenen van max-algebraïsche matrixfactorisaties, het construeren van matrices met een gegeven max-algebraïsche karakteristieke veelterm, het bepalen van toestandsruimtetransformaties voor max-lineaire tijdsinvariante discrete-gebeurtenissystemen, het berekenen van max-algebraïsche singuliere-waardenontbindingen of max-algebraïsche QR-ontbindingen van een gegeven matrix, enz.
Daarnaast hebben we ook nog enkele resultaten bekomen in verband met de complexiteit van het ULCP, de max-algebraïsche karakteristieke veelterm, toestandsruimtetransformaties voor max-lineaire tijdsinvariante discrete-gebeurtenissystemen en matrixontbindingen in de gesymmetriseerde max-plus-algebra.

Laat ons nu even de belangrijkste open problemen en enkele suggesties voor verder onderzoek vermelden.
Wij hebben nog geen algemene nodige en voldoende voorwaarden kunnen afleiden voor de coëfficiënten van de max-algebraïsche karakteristieke veelterm van een matrix waarvan de elementen tot $\mathbb{R}_\varepsilon$ behoren.
We weten niet of er toestandsruimtetransformaties bestaan die een willekeurige (minimale) toestandsruimterealisatie omzetten in een andere willekeurige (minimale) toestandsruimterealisatie.
We weten ook niet of het mogelijk is om, gegeven de impulsresponsie $\{G_k\}_{k=0}^{\infty}$ van een max-lineair tijdsinvariant discrete-gebeurtenissysteem, op een eenvoudige wijze het minimaal aantal termen N_0 te bepalen zodat elke realisatie van de rij $G_0, G_1, \ldots, G_{N_0-1}$ een realisatie is van de volledige impulsresponsie.
De techniek die gebruikt werd om het bestaan van de max-algebraïsche singuliere-waardenontbinding en de max-algebraïsche QR-ontbinding te bewijzen,

kan ook gebruikt worden om het bestaan van max-algebraïsche equivalenten van reële en complexe matrixontbindingen uit de lineaire algebra te bewijzen. We zouden nog moeten onderzoeken of en hoe deze max-algebraïsche matrixontbindingen kunnen gebruikt worden in de max-algebraïsche systeemtheorie.
In de praktijk kunnen we met het door ons ontwikkelde ULCP-algoritme slechts niet al te omvangrijke ULCP's oplossen aangezien de uitvoeringstijd van ons algoritme snel toeneemt als het aantal variabelen, vergelijkingen of ongelijkheden stijgt. Bovendien geeft ons algoritme *alle* oplossingen van een ULCP terwijl we voor vele problemen vaak reeds tevreden zijn met één oplossing. Daarom is één van de belangrijkste opdrachten voor verder onderzoek het ontwikkelen van efficiënte algoritmen voor het bepalen van één oplossing van een ULCP of van de max-algebraïsche problemen die wij in dit proefschrift behandeld hebben. Aangezien het ULCP en dus ook het bepalen van een oplossing van een stelsel multivariabele max-algebraïsche veeltermvergelijkingen en veeltermongelijkheden NP-harde problemen zijn, is het zeer onwaarschijnlijk dat we een algoritme met een polynomiale uitvoeringstijd zullen kunnen vinden voor het oplossen van deze problemen. We weten echter niet of andere max-algebraïsche problemen zoals het bepalen van een max-algebraïsche matrixfactorisatie, het berekenen van een toestandsruimterealisatie van de impulsresponsie van een max-lineair tijdsinvariant discrete-gebeurtenissysteem, het berekenen van een max-algebraïsche singuliere-waardenontbinding van een gegeven matrix, enz. ook NP-harde problemen zijn. Dit zou zeker onderzocht moeten worden. Met het oog op de verdere toepassing van onze resultaten is het van het hoogste belang dat er voor elk van deze problemen efficiënte algoritmen ontwikkeld worden. Hierbij zouden we gebruik kunnen maken van het feit dat wij nu de geometrische structuur van de oplossingsverzameling van deze problemen kennen.
Tenslotte zouden we ook kunnen nagaan in hoeverre de in dit proefschrift bekomen resultaten toepasbaar zijn op en uitbreidbaar zijn naar discrete-gebeurtenissystemen die niet met behulp van een max-lineair tijdsinvariant model kunnen beschreven worden.

Appendices

De appendices bevatten extra informatie, bijkomende voorbeelden en bewijzen die niet erg instructief zijn of die te lang waren om in de hoofdtekst te worden opgenomen.
In Appendix A bespreken we een alternatieve versie van de max-algebraïsche karakteristieke vergelijking van een matrix waarvan de elementen tot $\mathbb{R}_\varepsilon$ behoren.
Appendix B bevat de bewijzen van enkele stellingen uit Hoofdstuk 5. In deze appendix stellen we ook een hypothese voor in verband met de max-algebraïsche karakteristieke veelterm van een matrix waarvan de elementen tot $\mathbb{R}_\varepsilon$ behoren. Vertrekkende van deze hypothese ontwikkelen we vervolgens een heuristisch al-

goritme om een matrix met een gegeven max-algebraïsche karakteristieke veelterm te construeren.
In Appendix C bewijzen we enkele lemma's uit Hoofdstuk 6.
Appendix D bevat de bewijzen van enkele stellingen uit Hoofdstuk 7. Daarnaast stellen we in deze appendix ook enkele uitbreidingen van de max-algebraïsche SWO en de max-algebraïsche QR-ontbinding voor. We tonen aan dat deze uitgebreide max-algebraïsche matrixontbindingen eveneens met behulp van een ULCP kunnen berekend worden.
In Appendix E geven we een informele inleiding tot de gesymmetriseerde max-plus-algebra. In deze appendix geven we ook een aantal uitgewerkte voorbeeldjes waarin de eigenschappen van de belangrijkste bewerkingen en relaties uit de gesymmetriseerde max-plus-algebra geïllustreerd worden.

Chapter 1

Introduction and Motivation

In this chapter we first give a short introduction to discrete event systems. After giving some motivation as to why one would want to use mathematical models of discrete event systems, we concentrate on the max-plus algebra, the framework we use to model a certain class of discrete event systems. In order to give the reader an idea of the kind of discrete event systems that can be described by a max-algebraic model we present some worked examples in which we derive the equations that describe the behavior of the given discrete event system. To conclude this chapter we give an overview of the main contributions of this thesis. We discuss the overall structure of the thesis and the relations between the different problems that will be treated in it.

1.1 Discrete Event Systems

In recent years both industry and the academic world have become more and more interested in techniques to model, analyze and control complex systems such as flexible manufacturing systems, telecommunication networks, parallel processing systems, traffic control systems, logistic systems and so on. This kind of systems are typical examples of *discrete event systems* (DESs), the subject of an emerging discipline in system and control theory. The class of the DESs essentially contains man-made systems that consist of a finite number of resources (e.g. machines, communications channels, processors, ...) that are shared by several users (e.g. product types, information packets, jobs, ...) all of which contribute to the achievement of some common goal (e.g. the assembly of products, the end-to-end transmission of a set of information packets, a parallel computation, ...).

One of the most characteristic features of a DES is that its dynamics are *event-driven* as opposed to *time-driven*: the behavior of a DES is governed by events rather than by ticks of a clock. An *event* corresponds to the start or the end of an activity. If we consider a production system then possible events are: the completion of a part on a machine, a machine breakdown, a buffer becoming empty, and so on. Events occur at discrete time instants. Intervals between events are not necessarily identical; they can be deterministic or stochastic.

In general the dynamics of DESs are characterized by "synchronization" and "concurrency". Synchronization requires the availability of several resources at the same time (e.g. before we can assemble a product on a machine, the machine has to be idle and the various parts have to be available; before a specific job can be executed in a parallel processing system the processor and all the necessary input data have to be available, ...). Concurrency appears when at a certain time a user has to choose among several resources (e.g. in a production system a job may be executed on one of the several machines that can handle that job and that are idle at that time, in a data-driven parallel processing system a job may be executed on one of the several processors that are available at that time or that will soon become available, ...). DESs typically exhibit an asynchronous behavior with much parallelism and interaction with their environment, and they usually have a complex, hierarchical structure.

When we want to (re)design a system or develop a controller to ensure that a system meets certain requirements, or when we want to verify or optimize the behavior of a system, we should represent our knowledge about the properties and the behavior of the system in a model that allows us to study and to predict the performance of the system. There are many modeling and analysis techniques for DESs, such as queuing theory, (extended) state machines, max-plus algebra, formal languages, automata, temporal logic, generalized semi-Markov processes, Petri nets, perturbation analysis, computer simulation and so on. For more information and for tutorial articles on DESs and the techniques mentioned above the interested reader is referred to [3, 5, 13, 17, 20, 27, 80, 81, 94, 97, 112, 130, 131, 132, 155] and the references cited therein. All these modeling and analysis techniques have particular advantages and disadvantages and it really depends on the system we want to model and on the goals we want to achieve which one of the above methodologies best suits our needs. When we have to select the most appropriate method, then one of the most important trade-offs that we normally have to take into account is modeling power versus decision power: the more accurate the model is, the less we can analytically say about its properties. If we consider for example Petri nets, one of the most powerful mathematical modeling frameworks for DESs, then a complete analytic solution is usually not available. Furthermore, some of the problems encountered in Petri net analysis are "undecidable" [112, 131], i.e. for these problems there cannot exist generally applicable algorithms that solve the problem. On the other side, the max-algebraic approach allows us to determine and to analyze many properties of the system, but this approach can only be applied to the subclass of DESs that can be described by a max-linear time-invariant model. Loosely speaking we could say that this subclass corresponds to the class of deterministic time-invariant[1] DESs in which only synchronization and no concurrency occurs (In the next section we shall present some simple examples of DESs that can be modeled with the max-plus algebra in order to give the reader a more accurate idea of the types of DESs that can be described by a max-linear time-invariant model).

Up to now the most widely used technique to study DESs certainly is computer simulation. One of the major disadvantages of computer simulation is that it is computationally rather demanding since it requires a high degree of detail in the model. However, this also leads to a high degree of correspondence between the model and the real system. Another disadvantage of computer simulation is that it does not always give us a real understanding of the effects of parameter changes on properties such as robustness, stability, optimality of the system performance, and so on. Therefore, we often prefer to use mathematical models that are more suited for mathematical analysis. The main appeal of a mathematical description of a given system lies in the existence of efficient algorithms to evaluate the system performance. This is a significant advantage over time-consuming and expensive simulation, which is usually required to obtain the same information. Furthermore, analytic techniques also provide a better insight in the effects of parameter changes on the properties of the system. However, as we have already mentioned, we have to take into account that there is a trade-off between the accuracy of our model on the one hand and the techniques available for analysis on the other hand. Therefore, we normally use a mixture of analytic methods, approximations, heuristics and computer simulation when we want to study the properties and the behavior of a DES or when we want to design a controller for it.

There are three major levels of modeling for DESs: the logical level, the temporal level and the stochastic level. Logical models (such as (untimed) Petri nets and finite state machines) are used in order to study properties that concern event ordering only. If we are interested in the time instants at which the events occur then we use temporal models (such as timed Petri nets and max-algebraic models). Stochastic models (such as generalized semi-Markov processes) are used if we want to determine the expected behavior of the system under some given statistical conditions. The models we use in this thesis are situated on the temporal level.

Although in general DESs lead to a non-linear description in conventional algebra, there exists a subclass of DESs for which this model becomes "linear" when we formulate it in the max-plus algebra (See Section 1.2). Such a model will be called *max-linear*. In this thesis we concentrate on the class of DESs that can be described by max-linear time-invariant state space models.

The basic operations of the max-plus algebra are maximization and addition. There exists a remarkable analogy between the basic operations of the max-plus algebra on the one hand, and the basic operations of conventional algebra (addition and multiplication) on the other hand. As a consequence

[1] A system is said to be *time-invariant* if its response to a certain input sequence does not depend on absolute time.

many concepts and properties of conventional algebra (such as the Cayley-Hamilton theorem, eigenvectors and eigenvalues, Cramer's rule, ...) also have a max-algebraic analogue. Furthermore, this analogy also allows us to translate many concepts, properties and techniques from conventional linear system theory to system theory for max-linear time-invariant DESs. However, there are also some major differences that prevent a straightforward translation of properties, concepts and algorithms from conventional linear algebra and linear system theory to max-plus algebra and max-algebraic system theory for DESs.

In the early sixties the fact that certain classes of DESs can be described by max-linear models has been discovered independently by a number of researchers, among whom Cuninghame-Green [31, 32] and Giffler [60, 61, 62]. An account of the pioneering work of Cuninghame-Green on max-algebraic system theory for DESs has been given in [33]. Related work has been done by Gondran and Minoux [66, 67, 68, 69, 70]. In the late eighties the topic attracted new interest due to the research of Cohen, Dubois, Moller, Quadrat, Viot [19, 20, 22, 24, 26, 109, 110] and Olsder [115, 116, 117, 118, 122, 125, 126], which resulted in the publication of [3]. Another major contribution to this field is the work of Gaubert [53, 54, 55, 56, 57, 58].

It should be pointed out that compared to linear system theory the max-algebraic system theory for DESs is at present far from fully developed; much research on this topic is still needed in order to get a complete system theory. We hope that the research presented in this thesis will contribute to the extension and the enhancement of the max-algebraic system theory for DESs.

1.2 Discrete Event Systems and Max-Plus Algebra

Let us now show by some simple examples how certain classes of DESs can be modeled using the max-plus algebra. The main goal of this section is to give the reader an idea of what types of DESs can be described by a max-linear time-invariant model.

First we give a very short introduction to the basic concepts of the max-plus algebra that will be used in this section. A more elaborate introduction will be given in Section 2.2.

The elements of the max-plus algebra are the real numbers and $\varepsilon \stackrel{\text{def}}{=} -\infty$. The basic operations of the max-plus algebra are maximization (represented by $\oplus$) and addition (represented by $\otimes$). So we have

$$x \oplus y = \max(x, y)$$
$$x \otimes y = x + y$$

for $x, y \in \mathbb{R}_\varepsilon \stackrel{\text{def}}{=} \mathbb{R} \cup \{\varepsilon\}$. Note that $x \oplus \varepsilon = x = \varepsilon \oplus x$ for all $x \in \mathbb{R}_\varepsilon$.

The operations $\oplus$ and $\otimes$ are extended to matrices as follows:

$$(A \oplus B)_{ij} = a_{ij} \oplus b_{ij} = \max(a_{ij}, b_{ij})$$

$$(A \otimes B)_{ij} = \bigoplus_k a_{ik} \otimes b_{kj} = \max_k (a_{ik} + b_{kj})$$

for all i, j.
As has already been mentioned earlier on, one of the main reasons for choosing the symbols $\oplus$ and $\otimes$ to represent the basic operations of the max-plus algebra is that there exists a remarkable analogy between $\oplus$ and $+$, and between $\otimes$ and $\times$. In this section we shall see that by using these symbols we shall be able to write down a state space description of the form

$$x(k+1) = A \otimes x(k) \oplus B \otimes (k) \tag{1.1}$$

$$y(k) = C \otimes (k) \tag{1.2}$$

to describe the behavior of certain types of DESs.

Now we show how the behavior of some simple DESs such as various production systems, a railroad system and a queuing system can be described using the max-plus algebra.

Example 1.2.1 : A simple production system

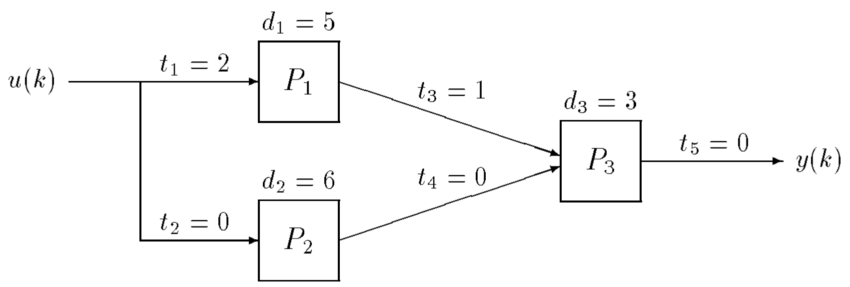


Figure 1.1: A simple production system.

Consider the system of Figure 1.1. This production system consists of 3 processing units: P_1, P_2 and P_3. Raw material is fed to P_1 and P_2, processed and sent to P_3 where assembly takes place. The processing times for P_1, P_2 and P_3 are respectively $d_1 = 5$, $d_2 = 6$ and $d_3 = 3$ time units. We assume that it takes $t_1 = 2$ time units for the raw material to get from the input source to P_1 and that it takes $t_3 = 1$ time unit for the finished products of processing unit P_1 to reach P_3. The other transportation times (t_2, t_4 and t_5) are assumed to be negligible. At the input of the system and between the processing units there are buffers with a capacity that is large enough to ensure that no buffer overflow will occur. Initially all buffers are empty and none of the processing units contains raw material or intermediate products.
A processing unit can only start working on a new product if it has finished processing the previous one. We assume that each processing unit starts working as soon as all parts are available. We define:

- $u(k)$: time instant at which raw material is fed to the system for the $(k+1)$st time,
- $x_i(k)$: time instant at which the ith processing unit starts working for the kth time,
- $y(k)$: time instant at which the kth finished product leaves the system.

Let us now determine the time instant at which processing unit P_1 starts working for the $(k+1)$st time. If we feed raw material to the system for the $(k+1)$st time, then this raw material is available at the input of processing unit P_1 at time $t = u(k)+2$. However, P_1 can only start working on the new batch of raw material as soon as it has finished processing the current, i.e. the kth batch. Since the processing time on P_1 is $d_1 = 5$ time units, the kth intermediate product will leave P_1 at time $t = x_1(k)+5$. Since P_1 starts working on a batch of raw material as soon as the raw material is available and the current batch has left the processing unit, this implies that we have

$$x_1(k+1) = \max(\, x_1(k) + 5,\ u(k) + 2\,) \tag{1.3}$$

for all $k \in \mathbb{N}_0$. The condition that initially processing unit P_1 is empty and idle corresponds to the initial condition $x_1(0) = \varepsilon$ since then it follows from (1.3) that $x_1(1) = u(0) + 2$, i.e. the first batch of raw material that is fed to the system will be processed immediately (after a delay of 2 time units needed to transport the raw material from the input to P_1).
Using a similar reasoning we find the following expressions for the time instants at which P_2 and P_3 start working for the $(k+1)$st time and for the time instant at which the kth finished product leaves the system:

$$\begin{aligned}
x_2(k+1) &= \max(\, x_2(k) + 6,\ u(k) + 0\,) && (1.4)\\
x_3(k+1) &= \max(\, x_1(k+1) + 5 + 1,\ x_2(k+1) + 6 + 0,\ x_3(k) + 3\,) && (1.5)\\
&= \max(\, x_1(k) + 11,\ x_2(k) + 12,\ x_3(k) + 3,\ u(k) + 8\,) && (1.6)\\
y(k) &= x_3(k) + 3 + 0 && (1.7)
\end{aligned}$$

for all $k \in \mathbb{N}_0$. The condition that initially all buffers are empty corresponds to the initial condition $x_1(0) = x_2(0) = x_3(0) = \varepsilon$.
Let us now rewrite the evolution equations of the production system using the symbols $\oplus$ and $\otimes$. It is easy to verify that (1.3) can be rewritten as

$$x_1(k+1) \;=\; 5 \otimes x_1(k) \;\oplus\; 2 \otimes u(k) \;\;.$$

Note that we do not need extra brackets in this expression to indicate the order of evaluation of the operations $\oplus$ and $\otimes$ since $\otimes$ has a higher priority than $\oplus$. Equations (1.4) – (1.7) result in

$$x_2(k+1) \;=\; 6 \otimes x_2(k) \;\oplus\; u(k)$$

$$\begin{aligned} x_3(k+1) &= 11 \otimes x_1(k) \oplus 12 \otimes x_2(k) \oplus 3 \otimes x_3(k) \oplus 8 \otimes u(k) \\ y(k) &= 3 \otimes x_3(k) \ . \end{aligned}$$

If we rewrite these evolution equations in max-algebraic matrix notation, we obtain

$$x(k+1) = \begin{bmatrix} 5 & \varepsilon & \varepsilon \\ \varepsilon & 6 & \varepsilon \\ 11 & 12 & 3 \end{bmatrix} \otimes x(k) \oplus \begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix} \otimes u(k)$$

$$y(k) = \begin{bmatrix} \varepsilon & \varepsilon & 3 \end{bmatrix} \otimes x(k)$$

where $x(k) = \begin{bmatrix} x_1(k) & x_2(k) & x_3(k) \end{bmatrix}^T$. Note that this is a model of the form (1.1) – (1.2).
In Section 2.4 we shall use this production system to illustrate some of the max-algebraic techniques that can be used to analyze max-linear time-invariant DESs. □

Example 1.2.2 Let us now discuss some other situations that occur in productions systems and that can be described by a model of the form (1.1) – (1.2). In the figures of this example we shall use circles to represent buffers with an "infinite" capacity, i.e. a capacity that is large enough to ensure that no buffer overflow will occur. Buffers with a finite capacity will be represented by rectangular boxes with rounded corners. The capacity of the buffer will be indicated above the box. Processing units will be represented by square boxes. The processing time will be indicated above the box.
A processing unit can only start working on a new product if it has finished processing the previous one. If a processing unit has finished processing a product and if the output buffer of the processing unit is full, then the finished product stays in the internal buffer of the processing unit. This implies that as long as its output buffer stays full, a processing unit cannot start processing a new product. Each processing unit starts processing a new product as soon as all parts are available and its internal buffer is empty. The transportation times are assumed to be negligible.
For Types 1 through 4 that will be discussed below, we assume that $u_i(k)$, $x_i(k)$ and $y_i(k)$ are defined as follows:

- $u_i(k)$: time instant at which raw material is fed to the ith input of the system for the $(k+1)$st time,
- $x_i(k)$: time instant at which the ith processing unit starts working for the kth time,
- $y_i(k)$: time instant at which a finished product leaves the ith output of the system for the kth time.

Now we discuss some elementary subsystems that can be found in production systems and we state the equations that describe the behavior of these subsystems. We assume that $x_i(k) = \varepsilon$ for all nonpositive k's, i.e. initially all the buffers are empty and all the processing units are idle.

Type 1: Serial production

Here we have two processing units P_1 and P_2 that are connected in series (See Figure 1.2). Between P_1 and P_2 there is a buffer with a finite capacity N_1.

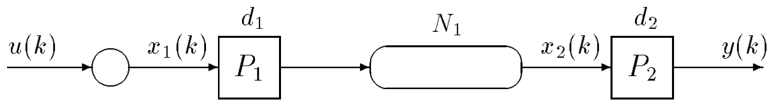


Figure 1.2: A production system with serial production.

Since the output buffer of processing unit P_1 has a capacity of N_1 parts, P_1 can only start processing the $(k+1)$st part if the $(k-N_1)$th part has left the output buffer of P_1, i.e. after processing unit P_2 has started processing the $(k-N_1)$th part. Therefore, we have

$$
\begin{aligned}
x_1(k+1) &= \max(\,u(k),\, x_1(k)+d_1,\, x_2(k-N_1)\,)\\
x_2(k+1) &= \max(\,x_2(k)+d_2,\, x_1(k+1)+d_1\,)\\
y(k) &= x_2(k)+d_2 \ .
\end{aligned}
$$

Type 2: Assembly

Now we consider a situation in which one processing unit (P_{n+1}) assembles the intermediate parts that come from the other processing units $(P_1, P_2, \ldots, P_n)$ (See Figure 1.3). Here we have

$$
x_i(k+1) = \max(\,x_i(k)+d_i,\, u_i(k),\, x_{n+1}(k-N_i)\,) \qquad \text{for } i = 1, 2, \ldots, n,
$$

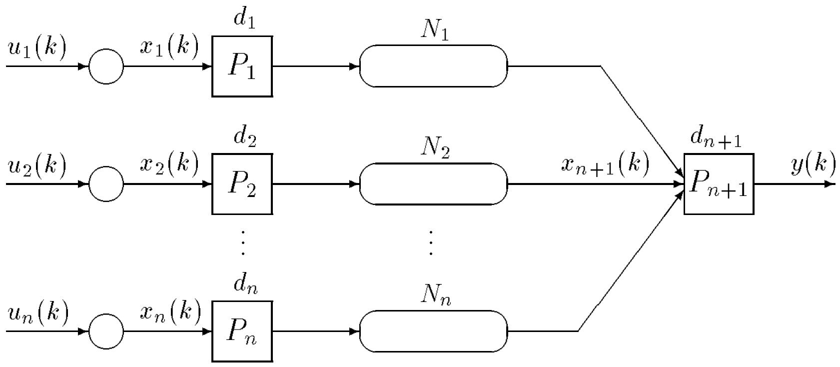


Figure 1.3: A production system in which assembly takes place.

$$
\begin{aligned}
x_{n+1}(k+1) &= \max(\, x_1(k+1)+d_1,\; x_2(k+1)+d_2, \ldots,\; x_n(k+1)+d_n, \\
&\qquad\qquad x_{n+1}(k)+d_{n+1}\,) \\
y(k) &= x_{n+1}(k)+d_{n+1} \;.
\end{aligned}
$$

Type 3: Splitting

Consider the production system of Figure 1.4. In this system the output of one processing unit (P_0) is distributed to the other processing units (P_1, P_2, ..., P_n). This situation can be described by

$$
\begin{aligned}
x_0(k+1) &= \max(\, x_0(k)+d_0,\; u(k),\; x_1(k-N_1),\; x_2(k-N_2), \ldots, \\
&\qquad\qquad x_n(k-N_n)\,) \\
x_i(k+1) &= \max(\, x_i(k)+d_i, x_0(k+1)+d_0) \qquad \text{for } i=1,2,\ldots,n, \\
y_i(k) &= x_i(k)+d_i \qquad\qquad\qquad\qquad\quad\;\; \text{for } i=1,2,\ldots,n\,.
\end{aligned}
$$

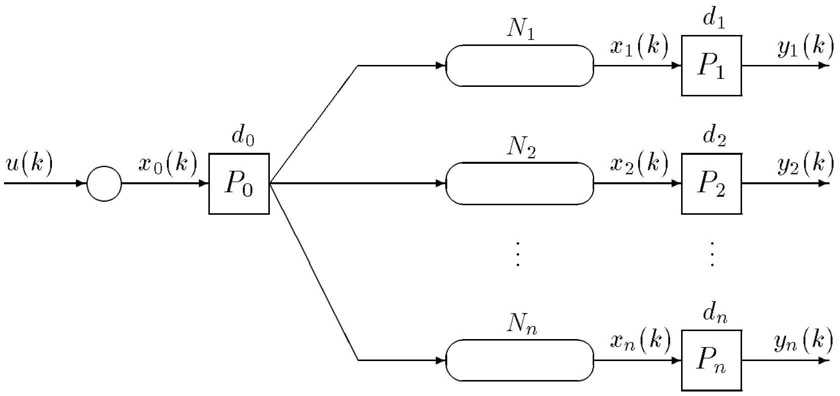


Figure 1.4: A production system with splitting.

Type 4: Parallel production

Assume that we have a system with 3 processing units (P_0, P_1 and P_2) and with the following routing rule:

- the odd numbered parts that leave processing unit P_0 go to processing unit P_1,
- the even numbered parts that leave processing unit P_0 go to processing unit P_2.

This system is represented in Figure 1.5. If we define

- $u^{\mathrm{o}}(k)$: time instant at which part $2k-1$ is fed to the system,
- $u^{\mathrm{e}}(k)$: time instant at which part $2k$ is fed to the system,
- $x_0^{\mathrm{o}}(k)$: time instant at which part $2k-1$ enters processing unit P_0,

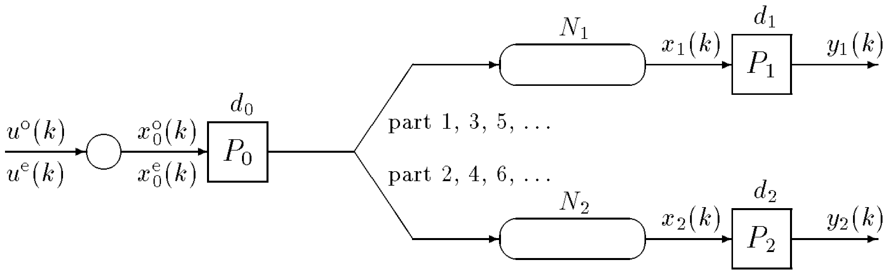


Figure 1.5: A production system with parallel production.

- $x_0^{\mathrm{e}}(k)$: time instant at which part $2k$ enters processing unit P_0,

then the behavior of the system can be described by

$$\begin{aligned}
x_0^{\mathrm{o}}(k+1) &= \max(\, x_0^{\mathrm{e}}(k) + d_0,\ u^{\mathrm{o}}(k+1),\ x_1(k-N_1)\,) \\
x_0^{\mathrm{e}}(k+1) &= \max(\, x_0^{\mathrm{o}}(k+1) + d_0,\ u^{\mathrm{e}}(k+1),\ x_2(k-N_2)\,) \\
x_1(k+1) &= \max(\, x_1(k) + d_1,\ x_0^{\mathrm{o}}(k+1) + d_0\,) \\
x_2(k+1) &= \max(\, x_2(k) + d_2,\ x_0^{\mathrm{e}}(k+1) + d_0\,) \\
y_1(k) &= x_1(k) + d_1 \\
y_2(k) &= x_2(k) + d_2\ .
\end{aligned}$$

Note that $u_0^{\mathrm{o}}(k) = u(2k-2)$ and $u_0^{\mathrm{e}}(k) = u(2k-1)$ for all $k \in \mathbb{N}_0$.

If there is a routing rule that imposes a fixed routing (such as e.g. the routing rule that has been used above), then the resulting model will be a max-linear and time-invariant model of the form (1.1)–(1.2). However, if there is no fixed routing (i.e. if e.g. a part that leaves a processing unit goes to the first idle processing unit of the next stage of the production process or to the first empty or non-full buffer) then we get in general a time-varying description that depends on the state of the system and that is not max-linear.

Type 5: Flexible production with a fixed sequence of activities

Here we shall not treat the general case but we shall consider a more specific example (See also Example 1.2.4).

Suppose that we have a system with 3 processing units (P_1, P_3 and P_4) in which two types of parts (T_1 and T_2) are produced (See Figure 1.6).

There are 4 different activities: A part of type T_1 is first processed on processing unit P_1 (activity 1) and then on processing unit T_3 (activity 3). A part of type T_2 is first processed on processing unit P_1 (activity 2) and then on processing unit P_4 (activity 4). The sequence of parts on processing unit P_1 is: P_1, P_2, P_1, P_2, ... The processing time for activity i is d_i.

If we define

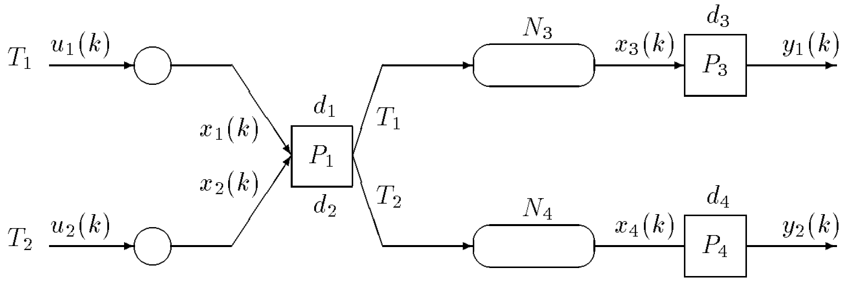


Figure 1.6: Flexible production with a fixed sequence of activities.

- $u_i(k)$: time instant at which raw material for a part of type T_i is fed to the system for the $(k+1)$st time,
- $x_i(k)$: time instant at which activity i starts for the kth time,
- $y_i(k)$: time instant at which a finished product of type T_i leaves the system,

then we have

$$\begin{aligned}
x_1(k+1) &= \max(\, x_2(k) + d_2,\, u_1(k),\, x_3(k - N_3)\,) \\
x_2(k+1) &= \max(\, x_1(k+1) + d_1,\, u_2(k),\, x_4(k - N_4)\,) \\
x_3(k+1) &= \max(\, x_3(k) + d_3,\, x_1(k+1) + d_1\,) \\
x_4(k+1) &= \max(\, x_4(k) + d_4,\, x_2(k+1) + d_2\,) \\
y_1(k) &= x_3(k) + d_3 \\
y_2(k) &= x_4(k) + d_4 \ .
\end{aligned}$$

In this case the condition that there has to be a fixed sequence of activities is also necessary to get a final model that is of the form (1.1) – (1.2).

If we have a system that consists of a combination of subsystems of Types 1 to 5 and for which there is a fixed sequence of activities, then the behavior of this system can in general be described by a model of the form

$$\begin{aligned}
x(k+1) &= A_0 \otimes x(k+1) \ \oplus\ A_1 \otimes x(k) \ \oplus\ \ldots \oplus \\
&\qquad\qquad A_q \otimes x(k-q) \ \oplus\ B \otimes u(k) \qquad (1.8)\\
y(k) &= C \otimes x(k) \ . \qquad (1.9)
\end{aligned}$$

Now we substitute the $x(k+1)$ on the right-hand side of (1.8) by the entire right-hand side and we keep on repeating this until the $x(k+1)$ disappears (which will always happen if the system contains no loops without delay). In the next example we shall present a numerical example of this procedure. If we also define a new augmented state vector $\tilde{x}(k) = \begin{bmatrix} x(k) \\ x(k-1) \\ \vdots \\ x(k-q) \end{bmatrix}$ then we get a description of the form (1.1)–(1.2). □

Example 1.2.3 : A railroad system

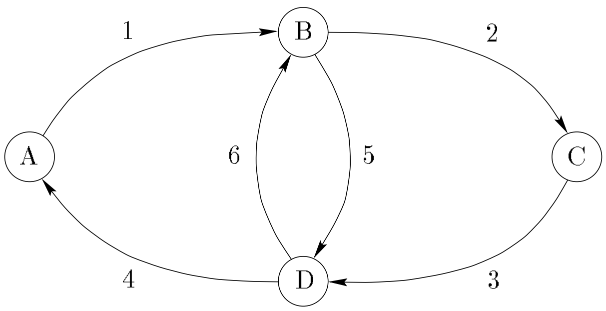


Figure 1.7: A railroad network.

Consider the railroad network of Figure 1.7. There are 4 stations in this railroad network (A, B, C and D) that are connected by 6 single tracks. There are two trains available. The first train follows the route $A \to B \to C \to D \to A$ and the second train follows the route $B \to D \to B$. We assume that there exists a periodic timetable that schedules the earliest departure times of the trains. The period of this timetable is $T = 60$ minutes. So if a departure of a train from station B is scheduled at 5.18 a.m., then there is also scheduled a departure of a train from station B at 6.18 a.m., 7.18 a.m., ... Table 1.1 summarizes the information in connection with the traveling and the departure times. All the times are measured in minutes. The indicated departure times are the earliest departure times in the initial station of the track expressed in minutes after the hour. The first period starts at time $t = 0$. At the beginning of the first period the first train is in station A and the second train is in station B.
Suppose that we have to guarantee the following connections in order to allow the passengers to change trains:

- the train on track 2 has to wait for the train on track 6,
- the train on track 4 has to wait for the train on track 5,
- the train on track 5 has to wait for the train on track 1,
- the train on track 6 has to wait for the train on track 3.

Track	From station	To station	Traveling time	Scheduled departure time modulo 60
1	A	B	15	00
2	B	C	10	18
3	C	D	10	30
4	D	A	12	45
5	B	D	18	20
6	D	B	18	50

Table 1.1: The traveling and the departure times for the railroad network of Example 1.2.3.

The passengers get 2 minutes to change trains. Each train departs as soon as all the connections are guaranteed, the passengers have gotten the opportunity to change over and the earliest departure time indicated in the timetable has passed. We assume that in the first period all the trains depart according to schedule. We define:

- $x_i(k)$: time instant at which the train departs from the initial station of track i for the kth time.

Now we write down the equations that describe the evolution of the $x_i(k)$'s. First we consider the train on track 1 and we determine $x_1(k)$, the time instant at which this train departs from station A for the kth time. At the beginning of the first period the train is in station A. So if k is equal to 1, the train departs from station A at time $t = 0$. If k is greater than 1, the train departs from station A (for the kth time) as soon as it has arrived in station A (for the $(k-1)$st time) and the earliest departure time indicated in the timetable has passed. The train arrives in station A for the $(k-1)$st time at the time instant given by $x_4(k-1) + 12$. Since the system operates under a periodic timetable with period T, the kth departure time of the train on track 1 according to the timetable is $0 + kT$. So if we set $x_4(0) = \varepsilon$, then we have

$$x_1(k) = \max(\, x_4(k-1) + 12,\, 0 + kT\,) \qquad \text{for all } k \in \mathbb{N}_0\,.$$

The train on track 1 will arrive for the kth time in station B at time $t = x_1(k) + 15$. If k is greater than 1, the train has to wait for the passengers of the train on track 6, which arrives in station B at time $t = x_6(k-1) + 18$. The passengers have 2 minutes to change trains. According to the timetable the train on track 2 can only depart after time $t = 18 + kT$. Hence,

$$x_2(k) = \max(\, x_1(k) + 15,\, x_6(k-1) + 18 + 2,\, 18 + kT\,) \qquad \text{for all } k \in \mathbb{N}_0$$

with $x_6(0) = \varepsilon$. Using a similar reasoning we find

$$\begin{aligned}
x_3(k) &= \max(\, x_2(k) + 10,\; 30 + kT\,) \\
x_4(k) &= \max(\, x_3(k) + 10,\; x_5(k) + 18 + 2,\; 45 + kT\,) \\
x_5(k) &= \max(\, x_1(k) + 15 + 2,\; x_6(k-1) + 18,\; 20 + kT\,) \\
x_6(k) &= \max(\, x_3(k) + 10 + 2,\; x_5(k) + 18,\; 50 + kT\,)
\end{aligned}$$

for all $k \in \mathbb{N}_0$. If we define

$$A_0 = \begin{bmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 15 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 10 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 10 & \varepsilon & 20 & \varepsilon \\ 17 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 12 & \varepsilon & 18 & \varepsilon \end{bmatrix} ,\; A_1 = \begin{bmatrix} \varepsilon & \varepsilon & \varepsilon & 12 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 20 \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 18 \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{bmatrix} ,$$

$$x(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k) \\ x_5(k) \\ x_6(k) \end{bmatrix} ,\; u(k) = \begin{bmatrix} 0 + kT \\ 18 + kT \\ 30 + kT \\ 45 + kT \\ 20 + kT \\ 50 + kT \end{bmatrix} ,\; x(0) = \begin{bmatrix} \varepsilon \\ \varepsilon \\ \varepsilon \\ \varepsilon \\ \varepsilon \\ \varepsilon \end{bmatrix} ,$$

then the evolution of the system is described by

$$x(k) = A_0 \otimes x(k) \;\oplus\; A_1 \otimes x(k-1) \;\oplus\; u(k) \tag{1.10}$$

for all $k \in \mathbb{N}_0$. Expression (1.10) is an implicit equation in $x(k)$. If we substitute the $x(k)$ on the right-hand side of this equation by the entire right-hand side, we get

$$\begin{aligned}
x(k) &= A_0 \otimes \big(\, A_0 \otimes x(k) \,\oplus\, A_1 \otimes x(k-1) \,\oplus\, u(k)\big) \,\oplus\, A_1 \otimes x(k-1) \,\oplus\, u(k) \\
&= {A_0}^{\otimes^2} \otimes x(k) \;\oplus\; \big(\, A_0 \,\oplus\, E\,\big) \otimes \big(\, A_1 \otimes x(k-1) \,\oplus\, u(k)\big)
\end{aligned}$$

where E is the max-algebraic identity matrix ($e_{ii} = 0$ for all i and $e_{ij} = \varepsilon$ for all i, j with $i \neq j$) and where $A^{\otimes^k}$ represents the kth max-algebraic matrix power of A.

After n substitutions we obtain

$$\begin{aligned}
x(k) = {A_0}^{\otimes^{n+1}} \otimes x(k) \;\oplus\; \Big(\, & {A_0}^{\otimes^n} \,\oplus\, {A_0}^{\otimes^{n-1}} \,\oplus\, \dots \\
& \oplus\, A_0 \,\oplus\, E \Big) \otimes \Big(A_1 \otimes x(k-1) \,\oplus\, u(k) \Big) \;.
\end{aligned} \tag{1.11}$$

Define $A_0^* = E \oplus A_0 \oplus {A_0}^{\otimes^2} \oplus \dots$. Since all the entries of the matrices

${A_0}^{\otimes^4}$, ${A_0}^{\otimes^5}$, ${A_0}^{\otimes^6}$, ... are equal to ε, we have

$$A_0^* = E \oplus A_0 \oplus {A_0}^{\otimes^2} \oplus {A_0}^{\otimes^3} = \begin{bmatrix} 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 15 & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 25 & 10 & 0 & \varepsilon & \varepsilon & \varepsilon \\ 37 & 20 & 10 & 0 & 20 & \varepsilon \\ 17 & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon \\ 37 & 22 & 12 & \varepsilon & 18 & 0 \end{bmatrix} .$$

So for any $n \geqslant 3$ equation (1.11) is equivalent to

$$x(k) = A_0^* \otimes A_1 \otimes x(k-1) \oplus A_0^* \otimes u(k) .$$

Hence,

$$x(k) = A \otimes x(k-1) \oplus B \otimes u(k)$$

with

$$A = A_0^* \otimes A_1 = \begin{bmatrix} \varepsilon & \varepsilon & \varepsilon & 12 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 27 & \varepsilon & 20 \\ \varepsilon & \varepsilon & \varepsilon & 37 & \varepsilon & 30 \\ \varepsilon & \varepsilon & \varepsilon & 49 & \varepsilon & 40 \\ \varepsilon & \varepsilon & \varepsilon & 29 & \varepsilon & 18 \\ \varepsilon & \varepsilon & \varepsilon & 49 & \varepsilon & 42 \end{bmatrix} \quad \text{and} \quad B = A_0^* .$$

Now we can determine various properties of the system such as stability, delays, settling time, sensitivity with respect to change-over time, shortest paths, critical paths, and so on (See [7, 8, 9]). This enables us to answer questions such as: How do perturbations propagate through the system and how long does it take before they have completely disappeared? What are the crucial parts of the system that determine the "speed" of the system? If extra trains become available, on which lines should they preferably be employed? What is the maximal amount with which the change-over time can be increased without endangering the stability of the system? □

The following example is taken from [3] (See also [19]). We have only changed the phrasing a little bit to make it consistent with the previous examples.

Example 1.2.4: A multi-product manufacturing system
We consider a manufacturing system that consists of three machines (M_1, M_2 and M_3). In this manufacturing system three different types of parts (T_1, T_2 and T_3) are produced according to a certain product mix. The routes followed by the various types of parts are depicted in Figure 1.8. Parts of type T_1 first visit machine M_2 and then they go to M_3. Parts of type T_2 enter the system via machine M_1, then they go to machine M_2 and they leave the system through machine M_3. Parts of type T_3 first visit machine M_1 and then they go to M_2. We make the following assumptions:

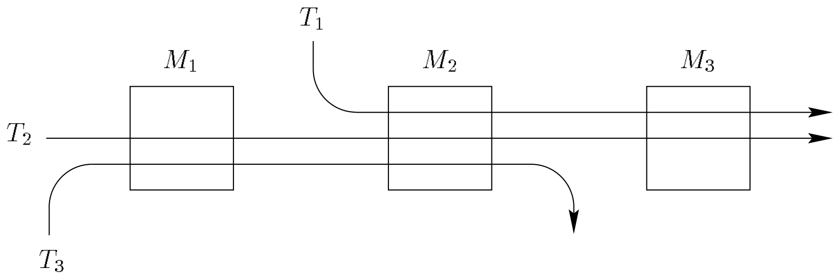


Figure 1.8: The routing of the various types of parts along the machines.

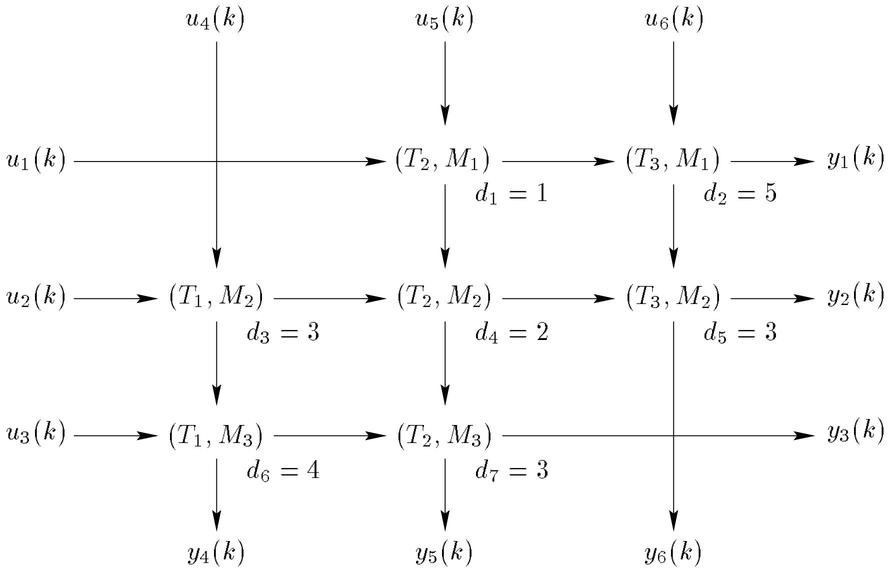


Figure 1.9: The sequence and the duration of the various activities.

- The parts are carried around on pallets. There is one pallet available for each part type.
- We assume that the transportation times are negligible and that there are no set-up times on the machines when they switch from one part type to another.
- The sequencing of the various part types on the machines is known: On machine M_1 we first process a part of type T_2 and then a part of type T_3. On machine M_2 the sequence is: T_1, T_2, T_3. On machine M_3 the sequence is: T_1, T_2.

The information about the sequence and the duration of the various activities is represented in Figure 1.9. In this figure we have represented the activities by

ordered pairs of the form (T_i, M_j) meaning that a part of type T_i is processed on machine M_j. The arcs represent the precedence constraints between activities. At the bottom right of the activities we have indicated the duration of the activity. Note that in the figure the activities are numbered from the left to the right and from the top to the bottom. So activity 1 is (T_2, M_1) and activity 7 is (T_2, M_3).
In order to simplify the process of deriving the evolution equations of this system, we shall first look at what happens in one cycle of the production process. We define:

- $u_i(k)$: time instant at which machine M_i is available for the first activity that should be performed on it in the kth production cycle for $i = 1, 2, 3$,
- $u_j(k)$: time instant at which the raw material for a part of type T_{j-3} is available in the kth production cycle for $j = 4, 5, 6$,
- $x_i(k)$: time instant at which activity i starts in the kth production cycle for $i = 1, 2, \ldots, 7$,
- $y_i(k)$: time instant at which machine M_i has finished processing the last part of the kth production cycle that should be processed on it for $i = 1, 2, 3$,
- $y_j(k)$: time instant at which the finished product of type T_{j-3} of the kth production cycle has been completed for $j = 4, 5, 6$.

Using a reasoning that is similar that the one that has been used in the previous examples we find that the relation between $u(k)$, $x(k)$ and $y(k)$ is given by

$$x(k) = A_0 \otimes x(k) \;\oplus\; B_0 \otimes u(k) \tag{1.12}$$

$$y(k) = C_0 \otimes x(k) \tag{1.13}$$

with

$$A_0 = \begin{bmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 1 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 1 & \varepsilon & 3 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 5 & \varepsilon & 2 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 3 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 2 & \varepsilon & 4 & \varepsilon \end{bmatrix}, \quad B_0 = \begin{bmatrix} 0 & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 \\ \varepsilon & 0 & \varepsilon & 0 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{bmatrix} \quad \text{and}$$

$$C_0 = \begin{bmatrix} \varepsilon & 5 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & 3 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 3 \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 4 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 3 \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & 3 & \varepsilon & \varepsilon \end{bmatrix} .$$

Since (1.12) is an implicit equation in $x(k)$, we apply the same technique as in Example 1.2.3 to eliminate the $x(k)$ on the right-hand side of this equation. This yields:

$$x(k) = A_0^* \otimes B_0 \otimes u(k) \ .$$

After a machine has finished a sequence of products it starts with the next sequence. Since we assume that there are no set-up times, this implies that the output time instants $y_1(k)$, $y_2(k)$, $y_3(k)$ of the kth cycle of the production process are the input time instants $u_1(k+1)$, $u_2(k+1)$, $u_3(k+1)$ of the $(k+1)$st cycle. If a pallet on which a part of type T_i was mounted leaves the last machine through which it should pass in the current production cycle, the finished product is removed and the empty pallet immediately goes back to the starting point to pick up a new part of type T_i. Since we assume that the transportation times are negligible, this implies that the output time instants $y_4(k)$, $y_5(k)$, $y_6(k)$ of the kth cycle of the production process are the input time instants $u_4(k+1)$, $u_5(k+1)$, $u_6(k+1)$ of the $(k+1)$st cycle. So we have $u(k+1) = y(k)$ for all $k \in \mathbb{N}_0$. Hence,

$$\begin{aligned} y(k+1) &= C_0 \otimes x(k+1) \\ &= C_0 \otimes A_0^* \otimes B_0 \otimes u(k+1) \\ &= A \otimes y(k) \end{aligned}$$

with

$$A = C_0 \otimes A_0^* \otimes B_0 = \begin{bmatrix} 6 & \varepsilon & \varepsilon & \varepsilon & 6 & 5 \\ 9 & 8 & \varepsilon & 8 & 9 & 8 \\ 6 & 10 & 7 & 10 & 6 & \varepsilon \\ \varepsilon & 7 & 4 & 7 & \varepsilon & \varepsilon \\ 6 & 10 & 7 & 10 & 6 & \varepsilon \\ 9 & 8 & \varepsilon & 8 & 9 & 8 \end{bmatrix} .$$

If we define a new state vector $\tilde{x}(k) = y(k)$, then we see that this cyclic production system can also be described by a model of the form (1.1) – (1.2) but now the model is autonomous i.e. there is no external input that controls the behavior of the system. So in this case the B matrix is equal to the max-algebraic zero matrix: $b_{ij} = \varepsilon$ for all i, j. □

Finally, we present an example of a DES that cannot be described by a *time-invariant* model of the form (1.1) – (1.2). This example has also been taken from [3]. We have only changed the explanation somewhat to make it consistent with the previous examples.

Example 1.2.5 : A queuing system

Consider the queuing system of Figure 1.10. This system consists of 4 servers (S_1, S_2, S_3 and S_4) that are connected in series. The system has an input

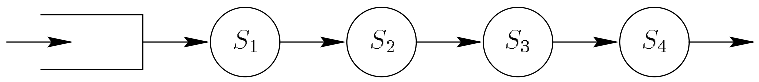


Figure 1.10: A queuing system with 4 servers and 1 input buffer.

buffer with a capacity that is large enough to ensure that the buffer will never be full. Each customer is to be served by S_1, S_2, S_3 and S_4, in this order.
If a customer arrives into the input buffer, if the input buffer is empty and if server S_1 is idle, then this customer is served immediately by S_1. Between the servers there are no buffers. This implies that if server S_i ($i = 1, 2$ or 3) has finished serving the kth customer but S_{i+1} is still busy serving the $(k-1)$st customer, then S_i cannot start serving the $(k+1)$st customer. This customer has to wait until S_{i+1} starts serving the kth customer. We assume that the traveling times between the servers are negligible. Define:

- $u(k)$: time instant at which the $(k+1)$st customer arrives into the input buffer of the queuing system,
- $x_i(k)$: time instant at which server S_i starts serving the kth customer,
- $\tau_i(k)$: number of time units it takes for server S_i to serve the kth customer.

Note that the service times are not constant but depend on the customer.
Server S_i can only start serving the $(k+1)$st customer if the following three conditions are fulfilled:

- server S_i has finished serving the kth customer,
- if $i \neq 4$ then S_{i+1} is idle,
- if $i \neq 1$ then S_{i-1} has finished serving the $(k+1)$st customer; and if $i = 1$ then the $(k+1)$st customer has arrived into the input buffer of the system.

Hence,

$$x(k+1) = A_0(k+1) \otimes x(k+1) \ \oplus \ A_1(k) \otimes x(k) \ \oplus \ B_0 \otimes u(k) \qquad (1.14)$$

with

$$A_0(k+1) = \begin{bmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \tau_1(k+1) & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \tau_2(k+1) & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \tau_3(k+1) & \varepsilon \end{bmatrix},$$

$$A_1(k) = \begin{bmatrix} \tau_1(k) & 0 & \varepsilon & \varepsilon \\ \varepsilon & \tau_2(k) & 0 & \varepsilon \\ \varepsilon & \varepsilon & \tau_3(k) & 0 \\ \varepsilon & \varepsilon & \varepsilon & \tau_4(k) \end{bmatrix} \text{ and } B_0 = \begin{bmatrix} 0 \\ \varepsilon \\ \varepsilon \\ \varepsilon \end{bmatrix}.$$

If we assume that initially all the servers are idle and the input buffer is empty then we have $x_1(0) = x_2(0) = x_3(0) = x_4(0) = \varepsilon$.
Since (1.14) is an implicit equation in $x(k+1)$ we apply the repeated-substitution technique that has also been used in the previous examples to eliminate the $x(k+1)$ on the right-hand side of this equation. This leads to

$$x(k+1) = A(k) \otimes x(k) \ \oplus \ B(k) \otimes u(k) \tag{1.15}$$

with

$$A(k) = (A_0(k+1))^* \otimes A_1(k) \ \text{ and } B(k) = (A_0(k+1))^* \otimes B$$

where

$$(A_0(k+1))^* =$$
$$\begin{bmatrix} 0 & \varepsilon & \varepsilon & \varepsilon \\ \tau_1(k+1) & 0 & \varepsilon & \varepsilon \\ \tau_1(k+1) \otimes \tau_2(k+1) & \tau_2(k+1) & 0 & \varepsilon \\ \tau_1(k+1) \otimes \tau_2(k+1) \otimes \tau_3(k+1) & \tau_2(k+1) \otimes \tau_3(k+1) & \tau_3(k+1) & 0 \end{bmatrix} .$$

Note that (1.15) resembles (1.1) but that in contrast to (1.1) the system matrices in (1.15) depend on k, i.e. they are time-varying. □

The results of the examples given above can be generalized: if we limit ourselves to time-invariant deterministic DESs in which the sequence of the events and the duration of the activities are fixed or can be determined in advance (such as repetitive production processes), then the behavior of the system can be described by equations of the form

$$x(k+1) = A \otimes x(k) \ \oplus \ B \otimes u(k) \tag{1.16}$$
$$y(k) = C \otimes x(k) \tag{1.17}$$

for all $k \in \mathbb{N}_0$ with an initial condition $x(0) = x_0$. We call (1.16) – (1.17) an nth order state space model. The vector x represents the state, u is the input vector and y is the output vector of the system. The description (1.16) – (1.17) closely resembles the conventional state space description

$$x(k+1) = Ax(k) \ + \ Bu(k) \tag{1.18}$$
$$y(k) = Cx(k) \tag{1.19}$$

for discrete linear time-invariant systems. Furthermore, in Section 2.4 we shall show that the input-output behavior of DESs that can be described by a model of the form (1.16) – (1.17) is *max-linear* in the sense that if the input sequences u_1 and u_2 yield the output sequences y_1 and y_2 respectively, then the input sequence $\alpha \otimes u_1 \oplus \beta \otimes u_2$ yields the output sequence $\alpha \otimes y_1 \oplus \beta \otimes y_2$. Therefore, we call DESs that can be described by a state space model of the form (1.16) – (1.17) *max-linear time-invariant* DESs. The analogy between (1.16) – (1.17) and (1.18) – (1.19) is also one of the reasons why the symbols $\oplus$ and $\otimes$ are used to represent the basic operations of the max-plus algebra.

The class of DESs that can be described by a max-linear time-invariant model consists of time-invariant deterministic DESs in which the sequence of the events and the duration of the activities are fixed or can be determined in advance. In this kind of DESs we essentially only have synchronization and no concurrency. Synchronization requires the availability of several resources at the same time and this typically leads to the appearance of the maximum operation in the description of the system as has been shown in the examples given above. When at a certain time a user has to choose among several resources we have concurrency. This occurs e.g. when we have a routing that is not fixed but that depends on the state of the system. This phenomenon cannot be described by a max-linear time-invariant model. This implies that the class of the DESs that can be described by a max-linear time-invariant model is only a small subclass of the class of all the DESs. On the other hand, for this kind of DESs there are many efficient analytic methods available to determine and to analyze the properties of the system since we can use the properties of the max-plus algebra to analyze max-linear time-invariant models in a very efficient way (as opposed to e.g. computer simulation where, before we can determine the steady-state behavior of a given DES, we first have to simulate the transient behavior, which in some cases might require a rather large amount of computation time). In Section 2.4 we shall present some analytic methods for DESs that can be described by a max-linear time-invariant model.

If we allow variable or stochastic processing and transportation times and variable routing, we can still describe the system by a max-algebraic model [2, 3, 119, 120, 123, 124] but it will not be time-invariant and max-linear any more, which means that it will become more difficult to analyze the properties of the system analytically.

For more information on max-algebraic modeling of production systems, timetable dependent transportation networks, queuing systems, array processors and other types of DESs the interested reader is referred to [3, 7, 8, 18, 19, 20, 21, 23, 25, 32, 95, 117].

1.3 General Survey of the Thesis and Chapter by Chapter Overview

1.3.1 General Survey

The overall goal of this thesis is to develop some tools that can be used in system theory for max-linear time-invariant DESs. One of the starting points of our research is the analogy between the state space description for linear time-invariant systems and the state space description for max-linear time-invariant

DESs (cf. Section 1.2). We have already mentioned that there exists a remarkable analogy between the basic operations of the max-plus algebra and the conventional operations addition and multiplication, and that they have many properties in common. Nevertheless, there are also some differences that prevent a straightforward translation of properties, concepts and algorithms from linear system theory to system theory for max-linear time-invariant DESs. This is one of the reasons why at present the max-algebraic system theory for DESs is not fully developed yet. Therefore, we have intensively studied one of the major open problems in max-algebraic system theory: the minimal state space realization problem for max-linear time-invariant DESs. In connection with this problem we have also studied the max-algebraic characteristic polynomial, max-algebraic state space transformations and max-algebraic matrix factorizations. Furthermore, as a first step towards the development of methods for the identification of DESs we have also studied max-algebraic analogues of matrix decompositions that are an important tool in many contemporary algorithms for the identification of linear systems (See e.g. [96, 98, 138, 139, 140, 142] and the references cited therein): the QR decomposition and the singular value decomposition. We found that many of the problems that arose in our research could be considered as special cases of a more general problem, viz. solving a system of multivariate max-algebraic polynomial equalities and inequalities, which could in its turn be reformulated as a mathematical programming problem: the Extended Linear Complementarity Problem. This problem consists in solving a system of linear equalities and inequalities in which there are some groups of inequalities where in each group at least one equality should hold with equality.

Let us now give an overview of the structure of this thesis and of the connections between the problems that will be treated in it.

This thesis consists of three major parts (See Figure 1.11). The first part of the thesis is oriented towards mathematical programming. In this part we introduce the Extended Linear Complementarity Problem (ELCP), which is a generalization of one of the fundamental problems in mathematical programming: the Linear Complementarity Problem. We shall show that many max-algebraic problems can be reformulated as an ELCP. Therefore, the first part of this thesis is the stepping stone for the next parts.

The second part of the thesis deals with the max-plus algebra. The main topic of this part is the minimal state space realization problem for max-linear time-invariant DESs. In this part we also treat the connection between the ELCP and max-algebraic problems. Furthermore, we present some results on the max-algebraic characteristic polynomial and on state space transformations for max-linear time-invariant DESs.

In Part 3 we discuss some matrix decompositions in the symmetrized max-plus algebra. These decompositions are max-algebraic versions of basic matrix decompositions from linear algebra such as the singular value decomposition and the QR decomposition.

Part 1 is linked to Parts 2 and 3 by the fact that many problems in the max-plus

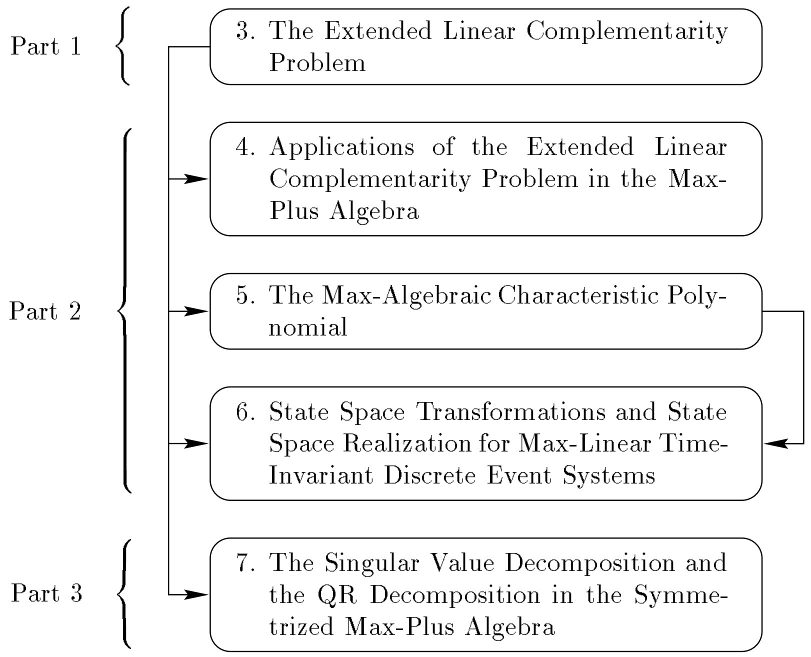


Figure 1.11: The relations between Chapters 3 to 7.

algebra and in the symmetrized max-plus algebra can be reformulated as an ELCP. We shall use the ELCP as a mathematical tool to describe and to solve many of the max-algebraic problems that will be treated in this thesis. We shall show that the problem of solving a system of multivariate max-algebraic polynomial equalities and inequalities can be reformulated as an ELCP and vice versa. This will enable us to solve many other max-algebraic problems such as:

- calculating max-algebraic matrix factorizations,
- performing state space transformations for max-linear time-invariant DESs,
- determining minimal or partial state space realizations of the impulse response of a max-linear time-invariant DES,
- constructing matrices with a given max-algebraic characteristic polynomial,
- calculating max-algebraic singular value decompositions and max-algebraic QR decompositions,
- ...

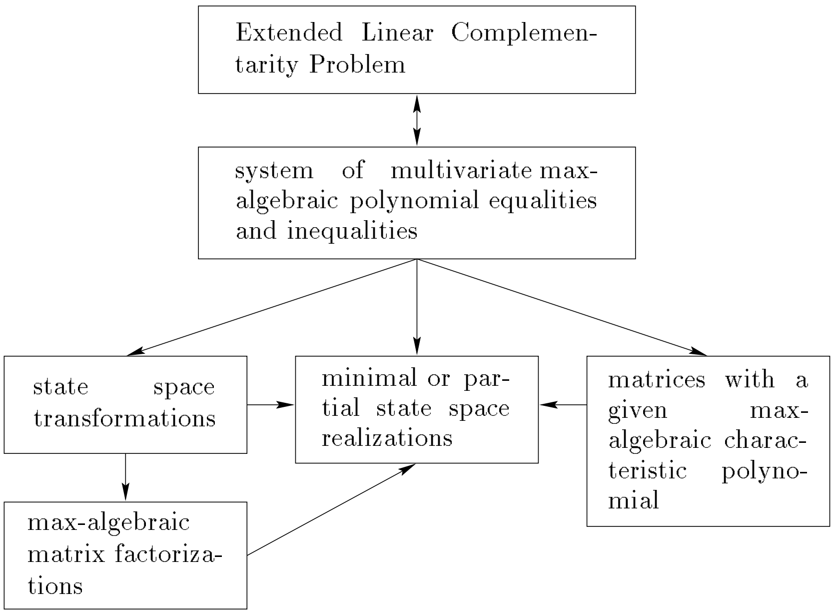


Figure 1.12: The connections between the ELCP and some of the max-algebraic problems that will be treated in this thesis.

Although the analogues of these problems in conventional algebra and linear system theory are easy to solve, the max-algebraic problems are not that easy to solve and for almost all of them the ELCP approach is at present to only way to solve the problem.

Let us now take a closer look at the connections between some of the problems mentioned above (See Figure 1.12). We have already mentioned that the ELCP and solving a system of multivariate max-algebraic polynomial equalities and inequalities are equivalent. Factorizing a given matrix as the max-algebraic product of two or more matrices can be considered as a special case of solving a system of multivariate max-algebraic polynomial equalities and inequalities, and therefore we can use the ELCP to compute max-algebraic matrix factorizations. We can transform a state space model that is characterized by the system matrices A, B, C and the initial condition x_0 into an equivalent state space model by determining a factorization of the matrix $\begin{bmatrix} A \\ C \end{bmatrix}$ or the matrix $\begin{bmatrix} A & B & x_0 \end{bmatrix}$. We shall also show that the ELCP can be used to compute matrices with a given max-algebraic characteristic polynomial.
We shall discuss several methods to determine minimal and partial state space realizations of the impulse response of a max-linear time-invariant DES. The first method can be considered as a kind of extended max-algebraic matrix factorization and can thus be solved using the ELCP approach. The second

method is a two-step method in which we first determine the A matrix starting from its max-algebraic characteristic polynomial; next the matrices B and C are determined using an extended max-algebraic matrix factorization that is a simplified variant of the matrix factorization that is used in the first method. Sometimes we can also use max-algebraic state space transformations to compute minimal state space realizations of the impulse response of a max-linear time-invariant DES (provided that we already have a state space description of the given system at our disposal).

1.3.2 Chapter by Chapter Overview

Now we briefly summarize the contents and main results of each chapter of this thesis.

Chapter 2: Background Material

In this introductory chapter we first present some of the notations we use in this thesis. Next we give an introduction to the max-plus algebra and the symmetrized max-plus algebra. We also discuss some of the relations between the max-plus algebra and graph theory. Finally we present some analysis techniques for DESs that can be described by a max-linear time-invariant state space model and we apply these techniques to an example.

Chapter 3: The Extended Linear Complementarity Problem

In this chapter we present a mathematical programming problem that we have called the Extended Linear Complementarity Problem (ELCP). This problem consists in finding a solution of a system of linear equalities and inequalities in which there are some groups of inequalities where in each group at least one equality should hold with equality. As has already been said earlier on, the ELCP is an important mathematical tool that will be used frequently in the next chapters.

First we show that the ELCP can be considered as a unifying framework for the Linear Complementarity Problem and some of its generalizations. We also study the general solution set of the ELCP and we give a geometrical characterization of this solution set. Next we derive an algorithm to find all the solutions of an ELCP. Finally, we discuss the performance of this algorithm and we show that the ELCP is an NP-hard problem.

Chapter 4: Applications of the Extended Linear Complementarity Problem in the Max-Plus Algebra

In this chapter we prove that the problem of finding all finite solutions of a system of multivariate max-algebraic polynomial equalities and inequalities is equivalent to an ELCP. We also show that many other max-algebraic problems such as calculating max-algebraic matrix factorizations, solving systems of max-linear "equations" in the symmetrized max-plus algebra, mixed max-min problems, max-max and max-min problems can be reformulated as an ELCP.

Chapter 5: The Max-Algebraic Characteristic Polynomial

The subject of this chapter is the max-algebraic characteristic polynomial of a matrix with entries in $\mathbb{R}_\varepsilon$ ($\mathbb{R}_\varepsilon = \mathbb{R} \cup \{-\infty\}$). First we derive necessary conditions for the coefficients of the max-algebraic characteristic polynomial of a matrix with entries in $\mathbb{R}_\varepsilon$. For max-algebraic polynomials with a degree that is less than or equal to 4 we derive necessary and sufficient conditions for the coefficients such that the given max-algebraic polynomial corresponds to the max-algebraic characteristic polynomial of a matrix with entries in $\mathbb{R}_\varepsilon$ and we also indicate how this matrix can be constructed. Finally, we show that the problem of constructing a matrix with entries in $\mathbb{R}_\varepsilon$ that has a given max-algebraic polynomial as its max-algebraic characteristic polynomial can be reformulated as an ELCP.

Chapter 6: State Space Transformations and State Space Realization for Max-Linear Time-Invariant Discrete Event Systems

In this chapter we use results of all the preceding chapters to develop a method to solve one of the basic problems in max-algebraic system theory for DESs: the minimal state space realization problem.

In the first part of this chapter we discuss max-algebraic state space transformations. These transformations allow us to transform a given max-algebraic state space model of a max-linear time-invariant DES into an equivalent state space model, i.e. a state space model that describes the same input-output behavior as the original model. We introduce two types of state space transformations that can be considered as an extension of max-algebraic similarity transformations and we show that these transformations can be computed using an ELCP.

In the second part of this chapter we treat the minimal state space realization problem for max-linear time-invariant DESs: we present a procedure to construct the system matrices of a state space model of a max-linear time-invariant DES starting from its impulse response and such that the dimension of the system matrices is as small as possible. Let us briefly discuss the various procedures to solve this problem that will be presented in this thesis (See Figure 1.13). We start from the impulse response of the DES. First we use the results of Chapter 5 to determine the minimal system order. The system matrices A, B and C will be determined by solving one or more ELCPs.

In the first method to determine the system matrices, A, B and C will be

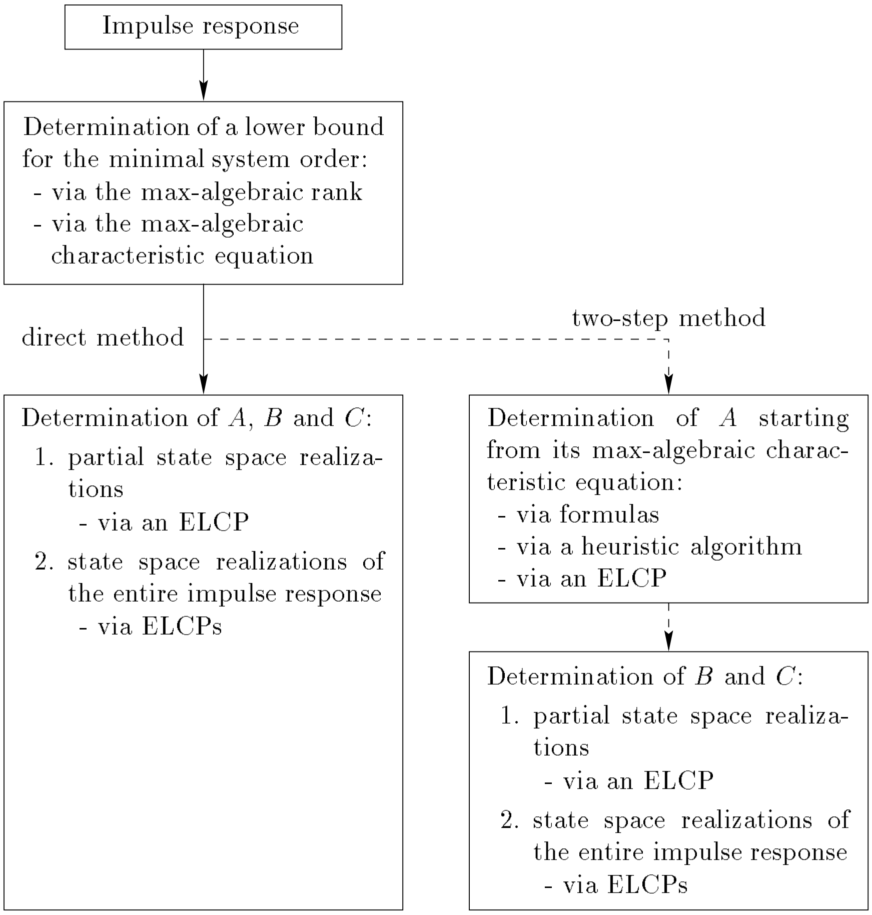


Figure 1.13: An overview of the methods to solve the minimal state space realization problem for max-linear time-invariant DESs that will be presented in this thesis.

computed simultaneously using a kind of extended max-algebraic matrix factorization. This method will be called the direct method. This method consists of two major steps: first we determine partial state space realizations of the given impulse response, i.e. we look for realizations of the first, say N, terms of the impulse response. In the second major step we determine how the set of the partial state space realizations evolves as N goes to ∞. This will result in state space realizations of the entire impulse response. We shall illustrate this procedure with some worked examples.
We also present another method to determine the system matrices. This method is a two-step method in which we first determine the A matrix starting

from its max-algebraic characteristic polynomial (using the formulas of Chapter 5, the heuristic algorithm of Appendix B or an ELCP). Next the matrices B and C are determined using an extended max-algebraic matrix factorization that is a simplified variant of the matrix factorization that is used in the first method. However, we shall show that this two-step method cannot always be used — that is why we have used dashed lines in Figure 1.13 for the arcs that correspond to the two-step method — since it is possible that for some matrices A that result from the first step it is impossible to determine matrices B and C such that the triple of system matrices (A, B, C) yields a state space model of the given impulse response.

Chapter 7: The Singular Value Decomposition and the QR Decomposition in the Symmetrized Max-Plus Algebra

In many linear algebra algorithms and in many contemporary algorithms for the identification of linear systems the QR decomposition (QRD) and the singular value decomposition (SVD) play an important role. In this chapter we shall develop a method to define and to prove the existence of max-algebraic analogues of these basic matrix factorizations. In [42] we have already proved the existence of the SVD in the symmetrized max-plus algebra. However, the alternative proof technique that will be presented in this chapter has the advantage that it can easily be adapted to prove the existence of max-algebraic analogues of many other matrix decompositions from linear algebra such as e.g. the eigenvalue decomposition for symmetric matrices, the LU decomposition, the Schur decomposition, the Hessenberg decomposition and so on. We shall also indicate how the max-algebraic SVD might be used in a procedure to solve the identification problem for max-linear time-invariant DESs.

In this chapter we first present a link between a ring of real functions and the symmetrized max-plus algebra. This leads to a further extension of the max-plus algebra that will correspond to a ring of complex functions. Then we use the link between the ring of real functions and the symmetrized max-plus algebra to define the SVD and the QRD in the symmetrized max-plus algebra and to prove the existence of these matrix decompositions. Next we derive some properties of the max-algebraic SVD. Finally we show that the problem of finding all the max-algebraic SVDs or all the max-algebraic QRDs of a given matrix can also be reformulated as an ELCP.

Appendices

The appendices contain additional information and examples. Proofs that are not instructive or that were too long to be included in the main body of the text have also been put in the appendices.

In Appendix A we discuss an alternative version of the max-algebraic characteristic equation of a matrix.

Appendix B contains the proofs of some of the propositions of Chapter 5. In

this appendix we also present a heuristic algorithm to construct matrices with a given max-algebraic characteristic polynomial.
In Appendix C we prove some of the lemmas of Chapter 6.
In Appendix D we prove some propositions of Chapter 7. We also propose some extensions of the max-algebraic SVD and the max-algebraic QRD. We show that these extended max-algebraic matrix decompositions can also be computed using the ELCP approach.
Appendix E contains an informal introduction to the symmetrized max-plus algebra. In this appendix we also give some extra worked examples of the basic operations and relations of the symmetrized max-plus algebra.

Chapter 2

Background Material

In this section we present some background material on the max-plus algebra, the symmetrized max-plus algebra and system theory for max-linear time-invariant discrete event systems. In Section 2.1 we introduce some notations and definitions that will be used in this thesis. Next we give a concise introduction to the max-plus algebra. In Section 2.3 we address the symmetrized max-plus algebra, which is an extension of the max-plus algebra. In Section 2.4 we give a short introduction to system theory for max-linear time-invariant discrete event systems.

Chapter 3 does not require any of the background material that will be presented in this section except maybe for the notation we use to denote submatrices of a given matrix (see Section 2.1). Especially in Chapter 5 the notations and definitions in connection with permutations that will be presented in Section 2.1 might be useful.

For Chapters 4 through 7 a basic knowledge of the max-plus algebra and an elementary knowledge of the symmetrized max-plus algebra is necessary. Readers that are familiar with the max-plus algebra, the symmetrized max-plus algebra and the basic concepts of max-algebraic system theory may want to skip Sections 2.2, 2.3 and 2.4 or go through them very quickly. Note that in Appendix E we shall give an informal introduction to the symmetrized max-plus algebra whereas in Section 2.3 a more formal derivation of the symmetrized max-plus algebra will be given. Therefore, readers that are not interested in a formal derivation of the symmetrized max-plus algebra might skip the first part of Section 2.3, read Appendix E instead and afterwards return to Section 2.3 to have a look at definitions of concepts like max-algebraic determinant, max-linear independence, etc. that will be presented at the end of that section.

2.1 Notations and definitions

We assume that the reader is familiar with symbols like $\emptyset$, $\subseteq$, $\subset$, $\#$, $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{R}$, $\mathbb{R}^n$, $\mathbb{R}^{m\times n}$, ... (The definitions of these and other symbols can be found on pp. v–vi).

Now we present some notations and definitions that are specific to this thesis.

We use $f : D \rightarrow T$ to indicate that f is a function with domain of definition D and target T. Sometimes we also use the symbol $f(\cdot)$ to represent a function. The domain of definition of the function f is denoted by $\operatorname{dom} f$ and the value of f in $x \in \operatorname{dom} f$ is denoted by $f(x)$. Since a function f can also be considered as a set of ordered pairs $\{\, (x,y) \mid x \in \operatorname{dom} f \text{ and } y = f(x) \,\}$, the union of two functions can be defined as follows: if f and g are functions and if $\operatorname{dom} f \cap \operatorname{dom} g = \emptyset$, then $f \cup g$ is a function with domain of definition $\operatorname{dom} f \cup \operatorname{dom} g$ and with

$$(f \cup g)(x) = \begin{cases} f(x) & \text{if } x \in \operatorname{dom} f\,, \\ g(x) & \text{if } x \in \operatorname{dom} g\,. \end{cases}$$

If f and g are two real functions and if ∞ is an accumulation point of both $\operatorname{dom} f$ and $\operatorname{dom} g$, then we say that $g = O(f)$ if $\displaystyle\limsup_{x\rightarrow\infty} \frac{|g(x)|}{f(x)}$ is finite. If $x \in \mathbb{R}$ then $\lfloor x \rfloor$ is the largest integer that is less than or equal to x.

We use "vector" as a synonym for "matrix with one column". So in this thesis we have $\mathbb{R}^n \equiv \mathbb{R}^{n\times 1}$.
Let a be a vector with n components. The ith component of a is denoted by a_i. If $\alpha \subseteq \{1, 2, \ldots, n\}$ then a_α is the subvector of a obtained by removing all the components of a that are not indexed by α.
Let A be an m by n matrix. Then a_{ij} or $(A)_{ij}$ is the entry on the ith row and the jth column of A. We use $A_{i,.}$ to denote the ith row of A and $A_{.,j}$ to denote the jth column of A. Let $\alpha \subseteq \{1, 2, \ldots, m\}$ and $\beta \subseteq \{1, 2, \ldots, n\}$. The submatrix of A obtained by removing all rows of A that are not indexed by α and all columns that are not indexed by β is denoted by $A_{\alpha\beta}$. The submatrix of A obtained by removing all rows of A except for those indexed by α is denoted by $A_{\alpha,.}$.
If the off-diagonal entries of a matrix $D \in \mathbb{R}^{m\times n}$ are equal to 0 then we say that D is a diagonal matrix. A matrix $R \in \mathbb{R}^{m\times n}$ is an upper triangular matrix if $r_{ij} = 0$ for all i, j with $i > j$. Note that D and R are not necessarily square matrices. A permutation matrix is square matrix with exactly one entry that is equal to 1 in each row and in each column and where the other entries are equal to 0.
Let a, $b \in \mathbb{R}^n$. The order relation for vectors is defined as follows: we have $a \leqslant b$ if and only if $a_i \leqslant b_i$ for $i = 1, 2, \ldots, n$. Furthermore, $a \geqslant 0$ means that $a_i \geqslant 0$ for $i = 1, 2, \ldots, n$. Likewise, $a = 0$ means that $a_i = 0$ for $i = 1, 2, \ldots, n$

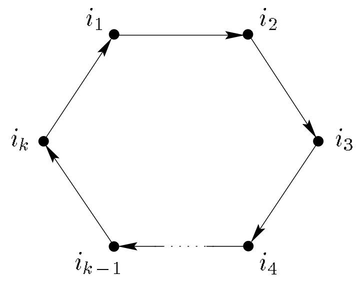


Figure 2.1: The graph of a cyclic permutation.

and $a \neq 0$ means that there exists an index $i \in \{1, 2, \ldots, n\}$ such that $a_i \neq 0$.

Let $k, n \in \mathbb{N}$. We use $\mathcal{C}_n^k$ to represent the set of all subsets with k elements of the set $\{1, 2, \ldots, n\}$. The number of elements of $\mathcal{C}_n^k$ is given by $\begin{pmatrix} n \\ k \end{pmatrix} = \dfrac{n!}{(n-k)!\,k!}$.
The set of all the permutations of $\{1, 2, \ldots, n\}$ is denoted by $\mathcal{P}_n$. An element σ of $\mathcal{P}_n$ will be considered as a mapping from $\{1, 2, \ldots, n\}$ to $\{1, 2, \ldots, n\}$.
Consider a subset $\{i_1, i_2, \ldots, i_k\}$ of $\{1, 2, \ldots, n\}$. If τ is a permutation of the set $\{i_1, i_2, \ldots, i_k\}$ such that

$$\tau(i_1) = i_2,\ \tau(i_2) = i_3,\ \ldots,\ \tau(i_{k-1}) = i_k,\ \tau(i_k) = i_1 \ ,$$

then we say that τ is a *cyclic permutation* or a *cycle* of length k. The graph of this permutation is represented in Figure 2.1.
If a permutation of $\{1, 2, \ldots, n\}$ is not a cycle, we can decompose it uniquely into r elementary cycles $\tau_1, \tau_2, \ldots, \tau_r$ for some $r > 1$ with $\operatorname{dom} \tau_i \cap \operatorname{dom} \tau_j = \emptyset$ for all i, j with $i \neq j$. If l_i is the length of τ_i for $i = 1, 2, \ldots, r$ then we have $\displaystyle\sum_{i=1}^{r} l_i = n$.

Example 2.1.1 Let $\sigma \in \mathcal{P}_4$ be defined by $\sigma(1) = 4$, $\sigma(2) = 2$, $\sigma(3) = 3$ and $\sigma(4) = 1$. The graph of this permutation is represented in Figure 2.2.

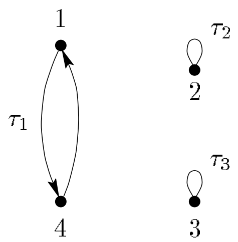


Figure 2.2: The graph of the permutation σ of Example 2.1.1.

This permutation can be decomposed into three elementary cycles: τ_1, τ_2 and τ_3 with $\operatorname{dom}\tau_1 = \{1, 4\}$, $\operatorname{dom}\tau_2 = \{2\}$, $\operatorname{dom}\tau_3 = \{3\}$ and $\tau_1(1) = 4$, $\tau_1(4) = 1$, $\tau_2(2) = 2$, $\tau_3(3) = 3$. □

The parity of a permutation can be determined in various ways. If we combine e.g. the results given on pp. 100–101 of [11], we get the following property, which for us will serve as a definition of the parity of a permutation:

Lemma 2.1.2 (Parity) *The parity of a permutation is equal to the parity of the number of its elementary cycles of even length.*

Let $n \in \mathbb{N}$. If n is even then a cyclic permutation $\sigma_c \in \mathcal{P}_n$ is odd since there is one elementary cycle of even length. If n is odd then a cyclic permutation $\sigma_c \in \mathcal{P}_n$ is even since there are no elementary cycles of even length.
The permutation σ of Example 2.1.1 is odd since it has one elementary cycle of even length, viz. τ_1.
The set of the even permutations of $\{1, 2, \ldots, n\}$ is denoted by $\mathcal{P}_{n,\mathrm{even}}$ and the set of the odd permutations of $\{1, 2, \ldots, n\}$ is denoted by $\mathcal{P}_{n,\mathrm{odd}}$.

2.2 The Max-Plus Algebra

In this section we give an introduction to the max-plus algebra. Most of the material presented in this section is selected from [3, 33], where a complete overview of the max-plus algebra can be found.

2.2.1 Terminology

First we have to make some remarks on the terminology used in this thesis:

- We use the term "algebra" or "algebraic structure" to indicate a set of elements with a number of operations that can be performed on these elements.
- Whenever we speak about conventional algebra, we refer to the algebra of the real (or the complex) numbers with addition and multiplication as basic operations.
- The term "max-algebraic" and the prefix "max-" will be used to indicate properties and concepts that pertain to the max-plus algebra $\mathbb{R}_{\max}$, which will be introduced in this section, and its extensions $\mathbb{S}_{\max}$ and $\mathbb{T}_{\max}$, which will be introduced in Sections 2.3 and 7.2 respectively.

2.2.2 Basic Operations

The basic max-algebraic operations are defined as follows:

$$x \oplus y = \max(x, y) \tag{2.1}$$

$$x \otimes y = x + y \tag{2.2}$$

for $x, y \in \mathbb{R} \cup \{-\infty\}$. The reason for using these symbols is that there is a remarkable analogy between $\oplus$ and addition and between $\otimes$ and multiplication: many concepts and properties from conventional linear algebra (such as the Cayley-Hamilton theorem, eigenvectors and eigenvalues, Cramer's rule, ...) can be translated to the (symmetrized) max-plus algebra by replacing $+$ by $\oplus$ and $\times$ by $\otimes$ (See also Section 7.2). Therefore, we also call $\oplus$ the max-algebraic addition, or max-addition for short. Likewise, we call $\otimes$ the max-algebraic multiplication or max-multiplication. The resulting algebraic structure $\mathbb{R}_{\max} = (\mathbb{R} \cup \{-\infty\}, \oplus, \otimes)$ is called the *max-plus algebra*. Define $\mathbb{R}_\varepsilon = \mathbb{R} \cup \{-\infty\}$. The zero element for $\oplus$ in $\mathbb{R}_\varepsilon$ is represented by $\varepsilon \stackrel{\text{def}}{=} -\infty$. So $x \oplus \varepsilon = x = \varepsilon \oplus x$ for all $x \in \mathbb{R}_\varepsilon$.
Let $r \in \mathbb{R}$. The rth max-algebraic power of $x \in \mathbb{R}$ is denoted by $x^{\otimes^r}$ and corresponds to rx in conventional algebra. If $x \in \mathbb{R}$ then $x^{\otimes^0} = 0$ and the inverse element of x w.r.t. $\otimes$ is $x^{\otimes^{-1}} = -x$. There is no inverse element for ε since ε is absorbing for $\otimes$. If $r > 0$ then $\varepsilon^{\otimes^r} = \varepsilon$. If $r < 0$ then $\varepsilon^{\otimes^r}$ is not defined. In this thesis we have $\varepsilon^{\otimes^0} = 0$ by definition.
The max-algebraic division operation is defined as follows:

$$\text{if } x, y \in \mathbb{R}_\varepsilon \text{ and } y \neq \varepsilon \text{ then } \frac{x}{y} = x \otimes y^{\otimes^{-1}} .$$

If y is equal to ε then $\frac{x}{y}$ is not defined .
The rules for the order of evaluation of the max-algebraic operators are similar to those of conventional algebra. So max-algebraic power has the highest priority, and max-algebraic multiplication and division have a higher priority than max-algebraic addition.

The matrix E_n is the n by n max-algebraic identity matrix:

$$\begin{aligned} (E_n)_{ii} &= 0 && \text{for } i = 1, 2, \ldots, n\,, \\ (E_n)_{ij} &= \varepsilon && \text{for } i = 1, 2, \ldots, n \text{ and } j = 1, 2, \ldots, n \text{ with } i \neq j\,. \end{aligned}$$

The m by n max-algebraic zero matrix is represented by $\mathcal{E}_{m \times n}$: we have $(\mathcal{E}_{m \times n})_{ij} = \varepsilon$ for all i, j. If the size of the max-algebraic identity matrix or the max-algebraic zero matrix is not specified, it should be clear from the context.
The off-diagonal entries of a max-algebraic diagonal matrix $D \in \mathbb{R}_\varepsilon^{m \times n}$ are equal to ε: $d_{ij} = \varepsilon$ for all i, j with $i \neq j$. A matrix $R \in \mathbb{R}_\varepsilon^{m \times n}$ is a max-algebraic upper triangular matrix if $r_{ij} = \varepsilon$ for all i, j with $i > j$. If we permute the rows or the columns of the max-algebraic identity matrix, we obtain a max-algebraic permutation matrix.
The max-algebraic operations are extended to matrices in the usual way. So if $\alpha \in \mathbb{R}_\varepsilon$ and if $A, B \in \mathbb{R}_\varepsilon^{m \times n}$ then

$$(\alpha \otimes A)_{ij} = \alpha \otimes a_{ij} \qquad \text{for } i = 1, 2, \ldots, m \text{ and } j = 1, 2, \ldots, n$$

and

$$(A \oplus B)_{ij} = a_{ij} \oplus b_{ij} \qquad \text{for } i = 1,2,\ldots,m \text{ and } j = 1,2,\ldots,n\,.$$

If $A \in \mathbb{R}_\varepsilon^{m\times p}$ and $B \in \mathbb{R}_\varepsilon^{p\times n}$ then

$$(A \otimes B)_{ij} = \bigoplus_{k=1}^{p} a_{ik} \otimes b_{kj} \qquad \text{for } i = 1,2,\ldots,m \text{ and } j = 1,2,\ldots,n\,.$$

Let $k \in \mathbb{N}$. The kth max-algebraic power of a matrix $A \in \mathbb{R}_\varepsilon^{n\times n}$ is defined recursively as follows:

$$\begin{aligned} A^{\otimes^k} &= A^{\otimes^{k-1}} \otimes A \qquad \text{if } k > 0\,, \\ A^{\otimes^0} &= E_n\ . \end{aligned}$$

One of the major differences between conventional algebra and max-plus algebra is that in general there do not exist inverse elements w.r.t. $\oplus$ in $\mathbb{R}_{\max}$. This also means that in general matrices are not invertible either.

Proposition 2.2.1 *A matrix $T \in \mathbb{R}_\varepsilon^{n\times n}$ is invertible in the max-plus algebra (or max-invertible for short) if and only if it can be factorized as $T = D \otimes P$ where $D \in \mathbb{R}_\varepsilon^{n\times n}$ is a max-algebraic diagonal matrix with non-ε diagonal entries and $P \in \mathbb{R}_\varepsilon^{n\times n}$ is a max-algebraic permutation matrix.*

Proof: See [33]. □

It is easy to verify that if a matrix $T \in \mathbb{R}_\varepsilon^{n\times n}$ can be factorized as $T = D \otimes P$ with D a max-algebraic diagonal matrix with non-ε diagonal entries and P a max-algebraic permutation matrix then it can also be factorized as $T = \tilde{P} \otimes \tilde{D}$ with $\tilde{D}$ a max-algebraic diagonal matrix with non-ε diagonal entries and $\tilde{P}$ a max-algebraic permutation matrix.

If D is a square max-algebraic diagonal matrix with non-ε diagonal entries then its max-algebraic inverse $D^{\otimes^{-1}}$ is a max-algebraic diagonal matrix with $(D^{\otimes^{-1}})_{ii} = -d_{ii}$ for all i. If P is a permutation matrix then $P^{\otimes^{-1}} = P^T$. If $T = D \otimes P$ then $T^{\otimes^{-1}} = P^{\otimes^{-1}} \otimes D^{\otimes^{-1}}$.

Example 2.2.2 Consider $T = \begin{bmatrix} \varepsilon & \varepsilon & 3 \\ -2 & \varepsilon & \varepsilon \\ \varepsilon & 0 & \varepsilon \end{bmatrix}$. This matrix is max-invertible since it can be written as $T = D \otimes P$ with $D = \begin{bmatrix} 3 & \varepsilon & \varepsilon \\ \varepsilon & -2 & \varepsilon \\ \varepsilon & \varepsilon & 0 \end{bmatrix}$ and $P = \begin{bmatrix} \varepsilon & \varepsilon & 0 \\ 0 & \varepsilon & \varepsilon \\ \varepsilon & 0 & \varepsilon \end{bmatrix}$. We have $D^{\otimes^{-1}} = \begin{bmatrix} -3 & \varepsilon & \varepsilon \\ \varepsilon & 2 & \varepsilon \\ \varepsilon & \varepsilon & 0 \end{bmatrix}$ and $P^{\otimes^{-1}} = \begin{bmatrix} \varepsilon & 0 & \varepsilon \\ \varepsilon & \varepsilon & 0 \\ 0 & \varepsilon & \varepsilon \end{bmatrix}$.

Hence, $T^{\otimes^{-1}} = P^{\otimes^{-1}} \otimes D^{\otimes^{-1}} = \begin{bmatrix} \varepsilon & 2 & \varepsilon \\ \varepsilon & \varepsilon & 0 \\ -3 & \varepsilon & \varepsilon \end{bmatrix}$. □

Let $A \in \mathbb{R}_\varepsilon^{n \times n}$ and $b \in \mathbb{R}_\varepsilon^n$. We say that $x \in \mathbb{R}_\varepsilon^n$ is a *subsolution* of the system of max-linear equations $A \otimes x = b$ if $A \otimes x \leqslant b$. Although the system $A \otimes x = b$ does not always have a solution, it always possible to determine the *greatest subsolution* if we allow components that are equal to ∞ in the solution and if we assume that $\varepsilon \otimes \infty = \infty \otimes \varepsilon = \varepsilon$ by definition (See also Section 4.2.4). For sake of simplicity and to avoid expressions like $\varepsilon - \varepsilon$, we assume from now on that all the components of b are finite. The greatest subsolution $\hat{x}$ of $Ax = b$ is given by

$$\hat{x}_j = \min_i \left(b_i - a_{ij} \right) \qquad \text{for } j = 1, 2, \ldots, n$$

where $\min \emptyset = \infty$ by definition.
A solution $\tilde{x}$ of the problem

$$\text{minimize } \max_i \left| b_i - (A \otimes x)_i \right|$$

is then given by

$$\tilde{x} = \hat{x} \otimes \frac{\delta}{2} \qquad \text{with } \delta = \max_i \left(b_i - (A \otimes \hat{x})_i \right) \ . \tag{2.3}$$

We have $\max_i \left| b_i - (A \otimes \tilde{x})_i \right| = \dfrac{\delta}{2}$.

The max-plus algebra is a typical example of a class of algebraic structures called *dioids*, which are defined as follows:

Definition 2.2.3 (Dioid) *A dioid is a set $\mathcal{D}$ endowed with two operations $\oplus$ (called "addition") and $\otimes$ (called "multiplication") such that*

$$\forall x, y, z \in \mathcal{D} : \ (x \oplus y) \oplus z = x \oplus (y \oplus z)$$

$$\forall x, y \in \mathcal{D} : \ x \oplus y = y \oplus x$$

$$\forall x \in \mathcal{D} : \ x \oplus x = x$$

$$\exists \epsilon \in \mathcal{D} : \forall x \in \mathcal{D} : \ x \oplus \epsilon = x = \epsilon \oplus x$$

$$\forall x, y, z \in \mathcal{D} : \ (x \otimes y) \otimes z = x \otimes (y \otimes z)$$

$$\exists e \in \mathcal{D} : \ x \otimes e = x = x \otimes e$$

$$\forall x \in \mathcal{D} : \ x \otimes \epsilon = \epsilon \otimes x = \epsilon$$

$$\forall x, y, z \in \mathcal{D} : \ x \otimes (y \oplus z) = (x \otimes y) \oplus (x \otimes z)$$

$$(x \oplus y) \otimes z = (x \otimes z) \oplus (y \otimes z) \ .$$

So in a dioid the additive operation $\oplus$ is associative, commutative and idempotent and it has a zero element; the multiplicative operation $\otimes$ is associative and its has an identity element; the additive zero element is absorbing for $\otimes$; and $\otimes$ is left and right distributive w.r.t. $\oplus$. Examples of dioids are: $(\mathbb{R}\cup\{\infty\}, \min, +)$, $([0,1], \max, \cdot)$, $(\mathbb{R}\cup\{-\infty,\infty\}, \max, \min)$ and $(\{\text{true}, \text{false}\}, \vee, \wedge)$ where $\vee$ is the logical "or" operator and $\wedge$ is the logical "and" operator. In the next section we shall encounter some more examples of dioids.
A dioid $(\mathcal{D}, \oplus, \otimes)$ is called commutative if the multiplicative operation $\otimes$ is commutative in $\mathcal{D}$. The max-plus algebra and the other dioids mentioned above are commutative dioids. The structure $(\mathbb{R}_\varepsilon^{n\times n}, \oplus, \otimes)$ with $n \in \mathbb{N}\setminus\{0,1\}$ is also a dioid, but it is not a commutative dioid.
More information on dioids can be found in [3, 33, 54, 66, 67, 68, 69, 70, 109, 110, 143, 144, 145] and the references given therein.

2.2.3 Connection with Graph Theory

There exists a close relation between dioids and graphs [3, 66, 68]: e.g. many algorithms for determining the shortest path in a graph have a nice interpretation in terms of the min-plus algebra $(\mathbb{R}\cup\{\infty\}, \min, +)$. In this subsection we first give a short introduction to graph theory and next we give a graph-theoretic interpretation of some basic max-algebraic operations and concepts.

A *graph* $\mathcal{G}$ is defined as an ordered pair $(\mathcal{V},\mathcal{E})$, where $\mathcal{V}$ is a set of elements called *vertices* and $\mathcal{E}$ is a set of (unordered) pairs of vertices. The elements of $\mathcal{E}$ are called *edges*. A *directed* graph $\mathcal{G}$ is defined as an ordered pair $(\mathcal{V},\mathcal{A})$, where $\mathcal{V}$ is a set of vertices and $\mathcal{A}$ is a set of ordered pairs of vertices. The elements of $\mathcal{A}$ are called *arcs*. A *loop* is an arc of the form (v, v).
When we make a drawing of a graph, we represent the vertices by dots that are labeled with the name of the vertex they represent. Edges and loops are represented by (curved) line segments that connect the dots that correspond to the initial and the final vertex. Arcs that are no loops are also represented by (curved) line segments that connect the dots that correspond to the initial and the final vertex, but now the line segment has an arrow at the end to indicate the direction of the arc. Let us illustrate this with an example. In Figure 2.3 we have drawn the directed graph $\mathcal{G}_1 = (\mathcal{V}_1, \mathcal{A}_1)$ with $\mathcal{V}_1 = \{v_1, v_2, v_3\}$ and $\mathcal{A}_1 = \{(v_1,v_1), (v_1,v_2), (v_2,v_1), (v_2,v_3), (v_3,v_1), (v_3,v_3)\}$.

Let $\mathcal{G} = (\mathcal{V},\mathcal{A})$ be a directed graph with $\mathcal{V} = \{v_1, v_2, \ldots, v_n\}$. A *path* of length l ($l \in \mathbb{N}_0$) is a sequence of vertices v_{i_1}, v_{i_2}, ..., v_{i_l} such that $(v_{i_k}, v_{i_{k+1}}) \in \mathcal{A}$ for $k = 1, 2, \ldots, l-1$. We represent this path by $v_{i_1} \to v_{i_2} \to \ldots \to v_{i_l}$. Vertex v_{i_1} is the initial vertex of the path and v_{i_l} is the final vertex of the path. When the initial and the final vertex of a path coincide, we have a *circuit*. An *elementary circuit* is a circuit in which no vertex appears more than once, except for the initial vertex, which appears exactly twice. A directed graph $\mathcal{G} = (\mathcal{V},\mathcal{A})$ is called *strongly connected* if for any two different vertices v_i, $v_j \in \mathcal{V}$ there exists a path from v_i to v_j.
In the directed graph $\mathcal{G}_1$ of Figure 2.3 $v_1 \to v_2 \to v_3 \to v_1$ is a path of length

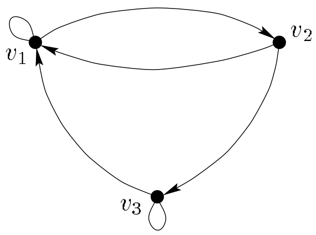


Figure 2.3: The graphical representation of the directed graph $\mathcal{G}_1$.

3. This path is also an elementary circuit. Clearly, $\mathcal{G}_1$ is strongly connected.

If we have a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{A})$ with $\mathcal{V} = \{1, 2, \ldots, n\}$ and if we associate a real number a_{ij} with each arc $(j, i) \in \mathcal{A}$, then we say that $\mathcal{G}$ is a *weighted* directed graph. We call a_{ij} the *weight* of the arc (j, i). Note that the first subscript of a_{ij} corresponds to the final (and not the initial) vertex of the arc (j, i). In the drawing of a weighted directed graph the arcs are labeled with their weights.

Definition 2.2.4 (Precedence graph) *Consider $A \in \mathbb{R}_\varepsilon^{n \times n}$. The precedence graph of A, denoted by $\mathcal{G}(A)$, is a weighted directed graph with vertices* 1, 2, $\ldots$, *n and an arc (j, i) with weight a_{ij} for each $a_{ij} \neq \varepsilon$.*

It easy to verify that every weighted directed graph corresponds to the precedence graph of an appropriately defined matrix with entries in $\mathbb{R}_\varepsilon$.

Let $\mathcal{G} = (\mathcal{V}, \mathcal{A})$ be a weighted directed graph with $\mathcal{V} = \{1, 2, \ldots, n\}$. The weight of a path $i_1 \to i_2 \to \ldots \to i_l$ is defined as the sum of the weights of the arcs that compose the path: $a_{i_2 i_1} + a_{i_3 i_2} + \ldots + a_{i_l i_{l-1}}$. The average weight of a circuit is defined as the weight of the circuit divided by the length of the circuit.

Now we can give a graph-theoretic interpretation of the max-algebraic matrix power. Let $A \in \mathbb{R}_\varepsilon^{n \times n}$. If $k \in \mathbb{N}_0$ then we have

$$(A^{\otimes^k})_{ij} = \max_{i_1, i_2, \ldots, i_{k-1}} (a_{i i_1} + a_{i_1 i_2} + \ldots + a_{i_{k-1} j})$$

for all i, j. Hence, $(A^{\otimes^k})_{ij}$ is the maximal weight of all paths of $\mathcal{G}(A)$ of length k that have j as their initial vertex and i as their final vertex — where we assume that if there does not exist a path of length k from j to i the maximal weight is equal to ε by definition.

Definition 2.2.5 (Irreducibility) *A matrix $A \in \mathbb{R}_\varepsilon^{n \times n}$ is called irreducible if its precedence graph is strongly connected.*

If we reformulate this in the max-plus algebra then a matrix $A \in \mathbb{R}_\varepsilon^{n \times n}$ is irreducible if

$$(A \oplus A^{\otimes^2} \oplus \ldots \oplus A^{\otimes^{n-1}})_{ij} \neq \varepsilon \qquad \text{for all } i, j \text{ with } i \neq j \, ,$$

since this condition means that for two arbitrary vertices i and j of $\mathcal{G}(A)$ with $i \neq j$ there exists at least one path (of length 1, 2, ... or $n-1$) from j to i. Note that it is not necessary to consider paths with a length that is greater than $n-1$.

Example 2.2.6 Consider

$$A = \begin{bmatrix} -2 & 1 & \varepsilon \\ 1 & 0 & 1 \\ \varepsilon & 0 & 2 \end{bmatrix} .$$

The precedence graph of A is represented in Figure 2.4.

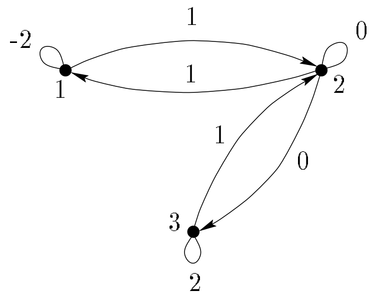


Figure 2.4: The precedence graph of the matrix A of Example 2.2.6.

We have

$$A^{\otimes^2} = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 2 & 4 \end{bmatrix} .$$

Since $(A^{\otimes^2})_{33}$ is equal to 4, the maximal weight of all paths of length 2 from vertex 3 to vertex 3 should be equal to 4. Let us verify this. There are two paths of length 2 from vertex 3 to vertex 3: $3 \rightarrow 3 \rightarrow 3$ with weight $2 + 2 = 4$ and $3 \rightarrow 2 \rightarrow 3$ with weight $1 + 0 = 1$. So the maximal weight of all paths of length 2 from vertex 3 to vertex 3 is really equal to 4.
The matrix A is irreducible since for any two different vertices i and j of $\mathcal{G}(A)$ there exists a path from i to j or since all the off-diagonal entries of $A \oplus A^{\otimes^2}$ are finite. $\Box$

Definition 2.2.7 (Max-algebraic eigenvalue, max-algebraic eigenvector) *Let $A \in \mathbb{R}_\varepsilon^{n\times n}$. If there exist a number $\lambda \in \mathbb{R}_\varepsilon$ and a vector $v \in \mathbb{R}_\varepsilon^n$ with $v \neq \mathcal{E}_{n\times 1}$ such that $A \otimes v = \lambda \otimes v$ then we say that λ is a max-algebraic eigenvalue of A and that v is a corresponding max-algebraic eigenvector of A.*

It can be shown that every square matrix with entries in $\mathbb{R}_\varepsilon$ has at least one eigenvalue (See e.g. [3]). However, in contrast to linear algebra, the number of max-algebraic eigenvalues of an n by n matrix is in general less than n. If a matrix is irreducible, it has is only one eigenvalue (See e.g. [20]).

The max-algebraic eigenvalue has the following graph-theoretic interpretation. Consider $A \in \mathbb{R}_\varepsilon^{n\times n}$. If $\lambda_{\max}$ is the maximal average weight over all elementary circuits of $\mathcal{G}(A)$, then $\lambda_{\max}$ is a max-algebraic eigenvalue of A. Every circuit of $\mathcal{G}(A)$ with an average weight that is equal to $\lambda_{\max}$ is called a *critical circuit*. For formulas and algorithms to determine max-algebraic eigenvalues and eigenvectors the interested reader is referred to [3, 10, 20, 91] and the references cited therein.

Theorem 2.2.8 *If $A \in \mathbb{R}_\varepsilon$ is irreducible, then*

$$\exists k_0 \in \mathbb{N},\ \exists c \in \mathbb{N}_0 \text{ such that } \forall k \geqslant k_0 : A^{\otimes^{k+c}} = \lambda^{\otimes^c} \otimes A^{\otimes^k}$$

where λ is the (unique) max-algebraic eigenvalue of A.

Proof: See e.g. [3, 20, 56]. □

Example 2.2.9 Consider again the matrix A of Example 2.2.6. The elementary circuits of $\mathcal{G}(A)$ are listed in Table 2.1.

Circuit	Length	Weight	Average weight
$1 \to 1$	1	-2	-2
$2 \to 2$	1	0	0
$3 \to 3$	1	2	2
$1 \to 2 \to 1$	2	2	1
$2 \to 3 \to 2$	2	1	0.5

Table 2.1: The elementary circuits of the precedence graph of the matrix A of Example 2.2.9.

The maximum average weight is 2. Hence, $\lambda_{\max} = 2$ is a max-algebraic eigenvalue of A. The vector $v = \begin{bmatrix} 0 & 1 & 2 \end{bmatrix}^T$ is a max-algebraic eigenvector of A that corresponds to the max-algebraic eigenvalue $\lambda_{\max}$ since

$$A \otimes v = \begin{bmatrix} -2 & 1 & \varepsilon \\ 1 & 0 & 1 \\ \varepsilon & 0 & 2 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = 2 \otimes \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \lambda_{\max} \otimes v \ .$$

Since

$$A^{\otimes^2} = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 2 & 4 \end{bmatrix}, \ A^{\otimes^3} = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 3 & 5 \\ 3 & 4 & 6 \end{bmatrix}, \ A^{\otimes^4} = \begin{bmatrix} 4 & 4 & 6 \\ 4 & 5 & 7 \\ 5 & 6 & 8 \end{bmatrix},$$

$$A^{\otimes^5} = \begin{bmatrix} 5 & 6 & 8 \\ 6 & 7 & 9 \\ 7 & 8 & 10 \end{bmatrix}, \; A^{\otimes^6} = \begin{bmatrix} 7 & 8 & 10 \\ 8 & 9 & 11 \\ 9 & 10 & 12 \end{bmatrix}, \; A^{\otimes^7} = \begin{bmatrix} 9 & 10 & 12 \\ 10 & 11 & 13 \\ 11 & 12 & 14 \end{bmatrix},$$

$$A^{\otimes^8} = \begin{bmatrix} 11 & 12 & 14 \\ 12 & 13 & 15 \\ 13 & 14 & 16 \end{bmatrix}, \; \ldots \; ,$$

we have $A^{\otimes^{k+1}} = 2 \otimes A^{\otimes^k}$ for $k = 5, 6, 7, \ldots$. □

2.3 The Symmetrized Max-Plus Algebra

One of the major differences between conventional algebra and the max-plus algebra is that there exist no inverse elements w.r.t. $\oplus$ in $\mathbb{R}_\varepsilon$: if $x \in \mathbb{R}_\varepsilon$ then there does not exist an element $y_x \in \mathbb{R}_\varepsilon$ such that $x \oplus y_x = \varepsilon = y_x \oplus x$, except when x is equal to ε. So $\mathbb{R}_{\max}$ is not a group. Therefore, we now introduce $\mathbb{S}_{\max}$ [3, 54, 106], which is a kind of symmetrization of the max-plus algebra. This can be compared with the extension of $\mathbb{N}$ to $\mathbb{Z}$. In Section 7.2 we shall show that $\mathbb{R}_{\max}$ corresponds to a set of nonnegative real functions with addition and multiplication as basic operations and that $\mathbb{S}_{\max}$ corresponds to a set of real functions with addition and multiplication as basic operations. Since the $\oplus$ operation is idempotent, we cannot use the conventional symmetrization technique since every idempotent group reduces to a trivial group [3, 106]. Nevertheless, it is possible to adapt the method of the construction of $\mathbb{Z}$ from $\mathbb{N}$ to obtain "balancing" elements rather than inverse elements.

In the following paragraphs we shall give a formal introduction to $\mathbb{S}_{\max}$. An more intuitive and informal introduction can be found in Appendix E. Readers that are not familiar with $\mathbb{S}_{\max}$ might wish to read this appendix first in order to get an idea of the purpose of this symmetrization and of the basic properties of the operators and relations that appear in it.

We shall restrict ourselves to a short introduction to the most important features of $\mathbb{S}_{\max}$. This introduction is based on [3, 54, 106]. First we introduce the "algebra of pairs". We consider the set of ordered pairs $\mathcal{P}_\varepsilon \stackrel{\text{def}}{=} \mathbb{R}_\varepsilon \times \mathbb{R}_\varepsilon$ with operations $\oplus$ and $\otimes$ that are defined as follows:

$$(x, y) \oplus (w, z) = (x \oplus w, \; y \oplus z)$$

$$(x, y) \otimes (w, z) = (x \otimes w \oplus y \otimes z, \; x \otimes z \oplus y \otimes w)$$

for $(x, y), (w, z) \in \mathcal{P}_\varepsilon$ and where the operations $\oplus$ and $\otimes$ on the right-hand sides correspond to maximization and addition as defined in (2.1) and (2.2). The reason for also using $\oplus$ and $\otimes$ on the left-hand sides is that these operations correspond to $\oplus$ and $\otimes$ as defined in $\mathbb{R}_\varepsilon$ as we shall see later on. It is easy to verify that in $\mathcal{P}_\varepsilon$ the $\oplus$ operation is associative, commutative and idempotent,

and its zero element is $(\varepsilon, \varepsilon)$; that the $\otimes$ operation is associative, commutative and distributive w.r.t. $\oplus$; that the identity element of $\otimes$ is $(0, \varepsilon)$; and that the zero element $(\varepsilon, \varepsilon)$ is absorbing for $\otimes$. Hence, the algebraic structure $(\mathcal{P}_\varepsilon, \oplus, \otimes)$ is a commutative dioid. We call the structure $(\mathcal{P}_\varepsilon, \oplus, \otimes)$ the *algebra of pairs*.
If $u = (x, y) \in \mathcal{P}_\varepsilon$ then we define the max-absolute value of u as $|u|_\oplus = x \oplus y$ and we introduce two unary operators $\ominus$ (the max-algebraic minus operator) and $(\cdot)^\bullet$ (the balance operator) such that $\ominus u = (y, x)$ and $u^\bullet = u \oplus (\ominus u) = (|u|_\oplus, |u|_\oplus)$. We have

$$u^\bullet = (\ominus u)^\bullet = (u^\bullet)^\bullet \tag{2.4}$$

$$u \otimes v^\bullet = (u \otimes v)^\bullet \tag{2.5}$$

$$\ominus(\ominus u) = u \tag{2.6}$$

$$\ominus(u \oplus v) = (\ominus u) \oplus (\ominus v) \tag{2.7}$$

$$\ominus(u \otimes v) = (\ominus u) \otimes v \tag{2.8}$$

for all $u, v \in \mathcal{P}_\varepsilon$. The last three properties allow us to write $u \ominus v$ instead of $u \oplus (\ominus v)$. Since the properties (2.6) – (2.8) resemble properties of the $-$ operator in conventional algebra, we could say that the $\ominus$ operator in the algebra of pairs can be considered as the analogue of the $-$ operator in conventional algebra (See also Section 7.2). As for the order of evaluation of the max-algebraic operators, the max-algebraic minus operator has the same, i.e. the lowest, priority as the max-algebraic addition operator. Furthermore, an expression of the form $\ominus \bigoplus_i \ldots$ should be interpreted as $\ominus\left(\bigoplus_i \ldots\right)$.
In conventional algebra we have $x - x = 0$ for all $x \in \mathbb{R}$, but in the algebra of pairs we have $u \ominus u = u^\bullet \neq (\varepsilon, \varepsilon)$ for all $u \in \mathcal{P}_\varepsilon$ unless u is equal to $(\varepsilon, \varepsilon)$, the zero element for $\oplus$ in $\mathcal{P}_\varepsilon$. Therefore, we introduce a new relation:

Definition 2.3.1 (Balance relation) *Consider $u = (x, y)$, $v = (w, z) \in \mathcal{P}_\varepsilon$. We say that u balances v, denoted by $u \nabla v$, if $x \oplus z = y \oplus w$.*

We have $u \ominus u = u^\bullet = (|u|_\oplus, |u|_\oplus) \nabla (\varepsilon, \varepsilon)$ for all $u \in \mathcal{P}_\varepsilon$. The balance relation is reflexive and symmetric but it is not transitive since e.g. $(2, 1) \nabla (2, 2)$ and $(2, 2) \nabla (1, 2)$ but $(2, 1) \not\nabla (1, 2)$. Hence, the balance relation is not an equivalence relation and we cannot use it to define the quotient set of $\mathcal{P}_\varepsilon$ by ∇ (as opposed to conventional algebra where $(\mathbb{N} \times \mathbb{N})/=$ yields $\mathbb{Z}$). Therefore, we introduce another relation that is closely related to the balance relation and that is defined as follows: if (x, y), $(w, z) \in \mathcal{P}_\varepsilon$ then

$$(x, y)\mathcal{B}(w, z) \quad \text{if} \quad \begin{cases} (x, y) \nabla (w, z) & \text{if } x \neq y \text{ and } w \neq z\,, \\ (x, y) = (w, z) & \text{otherwise.} \end{cases}$$

Note that if $u \in \mathcal{P}_\varepsilon$ then we have $u \ominus u \not\mathcal{B}\, (\varepsilon, \varepsilon)$ unless u is equal to $(\varepsilon, \varepsilon)$. It is easy to verify that $\mathcal{B}$ is an equivalence relation that is compatible with the $\oplus$ and $\otimes$ operations defined in $\mathcal{P}_\varepsilon$, with the balance relation ∇ and with the $\ominus$, $|\cdot|_\oplus$ and $(\cdot)^\bullet$ operators. We can distinguish between three kinds of equivalence classes generated by $\mathcal{B}$:

- $\overline{(w, -\infty)} = \{ (w, x) \in \mathcal{P}_\varepsilon \mid x < w \}$, called max-positive;
- $\overline{(-\infty, w)} = \{ (x, w) \in \mathcal{P}_\varepsilon \mid x < w \}$, called max-negative;
- $\overline{(w, w)} = \{ (w, w) \in \mathcal{P}_\varepsilon \}$, called balanced.

The class $\overline{(\varepsilon, \varepsilon)}$ is called the max-zero class.
Now we define the quotient set $\mathbb{S} = \mathcal{P}_\varepsilon / \mathcal{B}$. The algebraic structure $\mathbb{S}_{\max} = (\mathbb{S}, \oplus, \otimes)$ is also a commutative dioid. We call $\mathbb{S}_{\max}$ the symmetrized dioid of the max-plus algebra, or the *symmetrized max-plus algebra* for short. By associating $\overline{(w, -\infty)}$ with $w \in \mathbb{R}_\varepsilon$, we can identify $\mathbb{R}_\varepsilon$ with the set of max-positive or max-zero classes denoted by $\mathbb{S}^\oplus$. The set of max-negative or max-zero classes $\{\ominus w \mid w \in \mathbb{S}^\oplus\}$ will be denoted by $\mathbb{S}^\ominus$ and the set of balanced classes $\{w^\bullet \mid w \in \mathbb{S}^\oplus\}$ will be represented by $\mathbb{S}^\bullet$. This results in the following decomposition: $\mathbb{S} = \mathbb{S}^\oplus \cup \mathbb{S}^\ominus \cup \mathbb{S}^\bullet$. Note that the max-zero class $\overline{(\varepsilon, \varepsilon)}$ corresponds to ε. The max-positive elements, the max-negative elements and ε are called signed. Define $\mathbb{S}^\vee = \mathbb{S}^\oplus \cup \mathbb{S}^\ominus$. Note that $\mathbb{S}^\oplus \cap \mathbb{S}^\ominus \cap \mathbb{S}^\bullet = \{ \overline{(\varepsilon, \varepsilon)} \}$ and $\varepsilon = \ominus\varepsilon = \varepsilon^\bullet$. These notations allow us to write e.g. $2 \oplus (\ominus 4)$ instead of $\overline{(2, -\infty)} \oplus \overline{(-\infty, 4)}$. Since $\overline{(2, -\infty)} \oplus \overline{(-\infty, 4)} = \overline{(2, 4)} = \overline{(-\infty, 4)}$, we have $2 \oplus (\ominus 4) = \ominus 4$. In general, if $x, y \in \mathbb{R}_\varepsilon$ then we have

$$x \oplus (\ominus y) = x \qquad \text{if } x > y\,, \tag{2.9}$$

$$x \oplus (\ominus y) = \ominus y \qquad \text{if } x < y\,, \tag{2.10}$$

$$x \oplus (\ominus x) = x^\bullet\,. \tag{2.11}$$

Now we give some extra properties of balances that will be used in the next chapters.
An element with a $\ominus$ sign can be transferred to the other side of a balance as follows:

Proposition 2.3.2 $\forall a, b, c \in \mathbb{S} : a \ominus c \,\nabla\, b$ *if and only if* $a \,\nabla\, b \oplus c$.

If both sides of a balance are signed, we may replace the balance by an equality:

Proposition 2.3.3 $\forall a, b \in \mathbb{S}^\vee : a \,\nabla\, b \Rightarrow a = b$.

Let $a \in \mathbb{S}$. The max-positive part $a^\oplus$ and the max-negative part $a^\ominus$ of a are defined as follows:

- if $a \in \mathbb{S}^{\oplus}$ then $a^{\oplus} = a$ and $a^{\ominus} = \varepsilon$,
- if $a \in \mathbb{S}^{\ominus}$ then $a^{\oplus} = \varepsilon$ and $a^{\ominus} = \ominus a$,
- if $a \in \mathbb{S}^{\bullet}$ then there exists a number $x \in \mathbb{R}_{\varepsilon}$ such that $a = x^{\bullet}$ and then $a^{\oplus} = a^{\ominus} = x$.

So $a = a^{\oplus} \ominus a^{\ominus}$ and $a^{\oplus}, a^{\ominus} \in \mathbb{R}_{\varepsilon}$. Note that a decomposition of the form $a = x \ominus y$ with $x, y \in \mathbb{R}_{\varepsilon}$ is unique if it is required that either $x \neq \varepsilon$ and $y = \varepsilon$; $x = \varepsilon$ and $y \neq \varepsilon$; or $x = y$. Hence, the decomposition $a = a^{\oplus} \ominus a^{\ominus}$ is unique.
Note that $|a|_{\oplus} = a^{\oplus} \oplus a^{\ominus}$ for all $a \in \mathbb{S}$. We say that $a \in \mathbb{S}$ is *finite* if $|a|_{\oplus} \in \mathbb{R}$. If $|a|_{\oplus} = \varepsilon$ then we say that a is *infinite*.
Definition 2.3.1 can now be reformulated as follows:

Proposition 2.3.4 $\forall a, b \in \mathbb{S}$: $a \nabla b$ *if and only if* $a^{\oplus} \oplus b^{\ominus} = a^{\ominus} \oplus b^{\oplus}$.

We shall also use the following property:

Proposition 2.3.5 $\forall a, b \in \mathbb{S}$: $|a \oplus b|_{\oplus} = |a|_{\oplus} \oplus |b|_{\oplus}$ *and* $|a \otimes b|_{\oplus} = |a|_{\oplus} \otimes |b|_{\oplus}$.

The balance relation is extended to matrices in the usual way: if $A, B \in \mathbb{S}^{m \times n}$ then $A \nabla B$ if $a_{ij} \nabla b_{ij}$ for $i = 1, \ldots, m$ and $j = 1, \ldots, n$. Propositions 2.3.2 and 2.3.3 can be extended to the matrix case as follows:

Proposition 2.3.6 $\forall A, B, C \in \mathbb{S}^{m \times n}$: $A \ominus C \nabla B$ *if and only if* $A \nabla B \oplus C$.

Proposition 2.3.7 $\forall A, B \in (\mathbb{S}^{\vee})^{m \times n}$: $A \nabla B \Rightarrow A = B$.

We conclude this section with a few extra examples that illustrate the concepts defined above. Additional examples can be found in Appendix E.

Example 2.3.8 We have $2^{\oplus} = 2$, $2^{\ominus} = \varepsilon$ and $(5^{\bullet})^{\oplus} = (5^{\bullet})^{\ominus} = 5$. Hence, $2 \nabla 5^{\bullet}$ since $2^{\oplus} \oplus (5^{\bullet})^{\ominus} = 2 \oplus 5 = 5 = \varepsilon \oplus 5 = 2^{\ominus} \oplus (5^{\bullet})^{\oplus}$.
We have $2 \not\nabla \ominus 5$ since $2^{\oplus} \oplus (\ominus 5)^{\ominus} = 2 \oplus 5 = 5 \neq \varepsilon = \varepsilon \oplus \varepsilon = 2^{\ominus} \oplus (\ominus 5)^{\oplus}$. □

Example 2.3.9 Consider the balance $x \oplus 2 \nabla 5$. From Proposition 2.3.2 it follows that this balance can be rewritten as $x \nabla 5 \ominus 2$ or $x \nabla 5$ since $5 \ominus 2 = 5$ by (2.9).
If we want a signed solution, the balance $x \nabla 5$ becomes an equality by Proposition 2.3.3. This yields $x = 5$.
The balanced solutions of $x \nabla 5$ are of the form $x = t^{\bullet}$ with $t \in \mathbb{R}_{\varepsilon}$. We have $t^{\bullet} \nabla 5$ or equivalently $t = 5 \oplus t$ if and only if $t \geqslant 5$.
So the solution set of $x \oplus 2 \nabla 5$ is given by $\{ 5 \} \cup \{ t^{\bullet} \,|\, t \in \mathbb{R}_{\varepsilon}, t \geqslant 5 \}$. □

Definition 2.3.10 (Max-algebraic norm) *Let* $a \in \mathbb{S}^n$. *The max-algebraic norm of* a *is defined by*

$$\|a\|_{\oplus} = \bigoplus_{i=1}^{n} |a_i|_{\oplus} \quad .$$

The max-algebraic norm of a matrix $A \in \mathbb{S}^{m \times n}$ is defined by

$$\|A\|_{\oplus} = \bigoplus_{i=1}^{m} \bigoplus_{j=1}^{n} |a_{ij}|_{\oplus} \quad .$$

If $\alpha \in \mathbb{S}$, $a \in \mathbb{S}^n$ and $A \in \mathbb{S}^{m \times n}$ then we have $\|\alpha \otimes a\|_{\oplus} = |\alpha|_{\oplus} \otimes \|a\|_{\oplus}$ and $\|\alpha \otimes A\|_{\oplus} = |\alpha|_{\oplus} \otimes \|A\|_{\oplus}$.
The max-algebraic vector norm corresponds to the p-norms from linear algebra (See e.g. [65, 82]) since

$$\|a\|_{\oplus} = \left(\bigoplus_{i=1}^{n} |a_i|_{\oplus}^{\otimes^{p}} \right)^{\otimes^{\frac{1}{p}}} \qquad \text{for every } a \in \mathbb{S}^n \quad .$$

The max-algebraic matrix norm corresponds to both the Frobenius norm and the p-norms from linear algebra (See e.g. [65, 82]) since we have

$$\|A\|_{\oplus} = \left(\bigoplus_{i=1}^{m} \bigoplus_{j=1}^{n} |a_{ij}|_{\oplus}^{\otimes^{2}} \right)^{\otimes^{\frac{1}{2}}} \qquad \text{for every } A \in \mathbb{S}^{m \times n}$$

and also $\|A\|_{\oplus} = \max\limits_{\|x\|_{\oplus}=0} \|A \otimes x\|_{\oplus}$ (The maximum is reached for $x = O_{n \times 1}$).

Definition 2.3.11 (Max-algebraic signature) *If σ is a permutation, then the max-algebraic signature of σ is defined as follows:*

$$\operatorname{sgn}_{\oplus}(\sigma) = \begin{cases} 0 & \text{if } \sigma \text{ is even,} \\ \ominus 0 & \text{if } \sigma \text{ is odd.} \end{cases}$$

Definition 2.3.12 (Max-algebraic determinant) *Consider a matrix $A \in \mathbb{S}^{n \times n}$. The max-algebraic determinant of A is defined by*

$$\det_{\oplus} A = \bigoplus_{\sigma \in \mathcal{P}_n} \operatorname{sgn}_{\oplus}(\sigma) \otimes \bigotimes_{i=1}^{n} a_{i\sigma(i)} \quad .$$

If $\alpha \in \mathbb{S}$ and $A \in \mathbb{S}^{n \times n}$ then we have $\det_{\oplus} A^T = \det_{\oplus} A$ and $\det_{\oplus}(\alpha \otimes A) = \alpha^{\otimes^{n}} \otimes \det_{\oplus} A$.
Let us now illustrate Definition 2.3.12 by an example:

Example 2.3.13 Consider the matrix A of Example 2.2.6:

$$A = \begin{bmatrix} -2 & 1 & \varepsilon \\ 1 & 0 & 1 \\ \varepsilon & 0 & 2 \end{bmatrix} \quad .$$

The max-algebraic determinant of A can be determined as follows:

$$\begin{aligned}
\det{}_{\oplus} A &= a_{11} \otimes a_{22} \otimes a_{33} \ \oplus \ a_{12} \otimes a_{23} \otimes a_{31} \ \oplus \ a_{13} \otimes a_{21} \otimes a_{32} \ \ominus \\
&\qquad a_{11} \otimes a_{23} \otimes a_{32} \ \ominus \ a_{12} \otimes a_{21} \otimes a_{33} \ \ominus \ a_{13} \otimes a_{22} \otimes a_{31} \\
&= (-2) \otimes 0 \otimes 2 \ \oplus \ 1 \otimes 1 \otimes \varepsilon \ \oplus \ \varepsilon \otimes 1 \otimes 0 \ \ominus \\
&\qquad (-2) \otimes 1 \otimes 0 \ \ominus \ 1 \otimes 1 \otimes 2 \ \ominus \ \varepsilon \otimes 0 \otimes \varepsilon \\
&= 0 \oplus \varepsilon \oplus \varepsilon \ominus (-1) \ominus 4 \ominus \varepsilon \\
&= \ominus 4 \ .
\end{aligned}$$

□

Definition 2.3.14 (Max-algebraic minor rank) *Let $A \in \mathbb{S}^{m \times n}$. The max-algebraic minor rank of A, $\operatorname{rank}_{\oplus}(A)$, is the dimension of the largest square submatrix of A the max-algebraic determinant of which is not balanced.*

Theorem 2.3.15 *Let $A \in \mathbb{S}^{n \times n}$. The system of homogeneous max-linear balances $A \otimes x \nabla \mathcal{E}_{n \times 1}$ has a non-trivial signed solution if and only if $\det_{\oplus} A \nabla \varepsilon$.*

Proof: See [54]. □

Algorithms to solve systems of the form $A \otimes x \nabla b$ or $A \otimes x \nabla \mathcal{E}_{n\times 1}$ with $A \in \mathbb{S}^{n \times n}$ and $b \in \mathbb{S}^n$ can be found in [54, 106].
Consider m vectors $a_1, a_2, \ldots, a_m \in \mathbb{R}_\varepsilon^n$ and m numbers $\alpha_1, \alpha_2, \ldots, \alpha_m \in \mathbb{R}_\varepsilon$. A combination of the form $\displaystyle\bigoplus_{i=1}^{m} \alpha_i \otimes a_i$ is called a *max-linear combination* of the vectors $a_1, a_2, \ldots, a_m$.

Definition 2.3.16 (Max-linear independence) *We say that a set of vectors $\{ a_i \in \mathbb{S}^n \,|\, i = 1, 2, \ldots, m \}$ is max-linearly independent if the only signed solution of $\displaystyle\bigoplus_{i=1}^{m} \alpha_i \otimes a_i \ \nabla \ \mathcal{E}_{n \times 1}$ is $\alpha_1 = \alpha_2 = \ldots = \alpha_m = \varepsilon$. Otherwise, we say that the vectors $a_1, a_2, \ldots, a_m$ are max-linearly dependent.*

So if $A \in \mathbb{S}^{n \times n}$ then the columns of A are max-linearly independent if and only if $\det_{\oplus} A \nabla \varepsilon$. Since $\det_{\oplus} A = \det_{\oplus} A^T$ the rows of A are max-linearly independent if and only if $\det_{\oplus} A \nabla \varepsilon$.

2.4 An Introduction to System Theory for Max-Linear Time-Invariant Discrete Event Systems

In this section we give a short introduction to system theory for max-linear time-invariant DESs.

In Section 1.2 we have shown by some simple examples that time-invariant deterministic DESs in which the sequence of the events and the duration of the activities are fixed or can be determined in advance (such as repetitive production processes), can be described by an nth order state space model of the form

$$x(k+1) = A \otimes x(k) \;\oplus\; B \otimes u(k) \tag{2.12}$$

$$y(k) = C \otimes x(k) \tag{2.13}$$

for all $k \in \mathbb{N}_0$ with an initial condition $x(0) = x_0$ and where $A \in \mathbb{R}_\varepsilon^{n \times n}$, $B \in \mathbb{R}_\varepsilon^{n \times m}$ and $C \in \mathbb{R}_\varepsilon^{l \times n}$.

Now we present some analysis techniques for DESs that can be described by a model of the form (2.12) – (2.13).

First we determine the input-output behavior of the DES. We have

$$\begin{aligned}
x(1) &= A \otimes x(0) \;\oplus\; B \otimes u(0) \\
x(2) &= A \otimes x(1) \;\oplus\; B \otimes u(1) \\
&= A^{\otimes^2} \otimes x(0) \;\oplus\; A \otimes B \otimes u(0) \;\oplus\; B \otimes u(1) \\
&\;\;\vdots \\
x(k) &= A^{\otimes^k} \otimes x(0) \;\oplus\; \bigoplus_{i=0}^{k-1} A^{\otimes^{k-i-1}} \otimes B \otimes u(i) \qquad \text{for } k = 0, 1, 2, \ldots
\end{aligned}$$

where the empty max-algebraic sum $\bigoplus_{i=0}^{-1} \ldots$ is equal to $\mathcal{E}_{n \times 1}$ by definition. Hence,

$$y(k) = C \otimes A^{\otimes^k} \otimes x(0) \;\oplus\; \bigoplus_{i=0}^{k-1} C \otimes A^{\otimes^{k-i-1}} \otimes B \otimes u(i) \tag{2.14}$$

for $k = 0, 1, 2, \ldots$.

Consider two input sequences $u_1 = \{u_1(k)\}_{k=0}^{\infty}$ and $u_2 = \{u_2(k)\}_{k=0}^{\infty}$. Let $y_1 = \{y_1(k)\}_{k=1}^{\infty}$ be the output sequence that corresponds to the input sequence u_1 (with initial condition $x_{1,0}$) and let $y_2 = \{y_2(k)\}_{k=1}^{\infty}$ be the output sequence that corresponds to the input sequence u_2 (with initial condition $x_{2,0}$). Let $\alpha, \beta \in \mathbb{R}_\varepsilon$. From (2.14) it follows that the output sequence that corresponds to the input sequence $\alpha \otimes u_1 \oplus \beta \otimes u_2 = \{\alpha \otimes u_1(k) \oplus \beta \otimes u_2(k)\}_{k=0}^{\infty}$ (with initial condition $\alpha \otimes x_{1,0} \oplus \beta \otimes x_{2,0}$) is given by $\alpha \otimes y_1 \oplus \beta \otimes y_2$. This explains why DESs that can be described by a model of the form (2.12) – (2.13) are called *max-linear*.

Now we assume that $x(0) = \mathcal{E}_{n \times 1}$. For the production system of Example 1.2.1 this would mean that all the buffers are empty at the beginning and for the railroad system of Example 1.2.3 this would mean that in the first period the trains depart from each station at the time specified by the timetable. Let $p \in \mathbb{N}_0$. If we define $Y = \begin{bmatrix} y(1) & y(2) & \ldots & y(p) \end{bmatrix}^T$ and $U = \begin{bmatrix} u(0) & u(1) & \ldots & u(p-1) \end{bmatrix}^T$, then (2.14) results in

$$Y = H \otimes U$$

with

$$H = \begin{bmatrix} C \otimes B & \mathcal{E} & \ldots & \mathcal{E} \\ C \otimes A \otimes B & C \otimes B & \ldots & \mathcal{E} \\ \vdots & \vdots & \ddots & \vdots \\ C \otimes A^{\otimes^{p-1}} \otimes B & C \otimes A^{\otimes^{p-2}} \otimes B & \ldots & C \otimes B \end{bmatrix} .$$

For the production system of Example 1.2.1 this means that if we know the time instants at which raw material is fed to the system, we can compute the time instants at which the finished products will leave the system.
If we know the vector Y of latest times at which the finished products have to leave the system, we can compute the vector U of (latest) time instants at which raw material has to be fed to the system by solving the system of max-linear equations $H \otimes U = Y$, if this system has a solution, or by determining the greatest subsolution of $H \otimes U = Y$.
However, if we have perishable goods it is sometimes better to minimize the maximal deviation between the desired and the actual finishing times. In this case we have to solve the following problem:

$$\text{minimize } \max_i |\,(Y - H \otimes U)_i\,| \ .$$

This problem can be solved using formula (2.3).

Let $i \in \{1, 2, \ldots, m\}$. A max-algebraic unit impulse is a sequence that is defined as follows:

$$e(k) = \begin{cases} 0 & \text{if } k = 0\,, \\ \varepsilon & \text{if } k \neq 0\,, \end{cases}$$

for $k = 0, 1, 2, \ldots$. If we apply a max-algebraic unit impulse to the ith input of the system and if we assume that $x(0) = \mathcal{E}_{n \times 1}$ then we get

$$y(k) = C \otimes A^{\otimes^{k-1}} \otimes B_{.,i} \qquad \text{for } k = 1, 2, 3, \ldots$$

as the output of the DES. This output is called the impulse response due to a max-algebraic impulse at the ith input. Note that $y(k)$ corresponds to the ith column of the matrix $G_{k-1} \stackrel{\text{def}}{=} C \otimes A^{\otimes^{k-1}} \otimes B$ for $k = 1, 2, 3, \ldots$. Therefore, the sequence $\{G_k\}_{k=0}^{\infty}$ is called the *impulse response* of the DES. The G_k's are called the *impulse response matrices* or *Markov parameters.*

We can give the following physical interpretation to the impulse response of the production system of Example 1.2.1. At first all the internal buffers of the system are empty. Then we start feeding raw material to the input buffer and we keep on feeding raw material at such a rate that the input buffer never becomes empty. The time instants at which finished products leave the system correspond to the terms of the impulse response.

Now we consider the autonomous DES described by

$$x(k+1) = A \otimes x(k)$$

$$y(k) = C \otimes x(k)$$

with $x(0) = x_0$.
For the production system of Example 1.2.1 this means that we start from a situation in which some internal buffers and all the input buffer are not empty at the beginning (if $x_0 \neq \mathcal{E}_{n \times 1}$) and that afterwards the raw material is fed to the system at such a rate that the input buffers never become empty. For the railroad system of Example 1.2.3 autonomous behavior means that the trains depart from a station as soon as the train they should wait for has arrived and the passengers have changed trains, but they do not take the timetable into account.
If the system matrix A is irreducible, then we can calculate the "ultimate" behavior of the autonomous DES by solving the max-algebraic eigenvalue problem $A \otimes v = \lambda \otimes v$. By Theorem 2.2.8 there exist integers $k_0 \in \mathbb{N}$ and $c \in \mathbb{N}_0$ such that $x(k+c) = \lambda^{\otimes^c} \otimes x(k)$ for all $k \geqslant k_0$. This means that

$$x_i(k+c) - x_i(k) = c\lambda \qquad \text{for } i = 1, 2, \ldots, n \text{ and for all } k \geqslant k_0\,. \tag{2.15}$$

This "ultimate" behavior will be called *cyclic*.
From (2.15) it follows that for a production system λ will be the average duration of a cycle of the production process when the system has reached its cyclic behavior. The average production rate will then be equal to $\dfrac{1}{\lambda}$. This also enables us the calculate the utilization levels of the various machines in the production process. Furthermore, some algorithms to compute the eigenvalue also yield the critical paths of the production process and the bottleneck machines.
If the system matrix is not irreducible the analysis is more complicated (cf. Theorem 6.1.3 and Section C.1).

Now we apply the analysis techniques that have been discussed in this subsection to the production system of Example 1.2.1.

Example 2.4.1 Consider the production system of Example 1.2.1.
Define $Y = \begin{bmatrix} y(1) & y(2) & y(3) & y(4) \end{bmatrix}^T$ and $U = \begin{bmatrix} u(0) & u(1) & u(2) & u(3) \end{bmatrix}^T$.
If $x(0) = \mathcal{E}_{3\times 1}$ then we have $Y = H \otimes U$ with

$$H = \begin{bmatrix} 11 & \varepsilon & \varepsilon & \varepsilon \\ 16 & 11 & \varepsilon & \varepsilon \\ 21 & 16 & 11 & \varepsilon \\ 27 & 21 & 16 & 11 \end{bmatrix} .$$

If we feed raw material to the system at time instants $u(0) = 0$, $u(1) = 9$, $u(2) = 12$, $u(3) = 15$, the finished products will leave the system at time instants $y(1) = 11$, $y(2) = 20$, $y(3) = 25$ and $y(4) = 30$ since

$$H \otimes \begin{bmatrix} 0 \\ 9 \\ 12 \\ 15 \end{bmatrix} = \begin{bmatrix} 11 \\ 20 \\ 25 \\ 30 \end{bmatrix} .$$

If the finished parts should leave the system before time instants 17, 19, 24 and 27 and if we want to feed the raw material to the system as late as possible, then we should feed raw material to the system at time instants 0, 6, 11, 16 since the greatest subsolution of

$$H \otimes U = \begin{bmatrix} 17 \\ 19 \\ 24 \\ 27 \end{bmatrix}$$

is $\hat{U} = \begin{bmatrix} 0 & 6 & 11 & 16 \end{bmatrix}^T$. The actual output times $\hat{Y}$ are given by $\hat{Y} = H \otimes \hat{U} = \begin{bmatrix} 11 & 17 & 22 & 27 \end{bmatrix}^T$. Note that the largest deviation δ between the desired and the actual output times is equal to 6. The input times that minimize this deviation are given by $\tilde{U} = \hat{U} \otimes \dfrac{\delta}{2} = \hat{U} \otimes 3 = \begin{bmatrix} 3 & 9 & 14 & 19 \end{bmatrix}^T$. The corresponding output times are given by $\tilde{Y} = \begin{bmatrix} 14 & 20 & 25 & 30 \end{bmatrix}^T$. Note that the largest deviation between the desired finishing and the actual finishing times is now equal to $\dfrac{\delta}{2} = 3$.

The impulse response of the system is given by

$$\{g_k\}_{k=0}^{\infty} = 11, 16, 21, 27, 33, 39, 45, 51, 57, 63, 69, 75, \ldots .$$

Although the system matrix A is not irreducible, the system does exhibit an ultimately cyclic behavior of the form (2.15) with $\lambda = 6$ and $c = 1$. It is easy to verify that λ corresponds to the largest average circuit weight of the precedence graph of A (See Figure 2.5) and to the largest max-algebraic eigenvalue of A. If we feed raw material to the system at a rate such that the input buffer never becomes empty, then after a finite number of production cycles, the difference between $x_i(k+1)$ and $x_i(k)$ will be equal to 6 for all processing units P_i. So the average production rate of the system is $\dfrac{1}{6}$, i.e. every 6 times units a finished part leaves the production system.

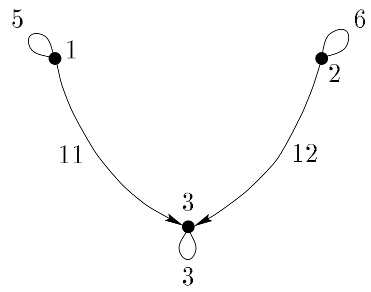


Figure 2.5: The precedence graph of the matrix A of Example 2.4.1.

The utilization level θ_i of processing unit P_i is defined by $\theta_i = \dfrac{d_i}{\lambda}$ for $i = 1, 2, 3$. Hence, $\theta_1 = \dfrac{5}{6} \approx 83\%$, $\theta_2 = \dfrac{6}{6} = 100\%$ and $\theta_3 = \dfrac{3}{6} = 50\%$. This means that when the system has reached its cyclic behavior, processing unit P_3 will be idle during half of the time, whereas processing unit P_2 will be fully occupied. This means that processing unit P_2 is the bottleneck machine. This is also revealed by the fact that the critical circuit of the precedence graph of the system matrix A is given by $2 \rightarrow 2$. $\Box$

For more information on the analysis of max-linear time-invariant DESs such as production systems, timetable dependent transportation networks, queuing systems, array processors and other types of DESs the interested reader is referred to [3, 7, 8, 9, 17, 18, 19, 20, 21, 23, 25, 26, 53, 54, 59, 117].

Remark: Clearly, one of the main advantages of an analytic (max-algebraic) model of a DES is that it enables us to derive some properties of the system (in particular the asymptotic behavior) fairly easily, whereas in some cases computer simulation might require a rather large amount of computation time (especially if the system exhibits a long transient behavior). $\diamond$

2.5 Conclusion

In this chapter we have presented some necessary background material in connection with the max-plus algebra, the symmetrized max-plus algebra, graph theory and max-algebraic system theory.
We have introduced the basic operations of max-plus algebra and stated some definitions, theorems and properties that will be used in the remainder of this thesis. We have also given a graph-theoretic interpretation of some max-algebraic concepts. Next we have given an introduction to the symmetrized max-plus algebra. Finally we have presented some elementary analysis techniques for max-linear time-invariant discrete event systems and applied these techniques to an example.

Chapter 3

The Extended Linear Complementarity Problem

In this chapter we introduce the Extended Linear Complementarity Problem (ELCP), which is an extension of the Linear Complementarity Problem, one of the fundamental problems of mathematical programming. We show that the ELCP can be viewed as a unifying framework for the Linear Complementarity Problem and its various extensions.

In the next chapters we shall show that many max-algebraic problems such as solving a system of multivariate max-algebraic polynomial equalities and inequalities, calculating max-algebraic matrix factorizations, performing max-algebraic state space transformations, determining state space realizations for max-linear time-invariant discrete event systems, calculating matrix decompositions in the symmetrized max-plus algebra and so on, can be reformulated as an ELCP. We derive an algorithm to find all solutions of an ELCP. This algorithm will yield a description of the complete solution set of an ELCP by finite points, generators for extreme rays and a basis for the linear subspace associated with the maximal affine subspaces of the solution set of the ELCP. In that way it provides a geometrical insight in the solution set of the max-algebraic problems mentioned above.

This chapter is organized as follows. In Section 3.1 we introduce the notations and some of the concepts and definitions that will be used in this chapter. We also present the Linear Complementarity Problem. In Section 3.2 we introduce the Extended Linear Complementarity Problem and show how it is linked to the LCP and its various extensions. In Section 3.3 we make a thorough study of the solution set of the general homogeneous ELCP and in Section 3.4 we develop an algorithm to find all solutions of an ELCP. Next we discuss the performance of this algorithm and the computational complexity of the ELCP. We conclude with two worked examples in which we illustrate our ELCP algorithm.

3.1 Introduction

In this section we give some definitions in connection with polyhedra and we present some examples that illustrate these definitions. We also introduce the Linear Complementarity Problem.

3.1.1 Definitions

The dimension of a vector space $\mathcal{V}$ is denoted by $\dim \mathcal{V}$.
Consider a set of vectors $S = \{a_1, a_2, \ldots, a_l\}$ with $a_1, a_2, \ldots, a_l \in \mathbb{R}^n$ and let $a = \sum_{i=1}^{l} \alpha_i a_i$. If $\alpha_i \in \mathbb{R}$ for all i then a is a *linear combination* of the vectors of S. If $\alpha_i > 0$ for all i, we have a *positive combination*. If $\alpha_i \geqslant 0$ for all i, we have a *nonnegative combination*. A nonnegative combination that also satisfies $\sum_{i=1}^{l} \alpha_i = 1$ is called a *convex combination*.

Definition 3.1.1 (Polyhedron) *A polyhedron is the solution set of a finite system of linear inequalities.*

Definition 3.1.2 (Polyhedral cone) *A polyhedral cone is the set of solutions of a finite system of homogeneous linear inequalities.*

The definitions of the remainder of this subsection are based on [134].
Let the polyhedron $\mathcal{P}$ be defined by $\mathcal{P} = \{\, x \in \mathbb{R}^n \,|\, Ax \geqslant b \,\}$. If $c \in \mathbb{R}^n$ with $c \neq 0$ and $\delta = \max\{\, c^T x \,|\, Ax \geqslant b \,\}$, then the hyperplane $\{\, x \,|\, c^T x = \delta \,\}$ is called a *supporting hyperplane* of $\mathcal{P}$.

Definition 3.1.3 (Face) *A subset F of a polyhedron $\mathcal{P}$ is called a face of $\mathcal{P}$ if $F = \mathcal{P}$ or if F is the intersection of $\mathcal{P}$ with a supporting hyperplane of $\mathcal{P}$.*

Note that each face of a nonempty polyhedron is also a nonempty polyhedron. The dimension of a face of a polyhedron is the dimension of the affine hull of that face. So if $\mathcal{P} = \{\, x \in \mathbb{R}^n \,|\, Ax \geqslant b \,\}$, then each k-dimensional face of $\mathcal{P}$ is the intersection of $\mathcal{P}$ and $n - k$ linearly independent hyperplanes from the system $Ax = b$.

Definition 3.1.4 (Minimal face) *A minimal face of a polyhedron $\mathcal{P}$ is a face that does not contain any other face of $\mathcal{P}$.*

Definition 3.1.5 (Lineality space) *Let $\mathcal{P}$ be a polyhedron defined by $\mathcal{P} = \{\, x \,|\, Ax \geqslant b \,\}$. The lineality space of $\mathcal{P}$, denoted by $\mathcal{L}(\mathcal{P})$, is defined by $\mathcal{L}(\mathcal{P}) = \{\, x \,|\, Ax = 0 \,\}$.*

So the lineality space of a polyhedron is the linear subspace associated with the largest affine subspace of $\mathcal{P}$. If $\mathcal{P} = \{\, x \mid Ax \geqslant b \,\}$ then we have $\dim \mathcal{L}(\mathcal{P}) = n - \operatorname{rank}(A)$. Let $t = \dim \mathcal{L}(\mathcal{P})$. If t is equal to 0 then $\mathcal{P}$ is called a *pointed* polyhedron. The minimal faces of $\mathcal{P}$ are translations of $\mathcal{L}(\mathcal{P})$. Hence, the dimension of a minimal face of $\mathcal{P}$ is equal to t. In this thesis points of $\mathcal{L}(\mathcal{P})$ are called *central generators* of $\mathcal{P}$. We say that a set of central generators is minimal and complete if it is a basis for $\mathcal{L}(\mathcal{P})$, i.e. if it consists of t linearly independent vectors. If G is a face of dimension $t+1$ of $\mathcal{P}$ then $G = \mathcal{L}(\mathcal{P}) + L$, where L is a line segment or a half-line. If $\mathcal{P}$ is pointed, then G is called an *edge* if L is a line segment, and an *extreme ray* if L is a half-line.

Now consider a polyhedral cone $\mathcal{K}$ defined by $\mathcal{K} = \{\, x \mid Ax \geqslant 0 \,\}$. Clearly, the only minimal face of $\mathcal{K}$ is its lineality space. Let t be the dimension of $\mathcal{L}(\mathcal{K})$. A face of $\mathcal{K}$ of dimension $t+1$ is called a *minimal proper face*. If G is a minimal proper face of $\mathcal{K}$ and if $e \in G$ with $e \notin \mathcal{L}(\mathcal{K})$, then any arbitrary point u of G can be represented as

$$u = \sum_{c_k \in \mathcal{C}} \lambda_k c_k + \kappa e \qquad \text{with } \lambda_k \in \mathbb{R} \text{ for all } k \text{ and } \kappa \geqslant 0$$

where $\mathcal{C}$ is a minimal complete set of central generators of $\mathcal{K}$. We call e an *extreme generator* that corresponds to G. The polyhedral cone $\mathcal{K}$ can be written as $\mathcal{K} = \mathcal{K}_{\text{red}} + \mathcal{L}(\mathcal{K})$ where $\mathcal{K}_{\text{red}}$ is a pointed polyhedral cone. Each point of an arbitrary extreme ray of $\mathcal{K}_{\text{red}}$ is an extreme generator of $\mathcal{K}$. If $\mathcal{C}$ is a minimal complete set of central generators of $\mathcal{K}$ and if $\mathcal{E}$ is a minimal complete set of extreme generators of $\mathcal{K}$, i.e. a set obtained by selecting exactly one point of each minimal proper face of $\mathcal{K}$, then any arbitrary point u of $\mathcal{K}$ can be represented uniquely as

$$u = \sum_{c_k \in \mathcal{C}} \lambda_k c_k + \sum_{e_k \in \mathcal{E}} \kappa_k e_k \qquad \text{with } \lambda_k \in \mathbb{R} \text{ and } \kappa_k \geqslant 0 \text{ for all } k\,.$$

Now consider the polyhedron $\mathcal{P} = \{\, x \mid Ax \geqslant b \,\}$. Let $\mathcal{K}_{\mathcal{P}}$ be the polyhedral cone defined by $\mathcal{K}_{\mathcal{P}} = \{\, y \mid Ay \geqslant 0 \,\}$. If $F_1, F_2, \ldots, F_r$ are the minimal faces of $\mathcal{P}$ and if we select a point x_k^{f} from F_k for $k = 1, 2, \ldots, r$ then for any point x of $\mathcal{P}$ there exists a point $u \in \mathcal{K}_{\mathcal{P}}$ such that x can be written as

$$x = u + \sum_{k=1}^{r} \mu_k x_k^{\mathrm{f}} \qquad \text{with } \mu_k \geqslant 0 \text{ for all } k \text{ and } \sum_{k=1}^{r} \mu_k = 1\,.$$

Let $k \in \{1, 2, \ldots, r\}$. We say that the point x_k^{f} is a *finite point* that corresponds to F_k. The set $\mathcal{X}^{\text{fin}} = \{x_1^{\mathrm{f}}, x_2^{\mathrm{f}}, \ldots, x_r^{\mathrm{f}}\}$ is called a minimal complete set of finite points of $\mathcal{P}$. Let $\mathcal{X}^{\text{cen}}$ be a minimal complete set of central generators of $\mathcal{K}_{\mathcal{P}}$ and let $\mathcal{X}^{\text{ext}}$ be a minimal complete set of extreme generators of $\mathcal{K}_{\mathcal{P}}$. Then any point x of $\mathcal{P}$ can be represented uniquely as

$$x = \sum_{x_k^{\mathrm{c}} \in \mathcal{X}^{\text{cen}}} \lambda_k x_k^{\mathrm{c}} + \sum_{x_k^{\mathrm{e}} \in \mathcal{X}^{\text{ext}}} \kappa_k x_k^{\mathrm{e}} + \sum_{x_k^{\mathrm{f}} \in \mathcal{X}^{\text{fin}}} \mu_k x_k^{\mathrm{f}}$$

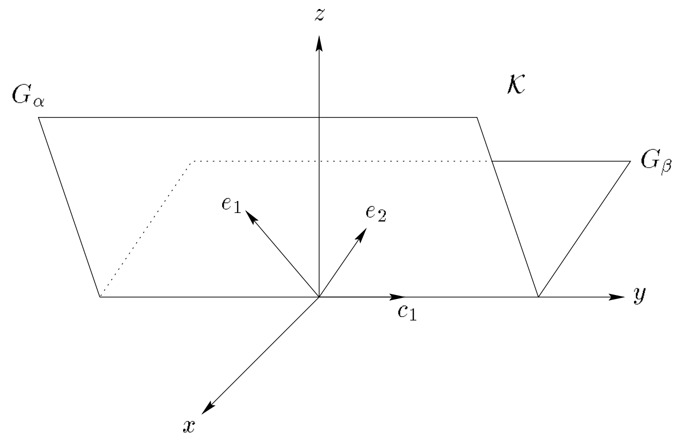


Figure 3.1: The polyhedral cone $\mathcal{K}$ of Example 3.1.7.

with $\lambda_k \in \mathbb{R}$, $\kappa_k \geqslant 0$, $\mu_k \geqslant 0$ for all k and $\sum_k \mu_k = 1$. We say that $\mathcal{X}^{\text{cen}}$ is a minimal complete set of central generators of $\mathcal{P}$ and that $\mathcal{X}^{\text{ext}}$ is a minimal complete set of extreme generators of $\mathcal{P}$. Note that $\mathcal{X}^{\text{cen}}$ is a basis of $\mathcal{L}(\mathcal{P})$.

Definition 3.1.6 (Adjacency) *Two minimal faces of a polyhedron $\mathcal{P}$ are called adjacent if they are contained in one face of dimension $t+1$, where $t = \dim \mathcal{L}(\mathcal{P})$.*
Two minimal proper faces of a polyhedral cone $\mathcal{K}$ are called adjacent if they are contained in one face of dimension $t+2$, where $t = \dim \mathcal{L}(\mathcal{K})$. Extreme generators corresponding to these faces are then also called adjacent.

Let us now illustrate the concepts that have been introduced above with a few examples.

Example 3.1.7 Consider the polyhedral cone $\mathcal{K}$ that is defined by

$$\mathcal{K} = \left\{ \begin{bmatrix} x & y & z \end{bmatrix}^T \in \mathbb{R}^3 \;\middle|\; -11x + 4z \geqslant 0 \text{ and } x + 3z \geqslant 0 \right\} .$$

Let α and β be the planes defined by

$$\alpha = \left\{ \begin{bmatrix} x & y & z \end{bmatrix}^T \in \mathbb{R}^3 \;\middle|\; -11x + 4z = 0 \right\}$$
$$\beta = \left\{ \begin{bmatrix} x & y & z \end{bmatrix}^T \in \mathbb{R}^3 \;\middle|\; x + 3z = 0 \right\}$$

Let G_α and G_β be the parts of respectively α and β that lie above the x-y plane. In Figure 3.1 we have represented the polyhedral cone $\mathcal{K}$ and the half-planes G_α and G_β.

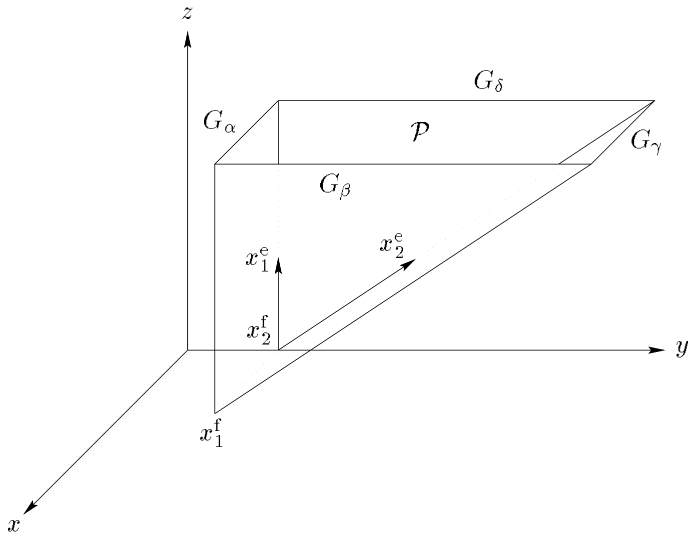


Figure 3.2: The polyhedron $\mathcal{P}$ of Example 3.1.8.

The x-y plane, α and β are supporting hyperplanes of $\mathcal{K}$. The y axis is a face of dimension 1 of $\mathcal{K}$. Clearly, the y axis is a minimal face of $\mathcal{K}$. Furthermore, the y axis is also the lineality space of $\mathcal{K}$. The half-planes G_α and G_β are faces of dimension 2 of $\mathcal{K}$. So they are minimal proper faces of $\mathcal{K}$.
The point $c_1 = \begin{bmatrix} 0 & 4 & 0 \end{bmatrix}^T$ belongs to $\mathcal{L}(\mathcal{K})$ and therefore it is a central generator of $\mathcal{K}$. The set $\{c_1\}$ is a minimal complete set of central generators of $\mathcal{K}$. The point $e_1 = \begin{bmatrix} 2 & -2 & 5.5 \end{bmatrix}^T$ lies in G_α and it does not belong to $\mathcal{L}(\mathcal{K})$. Therefore, it is an extreme generator of G_α. Likewise, the point $e_2 = \begin{bmatrix} -3 & 0 & 1 \end{bmatrix}^T$ is an extreme generator of G_β. The set $\{e_1, e_2\}$ is a minimal complete set of generators of $\mathcal{K}$.
The extreme generators e_1 and e_2 of $\mathcal{K}$ are adjacent since the corresponding minimal proper faces G_α and G_β are contained in one face of dimension 3, viz. $\mathcal{K}$ itself. □

Example 3.1.8 Consider the polyhedron $\mathcal{P}$ defined by

$$\mathcal{P} = \left\{ \begin{bmatrix} x & y & z \end{bmatrix}^T \in \mathbb{R}^3 \,\middle|\, y \geqslant 1,\ x \leqslant 1,\ 2y - 3z \leqslant 2 \text{ and } x \geqslant 0 \right\} .$$

Let α, β, γ and δ be the planes defined by

$$\alpha = \left\{ \begin{bmatrix} x & y & z \end{bmatrix}^T \in \mathbb{R}^3 \,\middle|\, y = 1 \right\}$$
$$\beta = \left\{ \begin{bmatrix} x & y & z \end{bmatrix}^T \in \mathbb{R}^3 \,\middle|\, x = 1 \right\}$$

$$\gamma = \left\{ \begin{bmatrix} x & y & z \end{bmatrix}^T \in \mathbb{R}^3 \,\middle|\, 2y - 3z = 2 \right\}$$

$$\delta = \left\{ \begin{bmatrix} x & y & z \end{bmatrix}^T \in \mathbb{R}^3 \,\middle|\, x = 0 \right\}$$

Let G_α, G_β, G_γ and G_δ be the intersection of $\mathcal{P}$ and α, β, γ and δ respectively. In Figure 3.2 we have represented the polyhedron $\mathcal{P}$ and the sets G_α, G_β, G_γ and G_δ. Note that in reality the polyhedron $\mathcal{P}$ extends infinitely in the direction of the positive z axis.

We have $\mathcal{L}(P) = \emptyset$. So $\mathcal{P}$ is a pointed polyhedron. Define $x_1^{\mathrm{f}} = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^T$ and $x_2^{\mathrm{f}} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$. The sets $\{x_1^{\mathrm{f}}\}$ and $\{x_2^{\mathrm{f}}\}$ are faces of dimension 0 of $\mathcal{P}$. The intersection of G_α and G_β is a face of dimension 1 of $\mathcal{P}$, and G_α, G_β, G_γ and G_δ are faces of dimension 2 of $\mathcal{P}$.

Define $x_1^{\mathrm{e}} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$ and $x_2^{\mathrm{e}} = \begin{bmatrix} 0 & 1.5 & 1 \end{bmatrix}^T$. The set $\{x_1^{\mathrm{f}}, x_2^{\mathrm{f}}\}$ is a minimal complete set of finite points of $\mathcal{P}$, and the set $\{x_1^{\mathrm{e}}, x_2^{\mathrm{e}}\}$ is a minimal complete set of extreme generators of $\mathcal{P}$.

The minimal faces $\{x_1^{\mathrm{f}}\}$ and $\{x_2^{\mathrm{f}}\}$ of $\mathcal{P}$ are adjacent since they are contained are contained in one face of dimension 1, viz. the line segment $x_1^{\mathrm{f}} x_2^{\mathrm{f}}$. □

3.1.2 The Linear Complementarity Problem

One of the possible formulations of the Linear Complementarity Problem (LCP) is the following [30]:

Given $M \in \mathbb{R}^{n \times n}$ and $q \in \mathbb{R}^n$, find w, $z \in \mathbb{R}^n$ such that

$$w, z \geqslant 0 \tag{3.1}$$

$$w = q + Mz \tag{3.2}$$

$$z^T w = 0 \ , \tag{3.3}$$

or show that no such w and z exist.

Note that if w and z are solutions of the LCP then it follows from (3.1) and (3.3) that

$$z_i w_i = 0 \qquad \text{for } i = 1, 2, \ldots, n \, .$$

As a consequence, we have

$$w_i > 0 \Rightarrow z_i = 0 \quad \text{and} \quad z_i > 0 \Rightarrow w_i = 0 \qquad \text{for } i = 1, 2, \ldots, n \, ,$$

i.e. the zero patterns of w and z are complementary. Therefore, condition (3.3) is called the *complementarity condition*.

The LCP has numerous applications such as quadratic programming problems, determination of the Nash equilibrium of a bimatrix game problem, the market equilibrium problem, the optimal invariant capital stock problem, the optimal stopping problem, etc. [30]. This makes the LCP one of the fundamental problems of mathematical programming.

3.2 The Extended Linear Complementarity Problem

In this section we introduce the Extended Linear Complementarity Problem and we establish a link between this problem and the Linear Complementarity Problem. We also show that many generalizations of the Linear Complementarity Problem can be considered as special cases of the Extended Linear Complementarity Problem.

3.2.1 Problem Formulation

Consider the following problem:

Given $A \in \mathbb{R}^{p \times n}$, $B \in \mathbb{R}^{q \times n}$, $c \in \mathbb{R}^p$, $d \in \mathbb{R}^q$ and m subsets ϕ_1, ϕ_2, ..., ϕ_m of $\{1, 2, \ldots, p\}$, find $x \in \mathbb{R}^n$ such that

$$\sum_{j=1}^{m} \prod_{i \in \phi_j} (Ax - c)_i = 0 \tag{3.4}$$

subject to

$$Ax \geqslant c \tag{3.5}$$

$$Bx = d \ , \tag{3.6}$$

or show that no such x exists.

In Section 3.2.3 we shall show that this problem is an extension of the Linear Complementarity Problem. Therefore, we call it the Extended Linear Complementarity Problem (ELCP). We shall show that (3.4) is a generalization of the complementarity condition (3.3) of the LCP. Therefore, we say that (3.4) represents the *complementarity condition* of the ELCP. One possible interpretation of this condition is the following. Since $Ax \geqslant c$, all the terms in (3.4) are nonnegative. Hence, condition (3.4) is equivalent to

$$\prod_{i \in \phi_j} (Ax - c)_i = 0 \qquad \text{for } j = 1, 2, \ldots, m \, .$$

So we could say that each set ϕ_j corresponds to a group of inequalities of $Ax \geqslant c$ and that in each group at least one inequality should hold with equality, i.e. the corresponding residue should be equal to 0:

$$\forall j \in \{1, 2, \ldots, m\} : \exists i \in \phi_j \text{ such that } (Ax - c)_i = 0 \ .$$

Let ψ_1, ψ_2, ..., ψ_r be subsets of $\{1, 2, \ldots, p\}$. If we have a condition of the form

$$\prod_{j=1}^{r} \left(\sum_{i \in \psi_j} (Ax - c)_i \right) = 0 \tag{3.7}$$

instead of (3.4), we still have an ELCP since (3.7) can always be rewritten as a condition of the form (3.4). Condition (3.7) can be interpreted as follows: there are r groups of linear inequalities and there should be at least one group in which *all* the inequalities hold with equality.

Example 3.2.1 Consider the following ELCP:

Find $x, y, z \in \mathbb{R}$ such that

$$(y-1)x \,+\, (-x+1)x \,+\, (-2y+3z+2)x \,=\, 0 \tag{3.8}$$

subject to

$$y \geqslant 1 \tag{3.9}$$

$$-x \geqslant -1 \tag{3.10}$$

$$-2y+3z \geqslant -2 \tag{3.11}$$

$$x \geqslant 0 \;. \tag{3.12}$$

Let the polyhedron $\mathcal{P}$ and its faces G_α, G_β, G_γ and G_δ be defined as in Example 3.1.8 (See also Figure 3.2). Note that $\mathcal{P}$ is the polyhedron defined by the system of linear inequalities (3.9) – (3.12). So any solution of the ELCP will belong to $\mathcal{P}$.
The complementarity condition (3.8) is satisfied for all the points of $\mathcal{P}$ that have an x component that is equal to 0. Hence, every point of G_δ is a solution of the ELCP defined by (3.8) – (3.12).
Let us now consider points of $\mathcal{P}$ that do not belong to G_δ, i.e. that have a positive x component. Clearly, for these points condition (3.8) can only be satisfied if we have

$$y-1=0 \;\text{ and }\; -x+1=0 \;\text{ and }\; -2y+3z+2=0 \;\;,$$

i.e. if the point lies on the intersection of G_α, G_β and G_γ. This implies that the point x_1^{f} is also a solution of the given ELCP.
So the solution set of the ELCP is given by $\{x_1^{\mathrm{f}}\} \cup G_\delta$. □

3.2.2 The Homogeneous ELCP

Consider the following problem:

Given $P \in \mathbb{R}^{p\times n}$, $Q \in \mathbb{R}^{q\times n}$ and $\phi_1, \phi_2, \ldots, \phi_m \subseteq \{1,2,\ldots,p\}$, find a non-trivial vector $u \in \mathbb{R}^n$ such that

$$\sum_{j=1}^{m} \prod_{i\in\phi_j} (Pu)_i \,=\, 0 \tag{3.13}$$

subject to $Pu \geqslant 0$ and $Qu = 0$, or show that no such u exists.

This problem is called the *homogeneous* ELCP. Note that this problem always has the trivial solution $u = O_{n\times 1}$.
Consider the ELCP defined by (3.4) – (3.6). If we introduce a real number $\alpha \geqslant 0$ and if we define

$$u = \begin{bmatrix} x \\ \alpha \end{bmatrix}, \; P = \begin{bmatrix} A & -c \\ O_{1\times n} & 1 \end{bmatrix} \text{ and } Q = \begin{bmatrix} B & -d \end{bmatrix},$$

then we get a homogeneous ELCP. In Section 3.4 we shall develop an algorithm to solve the homogeneous ELCP. Once we have computed the solutions of the homogeneous ELCP, we can extract the solutions of the original ELCP.
It is sometimes advantageous to use an alternative form of the complementarity condition: since $Pu \geqslant 0$, condition (3.13) is equivalent to

$$\prod_{i\in\phi_j} (Pu)_i = 0 \qquad \text{for } j = 1, 2, \ldots, m\,. \tag{3.14}$$

3.2.3 Link with the LCP

The LCP can be considered as a particular case of the ELCP: if we set $x = \begin{bmatrix} w \\ z \end{bmatrix}$, $A = I_{2n}$, $B = [\, I_n \;\; -M \,]$, $c = O_{2n\times 1}$, $d = q$ and $\phi_j = \{j, j+n\}$ for $j = 1, 2, \ldots, n$ in the formulation of the ELCP, we get an LCP.

3.2.4 Link with the Horizontal LCP

A problem that is slightly more general than the LCP is the Horizontal Linear Complementarity Problem (HLCP), which can be formulated as follows [30]:

Given $M,\, N \in \mathbb{R}^{n\times n}$ and $q \in \mathbb{R}^n$, find non-trivial $w,\, z \in \mathbb{R}^n$ such that

$$w, z \geqslant 0$$

$$Mz + Nw = q$$

$$z^T w = 0 \ .$$

The term *horizontal* is used to characterize the geometric shape of the matrix $\begin{bmatrix} M & N \end{bmatrix}$ since the number of rows of this matrix is less than the number of columns. It is obvious that the HLCP is also a particular case of the ELCP.

3.2.5 Link with the Vertical LCP

In [28] Cottle and Dantzig introduced a generalization of the LCP that is now called the Vertical Linear Complementarity Problem (VLCP) and that is defined as follows [30]:

Let $M \in \mathbb{R}^{m \times n}$ with $m \geqslant n$ and let $q \in \mathbb{R}^m$. Suppose that M and q are partitioned as follows:

$$M = \begin{bmatrix} M_1 \\ M_2 \\ \vdots \\ M_n \end{bmatrix} \quad \text{and} \quad q = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix}$$

with $M_i \in \mathbb{R}^{m_i \times n}$ and $q_i \in \mathbb{R}^{m_i}$ for $i = 1, 2, \ldots, n$ and with $\sum_{i=1}^{n} m_i = m$.

Now find $z \in \mathbb{R}^n$ such that

$$\begin{aligned} z &\geqslant 0 \\ q + Mz &\geqslant 0 \\ z_i \prod_{j=1}^{m_i} (q_i + M_i z)_j &= 0 \qquad \text{for } i = 1, 2, \ldots, n \, . \end{aligned}$$

Since the number of rows of M is greater than or equal to the number of columns of M, this problem is a *vertical* generalization of the LCP.
The VLCP is also a particular case of the ELCP: take $x = z$, $A = \begin{bmatrix} I_n \\ M \end{bmatrix}$, $B = [\,]$, $c = \begin{bmatrix} O_{n \times 1} \\ -q \end{bmatrix}$, $d = [\,]$ and $\phi_j = \{j, s_j + 1, s_j + 2, \ldots, s_j + m_j\}$ for $j = 1, 2, \ldots, n$ with $s_1 = m$ and $s_{j+1} = s_j + m_j$ for $j = 1, 2, \ldots, n-1$.

3.2.6 Link with the GLCP

In [39, 40] De Moor introduced the following Generalized Linear Complementarity Problem (GLCP):

> Given $Z \in \mathbb{R}^{p \times n}$ and m subsets ϕ_1, ϕ_2, $\ldots$, ϕ_m of $\{1, 2, \ldots, p\}$, find a non-trivial $u \in \mathbb{R}^n$ such that
>
> $$\sum_{j=1}^{m} \prod_{i \in \phi_j} u_i = 0$$
>
> subject to $u \geqslant 0$ and $Zu = 0$.

Now we show that the homogeneous ELCP and the GLCP are equivalent: that is, if we can solve the homogeneous ELCP, we can also solve the GLCP and vice versa.

Theorem 3.2.2 *The homogeneous ELCP and the GLCP are equivalent.*

Proof: The GLCP is a special case of the homogeneous ELCP since setting $P = I_n$ and $Q = Z$ in the formulation of the homogeneous ELCP yields a GLCP.

Now we prove that a homogeneous ELCP can be transformed into a GLCP. First we consider the sign decomposition of u: $u = u^+ - u^-$ with $u^+, u^- \geqslant 0$ and $(u^+)^T u^- = 0$. We define a vector of nonnegative slack variables $s \in \mathbb{R}^p$ such that $s = Pu = Pu^+ - Pu^-$. Clearly, the complementarity condition $\sum_{j=1}^{m} \prod_{i \in \phi_j} (Pu)_i = 0$ is equivalent to $\sum_{j=1}^{m} \prod_{i \in \phi_j} s_i = 0$. Since the components of u^+, u^- and s are nonnegative, we can combine the condition $\sum_{j=1}^{m} \prod_{i \in \phi_j} s_i = 0$ and the condition $(u^+)^T u^- = 0$. This yields a new complementarity condition: $\sum_{i=1}^{n} u_i^+ u_i^- + \sum_{j=1}^{m} \prod_{i \in \phi_j} s_i = 0$. Finally, we define $n+m$ sets $\phi'_1, \phi'_2, \ldots, \phi'_{n+m}$ such that

$$\phi'_j = \begin{cases} \{j, j+n\} & \text{if } j \in \{1, 2, \ldots, n\} \,, \\ \{i + 2n \,|\, i \in \phi_{j-n}\} & \text{if } j \in \{n+1, n+2, \ldots, n+m\} \,. \end{cases}$$

This leads to the following GLCP:

$$\text{Find } v = \begin{bmatrix} u^+ \\ u^- \\ s \end{bmatrix} \text{ such that } \sum_{j=1}^{n+m} \prod_{i \in \phi'_j} v_i = 0$$

$$\text{subject to } v \geqslant 0 \text{ and } \begin{bmatrix} P & -P & -I_p \\ Q & -Q & O_{q \times p} \end{bmatrix} v = 0 \,.$$

If we have determined a solution of this GLCP, we can obtain a solution u of the homogeneous ELCP by setting $u = u^+ - u^-$.

Hence, we have proved that the homogeneous ELCP and the GLCP are equivalent. □

In [39] an algorithm has been derived to find all solutions of a GLCP. Since the GLCP and the homogeneous ELCP are equivalent, we could use that algorithm to solve an ELCP. However, this approach has a few drawbacks:

- To convert the homogeneous ELCP into a GLCP we have introduced additional variables: the components of u^- and the slack variables (Since u^+ and u have the same dimension, we assume that u^+ replaces u and therefore we do not consider the components of u^+ as additional variables). The introduction of extra variables increases the complexity of the problem. The execution time of the algorithm of [39] increases rapidly as the number of unknowns increases (See also Sections 3.4.5 and 3.4.6). Therefore, it is not advantageous to have a large number of variables. Since the number of intermediate solutions and the required storage space also grow rapidly as the number of variables grows, the problem can even become intractable in practice if the number of variables is too large. Moreover, we do not need the extra slack variables, since they will be dropped at the end anyway.

- In Section 3.4 we shall show that the solution set of a GLCP can be characterized by a set $\mathcal{E}$ of generators of the extreme rays of the solution set and a set Γ of "cross-complementary" subsets of $\mathcal{E}$ (See also Remark 3.4.13). Even if there is no redundancy in the description of the solution set of the GLCP after dropping the slack variables, it is possible that the transition from u^+ and u^- to u results in redundant generators. It is also possible that some of the cross-complementary subsets can be merged. This means that in general we do not get a minimal description of the solution set of the ELCP.

We certainly do a great deal of unnecessary work if we use the detour via the GLCP. Therefore, we shall develop a separate algorithm to compute the complete solution set of an ELCP. If we use this algorithm, we do not have to introduce extra variables. Furthermore, this algorithm will yield a concise description of the solution set and it will also be much faster than the algorithm that uses the transformation into a GLCP.

3.2.7 Link with Other Generalizations of the LCP

In [72] Gowda and Sznajder have introduced the Generalized Order Linear Complementarity Problem (GOLCP) and the Extended Generalized Order Linear Complementarity Problem (EGOLCP). The EGOLCP is defined as follows:

> Given $k+1$ matrices $B_0, B_1, \ldots, B_k \in \mathbb{R}^{n\times n}$ and $k+1$ vectors $b_0, b_1, \ldots, b_k \in \mathbb{R}^n$, find $x \in \mathbb{R}^n$ such that
>
> $$(B_0x+b_0)\wedge(B_1x+b_1)\wedge\ldots\wedge(B_kx+b_k) = 0$$
>
> where $\wedge$ is the entrywise minimum: if $x,\ y \in \mathbb{R}^n$ then $x\wedge y \in \mathbb{R}^n$ and $(x\wedge y)_i = \min(x_i, y_i)$ for $i = 1,2,\ldots,n$.

If we take $B_0 = I_n$ and $b_0 = O_{n\times 1}$ then we have a GOLCP.
Now we show that the EGOLCP is also a special case of the ELCP:
Since the entrywise minimum of the vectors $B_0x+b_0,\ B_1x+b_1,\ \ldots,\ B_kx+b_k$ is equal to 0, we should have $B_ix \geqslant -b_i$ for $i = 1,2,\ldots,k$, and for every $j \in \{1,2,\ldots,n\}$ there should exist at least one index i such that $(B_ix+b_i)_j = 0$.
So if we put all matrices B_i in one large matrix $A = \begin{bmatrix} B_0 \\ \vdots \\ B_k \end{bmatrix}$ and all vectors b_i in one large vector $c = \begin{bmatrix} -b_0 \\ \vdots \\ -b_k \end{bmatrix}$ and if we define n sets $\phi_1,\ \phi_2,\ \ldots,\ \phi_n$ such

that $\phi_j = \{j, j+n, \ldots, j+kn\}$ for $j = 1, 2, \ldots, n$, we get an ELCP:

> Given A, c and $\phi_1, \phi_2, \ldots, \phi_n$, find $x \in \mathbb{R}^n$ such that $\displaystyle\sum_{j=1}^{n} \prod_{i \in \phi_j} (Ax-c)_i = 0$
>
> subject to $Ax \geqslant c$,

that is equivalent to the original EGOLCP.

The Extended Linear Complementarity Problem of Mangasarian and Pang [71, 102]:

> Given M, $N \in \mathbb{R}^{m \times n}$ and a polyhedral set $\mathcal{P} \subseteq \mathbb{R}^m$, find x, $y \in \mathbb{R}^n$ such that
>
> $$\begin{aligned} x, y &\geqslant 0 \\ Mx - Ny &\in \mathcal{P} \\ x^T y &= 0 \ , \end{aligned}$$

is also a special case of the ELCP:
We may assume without loss of generality that $\mathcal{P}$ can be represented as $\mathcal{P} = \{u \in \mathbb{R}^m \,|\, Au \geqslant b\}$ for some matrix $A \in \mathbb{R}^{l \times m}$ and some vector $b \in \mathbb{R}^l$. Hence, the condition $Mx - Ny \in \mathcal{P}$ is equivalent to $AMx - ANy \geqslant b$. So if we define $v = \begin{bmatrix} x \\ y \end{bmatrix}$, we get the following ELCP:

> Given A, M and N, find $v \in \mathbb{R}^{2n}$ such that $\displaystyle\sum_{i=1}^{n} v_i v_{i+n} = 0$ subject to
> $v \geqslant 0$ and
> $[\, AM \quad -AN \,]\, v \geqslant b$.

Furthermore, it is easy to show that the Generalized LCP of Ye [153]:

> Given A, $B \in \mathbb{R}^{m \times n}$, $C \in \mathbb{R}^{m \times k}$ and $q \in \mathbb{R}^m$, find x, $y \in \mathbb{R}^n$ and $z \in \mathbb{R}^k$ such that
>
> $$\begin{aligned} x, y, z &\geqslant 0 \\ Ax + By + Cz &= q \\ x^T y &= 0 \ ; \end{aligned}$$

the mixed LCP [30]:

> Given $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{m \times m}$, $C \in \mathbb{R}^{n \times m}$, $D \in \mathbb{R}^{m \times n}$, $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$, find $u \in \mathbb{R}^n$ and $v \in \mathbb{R}^m$ such that
>
> $$\begin{aligned} v &\geqslant 0 \\ a + Au + Cv &= 0 \\ b + Du + Bv &\geqslant 0 \\ v^T(b + Du + Bv) &= 0 \ ; \end{aligned}$$

the Extended Horizontal LCP of Sznajder and Gowda [137]:

> Given $k+1$ matrices $C_0, C_1, \ldots, C_k \in \mathbb{R}^{n \times n}$, $q \in \mathbb{R}^n$ and $k-1$ vectors $d_1, d_2, \ldots, d_{k-1} \in \mathbb{R}^n$ with positive components, find $x_0, x_1, \ldots, x_k \in \mathbb{R}^n$ such that
>
> $$\begin{aligned} & C_0 x_0 = q + \sum_{j=1}^{k} C_j x_j \\ & x_0, x_1, \ldots, x_k \geqslant 0 \\ & d_j - x_j \geqslant 0 && \text{for } j = 1, 2, \ldots, k-1 \\ & x_0^T x_1 = 0 \\ & (d_j - x_j)^T x_{j+1} = 0 && \text{for } j = 1, 2, \ldots, k-1 \, ; \end{aligned}$$

and the generalization of the LCP alluded to in [48]:

> Given n integers $m_1, m_2, \ldots, m_n \in \mathbb{N}_0$, n matrices $A_1, A_2, \ldots, A_n$ with $A_i \in \mathbb{R}^{p \times m_i}$ for all i and a vector $b \in \mathbb{R}^p$, find $x_1, x_2, \ldots, x_n$ with $x_i \in \mathbb{R}^{m_i}$ for all i such that
>
> $$\begin{aligned} & \sum_{i=1}^{n} \prod_{j=1}^{m_i} (x_i)_j = 0 \\ & \sum_{i=1}^{n} A_i x_i \leqslant b \\ & x_i \geqslant 0 \qquad \text{for } j = 1, 2, \ldots, n \, ; \end{aligned}$$

are also special cases of the ELCP.

Conclusion: As can be seen from this and the previous subsections, the ELCP can be considered as a unifying framework for the LCP and its various generalizations.

The underlying geometrical explanation for the fact that all these generalizations of the LCP are particular cases of the ELCP is that they all have a solution set that consists of the union of faces of a polyhedron, and that the union of any arbitrary set of faces of an arbitrary polyhedron can be described by an ELCP (See Theorem 3.4.16).
For more information on the LCP and the various generalizations discussed above and for applications, properties and methods to solve these problems the interested reader is referred to [28, 29, 30, 39, 40, 48, 49, 71, 72, 73, 84, 85, 101, 102, 107, 113, 136, 137, 141, 150, 153, 154] and the references therein.

Finally, we also want to remark that there also exist nonlinear extensions of the LCP, such as e.g. the Nonlinear Complementarity Problem (NCP), which is defined as follows:

> Given a mapping $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$, find $x \in \mathbb{R}^n$ such that $x \geqslant 0$, $f(x) \geqslant 0$ and $x^T f(x) = 0$.

More information on the NCP and other nonlinear extensions of the LCP can be found in [29, 30, 79, 88, 89, 128] and the references therein.

3.3 The Solution Set of the Homogeneous ELCP

In this section we discuss some properties of the solution set of the homogeneous ELCP:

> Given $P \in \mathbb{R}^{p \times n}$, $Q \in \mathbb{R}^{q \times n}$ and m subsets $\phi_1, \phi_2, \ldots, \phi_m$ of $\{1, 2, \ldots, p\}$, find a non-trivial vector $u \in \mathbb{R}^n$ such that
>
> $$\sum_{j=1}^{m} \prod_{i \in \phi_j} (Pu)_i = 0 \tag{3.15}$$
>
> subject to
>
> $$Pu \geqslant 0 \tag{3.16}$$
>
> $$Qu = 0 \ . \tag{3.17}$$

Note that a homogeneous ELCP can be considered as a system of homogeneous linear equalities and inequalities subject to a complementarity condition. The solution set of the system of homogeneous linear equalities and inequalities

(3.16) – (3.17) is a polyhedral cone $\mathcal{K}$. Let $\mathcal{C}$ be a minimal complete set of central generators of $\mathcal{K}$ and let $\mathcal{E}$ be a minimal complete set of extreme generators of $\mathcal{K}$. In Section 3.1 we have already said that any point u of $\mathcal{K}$ can be represented uniquely as

$$u = \sum_{c_k \in \mathcal{C}} \lambda_k c_k + \sum_{e_k \in \mathcal{E}} \kappa_k e_k \qquad \text{with } \lambda_k \in \mathbb{R} \text{ and } \kappa_k \geqslant 0 \text{ for all } k\,.$$

If c is a central generator then $Pc = 0$. By analogy we call all points $u \in \mathcal{K}$ that satisfy $Pu = 0$ *central solutions* of (3.16) – (3.17) and all points $u \in \mathcal{K}$ for which $Pu \neq 0$ *non-central solutions*. Note that if e is an extreme generator then we have $Pe \neq 0$.
Later on we shall show that any solution u of the homogeneous ELCP can be written as

$$u = \sum_{c_k \in \mathcal{C}} \lambda_k c_k + \sum_{e_k \in \mathcal{E}_s} \kappa_k e_k \qquad \text{with } \lambda_k \in \mathbb{R} \text{ and } \kappa_k \geqslant 0 \text{ for all } k$$

for some $\mathcal{E}_s \subseteq \mathcal{E}$ (See Theorem 3.4.11).
In the next section we shall present an algorithm to compute $\mathcal{C}$ and $\mathcal{E}$. But first we give some properties of the solution set of the homogeneous ELCP defined by (3.15) – (3.17).

Proposition 3.3.1 *If c is a central solution of the system of homogeneous linear inequalities and equalities (3.16) – (3.17), then λc is a solution of the homogeneous ELCP defined by (3.15) – (3.17) for every $\lambda \in \mathbb{R}$.*

Proof: Let $\lambda \in \mathbb{R}$. Since c is a central solution of (3.16) – (3.17), we have

$$\sum_{j=1}^{m} \prod_{i \in \phi_j} \left(P(\lambda c)\right)_i = \sum_{j=1}^{m} \prod_{i \in \phi_j} \lambda (Pc)_i = 0 \;.$$

Furthermore, $P(\lambda c) = \lambda(Pc) = 0 \geqslant 0$ and $Q(\lambda c) = \lambda(Qc) = 0$. So λc is a solution of the homogeneous ELCP. □

Note that every central solution of (3.16) – (3.17) automatically satisfies the complementarity condition.

Proposition 3.3.2 *If u is a solution of the homogeneous ELCP defined by (3.15) – (3.17) then κu is also a solution of the homogeneous ELCP for every $\kappa \geqslant 0$.*

Proof: Let $\kappa \geqslant 0$. We have

$$\sum_{j=1}^{m} \prod_{i \in \phi_j} \left(P(\kappa u)\right)_i = \sum_{j=1}^{m} \prod_{i \in \phi_j} \kappa (Pu)_i$$

$$= \sum_{j=1}^{m} \kappa^{\#\phi_j} \prod_{i \in \phi_j} (Pu)_i$$

$$= 0 \qquad\qquad \text{(by (3.14))} \ .$$

Since $Pu \geqslant 0$ and $\kappa \geqslant 0$, we have $P(\kappa u) = \kappa(Pu) \geqslant 0$. Furthermore, $Q(\kappa u) = \kappa(Qu) = 0$. Hence, κu is a solution of the homogeneous ELCP. □

Now we prove that extreme generators (or non-central solutions) that do not satisfy the complementarity condition cannot yield a solution of the ELCP. In our ELCP algorithm such generators will therefore immediately be discarded.

Proposition 3.3.3 *Let $\mathcal{C}$ be a minimal complete set of central generators of the polyhedral cone $\mathcal{K}$ defined by (3.16) – (3.17) and let $\mathcal{E}$ be a minimal complete set of generators of $\mathcal{K}$. Suppose that $e_l \in \mathcal{E}$ does not satisfy complementarity condition (3.15). Let $\mathcal{E}_s$ be an arbitrary subset of $\mathcal{E}$ with $e_l \in \mathcal{E}_s$. Then any combination of the form $u = \sum_{c_k \in \mathcal{C}} \lambda_k c_k + \sum_{e_k \in \mathcal{E}_s} \kappa_k e_k$ with $\lambda_k \in \mathbb{R}$ and $\kappa_k \geqslant 0$ for all k and $\kappa_l > 0$ will not satisfy the complementarity condition.*

Proof: If e_l does not satisfy the complementarity condition then we have

$$\sum_{j=1}^{m} \prod_{i \in \phi_j} (Pe_l)_i \neq 0 \ .$$

Since $Pe_l \geqslant 0$, this is only possible if

$$\exists j \in \{1, 2, \ldots, m\} \text{ such that } \forall i \in \phi_j : (Pe_l)_i \neq 0 \ . \tag{3.18}$$

Consider a combination of the form $u = \sum_{c_k \in \mathcal{C}} \lambda_k c_k + \sum_{e_k \in \mathcal{E}_s} \kappa_k e_k$ with $\lambda_k \in \mathbb{R}$ and $\kappa_k \geqslant 0$ for all k and $\kappa_l > 0$. Now we show by contradiction that u does not satisfy the complementarity condition.
Assume that u satisfies the complementarity condition. Then we have

$$\sum_{j=1}^{m} \prod_{i \in \phi_j} \left(P \left(\sum_{c_k \in \mathcal{C}} \lambda_k c_k + \sum_{e_k \in \mathcal{E}_s} \kappa_k e_k \right) \right)_i = 0$$

or equivalently

$$\sum_{j=1}^{m} \prod_{i \in \phi_j} \left(\sum_{c_k \in \mathcal{C}} \lambda_k (Pc_k)_i + \sum_{e_k \in \mathcal{E}_s} \kappa_k (Pe_k)_i \right) = 0 \ .$$

We have $Pc = 0$ for all $c \in \mathcal{C}$. Hence,

$$\sum_{j=1}^{m} \prod_{i \in \phi_j} \left(\kappa_l (Pe_l)_i + \sum_{e_k \in \mathcal{E}_s \setminus \{e_l\}} \kappa_k (Pe_k)_i \right) = 0 \ .$$

Since $Pe_k \geqslant 0$ and $\kappa_k \geqslant 0$ for all k and since $\kappa_l > 0$, this is only possible if

$$\forall j \in \{1, 2, \ldots, m\}, \exists i \in \phi_j \text{ such that } \kappa_l (Pe_l)_i + \sum_{e_k \in \mathcal{E}_s \setminus \{e_l\}} \kappa_k (Pe_k)_i = 0 \ .$$

Since $\kappa_l > 0$, this condition can only be satisfied if

$$\forall j \in \{1, 2, \ldots, m\}, \exists i \in \phi_j \text{ such that } (Pe_l)_i = 0 \ .$$

But this is in contradiction with (3.18). Hence, our assumption was wrong, which means that u does not satisfy the complementarity condition. □

Note that Proposition 3.3.3 also holds if $\mathcal{E}$ would have been a set of non-central solutions of (3.16) – (3.17).

3.4 An Algorithm to Find All Solutions of an ELCP

In this section we derive an algorithm to find all solutions of a general ELCP. As was already indicated in Section 3.2.2 we first solve the corresponding homogeneous ELCP and afterwards we extract the solutions of the original ELCP. So now we consider a homogeneous ELCP. To enhance the efficiency of the algorithm we first extract the inequalities of $Pu \geqslant 0$ that appear in the complementarity condition and put them in $P_1 u \geqslant 0$. The remaining inequalities are put in $P_2 u \geqslant 0$. If we also adapt the sets $\phi_1, \phi_2, \ldots, \phi_m$ accordingly, we get an ELCP of the following form:

Given $P_1 \in \mathbb{R}^{p_1 \times n}$, $Q \in \mathbb{R}^{q \times n}$, $P_2 \in \mathbb{R}^{p_2 \times n}$ and m subsets $\phi_1, \phi_2, \ldots, \phi_m$ of $\{1, 2, \ldots, p_1\}$, find a non-trivial $u \in \mathbb{R}^n$ such that

$$\sum_{j=1}^{m} \prod_{i \in \phi_j} (P_1 u)_i = 0 \tag{3.19}$$

subject to

$$P_1 u \geqslant 0 \tag{3.20}$$

$$Qu = 0 \tag{3.21}$$

$$P_2 u \geqslant 0 \ . \tag{3.22}$$

Note that $\bigcup_{j=1}^{m} \phi_j = \{1, 2, \ldots, p_1\}$.

The ELCP algorithm consists of 3 parts:

Part 1: Find all solutions of $P_1 u \geqslant 0$ that satisfy the complementarity condition. We describe the solution set of this problem with central and extreme generators.

Part 2: Take the conditions $Qu = 0$ and $P_2 u \geqslant 0$ into account.

Part 3: Determine which combinations of the central and the extreme generators are solutions of the ELCP: i.e. determine the "cross-complementary" sets.

Now we go through the algorithm part by part. We shall give the different parts of the algorithm in their most rudimentary form. In the remarks after each algorithm we shall indicate how the numerical stability and the performance of the algorithm can be improved.

3.4.1 Determining All Solutions of a System of Linear Inequalities that Also Satisfy a Complementarity Condition

The algorithm of this subsection is an extension and adaptation of the double description method of [111] to find all solutions of a system of linear inequalities. We have adapted it to get a more concise description of the solution set and we have added tests to reject solutions that do not satisfy the complementarity condition. In this algorithm we take a new inequality into account in each major step and we determine the intersection of the current polyhedral cone — described by a minimal complete set of central generators $\mathcal{C}$ and a minimal complete set of extreme generators $\mathcal{E}$ — with the half-space determined by the new inequality. Generators that do not satisfy the complementarity condition are immediately removed.

We give the algorithm to compute $\mathcal{C}$ and $\mathcal{E}$ in a pseudo programming language. We use $\leftarrow$ to indicate an assignment. Italic text inside braces is meant to be a comment.

Algorithm 1: Solve a system of linear inequalities subject to a complementarity condition.

Input: p_1, n, $P_1 \in \mathbb{R}^{p_1 \times n}$, m, $\{\phi_j\}_{j=1}^m$

Initialization:

$\mathcal{C} \leftarrow \{ c_i \,|\, c_i = (I_n)_{.,i} \text{ for } i = 1, 2, \ldots, n \}$

$\mathcal{E} \leftarrow \emptyset$

$P_{\text{nec}} \leftarrow [\,]$

Main loop:

for $k = 1, 2, \ldots, p_1$ **do** { *The rows of* P_1 *are taken one by one.* }

$\forall s \in \mathcal{C} \cup \mathcal{E} : \text{res}(s) \leftarrow (P_1)_{k,.}\, s$ { *Compute the residues.* }

$\mathcal{C}^+ \leftarrow \{ c \in \mathcal{C} \,|\, \text{res}(c) > 0 \}$

$\mathcal{C}^- \leftarrow \{ c \in \mathcal{C} \,|\, \mathrm{res}(c) < 0 \}$
$\mathcal{C}^0 \leftarrow \{ c \in \mathcal{C} \,|\, \mathrm{res}(c) = 0 \}$
$\mathcal{E}^+ \leftarrow \{ e \in \mathcal{E} \,|\, \mathrm{res}(e) > 0 \}$
$\mathcal{E}^- \leftarrow \{ e \in \mathcal{E} \,|\, \mathrm{res}(e) < 0 \}$
$\mathcal{E}^0 \leftarrow \{ e \in \mathcal{E} \,|\, \mathrm{res}(e) = 0 \}$
if $\mathcal{C}^+ = \emptyset$ **and** $\mathcal{C}^- = \emptyset$ **and** $\mathcal{E}^- = \emptyset$ **then** { *Case 1* }
  { *The kth inequality is redundant.* }
  $\mathcal{E} \leftarrow \mathcal{E}^0 \cup \{ e \in \mathcal{E}^+ \mid e$ satisfies the partial complementarity condition $\}$
else
  if $\mathcal{C}^+ = \emptyset$ **and** $\mathcal{C}^- = \emptyset$ **then** { *Case 2* }
    $\mathcal{E} \leftarrow \mathcal{E}^0 \cup \{ e \in \mathcal{E}^+ \mid e$ satisfies the partial complementarity condition $\}$
    for all pairs $(e^+, e^-) \in \mathcal{E}^+ \times \mathcal{E}^-$ **do**
      if e^+ and e^- are adjacent **then**
        $e^{\mathrm{new}} \leftarrow \mathrm{res}(e^+)e^- - \mathrm{res}(e^-)e^+$
        if e^{new} satisfies the partial complementarity condition **then**
          $\mathcal{E} \leftarrow \mathcal{E} \cup \{ e^{\mathrm{new}} \}$
        end if
      end if
    end for
  else { *Case 3* }
    $\mathcal{C} \leftarrow \mathcal{C}^0$
    $\mathcal{E} \leftarrow \mathcal{E}^0$
    $\mathcal{C}^+ \leftarrow \mathcal{C}^+ \cup \{ -s \,|\, s \in \mathcal{C}^- \}$
    $\forall s \in \mathcal{C}^- : \mathrm{res}(-s) \leftarrow -\mathrm{res}(s)$ { *Adapt the residues.* }
    Take an arbitrary generator $c \in \mathcal{C}^+$.
    if c satisfies the partial complementarity condition **then**
      $\mathcal{E} \leftarrow \mathcal{E} \cup \{c\}$
    end if
    $\forall c^+ \in \mathcal{C}^+ \setminus \{c\} : \mathcal{C} \leftarrow \mathcal{C} \cup \{ \mathrm{res}(c^+)c - \mathrm{res}(c)c^+ \}$
    for all $e \in \mathcal{E}^+ \cup \mathcal{E}^-$ **do**
      $e^{\mathrm{new}} \leftarrow \mathrm{res}(c)e - \mathrm{res}(e)c$ **do**
      if e^{new} satisfies the partial complementarity condition **then**
        $\mathcal{E} \leftarrow \mathcal{E} \cup \{ e^{\mathrm{new}} \}$

end if
end for
end if
Add the kth row of P_1 to P_{nec}.
end if
end for

Output: $\mathcal{C}$, $\mathcal{E}$, P_{nec}

Remarks

1. If s_1 and s_2 are two generators at the beginning of the kth pass through the main loop of the algorithm and if $\text{res}(s_1)\,\text{res}(s_2) < 0$, then the vector

$$s = |\text{res}(s_1)|\, s_2 + |\text{res}(s_2)|\, s_1 \tag{3.23}$$

will satisfy $(P_1)_{k,.}s = 0$. In other words, s will lay in the hyperplane defined by the kth row of P_1.

Proof: Without loss of generality we may assume that $\text{res}(s_1) > 0$ and $\text{res}(s_2) < 0$. Then we have

$$\begin{aligned}(P_1)_{k,.}s &= (P_1)_{k,.}\left(|\text{res}(s_1)|s_2 + |\text{res}(s_2)|s_1\right)\\ &= (P_1)_{k,.}\left(\text{res}(s_1)s_2 - \text{res}(s_2)s_1\right)\\ &= \text{res}(s_1)(P_1)_{k,.}s_2 - \text{res}(s_2)(P_1)_{k,.}s_1\\ &= \text{res}(s_1)\,\text{res}(s_2) - \text{res}(s_2)\,\text{res}(s_1)\\ &= 0\ . \end{aligned}$$ □

Note that s will also belong to the polyhedral cone that is defined by the first $k-1$ inequalities of the system $P_1 u \geqslant 0$ since it is a positive combination of s_1 and s_2.
In our algorithm we have worked out the absolute values in (3.23), which leads to the different expressions for constructing new generators.

2. In each pass through the main loop we have to combine intermediate generators. Therefore, it is advantageous to have as few intermediate generators as possible. The complementarity test is one way to reject generators. We cannot use the full complementarity condition (3.13) when we are processing the kth inequality since this complementarity condition takes all inequalities into account. However, if we consider the equivalent complementarity condition (3.14) then it is obvious that

we can apply the condition for ϕ_j as soon as we have considered all inequalities that correspond to that particular ϕ_j. That is why we use a partial complementarity test. In the kth pass the *partial complementarity condition* is:

$$\prod_{i\in\phi_j}(P_1 u)_i = 0 \qquad \text{for all } j \text{ such that } \phi_j \subseteq \{1,2,\ldots,k\} \ . \tag{3.24}$$

If there are no sets ϕ_j such that $\phi_j \subseteq \{1,2,\ldots,k\}$ then the partial complementarity condition is satisfied by definition.
New generators are constructed by taking positive combinations of other generators as is indicated by (3.23). By Proposition 3.3.3, which is also valid for the partial complementarity condition, any generator that does not satisfy the (partial) complementarity condition cannot yield a generator that satisfies the complementarity condition. Therefore, we can reject such generators immediately.
Since central generators automatically satisfy the complementarity condition, we only have to check the extreme generators. We can even be more specific. In fact, we only have to test new extreme generators and extreme generators that have a non-zero residue:

Proposition 3.4.1 *If $e \in \mathcal{E}^0$ in pass k of Algorithm 1 and if e satisfies the partial complementarity condition of pass $k-1$, then e will also satisfy the partial complementarity condition of pass k.*

Proof: If $e \in \mathcal{E}^0$ then $(P_1)_{k,.}e = 0$ or equivalently $(P_1 e)_k = 0$ and thus

$$\prod_{i\in\phi_j}(P_1 e)_i = 0 \qquad \text{for all } j \text{ such that } k \in \phi_j \ . \tag{3.25}$$

Since e satisfies the partial complementarity condition of pass $k-1$, we have

$$\prod_{i\in\phi_j}(P_1 e)_i = 0 \qquad \text{for all } j \text{ such that } \phi_j \subseteq \{1,2,\ldots,k-1\} \ . \tag{3.26}$$

If we combine (3.25) and (3.26), we obtain

$$\prod_{i\in\phi_j}(P_1 u)_i = 0 \qquad \text{for all } j \text{ such that } \phi_j \subseteq \{1,2,\ldots,k\} \ .$$

So e satisfies the partial complementarity condition of pass k. □

3. The matrix P_{nec} is used to determine whether two extreme generators are adjacent. Since we do not want any redundancy in the description of the solution set, we only combine adjacent extreme generators. Note that at

the beginning of the kth pass P_{nec} contains all the inequalities that define the current polyhedral cone. It is obvious that we do not have to include redundant inequalities in P_{nec}.
Let $\mathcal{K}$ be the polyhedral cone defined by $\mathcal{K} = \{\, u \,|\, P_{\mathrm{nec}} u \geqslant 0 \,\}$ at the beginning of pass k. Let $\mathcal{C}_{\mathcal{K}}$ be a minimal complete set of central generators of $\mathcal{K}$, let $\mathcal{E}_{\mathcal{K}}$ be a minimal complete set of extreme generators of $\mathcal{K}$ and let $t = \dim \mathcal{L}(\mathcal{K}) = \#\mathcal{C}_{\mathcal{K}}$.
The zero index set $\mathcal{I}_0(e)$ of an extreme generator $e \in \mathcal{E}_{\mathcal{K}}$ is defined as follows:

$$\mathcal{I}_0(e) \;=\; \{\, i \,|\, (P_{\mathrm{nec}}\, e)_i = 0 \,\} \;.$$

Now we derive necessary and sufficient conditions for two extreme generators e_1 and e_2 of the polyhedral cone $\mathcal{K}$ to be adjacent.
If e_1 and e_2 are adjacent then by Definition 3.1.6 there exist two minimal proper faces G_1 and G_2 of $\mathcal{K}$ with $e_1 \in G_1$ and $e_2 \in G_2$ and a $(t+2)$-dimensional face F of $\mathcal{K}$ such that $G_1 \subset F$ and $G_2 \subset F$. This means that both e_1 and e_2 have to belong to the same $(t+2)$-dimensional face F of $\mathcal{K}$. Since F is the intersection of $\mathcal{K}$ and $n-t-2$ linearly independent hyperplanes from the system $P_{\mathrm{nec}}\, u = 0$, this leads to the following proposition:

Proposition 3.4.2 (Necessary condition for adjacency) *A necessary condition for two extreme generators e_1 and e_2 of the polyhedral cone $\mathcal{K} = \{\, u \in \mathbb{R}^n \,|\, P_{\mathrm{nec}}\, u \geqslant 0 \,\}$ to be adjacent is that the zero index sets of e_1 and e_2 contain at least $n-t-2$ common indices where $t = \dim \mathcal{L}(\mathcal{K})$.*

Since F is a $(t+2)$-dimensional face of $\mathcal{K}$ and since $e_1 \in G_1 \subset F$ and $e_2 \in G_2 \subset F$, we have

$$F \;=\; \left\{ x \;\middle|\; x = \sum_{c_k \in \mathcal{C}} \lambda_k c_k + \kappa_1 e_1 + \kappa_2 e_2 \ \text{ with } \lambda_k \in \mathbb{R} \text{ for all } k \right.$$
$$\left. \text{and with } \kappa_1, \kappa_2 \geqslant 0 \right\} \;.$$

Since e_1 and e_2 belong to F and since there is exactly one extreme generator in $\mathcal{E}_{\mathcal{K}}$ for each minimal proper face of $\mathcal{K}$, there are no other extreme generators in $\mathcal{E}_{\mathcal{K}}$ that also belong to F. Hence,

Proposition 3.4.3 (Necessary and sufficient condition for adjacency) *Let $\mathcal{E}_{\mathcal{K}}$ be a minimal complete set of extreme generators of the polyhedral cone $\mathcal{K} = \{\, u \,|\, P_{\mathrm{nec}}\, u \geqslant 0 \,\}$. Two extreme generators $e_1,\ e_2 \in \mathcal{E}_{\mathcal{K}}$ are adjacent if and only if there is no extreme generator $e \in \mathcal{E}_{\mathcal{K}} \setminus \{e_1, e_2\}$ such that $\mathcal{I}_0(e_1) \cap \mathcal{I}_0(e2) \subseteq \mathcal{I}_0(e)$.*

The conditions of Propositions 3.4.2 and 3.4.3 can be considered as an extension and a generalization of the necessary and/or sufficient conditions for the adjacency of two extreme generators of a *pointed* polyhedral cone that are given in [105] and in [39, 40, 141].
In general $\mathcal{E}$ and $\mathcal{E}_{\mathcal{K}}$ do not always coincide since it is possible that some of the elements of $\mathcal{E}_{\mathcal{K}}$ have already been eliminated from $\mathcal{E}$ in a previous pass through the main loop of the ELCP algorithm because they did not satisfy the (partial) complementarity condition. This means that in general the condition of Proposition 3.4.3 with $\mathcal{E}_{\mathcal{K}}$ replaced by $\mathcal{E}$ is not a sufficient condition any more since we do not consider *all* the elements of $\mathcal{E}_{\mathcal{K}}$.
Therefore, we apply the following procedure in our ELCP algorithm to determine whether 2 extreme generators $e_1, e_2 \in \mathcal{E}$ are adjacent:

Adjacency Test 1: First we determine the common zero indices. If there are less than $n-t-2$ common zero indices then e_1 and e_2 are not adjacent.

Adjacency Test 2: Next we test whether there is an extreme generator $e \in \mathcal{E} \setminus \{e_1, e_2\}$ such that $\mathcal{I}_0(e_1) \cap \mathcal{I}_0(e2) \subseteq \mathcal{I}_0(e)$. If such a generator exists then e_1 and e_2 are not adjacent.

Note that the first test takes far less time to perform than the second especially if the number of extreme generators is large. That is why we use Adjacency Test 1 first.
Central generators are never rejected because they always satisfy the (partial) complementarity condition. This means that at the beginning of pass k the current set $\mathcal{C}$ is a basis of $\mathcal{L}(\mathcal{K})$. So the number t that is used in Adjacency Test 1 is equal to $\#\mathcal{C}$.
It is possible that two non-adjacent extreme generators pass Adjacency Test 2 if some other extreme generators of $\mathcal{K}$ have already been eliminated. However, in that case the following proposition provides a sufficient condition for adjacency.

Proposition 3.4.4 *Let $\mathcal{E}$ be the set of extreme generators at the beginning of pass k of Algorithm 1. If two non-adjacent extreme generators $e_1, e_2 \in \mathcal{E}$ pass Adjacency Test 2, then any arbitrary positive combination of e_1 and e_2 will not satisfy the (partial) complementarity condition.*

Proof: Since adjacency only depends on the extreme generators, we may assume without loss of generality that there are no central generators.
Let $\psi = \{1, 2, \ldots, k-1\}$ and let $\mathcal{K}$ be the polyhedral cone defined by $\mathcal{K} = \left\{ u \,\middle|\, (P_1)_{\psi,.}\, u \geqslant 0 \right\} = \left\{ u \,\middle|\, P_{\mathrm{nec}}\, u \geqslant 0 \right\}$.
If $\phi_j \cap \psi = \emptyset$ for all j, then the partial complementarity condition was always satisfied by definition in the previous passes. Hence, no extreme

generator of $\mathcal{K}$ has been eliminated, which means that $\mathcal{E}$ is still a minimal complete set of extreme generators of $\mathcal{K}$ and that the condition of Proposition 3.4.3 is still a necessary and sufficient condition for adjacency. Therefore, non-adjacent extreme generators cannot pass Adjacency Test 2.

If at least one set ϕ_j has a nonempty intersection with ψ, then there was a partial complementarity condition in pass $k-1$. Since this partial complementarity condition requires that some of the inequalities of the system $(P_1)_{\psi,.}\, u \geqslant 0$ hold with equality, only points on the border of $\mathcal{K}$ satisfy the partial complementarity condition; interior points of $\mathcal{K}$ do not satisfy the partial complementarity condition. Now consider a face F of $\mathcal{K}$ that contains both e_1 and e_2. Note that F itself is also a polyhedral cone. The non-adjacent generators e_1 and e_2 can only pass Adjacency Test 2 if another extreme generator of F has already been eliminated because it did not satisfy the partial complementarity condition of one of the previous passes. Since either all points of F satisfy the partial complementarity condition of pass $k-1$ or only points on the border of F satisfy the partial complementarity condition of pass $k-1$, this means that any arbitrary positive combination e of the non-adjacent extreme generators e_1 and e_2 — which always lies in the interior of F — does not satisfy the partial complementarity condition of pass $k-1$. As a consequence, e will not satisfy the partial complementarity condition of pass k either. $\square$

So if two non-adjacent generators e_1 and e_2 pass both adjacency tests, then the combination $e^{\text{new}} = |\operatorname{res}(e_1)|\, e_2 + |\operatorname{res}(e_2)|\, e_1$ will not satisfy the (partial) complementarity condition and as a consequence, it will be rejected. Therefore, no redundant generators will be created. Hence, Algorithm 1 will result in a minimal complete set of extreme generators of the solution set of the homogeneous ELCP defined by (3.19) and (3.20). Together the two adjacency tests and the test on the partial complementarity condition provide necessary and sufficient conditions for adjacency. The final P_{nec} is also considered as an output of this algorithm since we need it in the second part of the ELCP algorithm, when we process $P_2 u \geqslant 0$.

4. Let $\psi = \{1, 2, \ldots, k-1\}$. If $c \in \mathcal{C}$ at the beginning of pass k then both c and $-c$ are solutions of $(P_1)_{\psi,.}\, u \geqslant 0$. We have $\operatorname{res}(-c) = (P_1)_{k,.}\,(-c) = -(P_1)_{k,.}\, c = -\operatorname{res}(c)$. So if $c \in \mathcal{C}^+$ then $-c \in \mathcal{C}^-$ and vice versa. This explains why we may set $\mathcal{C}^+ \leftarrow \mathcal{C}^+ \cup \{\, -s \,|\, s \in \mathcal{C}^- \,\}$ in the third step of Case 3 and why we have adapted the residues in the fourth step. After that step all the central generators have a nonnegative residue.

5. If we multiply a central or an extreme generator by a positive real number, it stays a central or an extreme generator by Propositions 3.3.1 and 3.3.2. This means that we can normalize all new generators after each pass

through the main loop in order to avoid problems such as overflow.
To avoid problems arising from round-off errors it is better to test the residues against a threshold $\tau > 0$ instead of against 0 when determining the sets $\mathcal{C}^+$, $\mathcal{C}^-$, $\mathcal{C}^0$, $\mathcal{E}^+$, $\mathcal{E}^-$ and $\mathcal{E}^0$.

6. If both $\mathcal{C}$ and $\mathcal{E}$ are empty after a pass through the main loop, we may stop the algorithm. In that case the homogeneous ELCP will not have any solutions except for the trivial solution $u = O_{n\times 1}$.

For more information on the method used to find all solutions of a system of linear inequalities the interested reader is referred to [111]. One of the main differences between our algorithm and the double description method of [111] is that we only store one version of each central generator c, whereas in the double description method both c and $-c$ are stored. We have also added the test on the (partial) complementarity condition to eliminate as many generators as soon as possible.

3.4.2 The Remaining Equality and Inequality Constraints

The next algorithm is an adaptation of Algorithm 1. Since we have already processed all rows of P_1 in Algorithm 1, we can now test for the full complementarity condition.

Algorithm 2: Take the equality constraints into account.

Input: p_1, q, n, m, $P_1 \in \mathbb{R}^{p_1\times n}$, $Q \in \mathbb{R}^{q\times n}$, $\{\phi_j\}_{j=1}^m$, $\mathcal{C}$, $\mathcal{E}$, P_{nec}

Main loop:

for $k = 1, 2, \ldots, q$ **do** { *The rows of Q are taken one by one.* }
 $\forall s \in \mathcal{C} \cup \mathcal{E} : \mathrm{res}(s) \leftarrow Q_{k,.}\, s$ { *Compute the residues.* }
 $\mathcal{C}^+ \leftarrow \{ c \in \mathcal{C} \,|\, \mathrm{res}(c) > 0 \}$
 $\mathcal{C}^- \leftarrow \{ c \in \mathcal{C} \,|\, \mathrm{res}(c) < 0 \}$
 $\mathcal{C}^0 \leftarrow \{ c \in \mathcal{C} \,|\, \mathrm{res}(c) = 0 \}$
 $\mathcal{E}^+ \leftarrow \{ e \in \mathcal{E} \,|\, \mathrm{res}(e) > 0 \}$
 $\mathcal{E}^- \leftarrow \{ e \in \mathcal{E} \,|\, \mathrm{res}(e) < 0 \}$
 $\mathcal{E}^0 \leftarrow \{ e \in \mathcal{E} \,|\, \mathrm{res}(e) = 0 \}$
 if $\mathcal{C}^+ = \emptyset$ **and** $\mathcal{C}^- = \emptyset$ **and** $\mathcal{E}^+ = \emptyset$ **and** $\mathcal{E}^- = \emptyset$
 then { *Case 1* }
  { *The kth equation is redundant.* }
 else
  if $\mathcal{C}^+ = \emptyset$ **and** $\mathcal{C}^- = \emptyset$ **then** { *Case 2* }
   $\mathcal{E} \leftarrow \mathcal{E}^0$
   for all pairs $(e^+, e^-) \in \mathcal{E}^+ \times \mathcal{E}^-$ **do**
    if e^+ and e^- are adjacent **then**

$e^{\text{new}} \leftarrow \text{res}(e^+)e^- - \text{res}(e^-)e^+$
if e^{new} satisfies the complementarity condition
then
$\mathcal{E} \leftarrow \mathcal{E} \cup \{ e^{\text{new}} \}$
end if
end if
end for
else { *Case 3* }
$\mathcal{C} \leftarrow \mathcal{C}^0$
$\mathcal{E} \leftarrow \mathcal{E}^0$
$\mathcal{C}^+ \leftarrow \mathcal{C}^+ \cup \{ -s \,|\, s \in \mathcal{C}^- \}$
$\forall s \in \mathcal{C}^- : \text{res}(-s) \leftarrow -\text{res}(s)$ { *Adapt the residues.* }
Take an arbitrary generator $c \in \mathcal{C}^+$.
$\forall c^+ \in \mathcal{C}^+ \setminus \{c\} : \mathcal{C} \leftarrow \mathcal{C} \cup \{\text{res}(c^+)c - \text{res}(c)c^+\}$
for all $e \in \mathcal{E}^+ \cup \mathcal{E}^-$ **do**
$e^{\text{new}} \leftarrow \text{res}(c)e - \text{res}(e)c$
if e^{new} satisfies the complementarity condition **then**
$\mathcal{E} \leftarrow \mathcal{E} \cup \{ e^{\text{new}} \}$
end if
end for
end if
end if
end for
Output: $\mathcal{C}$, $\mathcal{E}$

Remarks

1. Now we do not have to add rows of Q to P_{nec} since after the kth pass through the main loop every generator s will satisfy $Q_{\{1,\ldots,k\},.}s = 0$. So adding the kth row of Q to P_{nec} would yield the same extra element in all zero index sets. As a consequence, Adjacency Test 1 for the kth pass of Algorithm 2 becomes: if there are less than $n-(k-1)-t-2$ common indices in the zero index sets of e_1 and e_2 then e_1 and e_2 are not adjacent.

2. The main difference with Algorithm 1 is that now we have to satisfy equality constraints. That is why we only keep those generators that have a zero residue, whereas in Algorithm 1 we kept all generators with a positive or a zero residue.

3. If we construct new generators, we immediately test whether the full complementarity condition is satisfied. We do not have to test the generators that are copied from the previous loop since they already satisfy

the complementarity condition. Since each new central generator c will still satisfy $P_1 c = 0$ and thus also $\sum_{j=1}^{m} \prod_{i \in \phi_j} (P_1 c)_i = 0$, we only have to test new extreme generators.

4. If we would only have been interested in obtaining one solution of the homogeneous ELCP, we could have used the equality constraints to eliminate some of the variables. However, since we want a minimal description of the entire solution set of the ELCP with central and extreme generators, we do not eliminate any variables.

To take the remaining inequalities into account we apply Algorithm 1 again but we skip the initialization step and continue with the sets $\mathcal{C}$ and $\mathcal{E}$ that resulted from Algorithm 2 and the matrix $P_{\rm nec}$ from Algorithm 1. Adjacency Test 1 now becomes: if there are less than $n - q - t - 2$ common indices in $\mathcal{I}_0(e_1)$ and $\mathcal{I}_0(e_2)$ then e_1 and e_2 are not adjacent. In the main loop we only have to test whether newly constructed extreme generators satisfy the full complementarity condition.
The resulting sets $\mathcal{C}$ and $\mathcal{E}$ are called respectively a minimal complete set of central generators of (the solution set of) the homogeneous ELCP and a minimal complete set of extreme generators of (the solution set of) the homogeneous ELCP.

To avoid unnecessary computations and to limit the required amount of storage space, it is advantageous to have as few intermediate generators as possible. That is why we have split the inequalities of $Pu \geqslant 0$ up into two groups and why we process $Qu = 0$ before $P_2 u \geqslant 0$:

- The complementarity condition is one way to reject generators. Therefore, we already use a partial complementarity condition in Algorithm 1. This also explains why we have removed the inequalities that did not appear in the complementarity condition and put them in $P_2 u \geqslant 0$: this allows us to apply the partial complementarity test as soon as possible.

- Next we further reduce the solution set by taking the extra equality and inequality constraints into account. Unless we have a priori knowledge about the coefficients of the equalities and the inequalities and about the structure of the solution set of the system of equalities and inequalities, it is quite reasonable to assume that an equality will yield less intermediate generators than an inequality, since for an equality we only retain existing extreme generators with a zero residue in the first step of Case 2, whereas for an inequality we retain all existing extreme generators with a positive or a zero residue. That is why we first take $Qu = 0$ into account instead of $P_2 u \geqslant 0$.

Since a minimal complete set of central generators of the homogeneous ELCP is a basis for the null space of the matrix $\begin{bmatrix} P_1 \\ Q \\ P_2 \end{bmatrix}$, we could determine the set $\mathcal{C}$ before executing Algorithms 1 and 2. Once $\mathcal{C} = \{c_1, c_2, \ldots, c_r\}$ has been determined, the lineality space of the solution set of the ELCP can be removed by adding the additional constraint

$$\begin{bmatrix} c_1 & c_2 & \ldots & c_r \end{bmatrix}^T u = 0 \tag{3.27}$$

to the system of linear equalities and inequalities of the ELCP. Then we could set

$$\mathcal{C} \leftarrow \text{a basis for the null space of } \begin{bmatrix} c_1 & c_2 & \ldots & c_r \end{bmatrix}^T \tag{3.28}$$

in the initialization step of Algorithm 1. Since (3.27) is a system of r homogeneous linear equalities, we also have to include an extra term $-r$ in Adjacency Test 1. This should be compared with the term $-q$ that is added in Adjacency Test 1 when we take the inequalities of the system $P_2 u \geqslant 0$ into account. Note that we could select an orthonormal basis in (3.28) to augment the numerical stability of the algorithm.

In order to reduce the effects of error propagation we could regularly recompute the central and extreme generators. This could be done as follows. For each central and extreme generator s we first construct a system $\mathcal{S}(s)$ of homogeneous linear equalities and inequalities: for a central generator $\mathcal{S}(s)$ consists of all the equalities and inequalities considered so far but with the inequalities transformed into equalities; for an extreme generator $\mathcal{S}(s)$ consists of all the equalities and inequalities considered so far but with the inequalities indexed by the zero index set of s transformed into equalities. Next we determine the point of the solution set of $\mathcal{S}(s)$ that is the nearest to s. Note that we could use an iterative algorithm to solve $\mathcal{S}(s)$ with s as the starting point for the algorithm.

3.4.3 The Cross-Complementary Sets

Let $\mathcal{K}$ be the polyhedral cone defined by $P_1 u \geqslant 0$, $Qu = 0$ and $P_2 u \geqslant 0$. As a direct consequence of the way in which $\mathcal{C}$ and $\mathcal{E}$ are constructed, every combination of the form

$$u = \sum_{c_k \in \mathcal{C}_s} \lambda_k c_k + \sum_{e_k \in \mathcal{E}_s} \kappa_k e_k \qquad \text{with } \lambda_k \in \mathbb{R} \text{ and } \kappa_k \geqslant 0 \text{ for all } k \tag{3.29}$$

where $\mathcal{C}_s$ is an arbitrary subset of $\mathcal{C}$ and $\mathcal{E}_s$ is an arbitrary subset of $\mathcal{E}$, belongs to $\mathcal{K}$. The complementarity condition requires that in each group of inequalities of $P_1 u \geqslant 0$ that corresponds to some ϕ_j at least one inequality holds with equality. As a consequence, the complementarity condition is satisfied either by all the points of $\mathcal{K}$ (if all the ϕ_j's are empty) or only by points that lie on the border of $\mathcal{K}$ (if at least one of the ϕ_j's is not empty). Since we have only rejected generators that did not satisfy the complementarity condition and hence certainly would not yield solutions of the ELCP, any arbitrary solution of the homogeneous ELCP can be represented by (3.29) for some $\mathcal{C}_s \subseteq \mathcal{C}$ and some $\mathcal{E}_s \subseteq \mathcal{E}$.

However, if we take arbitrary subsets of $\mathcal{C}$ and $\mathcal{E}$ then in general not every combination of the form (3.29) will be a solution of the ELCP. The complementarity condition determines for which subsets of $\mathcal{C}$ and $\mathcal{E}$ (3.29) will yield a solution of the homogeneous ELCP. This is where the concept "cross-complementarity" arises.
In [39, 40] two solutions of a GLCP are called cross-complementary if every nonnegative combination of these solutions satisfies the complementarity condition. This definition can be extended to an arbitrary number of solutions. However, for the ELCP we have to adapt this definition as follows.

Definition 3.4.5 (Cross-complementarity) *A set $\mathcal{S}$ of solutions of the homogeneous ELCP defined by (3.19) – (3.22) is called cross-complementary if every sum of an arbitrary linear combination of the central solutions in $\mathcal{S}$ and an arbitrary nonnegative combination of the non-central solutions in $\mathcal{S}$:*

$$u = \sum_{s_k \in \mathcal{S}^{\mathrm{c}}} \lambda_k s_k + \sum_{s_k \in \mathcal{S}^{\mathrm{nc}}} \kappa_k s_k \qquad \text{with } \lambda_k \in \mathbb{R} \text{ and } \kappa_k \geqslant 0 \text{ for all } k \tag{3.30}$$

where $\mathcal{S}^{\mathrm{c}} = \{\, s \in \mathcal{S} \mid P_1 s = 0 \text{ and } P_2 s = 0 \,\}$ *and* $\mathcal{S}^{\mathrm{nc}} = \{\, s \in \mathcal{S} \mid P_1 s \neq 0 \text{ or } P_2 s \neq 0 \,\}$, *satisfies the complementarity condition.*

If all the elements of a set $\mathcal{S}$ are cross-complementary then we say that $\mathcal{S}$ is a *cross-complementary set*.
Note that every combination of the form (3.30) always belongs to $\mathcal{K}$. So if $\mathcal{S}$ is a cross-complementary set then every combination of the form

$$u = \sum_{s_k \in \mathcal{S}^{\mathrm{c}}} \lambda_k s_k + \sum_{s_k \in \mathcal{S}^{\mathrm{nc}}} \kappa_k s_k \qquad \text{with } \lambda_k \in \mathbb{R} \text{ and } \kappa_k \geqslant 0 \text{ for all } k$$

where $\mathcal{S}^{\mathrm{c}} = \{\, s \in \mathcal{S} \mid P_1 s = 0 \text{ and } P_2 s = 0 \,\}$ and $\mathcal{S}^{\mathrm{nc}} = \{\, s \in \mathcal{S} \mid P_1 s \neq 0 \text{ or } P_2 s \neq 0 \,\}$ is a solution of the ELCP.
Now we present a method to determine the maximal sets of cross-complementary solutions. The following proposition tells us that we can always set $\mathcal{C}_s = \mathcal{C}$ in (3.29).

Proposition 3.4.6 *Let $\mathcal{C}$ be a minimal complete set of central generators of a homogeneous ELCP. If u_1 is a solution of the homogeneous ELCP then the set $\mathcal{C} \cup \{u_1\}$ is cross-complementary.*

Proof: Assume that the homogeneous ELCP is defined by (3.19) – (3.22). Define a set A such that

$$A = \begin{cases} \mathbb{R} & \text{if } u_1 \text{ is a central solution of (3.20) – (3.22),} \\ \mathbb{R}^+ & \text{if } u_1 \text{ is a non-central solution of (3.20) – (3.22).} \end{cases}$$

So now we have to prove that any combination of the form $u = \sum_{c_k \in \mathcal{C}} \lambda_k c_k + \mu u_1$ with $\lambda_k \in \mathbb{R}$ for all k and with $\mu \in A$ satisfies the complementarity condition. We have

$$\begin{aligned}
\sum_{j=1}^{m} \prod_{i \in \phi_j} & \left(P_1 \left(\sum_{c_k \in \mathcal{C}} \lambda_k c_k + \mu u_1 \right) \right)_i \\
&= \sum_{j=1}^{m} \prod_{i \in \phi_j} \left(\sum_{c_k \in \mathcal{C}} \lambda_k (P_1 c_k)_i + \mu (P_1 u_1)_i \right) \\
&= \sum_{j=1}^{m} \prod_{i \in \phi_j} \left(0 + \mu (P_1 u_1)_i \right) \qquad \text{(since } P_1 c_k = 0\text{)} \\
&= \sum_{j=1}^{m} \mu^{\# \phi_j} \prod_{i \in \phi_j} (P_1 u_1)_i \\
&= 0 \qquad \text{(by (3.14) with } P \text{ replaced by } P_1\text{)} \ .
\end{aligned}$$

This means that $\mathcal{C} \cup \{u_1\}$ is a cross-complementary set. □

So now we only have to consider the extreme generators. The following proposition tells us that we only have to test one positive combination to determine whether a set of extreme generators (or non-central solutions) is cross-complementary or not:

Proposition 3.4.7 *Let $e_1, e_2, \ldots, e_k$ be arbitrary extreme generators (or non-central solutions) of a homogeneous ELCP. Then $\kappa_1 e_1 + \kappa_2 e_2 + \ldots + \kappa_k e_k$ satisfies the complementarity condition for all $\kappa_1, \kappa_2, \ldots, \kappa_k \geqslant 0$ if and only if there exist $\mu_1, \mu_2, \ldots, \mu_k > 0$ such that $\mu_1 e_1 + \mu_2 e_2 + \ldots + \mu_k e_k$ satisfies the complementarity condition.*

Proof: Since the proof of the "only if" part is trivial, we only prove the "if" part.
Assume that the homogeneous ELCP is defined by (3.19) – (3.22). If there exist positive real numbers $\mu_1, \mu_2, \ldots, \mu_k$ such that $\mu_1 e_1 + \mu_2 e_2 + \ldots + \mu_k e_k$ satisfies the complementarity condition then we have

$$\prod_{i \in \phi_j} \left(P_1 \left(\sum_{l=1}^{k} \mu_l e_l \right) \right)_i = 0 \qquad \text{for } j = 1, 2 \ldots, m$$

or equivalently

$$\prod_{i \in \phi_j} \left(\sum_{l=1}^{k} \mu_l (P_1 e_l)_i \right) = 0 \qquad \text{for } j = 1, 2, \ldots, m$$

and thus

$$\sum_{(\psi_1,\psi_2,\ldots,\psi_k)\in\Psi(k,j)} \prod_{l=1}^{k} \prod_{i\in\psi_l} \mu_l (P_1 e_l)_i \;=\; 0 \qquad \text{for } j = 1,2,\ldots,m \tag{3.31}$$

where $\Psi(k,j)$ is the set of all possible k-tuples of k disjoint subsets of ϕ_j the union of which is equal to ϕ_j:

$$\Psi(k,j) = \left\{ (\psi_1,\psi_2,\ldots,\psi_k) \;\middle|\; \psi_1,\psi_2,\ldots,\psi_k \subseteq \phi_j\,;\, \bigcup_{l=1}^{k} \psi_l = \phi_j \text{ and } \forall l_1,l_2 \in \{1,2,\ldots,k\} : \text{ if } l_1 \neq l_2 \text{ then } \psi_{l_1} \cap \psi_{l_2} = \emptyset \right\} ,$$

and where the empty product $\prod_{i\in\emptyset} \ldots$ is equal to 1 by definition. Note that we also allow empty subsets ψ_l in the definition of $\Psi(k,j)$.
From (3.31) it follows that

$$\sum_{(\psi_1,\psi_2,\ldots,\psi_k)\in\Psi(k,j)} \left(\left(\prod_{l=1}^{k} \mu_l^{\#\psi_l} \right) \left(\prod_{l=1}^{k} \prod_{i\in\psi_l} (P_1 e_l)_i \right) \right) \;=\; 0$$

for $j = 1,2,\ldots,m$. Since $\mu_l > 0$ and $(P_1 e_l)_i \geqslant 0$ for $l = 1,2,\ldots,k$, this is only possible if

$$\forall (\psi_1,\psi_2,\ldots,\psi_k) \in \Psi(k,j) \;:\; \prod_{l=1}^{k} \prod_{i\in\psi_l} (P_1 e_l)_i \;=\; 0 \tag{3.32}$$

for $j = 1,2,\ldots,m$.
Now we show that $\kappa_1 e_1 + \kappa_2 e_2 + \ldots + \kappa_k e_k$ also satisfies the complementarity condition for all nonnegative real numbers κ_1, κ_2, $\ldots$, κ_k. Using the same reasoning as for $\mu_1 e_1 + \mu_2 e_2 + \ldots + \mu_k e_k$ we find

$$\begin{aligned}
&\sum_{j=1}^{m} \prod_{i\in\phi_j} \left(P_1 \left(\sum_{l=1}^{k} \kappa_l e_l \right) \right)_i \\
&\qquad = \sum_{j=1}^{m} \sum_{(\psi_1,\psi_2,\ldots,\psi_k)\in\Psi(k,j)} \left(\left(\prod_{l=1}^{k} \kappa_l^{\#\psi_l} \right) \left(\prod_{l=1}^{k} \prod_{i\in\psi_l} (P_1 e_l)_i \right) \right) \\
&\qquad = 0 \qquad\qquad (\text{by } (3.32)) \;. \qquad\qquad \square
\end{aligned}$$

To determine whether a set of extreme generators of a homogeneous ELCP is cross-complementary we take an arbitrary positive combination of these generators. If the combination satisfies the complementarity condition then the

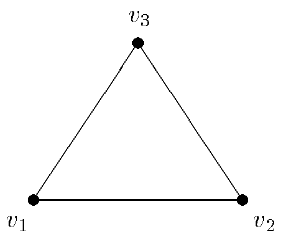


Figure 3.3: The cross-complementarity graph $\mathcal{G}_c$ of Example 3.4.8.

generators are cross-complementary. If the combination does not satisfy the complementary condition then the generators cannot be cross-complementary.

The *cross-complementarity graph* $\mathcal{G}_c$ that corresponds to a set of extreme generators $\mathcal{E}$ of a homogeneous ELCP is defined as follows. The set of vertices of $\mathcal{G}_c$ is $\{v_1, v_2, \ldots, v_{\#\mathcal{E}}\}$. So we have one vertex v_k for each extreme generator $e_k \in \mathcal{E}$. There is an edge between two different vertices v_k and v_l if the corresponding extreme generators e_k and e_l are cross-complementary. A subset $\mathcal{V}$ of vertices of a graph such that any two vertices of $\mathcal{V}$ are connected by an edge is called a *clique*. A *maximal clique* is a clique that is not a subset of any other clique of the graph. In contrast to what has been suggested in [40], finding all cross-complementary solutions does not amount to detecting all maximal cliques of the cross-complementarity graph $\mathcal{G}_c$, as will be shown by the following trivial example.

Example 3.4.8 Consider the following GLCP:

Find $x \in \mathbb{R}^4$ such that $x_1\, x_2\, x_3\, x_4 = 0$ subject to $x_1 - x_2 = 0$ and $x \geqslant 0$.

Define

$$e_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \; e_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \text{ and } e_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

The set $\{e_1, e_2, e_3\}$ is a minimal complete set of extreme generators of the GLCP. The set $\{e_1, e_2\}$ is a set of cross-complementary solutions, and the same goes for $\{e_2, e_3\}$ and $\{e_3, e_1\}$.
The cross-complementarity graph $\mathcal{G}_c$ that corresponds to the set $\{e_1, e_2, e_3\}$ is represented in Figure 3.3. Clearly, $\{v_1, v_2, v_3\}$ is a clique of this graph. However, the corresponding set of extreme generators $\{e_1, e_2, e_3\}$ is not a cross-complementary set since $e_1 + e_2 + e_3 = [\, 1 \;\; 1 \;\; 1 \;\; 1 \,]^T$ does not satisfy the complementarity condition. $\square$

To find all cross-complementary solutions we have to construct all maximal cross-complementary subsets $\mathcal{E}_s$ of $\mathcal{E}$. For Example 3.4.8 this would yield $\mathcal{E}_1 =$

$\{e_1, e_2\}$, $\mathcal{E}_2 = \{e_2, e_3\}$ and $\mathcal{E}_3 = \{e_3, e_1\}$.
We can save much time if we make some extra provisions, as will be shown by the following propositions.

Proposition 3.4.9 *Let $\mathcal{E}$ be a minimal complete set of extreme generators of a homogeneous ELCP. If $e_1 \in \mathcal{E}$ satisfies $P_1 e_1 = 0$ then e_1 belongs to every maximal cross-complementary subset of $\mathcal{E}$.*

Proof: Suppose that the homogeneous ELCP is defined by (3.19) – (3.22). Assume that $\mathcal{E}_s \subseteq \mathcal{E} \setminus \{e_1\}$ is a cross-complementary set. Now we show that $\mathcal{E}_s \cup \{e_1\}$ is also a cross-complementary set. We have to prove that every nonnegative combination of the elements of $\mathcal{E}_s \cup \{e_1\}$:

$$u = \kappa e_1 + \sum_{e_k \in \mathcal{E}_s} \kappa_k e_k \qquad \text{with } \kappa \geqslant 0 \text{ and } \kappa_k \geqslant 0 \text{ for all } k$$

satisfies the complementarity condition.
Since $\mathcal{E}_s$ is a cross-complementary set, we have

$$\sum_{j=1}^{m} \prod_{i \in \phi_j} \left(P_1 \left(\sum_{e_k \in \mathcal{E}_s} \kappa_k e_k \right) \right)_i = 0 \ . \tag{3.33}$$

We have

$$\begin{aligned}
&\sum_{j=1}^{m} \prod_{i \in \phi_j} (P_1 u)_i \\
&\quad = \sum_{j=1}^{m} \prod_{i \in \phi_j} \left(P_1 \left(\kappa e_1 + \sum_{e_k \in \mathcal{E}_s} \kappa_k e_k \right) \right)_i \\
&\quad = \sum_{j=1}^{m} \prod_{i \in \phi_j} \left(\kappa (P_1 e_1)_i + \left(P_1 \left(\sum_{e_k \in \mathcal{E}_s} \kappa_k e_k \right) \right)_i \right) \\
&\quad = \sum_{j=1}^{m} \prod_{i \in \phi_j} \left(P_1 \left(\sum_{e_k \in \mathcal{E}_s} \kappa_k e_k \right) \right)_i && \text{(since } P_1 e_1 = 0\text{)} \\
&\quad = 0 && \text{(by (3.33))} \ .
\end{aligned}$$

So $\mathcal{E}_s \cup \{e_1\}$ is a cross-complementary set. Hence, e_1 will belong to every maximal cross-complementary subset of $\mathcal{E}$. □

Proposition 3.4.10 *Let $\mathcal{E}$ be a minimal complete set of extreme generators of the homogeneous ELCP defined by (3.19) – (3.22) and let e_1, $e_2 \in \mathcal{E}$. If we have*

$$\forall i \in \{1, 2, \ldots, p_1\} \ : \ (P_1 e_1)_i = 0 \ \text{ if and only if } \ (P_1 e_2)_i = 0 \ , \tag{3.34}$$

then e_1 will belong to a maximal cross-complementary subset of $\mathcal{E}$ if and only if e_2 also belongs to that subset.

Proof: Consider an arbitrary subset $\mathcal{E}_s$ of $\mathcal{E} \setminus \{e_1, e_2\}$. First we prove that if the set $\mathcal{E}_s \cup \{e_1\}$ is cross-complementary then $\mathcal{E}_s \cup \{e_1, e_2\}$ is also cross-complementary.
If the set $\mathcal{E}_s \cup \{e_1\}$ is cross-complementary then every nonnegative combination of its elements satisfies the complementarity condition. This means that

$$u = e_1 + \sum_{e_k \in \mathcal{E}_s} e_k = e_1 + v_s$$

with $v_s = \sum_{e_k \in \mathcal{E}_s} e_k$, satisfies the complementarity condition:

$$\prod_{i \in \phi_j} \left(P_1(e_1 + v_s)\right)_i = 0 \qquad \text{for } j = 1, 2, \ldots, m$$

or

$$\prod_{i \in \phi_j} \left((P_1 e_1)_i + (P_1 v_s)_i\right) = 0 \qquad \text{for } j = 1, 2, \ldots, m$$

and thus

$$\sum_{\psi \in \mathcal{P}(\phi_j)} \prod_{i \in \psi} (P_1 e_1)_i \prod_{i \in \phi_j \setminus \psi} (P_1 v_s)_i = 0 \qquad \text{for } j = 1, 2, \ldots, m \tag{3.35}$$

where $\mathcal{P}(\phi_j)$ is the set of all the subsets of ϕ_j. Since $(P_1 e_1)_i \geqslant 0$ and $(P_1 v_s)_i \geqslant 0$ for all i, (3.35) can only hold if

$$\forall \psi \in \mathcal{P}(\phi_j) \, : \, \prod_{i \in \psi} (P_1 e_1)_i \prod_{i \in \phi_j \setminus \psi} (P_1 v_s)_i = 0 \qquad \text{for } j = 1, 2, \ldots, m$$

or equivalently

$$\begin{aligned} \forall \psi \in \mathcal{P}(\phi_j) : \big(\, \exists i \in \psi \text{ such that } (P_1 e_1)_i = 0 \,\big) &\text{ or} \\ \big(\, \exists i \in \phi_j \setminus \psi \text{ such that } (P_1 v_s)_i = 0 \,\big)& \end{aligned}$$

for $j = 1, 2, \ldots, m$. But if $(P_1 e_1)_i = 0$ then also $(P_1 e_2)_i = 0$ and thus $\left(P_1(e_1 + e_2)\right)_i = 0$. This leads to

$$\begin{aligned} \forall \psi \in \mathcal{P}(\phi_j) : \big(\, \exists i \in \psi \text{ such that } \left(P_1(e_1 + e_2)\right)_i = 0 \,\big) &\text{ or} \\ \big(\, \exists i \in \phi_j \setminus \psi \text{ such that } (P_1 v_s)_i = 0 \,\big)& \end{aligned}$$

for $j = 1, 2, \ldots, m$, and consequently

$$\prod_{i \in \phi_j} \left(P_1(e_1 + e_2 + v_s)\right)_i = 0 \qquad \text{for } j = 1, 2, \ldots, m\,.$$

Hence, the positive combination

$$v = e_1 + e_2 + v_s = e_1 + e_2 + \sum_{e_k \in \mathcal{E}_s} e_k$$

of the elements of $\mathcal{E}_s \cup \{e_1, e_2\}$ satisfies the complementarity condition. From Proposition 3.4.7 it follows that the set $\mathcal{E}_s \cup \{e_1, e_2\}$ is cross-complementary.
To prove the "only if" part we interchange e_1 and e_2 and repeat the above reasoning. $\square$

This leads to the following procedure for determining the maximal cross-complementary subsets of a set of extreme generators $\mathcal{E}$ of a homogeneous ELCP. First we put all the generators $e \in \mathcal{E}$ that satisfy $P_1 e = 0$ in a set $\mathcal{E}_0$. By Proposition 3.4.9 these generators will belong to every maximal cross-complementary subset of $\mathcal{E}$. Next we define an equivalence relation $\sim$ on $\mathcal{E} \setminus \mathcal{E}_0$:

$$e_1 \sim e_2 \quad \text{if} \quad \forall i \in \{1, 2, \ldots, p_1\} : \big((P_1 e_1)_i = 0 \big) \Leftrightarrow \big((P_1 e_2)_i = 0 \big) \ ,$$

and we construct the corresponding equivalence classes. We take one representative out of each equivalence class and we put all the representatives in a set $\mathcal{E}_{\text{red}}$. Suppose that $\mathcal{E}_{\text{red}} = \{e_1, e_2, \ldots, e_r\}$. If we define $s_l = P_1 e_l$ for $l = 1, 2, \ldots, r$ and $\mathcal{S} = \{s_1, s_2, \ldots, s_r\}$, then $s_l \geqslant 0$ for all $s_l \in \mathcal{S}$. For any subset $\{e_1, e_2, \ldots, e_k\}$ of $\mathcal{E}_{\text{red}}$, for any $i \in \{1, 2, \ldots, p_1\}$ and for any μ_1, μ_2, $\ldots$, $\mu_k \geqslant 0$, we have

$$\left(P_1 \left(\sum_{l=1}^{k} \mu_l e_l \right) \right)_i = \sum_{l=1}^{k} \mu_l (P_1 e_l)_i = \sum_{l=1}^{k} \mu_l (s_l)_i \ .$$

Therefore, there is a one-to-one correspondence between the maximal cross-complementary subsets of $\mathcal{E}_{\text{red}}$ and the maximal cross-complementary subsets of $\mathcal{S}$. Note that $\mathcal{S}$ is a minimal complete set of extreme generators of the following GLCP:

> Given ϕ_1, ϕ_2, $\ldots$, ϕ_m, find $s \in \mathbb{R}^{p_1}$ such that $\displaystyle\sum_{j=1}^{m} \prod_{i \in \phi_j} s_i = 0$ subject to $s \geqslant 0$.

Therefore, we now present an algorithm to determine the set Γ of the maximal cross-complementary subsets of a set $\mathcal{S}$ of extreme generators of the solution set of a GLCP. In the description of this algorithm we also use $a(i)$ to represent the ith component of a vector a and $A(i, j)$ to represent the entry on the ith row and the jth column of a matrix A. The operator $\vee$ represents the entrywise binary "or".

Algorithm 3: Determine the maximal cross-complementary subsets of a set of extreme generators of a GLCP.

Input: m, $\mathcal{S}$, $\{\phi_j\}_{j=1}^{m}$

Initialization:

$\Gamma \leftarrow \emptyset$

{ *Construct the binary equivalents:* }

$\mathcal{B} \leftarrow \{\, b_k \mid b_k = \mathrm{binary}(s_k),\ s_k \in \mathcal{S} \,\}$

{ *Construct the cross-complementarity matrix:* }

$cross \leftarrow O_{\#\mathcal{B} \times \#\mathcal{B}}$

for $k = 1, 2, \ldots, \#\mathcal{B} - 1$

  for $l = k+1, k+2, \ldots, \#\mathcal{B}$

    if $(\, b_k \vee b_l \,)$ satisfies the binary complementarity condition

    then

      $cross(k,l) \leftarrow 1$

    else

      $cross(k,l) \leftarrow 0$

    end if

  end for

end for

$depth \leftarrow 1$

$start \leftarrow O_{(\#\mathcal{B}+1) \times 1}$

$last \leftarrow O_{(\#\mathcal{B}+1) \times 1}$

$last(1) \leftarrow \#\mathcal{B}$

$vertices \leftarrow O_{\#\mathcal{B} \times \#\mathcal{B}}$

$\forall k \in \{1, 2, \ldots, \#\mathcal{B}\} :\ vertices(1,k) \leftarrow k$

Main loop:

while $depth > 0$ **do**

  $start(depth) \leftarrow start(depth) + 1$

  $b \leftarrow \bigvee_{d=1}^{depth} b_{vertices(d, start(d))}$

  { *Determine the vertices for the next depth:* }

  $current\_vertex \leftarrow vertices(depth, start(depth))$

  $next\_depth \leftarrow depth + 1$

  $start(next\_depth) \leftarrow 0$

  $last(next\_depth) \leftarrow 0$

  for $k = start(depth) + 1, \ldots, last(depth)$ **do**

    $new\_vertex \leftarrow vertices(depth, k)$

```
        if  cross(current_vertex, new_vertex) = 1  then
            if  ( b ∨ b_new_vertex )  satisfies the binary complementarity
            condition then
                last(next_depth) ← last(next_depth) + 1
                vertices(next_depth, last(next_depth)) ← new_vertex
            end if
        end if
    end for
    { If the next depth does not contain any vertices, then the current }
    { cross-complementary set cannot be extended any more.              }
    if  last(next_depth) > 0  then
        depth ← next_depth
    else
        { If the current set is a maximal cross-complementary set, }
        { we add it to Γ. }
        S^new ← ⋃_{d=1}^{depth} {s_vertices(d,start(d))}
        if  ∀S_s ∈ Γ : S^new ⊈ S_s  then
            Γ ← Γ ∪ {S^new}
        end if
        { Check whether the current set contains all the remaining }
        { vertices, otherwise return to the previous point where a  }
        { choice was made: }
        if  start(1) + depth − 1 = #B  then
            depth ← 0
        else
            while  start(depth) = last(depth)  do
                depth ← depth − 1
            end while
        end if
    end if
end while
Output: Γ = {S_1, S_2, ...}
```

Remarks

1. This algorithm is an adaptation of the enumerative algorithm of [12] to determine a *maximum clique* of a graph, i.e. a clique of maximum cardinality. Note that each maximum clique of a graph is also a maximal

clique, but in general the reverse is not true.

2. In this algorithm we present the information about which pairs of extreme generators are cross-complementary by the cross-complementarity matrix *cross*. We have

$$cross(k,l) = \begin{cases} 1 & \text{if } e_k \text{ and } e_l \text{ are cross-complementary and if } k > l\,, \\ 0 & \text{otherwise.} \end{cases}$$

   Note that when we test whether $cross(current\_vertex, new\_vertex)$ is equal to 1, we always have $new\_vertex > current\_vertex$. Therefore, we have only constructed the strictly upper triangular part of the cross-complementarity matrix.
   In order to make the explanation easier to follow, we shall not use the cross-complementarity matrix *cross* when we describe the main steps of this algorithm in the next remarks and in Example 3.6.2, but we shall explain everything in terms of the cross-complementarity graph $\mathcal{G}_c$ that corresponds to $\mathcal{S}$.

3. We start with a set that contains one vertex of $\mathcal{G}_c$ and we keep adding extra vertices as long as the corresponding set of extreme generators stays cross-complementary. If no vertices can be added without violating the cross-complementarity, we have found a maximal cross-complementary set. Then we go back to the last point were a choice was made and we repeat the procedure.

4. If we encounter a set that is not cross-complementary, then any superset of that set cannot be cross-complementary by Proposition 3.3.3, which is also valid if e_l is a nonnegative combination of extreme generators or equivalently if e_l is a non-central solution of the system of equalities and inequalities of the ELCP. So once we have found a set that is not cross-complementary, we do not have to add extra vertices any more.

5. If the current set cannot be extended any more, we check whether it is a maximal cross-complementary set. Because of the order in which the maximal cross-complementary sets are processed it is impossible that the current set is a superset of a set that is already in Γ. So we only have to test whether the new set is a subset of one of the sets in Γ.

6. As was mentioned in [39] the cross-complementarity test can be done in binary arithmetic only:
   First we replace each generator $s \in \mathcal{S}$ by its binary equivalent $\text{binary}(s)$, which is defined as follows:

$$\text{if } s \in \mathbb{R}^n \text{ and if } b = \text{binary}(s) \in \mathbb{R}^n \text{ then } b_i = \begin{cases} 0 & \text{if } |s_i| \leqslant \tau\,, \\ 1 & \text{if } |s_i| > \tau\,, \end{cases}$$

where $\tau > 0$ is a threshold.
We also adapt the complementarity condition. In the binary complementarity condition we use binary "and" ($\wedge$) instead of multiplication and binary "or" ($\vee$) instead of addition. So if the complementarity condition is $\sum_{j=1}^{m} \prod_{i \in \phi_j} s_i = 0$, then the binary complementarity condition is $\bigvee_{j=1}^{m} \bigwedge_{i \in \phi_j} (b_i = 0)$.
We have already included this technique in our algorithm since it will be much faster than doing everything in floating point arithmetic. Note that we can also use this technique in Algorithms 1 and 2.
To determine whether two (or more) solutions are cross-complementary we first construct a new vector by taking the componentwise binary "or" of the binary equivalents of the solutions and then we test whether this vector satisfies the binary complementarity condition.

7. If we are only interested in obtaining one solution of an ELCP, we can skip Algorithm 3. However, this is certainly not the most efficient way to get one solution of an ELCP (See also Section 3.4.5).

For additional information about this algorithm the interested reader is referred to [12].

Once we have found the maximal cross-complementary sets of extreme generators of the GLCP, we reconstruct the corresponding maximal cross-complementary subsets of $\mathcal{E}$ by replacing each s_k in each subset $\mathcal{S}_s$ by the corresponding e_k and all the other members of the equivalence class of e_k. If we also add all the elements of $\mathcal{E}_0$ to each subset, we finally get $\tilde{\Gamma}$, the set of maximal cross-complementary subsets $\mathcal{E}_s$ of extreme generators of the homogeneous ELCP. Now we can characterize the solution set of the homogeneous ELCP:

Theorem 3.4.11 *Let $\mathcal{C}$ be a minimal complete set of central generators of a homogeneous ELCP, let $\mathcal{E}$ be a minimal complete set of extreme generators of the homogeneous ELCP and let $\tilde{\Gamma}$ be the set of the maximal cross-complementary subsets of $\mathcal{E}$. Then u is a solution of the homogeneous ELCP if and only if there exists a set $\mathcal{E}_s \in \tilde{\Gamma}$ such that*

$$u = \sum_{c_k \in \mathcal{C}} \lambda_k c_k + \sum_{e_k \in \mathcal{E}_s} \kappa_k e_k \qquad \textit{with } \lambda_k \in \mathbb{R} \textit{ and } \kappa_k \geqslant 0 \textit{ for all } k\,.$$

This leads to:

Proposition 3.4.12 *In general the solution set of a homogeneous ELCP consists of the union of faces of a polyhedral cone.*

Remark 3.4.13 The main difference between the ELCP and the GLCP is that the solution set of a homogeneous ELCP consists of the union of faces

of a polyhedral cone — which means that it can contain a linear subspace — whereas the solution set of a GLCP is the union of faces of a *pointed* polyhedral cone, which means that it cannot contain a linear subspace. Hence, there never are (non-trivial) central solutions in the solution set of a GLCP.
The algorithm of [39, 40, 141] to compute the solution set of a GLCP starts with

$$\mathcal{C} \leftarrow \emptyset$$
$$\mathcal{E} \leftarrow \{\, e_i \,|\, e_i = (I_n)_{.,i} \text{ for } i = 1, 2, \ldots, n \,\}$$
$$P_{\text{nec}} \leftarrow I_n$$

and directly goes to Algorithm 2 and skips all the steps that deal with central generators. ◇

3.4.4 Solutions of the Original General ELCP

Now we explain how the solutions of the original general ELCP defined by (3.4) – (3.6) can be retrieved from the solutions of the corresponding homogeneous ELCP.
Any solution u of the homogeneous ELCP has the following form: $u = \begin{bmatrix} x_u \\ \alpha_u \end{bmatrix}$ with $\alpha_u \geqslant 0$. First we normalize all the central and the extreme generators that have a non-zero α component:

- If c is a central generator then we automatically have $\alpha_c = 0$.
- For an extreme generator e there are two possibilities: either $\alpha_e = 0$ or $\alpha_e > 0$. If $\alpha_e = 0$, we leave e as it is. If $\alpha_e > 0$, we divide each component of e by α_e such that the α component of e becomes 1. By Proposition 3.3.2 the new e will still be a solution of the homogeneous ELCP.
  This results in two groups of extreme generators: $\mathcal{E}^{\text{ext}} = \{\, e \in \mathcal{E} \,|\, \alpha_e = 0 \,\}$ and $\mathcal{E}^{\text{fin}} = \{\, e \in \mathcal{E} \,|\, \alpha_e = 1 \,\}$. Later on we shall see that the elements of $\mathcal{E}^{\text{ext}}$ correspond to extreme generators of the solution set of the original ELCP, whereas the elements of $\mathcal{E}^{\text{fin}}$ correspond to finite solutions of the original ELCP.

Let $\mathcal{X}^{\text{cen}} = \{\, c_u \,|\, c \in \mathcal{C} \,\}$, $\mathcal{X}^{\text{ext}} = \{\, e_u \,|\, e \in \mathcal{E}^{\text{ext}} \,\}$ and $\mathcal{X}^{\text{fin}} = \{\, e_u \,|\, e \in \mathcal{E}^{\text{fin}} \,\}$. For each $\mathcal{E}_s \in \tilde{\Gamma}$ we construct the corresponding sets $\mathcal{X}_s^{\text{ext}} \subseteq \mathcal{X}^{\text{ext}}$ and $\mathcal{X}_s^{\text{fin}} \subseteq \mathcal{X}^{\text{fin}}$. All the ordered pairs $\left(\mathcal{X}_s^{\text{ext}}, \mathcal{X}_s^{\text{fin}}\right)$ for which $\mathcal{X}_s^{\text{fin}}$ is not empty, are put in a set Λ. Finally, we remove all the generators $x_k^{\text{e}} \in \mathcal{X}^{\text{ext}}$ that do not appear in one of the ordered pairs $\left(\mathcal{X}_s^{\text{ext}}, \mathcal{X}_s^{\text{fin}}\right) \in \Lambda$.
Let $\mathcal{P}$ be the polyhedron defined by the system of linear equalities and inequalities of the original ELCP. There exists a pointed polyhedron $\mathcal{P}_{\text{red}}$ with $\mathcal{P} = \mathcal{P}_{\text{red}} + \mathcal{L}(\mathcal{P})$ such that the elements of $\mathcal{X}^{\text{ext}}$ are extreme generators of $\mathcal{P}_{\text{red}}$ and such that the elements of $\mathcal{X}^{\text{fin}}$ are finite points of $\mathcal{P}_{\text{red}}$. Now we can give the following geometrical interpretation to the sets $\mathcal{X}^{\text{cen}}$, $\mathcal{X}^{\text{ext}}$, $\mathcal{X}^{\text{fin}}$ and Λ:

- $\mathcal{X}^{\text{cen}}$ is a basis for the lineality space of $\mathcal{P}$. We call $\mathcal{X}^{\text{cen}}$ a minimal complete set of *central generators* of (the solution set of) the ELCP.
- $\mathcal{X}^{\text{ext}}$ is a set of extreme generators of $\mathcal{P}_{\text{red}}$ that satisfy the complementarity condition. We say that $\mathcal{X}^{\text{ext}}$ is a minimal complete set of *extreme generators* of (the solution set of) the ELCP.
- $\mathcal{X}^{\text{fin}}$ is the set of the finite vertices of $\mathcal{P}_{\text{red}}$ that satisfy the complementarity condition. We call $\mathcal{X}^{\text{fin}}$ a minimal complete set of *finite points* of (the solution set of) the ELCP.
- Λ is the set of *ordered pairs of maximal cross-complementary subsets* of $\mathcal{X}^{\text{ext}}$ and $\mathcal{X}^{\text{fin}}$. Each ordered pair $\left(\mathcal{X}_s^{\text{ext}}, \mathcal{X}_s^{\text{fin}}\right) \in \Lambda$ determines a face $\mathcal{F}_s$ of $\mathcal{P}_{\text{red}}$ that belongs to the solution set of the ELCP: $\mathcal{X}_s^{\text{ext}}$ is a minimal complete set of extreme generators of $\mathcal{F}_s$ and $\mathcal{X}_s^{\text{fin}}$ is the set of the finite vertices of $\mathcal{F}_s$.

For the ELCP of Example 3.2.1 we have $\mathcal{X}^{\text{cen}} = \emptyset$, $\mathcal{X}^{\text{ext}} = \{x_1^{\text{e}}, x_2^{\text{e}}\}$, $\mathcal{X}^{\text{fin}} = \{x_1^{\text{f}}, x_2^{\text{f}}\}$ and $\Lambda = \Big\{ \left(\emptyset, \{x_1^{\text{f}}\}\right),\ \left(\{x_1^{\text{e}}, x_2^{\text{e}}\}, \{x_2^{\text{f}}\}\right) \Big\}$.

Theorem 3.4.14 *Let $\mathcal{X}^{\text{cen}}$ be a minimal complete set of central generators of a general ELCP, let $\mathcal{X}^{\text{ext}}$ be a minimal complete set of extreme generators of the ELCP, let $\mathcal{X}^{\text{fin}}$ be a minimal complete set of finite points of the ELCP and let Λ be the set of ordered pairs of maximal cross-complementary subsets of $\mathcal{X}^{\text{ext}}$ and $\mathcal{X}^{\text{fin}}$. Then x is a solution of the ELCP if and only if there exists an ordered pair $\left(\mathcal{X}_s^{\text{ext}}, \mathcal{X}_s^{\text{fin}}\right) \in \Lambda$ such that*

$$x = \sum_{x_k^{\text{c}} \in \mathcal{X}^{\text{cen}}} \lambda_k x_k^{\text{c}} + \sum_{x_k^{\text{e}} \in \mathcal{X}_s^{\text{ext}}} \kappa_k x_k^{\text{e}} + \sum_{x_k^{\text{f}} \in \mathcal{X}_s^{\text{fin}}} \mu_k x_k^{\text{f}} \tag{3.36}$$

with $\lambda_k \in \mathbb{R}$, $\kappa_k \geqslant 0$, $\mu_k \geqslant 0$ for all k and $\displaystyle\sum_k \mu_k = 1$.

Proof: Suppose that the ELCP is defined by (3.4) – (3.6) and that $\mathcal{X}^{\text{cen}} = \{x_1^{\text{c}}, x_2^{\text{c}}, \ldots, x_r^{\text{c}}\}$, $\mathcal{X}^{\text{ext}} = \{x_1^{\text{e}}, x_2^{\text{e}}, \ldots, x_t^{\text{e}}\}$, $\mathcal{X}^{\text{fin}} = \{x_1^{\text{f}}, x_2^{\text{f}}, \ldots, x_v^{\text{f}}\}$ and $\Lambda = \big\{ (\mathcal{X}_1^{\text{ext}}, \mathcal{X}_1^{\text{fin}}), (\mathcal{X}_2^{\text{ext}}, \mathcal{X}_2^{\text{fin}}), \ldots, (\mathcal{X}_w^{\text{ext}}, \mathcal{X}_w^{\text{fin}}) \big\}$. Define

$$
\begin{aligned}
c_k &= \begin{bmatrix} x_k^{\text{c}} \\ 0 \end{bmatrix} && \text{for } k = 1, 2, \ldots, r\,, \\
e_k^{\text{e}} &= \begin{bmatrix} x_k^{\text{e}} \\ 0 \end{bmatrix} && \text{for } k = 1, 2, \ldots, t\,, \\
e_k^{\text{f}} &= \begin{bmatrix} x_k^{\text{f}} \\ 1 \end{bmatrix} && \text{for } k = 1, 2, \ldots, v\,, \\
\mathcal{E}_s^{\text{ext}} &= \{\, e_k^{\text{e}} \mid x_k^{\text{e}} \in \mathcal{X}_s^{\text{ext}} \,\} && \text{for } s = 1, 2, \ldots, w\,,
\end{aligned}
$$

$$\mathcal{E}_s^{\text{fin}} = \{\, e_k^{\text{f}} \mid x_k^{\text{f}} \in \mathcal{X}_s^{\text{fin}} \,\} \qquad \text{for } s = 1, 2, \ldots, w \,.$$

If we set $\mathcal{C} = \{c_1, c_2, \ldots, c_t\}$, $\mathcal{E}^{\text{ext}} = \{e_1^{\text{e}}, e_2^{\text{e}}, \ldots, e_t^{\text{e}}\}$, $\mathcal{E}^{\text{fin}} = \{e_1^{\text{f}}, e_2^{\text{f}}, \ldots, e_v^{\text{f}}\}$, and $\tilde{\Gamma} = \{\mathcal{E}_1^{\text{ext}} \cup \mathcal{E}_1^{\text{fin}}, \mathcal{E}_2^{\text{ext}} \cup \mathcal{E}_2^{\text{fin}}, \ldots, \mathcal{E}_w^{\text{ext}} \cup \mathcal{E}_w^{\text{fin}}\}$, then $\mathcal{C}$ is a minimal complete set of central generators of the homogeneous ELCP $\mathcal{H}$ that corresponds to the original general ELCP, $\mathcal{E}^{\text{ext}}$ is a set of extreme generators of $\mathcal{H}$ that have an α component that is equal to 0, $\mathcal{E}^{\text{fin}}$ is a minimal complete set of extreme generators of $\mathcal{H}$ that have an α component that is equal to 1, and $\tilde{\Gamma}$ is the set of maximal cross-complementary subsets of $\mathcal{E}^{\text{ext}} \cup \mathcal{E}^{\text{fin}}$ such that in each set $\mathcal{E}_s^{\text{ext}} \cup \mathcal{E}_s^{\text{fin}} \in \tilde{\Gamma}$ there is at least one generator with a non-zero α component. Note that $\mathcal{E}^{\text{ext}} \cup \mathcal{E}^{\text{fin}}$ is not necessarily a *complete* set of extreme generators of $\mathcal{H}$ since in the construction of $\mathcal{X}^{\text{ext}}$ we have removed all the extreme generators that were derived from extreme generators of $\mathcal{H}$ that had an α component that was equal to 0, but that did not appear in one of the pairs $(\mathcal{X}_s^{\text{ext}}, \mathcal{X}_s^{\text{fin}}) \in \Lambda$.

First we prove the "if" part of the theorem.
Let $\tau = \{\, k \mid x_k^{\text{e}} \in \mathcal{X}_s^{\text{ext}} \,\}$ and let $\varphi = \{\, k \mid x_k^{\text{f}} \in \mathcal{X}_s^{\text{fin}} \,\}$. Now we have to prove that any combination of the form

$$x \;=\; \sum_{k=1}^{r} \lambda_k x_k^{\text{c}} + \sum_{k \in \tau} \kappa_k x_k^{\text{e}} + \sum_{k \in \varphi} \mu_k x_k^{\text{f}}$$

with $\lambda_k \in \mathbb{R}$, $\kappa_k \geqslant 0$, $\mu_k \geqslant 0$ for all k and $\sum\limits_{k \in \psi} \mu_k = 1$ is a solution of the original ELCP.
If we define

$$u \;=\; \sum_{k=1}^{r} \lambda_k c_k + \sum_{k \in \tau} \kappa_k e_k^{\text{e}} + \sum_{k \in \varphi} \mu_k e_k^{\text{f}} \;\;,$$

then u is a solution of $\mathcal{H}$. Moreover, if $u = \begin{bmatrix} x_u \\ \alpha_u \end{bmatrix}$, then $x_u = x$ and

$$\alpha_u \;=\; \sum_{k=1}^{r} \lambda_k \cdot 0 + \sum_{k \in \tau} \kappa_k \cdot 0 + \sum_{k \in \varphi} \mu_k \cdot 1 \;=\; \sum_{k \in \varphi} \mu_k \;=\; 1 \;\;.$$

Since $\mathcal{H}$ is the homogeneous ELCP that corresponds to the original ELCP and since $x_u = x$ and $\alpha_u = 1$, this implies that

$$\sum_{j=1}^{m} \prod_{i \in \phi_j} (Ax - c)_i \;=\; 0 \,,\; Ax - c \geqslant 0 \;\text{ and }\; Bx - d = 0 \;\;.$$

Hence, x is a solution of the original ELCP.

Now we prove the "only if" part of the theorem.
We have to show that any solution of the general ELCP can be written as a

combination of the form (3.36) for some ordered pair $\left(\mathcal{X}_s^{\text{ext}}, \mathcal{X}_s^{\text{fin}}\right) \in \Lambda$ with $\lambda_k \in \mathbb{R}$, $\kappa_k \geqslant 0$, $\mu_k \geqslant 0$ for all k and $\sum_k \mu_k = 1$. Consider an arbitrary solution x of the general ELCP and construct $u = \begin{bmatrix} x \\ 1 \end{bmatrix}$. Since x is a solution of the original general ELCP, u is a solution of $\mathcal{H}$. Since u has a non-zero α component, there exists an index $s \in \{1, 2, \ldots, w\}$ such that

$$u = \sum_{k=1}^{r} \lambda_k c_k + \left(\sum_{k \in \chi} \kappa_k e_k^{\text{e}} + \sum_{k \in \psi} \mu_k e_k^{\text{f}} \right)$$

with $\lambda_k \in \mathbb{R}$, $\kappa_k \geqslant 0$, $\mu_k \geqslant 0$ for all k and where $\chi = \{\, k \mid e_k^{\text{e}} \in \mathcal{E}_s^{\text{ext}} \,\}$ and $\psi = \{\, k \mid e_k^{\text{f}} \in \mathcal{E}_s^{\text{fin}} \,\}$. Hence,

$$x = \sum_{k=1}^{r} \lambda_k x_k^{\text{c}} + \sum_{k \in \chi} \kappa_k x_k^{\text{e}} + \sum_{k \in \psi} \mu_k x_k^{\text{f}} \ .$$

Furthermore, since

$$\alpha_u = 1 = \sum_{k=1}^{r} \lambda_k \cdot 0 + \sum_{k \in \chi} \kappa_k \cdot 0 + \sum_{k \in \psi} \mu_k \cdot 1 \ ,$$

we have $\sum_{k \in \psi} \mu_k = 1$. □

The following proposition is a direct consequence of Theorem 3.4.14 and of the way in which the sets $\mathcal{X}^{\text{cen}}$, $\mathcal{X}^{\text{ext}}$, $\mathcal{X}^{\text{fin}}$ and Λ have been constructed.

Proposition 3.4.15 *In general the solution set of an ELCP consists of the union of faces of a polyhedron.*

We can also reverse this proposition:

Theorem 3.4.16 *The union of any arbitrary set $\mathcal{F}$ of faces of an arbitrary polyhedron $\mathcal{P}$ can be described by an ELCP.*

Proof: First we assume that $\mathcal{P}$ is nonempty. As a consequence, $\mathcal{F}$ is also nonempty. Let $\mathcal{P}$ be defined by $\mathcal{P} = \{\, x \mid Ax \geqslant c \,\}$ with $A \in \mathbb{R}^{p \times n}$ and $c \in \mathbb{R}^p$, and let F be the union of the faces in $\mathcal{F}$: $F = \bigcup_{F_i \in \mathcal{F}} F_i$. Define $m = \#\mathcal{F}$.
Consider an arbitrary face $F_i \in \mathcal{F}$. If k_i is the dimension of F_i, then F_i is the intersection of $\mathcal{P}$ and $n - k_i$ linearly independent hyperplanes from the system $Ax = c$. Let ϕ_i be the set of the indices that correspond to these hyperplanes. Hence, $F_i = \{\, x \mid Ax \geqslant c \text{ and } (Ax - c)_j = 0 \text{ for all } j \in \phi_i \,\}$. Since $Ax - c \geqslant 0$,

this can be rewritten as $F_i = \left\{ x \,\middle|\, Ax \geqslant c \text{ and } \sum_{j\in\phi_i} (Ax-c)_j = 0 \right\}$.

If we repeat this reasoning for each face $F_i \in \mathcal{F}$, then we find that F coincides with the solution set of the following ELCP:

Given A, c and $\phi_1, \phi_2, \ldots, \phi_m$, find $x \in \mathbb{R}^n$ such that

$$\prod_{i=1}^{m} \left(\sum_{j\in\phi_i} (Ax-c)_j \right) = 0 \tag{3.37}$$

subject to $Ax \geqslant c$.

Equation (3.37) can always be rewritten as a sum of products. So it really represents a complementarity condition.

If $\mathcal{P}$ and thus also $\mathcal{F}$ are empty, we can take an infeasible system of linear inequalities for $Ax \geqslant c$ and $(Ax-c)_1 = 0$ as complementarity condition. □

One of the most important common characteristics of the LCP and the various generalizations discussed in Section 3.2 is that in general the solution set of all these problems is the union of faces of a polyhedron. Proposition 3.4.15 and Theorem 3.4.16 state that with every union of faces of a polyhedron there corresponds an ELCP and vice versa. Therefore, we claim that ELCP can be considered as the most general linear extension of the LCP.

Remark 3.4.17 For a homogeneous ELCP that corresponds to a general ELCP we can already take the constraint $\alpha \geqslant 0$ into account at the beginning of the ELCP algorithm by setting

$$\begin{aligned}
&\mathcal{C} \leftarrow \{\, c_i \mid c_i = (I_n)_{.,i} \text{ for } i = 1, 2, \ldots, n-1 \,\} \\
&\mathcal{E} \leftarrow \{\, e_1 \mid e_1 = (I_n)_{.,n} \,\} \\
&P_{\text{nec}} \leftarrow (I_n)_{1,.}
\end{aligned}$$

in the initialization step of Algorithm 1.

Moreover, if we determine the maximal cross-complementary subsets of the set $\mathcal{E}$ of extreme generators of the homogeneous ELCP that corresponds to the original general ELCP, we are only interested in maximal cross-complementary subsets $\mathcal{E}_s$ of $\mathcal{E}$ that contain at least one generator with a non-zero α component since then $\mathcal{E}_s^{\text{fin}}$ will be nonempty. Therefore, it is advantageous to order the extreme generators of the homogeneous ELCP such that all the generators with a non-zero α component have a lower index than the generators with a zero α component. In that case we do not have to determine all the sets of cross-complementary extreme generators, but we can stop Algorithm 3 as soon as we have considered all the sets that contain at least one generator with a non-zero α component. ◇

3.4.5 The Performance of the ELCP Algorithm

In each pass through the main loops of Algorithms 1 and 2 we have to make combinations of intermediate generators. This means that the execution time of these algorithms strongly depends on the number of new generators that are created in each pass. Experiments show that the execution time and the required amount of storage space grow rapidly as the number of variables and (in)equalities grows. However, the execution time and the storage space requirements of the ELCP algorithm do not only depend on the number of variables and (in)equalities but also on the structure of the solution set and the order in which the (in)equalities are processed.

In [105] Mattheiss and Rubin have given a survey and a comparison of methods for finding all vertices of polytopes or polyhedra with an empty lineality space. The worst case behavior of Algorithms 1 and 2 can be compared with these algorithms if we would take $\prod_{i=1}^{p} (Ax - c)_i = 0$ as complementarity condition since this means that at least one inequality should hold with equality or that every point that lies on the border of the polyhedron defined by $Ax \geqslant c$ is a solution of the ELCP. Note that we may assume that there are no central generators since we can first determine a minimal complete set of central generators by solving the system of homogeneous linear equations $Ax = 0$ and $Bx = 0$, and then remove the central generators from the solution set of the ELCP by imposing the condition that the other solutions have to be orthogonal to the central generators. Mattheiss and Rubin report execution times of the order $O(v^\rho)$ with $\rho = 1.418$ and v the number of vertices of the polyhedron for the Chernikova algorithm, which is a special case of the double description method: the Chernikova algorithm requires the additional constraint $x \geqslant 0$ (Note that this implies that the polyhedron is pointed; so there are no central generators). By the upper bound conjecture [105] we have the following least upper bound for the number of vertices of a polytope defined by p (irredundant) inequality constraints in an n-dimensional space:

$$\binom{p - \left\lfloor \frac{n+1}{2} \right\rfloor}{p-n} + \binom{p - \left\lfloor \frac{n+2}{2} \right\rfloor}{p-n} .$$

This means that in the worst case the number of vertices v can be $O\left(p^{\lfloor \frac{n}{2} \rfloor}\right)$ if $p \gg n \gg 1$.

Fortunately, we can already use Proposition 3.3.3 to reject extreme generators that do not satisfy the (partial) complementarity condition during the iteration process. This means that on average the execution times of our algorithm will be considerably less than the ones reported in [105].

The execution time of Algorithm 3 depends strongly on the structure of the solution set and on the number of extreme generators of the GLCP.

Since the execution time of our ELCP algorithm depends on so many factors, it is difficult to give a neat characterization of the computational complexity as a function of the number of variables and (in)equalities. In the next subsection we present the results of some representative experiments that confirm what we have said above.

3.4.6 Some Computational Results

For the first three series of experiments discussed in this subsection we have used the following procedure to construct random ELCPs that had at least one solution. Suppose that we have to construct an ELCP with n variables, p inequalities and at most m terms in the complementarity condition. First we determine a random matrix $A \in \mathbb{R}^{p \times n}$ and a random vector $u \in \mathbb{R}^n$ with entries that are uniformly distributed in the interval $(-1, 1)$, a random vector $\delta \in \mathbb{R}^p$ with entries that are uniformly distributed in the interval $(0, 1)$, and m random nonempty subsets $\phi_1, \phi_2, \ldots, \phi_m$ of $\{1, 2, \ldots, p\}$ (It is possible that $\phi_i \subseteq \phi_j$ for some $i, j \in \{1, 2, \ldots, m\}$ with $i \neq j$). Next we define $\Phi = \bigcup_{j=1}^{m} \phi_j$ and a vector $c \in \mathbb{R}^p$ such that

$$c_i = \begin{cases} (Au)_i & \text{if } i \in \Phi\,, \\ (Au)_i - \delta & \text{otherwise.} \end{cases}$$

This results in the following ELCP:

Given A, c and $\phi_1, \phi_2, \ldots, \phi_m$, find $x \in \mathbb{R}^n$ such that

$$\sum_{j=1}^{m} \prod_{i \in \phi_j} (Ax - c)_i = 0$$

subject to $Ax \geqslant c$.

The vector u will be a solution of this ELCP.

In all the experiments we have taken Remark 3.4.17 into account. The experiments have been performed on an HP 9000 Model 712/80 workstation with 64 MB internal memory and with the algorithms implemented in C and called from MATLAB (using the MEX-file facility). In order to augment the legibility of the plots that show the results of the first three series of experiments and to emphasize the evolution of the average CPU time in function of the various parameters, we have connected adjacent measurement points by straight lines instead of plotting the individual measurement points.

In the first experiment we have constructed 3000 random ELCPs using the method described above with a fixed number of variables $n = 5$ and at most $m = 4$ terms in the complementarity condition. For each ELCP the number of inequalities p was chosen randomly from the set $\{1, 2, \ldots, 20\}$. In Figure 3.4 we have plotted the average CPU time used by our ELCP algorithm to determine the central and extreme generators of the random ELCPs as a function

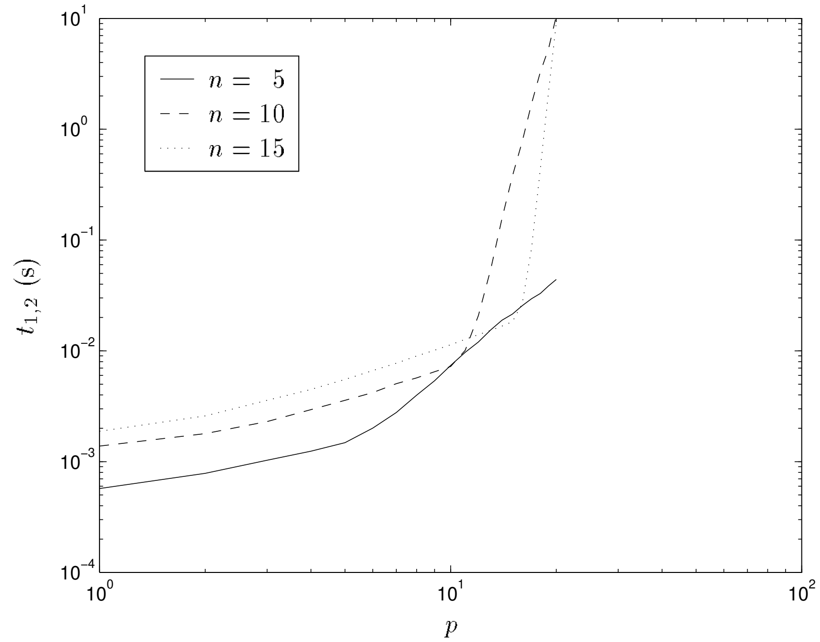


Figure 3.4: The average CPU time $t_{1,2}$ (in seconds) used by Parts 1 and 2 of the ELCP algorithm as a function of the total number of inequalities p for random ELCPs with $n = 5$, 10 or 15 variables.

of the number of inequalities p. Next we have repeated this experiment with $n = 10$ and $n = 15$. The results of these experiments have also been plotted in Figure 3.4.

Note that each curve consists of two pieces with a different slope and that the change of slope occurs in the neighborhood of the point where the number of inequalities is equal to the number of variables. This can be explained as follows. In the first passes through the main loop of the ELCP algorithm the number of inequalities that have already been processed is less than the number of variables, which implies that there are still central generators. Therefore, we are most of the time in Case 3 of the algorithm, where we have to combine *one* central generator with all the other central and extreme generators that have a non-zero residue. However, if the number of inequalities that have been processed exceeds the number of variables there are in general no more central generators left, which means that we are in Case 2 of the algorithm. In this case we have to combine *all the pairs* of extreme generators with positive and negative residues, which takes more time. Moreover, in Case 3 each central or extreme generator can generate at most one new central or extreme generator, whereas in Case 2 an extreme generator can generate more than one new generator, which results in a cumulative effect since this means that in the next

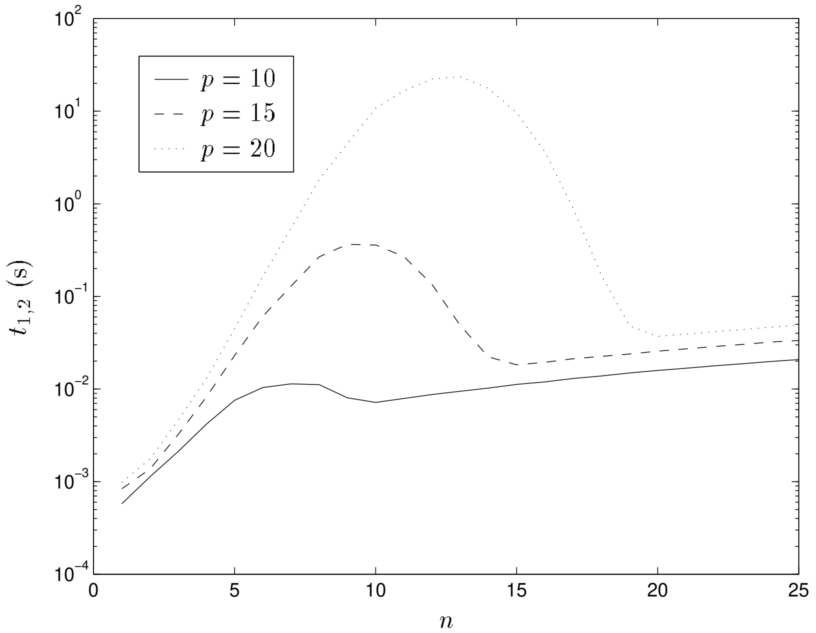


Figure 3.5: The average CPU time $t_{1,2}$ (in seconds) used by Parts 1 and 2 of the ELCP algorithm as a function of the total number of variables n for random ELCPs with $p = 10$, 15 or 20 inequalities.

pass — in which we will probably also be in Case 2 — the number of possible combinations will in general be much larger than in the current pass.

In the second series of experiments we have first constructed 3000 random ELCPs using the method described above but now with a fixed number of inequalities $p = 10$ and at most $m = 4$ terms in the complementarity condition. The number of variables n was chosen randomly from the set $\{1, 2, \ldots, 25\}$. This experiment was repeated for $p = 15$ and $p = 20$. The results of these experiments have been plotted in Figure 3.5. Note that there also is a change of slope in this case.

The first and the second series of experiments show that if the number of inequalities p is (much) smaller than the number of variables n or if p is (much) larger than n, the average execution time of Parts 1 and 2 of our ELCP algorithm for random ELCPs depends polynomially on p (for a fixed n) and more or less exponentially on n (for a fixed p).

We have also examined the performance of Algorithm 3. Recall that in order to determine the maximal sets of cross-complementary extreme generators of a homogeneous ELCP we transform the extreme generators of the homogeneous ELCP into extreme generators of a GLCP by multiplying them with the matrix P_1 (cf. Section 3.4.3). As a consequence, the number of variables n of the ELCP will not influence the execution time of Algorithm 3. Therefore, we have

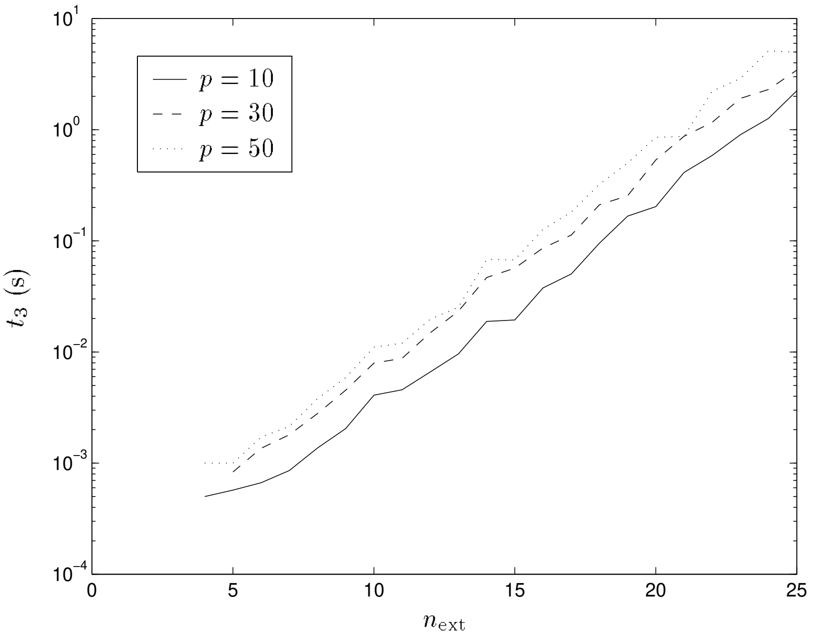


Figure 3.6: The average CPU time t_3 (in seconds) used by Algorithm 3 as a function of the number n_{ext} of extreme generators of the corresponding GLCP for random ELCPs with $p = 10$, 30 or 50 inequalities.

taken a fixed number of variables $n = 5$ for all the random ELCPs of this series of experiments. We have only determined the maximal cross-complementary sets of extreme generators of the homogeneous ELCP that contained at least one extreme generator with a non-zero α component (cf. Remark 3.4.17). In order to limit the time needed to perform the experiments we have only determined the cross-complementary sets if the number of extreme generators of the corresponding GLCP did not exceed 25. For all the ELCPs there were at most $m = 4$ terms in the complementarity condition. For each value of $p \in \{10, 30, 50\}$ we have continued constructing random ELCPs until we had obtained 3000 ELCPs for which the corresponding homogeneous ELCP had at least 1 extreme generator with a non-zero α component and for which the corresponding GLCP had at most 25 extreme generators. In Figure 3.6 we have plotted the average CPU time used by Algorithm 3 as a function of the number of extreme generator of the GLCP. Clearly, the execution time of the algorithm to determine the maximal cross-complementary sets of extreme generators of a GLCP depends more or less exponentially on the number of extreme generators. The experiments show that the influence of the number of inequalities on the execution time of Algorithm 3 is not very strong.

The order in which the inequalities are processed is also important. We shall illustrate this for the ELCP of Example 7.5.5 (with $\xi = 1000$). First

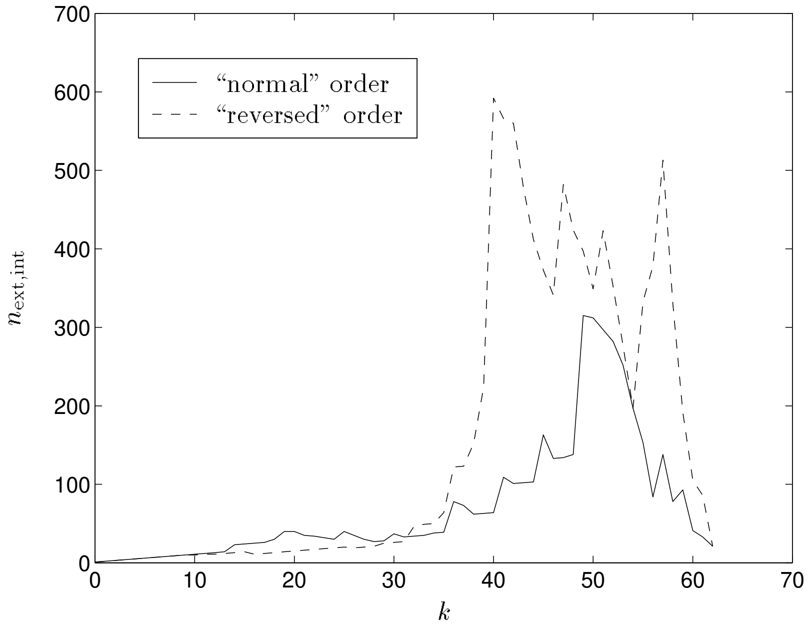


Figure 3.7: The number of intermediate extreme generators $n_{\mathrm{ext,int}}$ as a function of the pass k for Parts 1 and 2 of our ELCP algorithm applied to the ELCP of Example 7.5.5 with the inequalities processed in "normal" and in "reversed" order.

we have used Parts 1 and 2 of our ELCP algorithm to determine the central and extreme generators of the corresponding homogeneous ELCP. Next we have applied the ELCP algorithm again but in both Part 1 and Part 2 of the algorithm we have processed the inequalities in the reversed order, i.e. starting with the last inequality of respectively $P_1 u \geqslant 0$ and $P_2 u \geqslant 0$ instead of with the first inequality. In Figure 3.7 we have plotted the number of intermediate extreme generators $n_{\mathrm{ext,int}}$ as a function of the pass k. In order to augment the legibility of the plot we have also connected adjacent measurement points by straight lines in this plot instead of plotting the sequence of discrete points. For both the "normal" and the "reversed" order we have processed the inequality $\alpha \geqslant 0$ first (cf. Remark 3.4.17). The average number of intermediate extreme generators for the "normal" order is about 70.5 compared to 149.7 for the "reversed" order. As a consequence, the CPU time needed to process the inequalities in the "reversed" order is much larger than the CPU time needed for the "normal" order: the average execution times over 10 runs of the algorithm on an HP 9000 Model 712/80 workstation are respectively 13.60 s and 1.727 s (with standard deviations 0.18 s and 0.013 s).
Note that we have not plotted the number of intermediate central generators since this number can only decrease during the execution of the algorithm

and since the number of intermediate central generators hardly influences the execution time of our ELCP algorithm.
This experiment shows that the order in which the inequalities and equalities are processed can strongly influence the execution time of the ELCP algorithm and the number of intermediate extreme generators (and thus also the required amount of storage space). It is still an open question how the optimal order can be determined.

Remark 3.4.18 For the ELCPs that correspond to the max-algebraic problems that will be treated in the next chapters, we have noticed the same dependence of the execution time of our ELCP algorithm on the number of variables, (in)equalities and extreme generators as for random ELCPs. However, in general the structure of the solution set of these problems is more regular than that of random ELCPs. This also means that for most of these problems it is possible to determine an order in which the (in)equalities should be processed that is more or less optimal for all instances of the given problem. As a consequence, our ELCP algorithm on average performs much better for ELCPs derived from these max-algebraic problems than for random ELCPs. $\diamond$

3.4.7 Some Alternative Solution Methods

From the above we can conclude that our ELCP algorithm is not well suited for large ELCPs with a large number of variables and (in)equalities or a complex solution set. For such kind of systems one could try to develop algorithms that only search one solution, since in many cases we do not need all solutions. Now we briefly discuss some possible approaches to find one solution of the ELCP defined by (3.4) – (3.6):

- Global minimization [103]:
  The ELCP can be reformulated as a constrained optimization problem:

  Given A, B, c, d and $\phi_1, \phi_2, \ldots, \phi_m$, find a vector x that minimizes

  $$\sum_{j=1}^{m} \prod_{i \in \phi_j} (Ax - c)_i \tag{3.38}$$

  subject to $Ax \geqslant c$ and $Bx = d$.

  So we have to minimize the left-hand side of the complementarity condition over the equality and inequality constraints. Since all the factors $(Ax-c)_i$ in (3.38) are nonnegative, the object function is always nonnegative. The value of the object function will be equal to 0 in a solution of the ELCP.
  However, in practice there appear to be many local minima that are not a global minimum. In general the solutions of the ELCP always lie on

the border of the polyhedron defined by $Ax \geqslant c$ and $Bx = d$. This causes extra problems in connection with convergence to a global minimum. The interested reader is referred to e.g. [63, 129] for methods and algorithms for constrained optimization.

- Systems of polynomial equations:
By introducing dummy variables the ELCP can be transformed into a system of multivariate polynomial equations. Consider an arbitrary index $i \in \{1, 2, \ldots, p\}$. If we introduce a dummy variable s_i then the ith inequality of the system $Ax \geqslant c$ can be transformed into an equality: $A_{i,.}x - s_i^2 = c_i$. Note that $s_i = 0$ if and only if $A_{i,.}x = c_i$. If we repeat this reasoning for each inequality, then we find that the complementarity condition results in
$$\sum_{j=1}^{m} \prod_{i \in \phi_j} s_i = 0 \ .$$
So the ELCP can be reformulated as the following unconstrained system of polynomial equations:

    Given A, B, c, d and ϕ_1, ϕ_2, ..., ϕ_m, find $x \in \mathbb{R}^n$ and $s \in \mathbb{R}^p$ such that
$$\begin{aligned} \sum_{j=1}^{m} \prod_{i \in \phi_j} s_i &= 0 \\ A_{i,.}x - s_i^2 &= c_i \qquad \text{for } i = 1, 2, \ldots, p \,, \\ Bx &= d \ . \end{aligned}$$
An advantage of this approach is that the Jacobian of this system of equations can be calculated analytically and evaluated efficiently, which means that we can apply algorithms that use derivatives to solve the system of nonlinear equations. In general such algorithms are faster and more reliable than algorithms that do not use information on the derivatives or that compute the derivatives numerically. But on the other hand we have introduced extra variables, which makes the problem more difficult to solve.

- Combinatorial solution:
We select one index i_j out of each set ϕ_j for $j = 1, 2, \ldots, m$. Each index i_j corresponds to an inequality of $Ax \geqslant c$ that should hold with equality. Define $\alpha = \{i_1, i_2, \ldots, i_m\}$ and $\alpha^c = \{1, 2, \ldots, p\} \setminus \alpha$. Now we have to find a solution of the following system of linear equalities and inequalities:
$$\begin{aligned} A_{\alpha,.}x &= c_\alpha \\ A_{\alpha^c,.}x &\geqslant c_{\alpha^c} \\ Bx &= d \ . \end{aligned}$$

If this system of equalities and inequalities has no solution, we select another combination of indices $i_1, i_2, \ldots, i_m$ that has not yet been considered and we repeat the procedure.
Since this algorithm is essentially combinatorial, it will not be very efficient in practice.

- Adaptations and extensions of the existing methods for LCPs and GLCPs of e.g. [1, 30, 87, 107, 108, 128, 136, 150, 153, 154]. We could also try to adapt and extend these algorithms to solve special cases of the general ELCP.

However, in the next section we shall show that the ELCP is intrinsically a computationally hard problem.

3.5 The Complexity of the ELCP

In this section we discuss the complexity of the ELCP and we show that developing efficient algorithms to solve the general ELCP is not an easy (and maybe even an impossible) task. But first we give an elementary and informal description of some basic concepts of the theory of "NP-completeness". The interested reader is referred to [52] for an extensive treatment of NP-completeness.

A *decision problem* is a problem that has only two possible solutions: either the answer "yes" or the answer "no". A *search problem* is a problem for which we either have to give a solution or have to establish that the problem has no solution. The ELCP is an example of a search problem. We say that a problem can be solved in *polynomial time* if there exists an algorithm to solve the problem such that the execution time of the algorithm is bounded from above by a polynomial in the size (or the input length) of the given problem instance.
Loosely speaking, the *class P* consists of the decision problems that can be solved by a polynomial time algorithm. A decision problem belongs to the *class NP* if a nondeterministic algorithm can guess a "proof" that would show that the answer to the decision problem is "yes" and then verify in polynomial time whether this guess really proves that the answer to the problem is "yes". The *NP-complete* problems are the "hardest" problems in NP in the sense that an NP-complete problem can only be solved in polynomial time *if* the class P would coincide with the class NP. Furthermore, if any single NP-complete problem can be solved in polynomial time, then *all* problems in NP can be solved in polynomial time. With the present state of knowledge it is still an open question whether the class P coincides with the class NP. However, since no NP-complete problem is known to be solvable in polynomial time despite the efforts of many excellent researchers, it is widely conjectured that no NP-complete problem can be solved by a polynomial time algorithm.
If the decision problem that corresponds to a search problem is NP-complete then the search problem is called *NP-hard*. NP-hard problems are even harder

to solve than NP-complete problems: they cannot not be solved in polynomial time *unless* the class P would coincide with the class NP.

Theorem 3.5.1 *In general the ELCP with rational data is an NP-hard problem.*

Proof: First we consider the decision problem that corresponds to an ELCP with rational data:

> Given $A \in \mathbb{Q}^{p \times n}$, $B \in \mathbb{Q}^{q \times n}$, $c \in \mathbb{Q}^p$, $d \in \mathbb{Q}^q$ and m subsets $\phi_1, \phi_2, \ldots, \phi_m$ of $\{1, 2, \ldots, p\}$, does there exist a vector $x \in \mathbb{Q}^n$ such that
> $$\sum_{j=1}^{m} \prod_{i \in \phi_j} (Ax - c)_i = 0$$
> subject to $Ax \geqslant c$ and $Bx = d$?

This problem will be called the *ELCP decision problem* (EDP). The EDP belongs to NP: a nondeterministic algorithm can guess a vector x and then check in polynomial time whether x satisfies the complementarity condition and the system of linear equalities and inequalities. Chung [16] has proved that the decision problem that corresponds to the LCP with rational data is in general an NP-complete problem. Since the LCP is a special case of the ELCP, the EDP is also NP-complete. This means that in general the ELCP with rational data is NP-hard. □

Since Chung [16] has also proved that the decision problem that corresponds to the LCP with rational data is in general NP-complete in the strong sense, the general ELCP with rational data is NP-hard in the strong sense. This means that the general ELCP with rational data cannot be solved by a pseudo-polynomial time algorithm — i.e. an algorithm that runs in time bounded from above by a polynomial in the size of the given instance of the problem and the maximum absolute value of the numerators and the denominators occurring in the given instance of the problem [51, 52] — unless P = NP.

Now we show that in general the ELCP with rational data can be solved in polynomial time *if and only if* P = NP.

Theorem 3.5.2 *The general ELCP with rational data is an NP-easy problem, i.e. if P = NP then the general ELCP with rational data can be solved in polynomial time.*

Proof: Consider the following ELCP with rational data (ELCP$_{\mathbb{Q}}$ for short):

> Given $A \in \mathbb{Q}^{p \times n}$, $B \in \mathbb{Q}^{q \times n}$, $c \in \mathbb{Q}^p$, $d \in \mathbb{Q}^q$ and m subsets $\phi_1, \phi_2, \ldots, \phi_m$ of $\{1, 2, \ldots, p\}$, find $x \in \mathbb{Q}^n$ such that
> $$\sum_{j=1}^{m} \prod_{i \in \phi_j} (Ax - c)_i = 0$$
> subject to $Ax \geqslant c$ and $Bx = d$.

If $x \in \mathbb{Q}^n$ is a solution of this problem then in each subset ϕ_j with $j \in \{1, 2, \ldots, m\}$ there is at least one index i_j such that $(Ax)_{i_j} = c_{i_j}$. We select exactly one index i_j that satisfies this condition out of each set ϕ_j for $j = 1, 2, \ldots, m$ and we put these indices in the set θ. If we know θ then we can construct a solution of the $\mathrm{ELCP}_{\mathbb{Q}}$ by solving the following system of linear equalities and inequalities:

$$A_{\theta,.}\, x = c_\theta \tag{3.39}$$

$$A_{\theta^c,.}\, x \geqslant c_{\theta^c} \tag{3.40}$$

$$Bx = d \tag{3.41}$$

where $\theta^c = \{1, 2, \ldots, p\} \setminus \theta$. Note that a system of linear equalities and inequalities with rational data, if solvable, always has a solution with rational components.

Now we introduce the *ELCP extension problem* (EEP), which is defined as follows:

> Given $A \in \mathbb{Q}^{p\times n}$, $B \in \mathbb{Q}^{q\times n}$, $c \in \mathbb{Q}^p$, $d \in \mathbb{Q}^q$, m subsets $\phi_1, \phi_2, \ldots, \phi_m$ of $\{1, 2, \ldots, p\}$, $k \in \{1, 2, \ldots, m\}$, and a set $\theta_k \subseteq \{1, 2, \ldots, p\}$ such that
>
> $$\forall j \in \{1, 2, \ldots, k\} : \exists i_j \in \phi_j \text{ such that } i_j \in \theta_k$$
>
> and such that the system of linear equalities and inequalities
>
> $$A_{\theta_k,.}\, x = c_{\theta_k}$$
> $$A_{\theta_k^c,.}\, x \geqslant c_{\theta_k^c}$$
> $$Bx = d$$
>
> where $\theta_k^c = \{1, 2, \ldots, p\} \setminus \theta_k$, has a solution,
> can θ_k be completed to a complete set θ with $\theta_k \subseteq \theta \subseteq \{1, 2, \ldots, p\}$ such that
>
> $$\forall j \in \{1, 2, \ldots, m\} : \exists i_j \in \phi_j \text{ such that } i_j \in \theta$$
>
> and such that the system of linear equalities and inequalities (3.39) – (3.41) has a solution?

It is obvious that if the solution of this decision problem is "yes" if and only if the corresponding $\mathrm{ELCP}_{\mathbb{Q}}$ has a solution. In that case we call θ_k an *extendible partial set*.

Note that the EEP belongs to NP: a nondeterministic algorithm can guess a set θ and then check in polynomial time whether this set satisfies the conditions stated above.

Now we need the concept of Turing reducibility [52, p. 111]: "A *polynomial time Turing reduction* (or simply Turing reduction) from a search problem Π to a search problem Π' is an algorithm A that solves Π by using a hypothetical subroutine S for solving Π' such that, if S were a polynomial time algorithm for Π', then A would be a polynomial time algorithm for Π." This will be denoted by $\Pi \propto_{\mathrm{T}} \Pi'$. This definition can also be applied to decision problems, since a decision problem can be considered as a search problem with "yes" and "no" as possible answers. Note that the relation $\propto_{\mathrm{T}}$ is transitive.

If a problem Π is in NP and if problem Π' is NP-complete, then we have $\Pi \propto_{\mathrm{T}} \Pi'$ [52]. Since EDP is NP-complete (cf. the proof of Theorem 3.5.1), we have EEP $\propto_{\mathrm{T}}$ EDP.

Now we are going to prove that ELCP$_{\mathbb{Q}}$ $\propto_{\mathrm{T}}$ EEP.

Suppose that S is a subroutine for solving the EEP. Now we are going to use this subroutine to solve a given instance of the ELCP$_{\mathbb{Q}}$ either by determining that the ELCP has no solution or by constructing a sequence of extendible partial sets $\theta_1, \theta_2, \ldots, \theta_m$ such that $\theta = \theta_m$ is a set that contains at least one index out of each set $\phi_1, \phi_2, \ldots, \phi_m$ and such that the system of linear equalities and inequalities (3.39) – (3.41) has a solution.

We start with $\theta_1 = \{r_1\}$ where $r_1 \in \phi_1$. Then we use the subroutine S to determine whether the set θ_1 is an extendible partial set. If the answer is "yes", we continue with the next step in which we shall determine θ_2. If the answer is "no", we set $\theta_1 = \{r_2\}$ with $r_2 \in \phi_1 \setminus \{r_1\}$ and we use the subroutine S to determine whether the set θ_1 now is an extendible partial set. If the answer is again "no", we set $\theta_1 = \{r_3\}$ with $r_3 \in \phi_1 \setminus \{r_1, r_2\}$ and so on. We continue until we encounter an extendible partial set or until all the elements of ϕ_1 have been considered. In the latter case, we know that the ELCP$_{\mathbb{Q}}$ does not have a solution. Note that in the worst case scenario, we have to use the subroutine S $\#\phi_1$ times where $\#\phi_1 \leqslant p$.

Assume that we have found an extendible partial set $\theta_1 = \{i_1\}$. If $i_1 \in \phi_2$, we set $\theta_2 = \theta_1$ and we go to the next step. Otherwise, we select an index $s_1 \in \phi_2$ and we set $\theta_2 = \theta_1 \cup \{s_1\}$. Then we use the subroutine S to determine whether the set θ_2 is an extendible partial set. If the answer is "yes", we go to the next step and if the answer is "no", we continue with $\theta_2 = \theta_1 \cup \{s_2\}$ where $s_2 \in \phi_2 \setminus \{s_1\}$ and so on. Since we already know that θ_1 is an extendible partial set there will exist an index $i_2 \in \phi_2$ such that $\theta_2 = \theta_1 \cup \{i_2\}$ is an extendible partial set. So if the first $\#\phi_2 - 1$ elements of ϕ_2 do not yield an extendible partial set, the remaining element must yield an extendible partial set. This means that in order to determine θ_2 we have to use the subroutine S at most $\#\phi_2 - 1$ times where $\phi_2 - 1 \leqslant p$.

We continue until we finally get a complete set $\theta = \theta_m$. If the ELCP$_{\mathbb{Q}}$ has a solution, the set θ can now be used to determine a solution of the ELCP$_{\mathbb{Q}}$ by solving the system of linear equalities and inequalities (3.39) – (3.41). This can be done in polynomial time (See e.g. [90, 92, 114]).

Since we have made less than mp calls to the subroutine S, we may conclude that if S were a polynomial time algorithm for the EEP, then the algorithm described above would be a polynomial time algorithm for the ELCP$_{\mathbb{Q}}$. Hence, ELCP$_{\mathbb{Q}}$ $\propto_{\mathrm{T}}$ EEP. Since EEP $\propto_{\mathrm{T}}$ EDP and since $\propto_{\mathrm{T}}$ is transitive, we have ELCP$_{\mathbb{Q}}$ $\propto_{\mathrm{T}}$ EDP.

Since EDP is NP-complete, this means that the $\text{ELCP}_{\mathbb{Q}}$ can be solved in polynomial time if P = NP. □

Corollary 3.5.3 *The general ELCP with rational data is an NP-equivalent problem, i.e. the general ELCP with rational data can be solved in polynomial time if and only if P = NP.*

Proof: This is a direct consequence of Theorems 3.5.1 and 3.5.2. □

3.6 Worked Examples of the ELCP Algorithm

In this section we give two worked examples that illustrate our ELCP algorithm. In the first example we solve a simple ELCP with Algorithms 1 and 2. However, for this small-sized ELCP we do not get enough extreme generators to demonstrate Algorithm 3. Therefore, we give a second example in which we show how to determine all maximal sets of cross-complementary extreme generators of a GLCP.

Example 3.6.1: Determination of the central and the extreme generators
Consider the following ELCP:

Given

$$P = \begin{bmatrix} 1 & 0 & 0 & 1 \\ -1 & 2 & -1 & -1 \\ 0 & -3 & 1 & -2 \end{bmatrix} \text{ and } Q = \begin{bmatrix} 2 & 1 & 0 & 4 \end{bmatrix} ,$$

find $u \in \mathbb{R}^4$ such that

$$(Pu)_1 \, (Pu)_2 + (Pu)_2 \, (Pu)_3 \; = \; 0 \tag{3.42}$$

subject to

$$Pu \geqslant 0 \tag{3.43}$$

$$Qu = 0 \ . \tag{3.44}$$

Since all the inequalities of $Pu \geqslant 0$ appear in the complementarity condition, we do not have to split $Pu \geqslant 0$.
First we process the inequalities of $Pu \geqslant 0$:

<u>Initialization</u>

We set

$$c_{0,1} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} , \quad c_{0,2} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} , \quad c_{0,3} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} , \quad c_{0,4} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} .$$

<u>$k = 1$</u>

The residues of the central generators are given by

$$\mathrm{res}(c_{0,1}) = 1\,,\ \mathrm{res}(c_{0,2}) = 0\,,\ \mathrm{res}(c_{0,3}) = 0\,,\ \mathrm{res}(c_{0,4}) = 1\ \ .$$

Hence, $\mathcal{C}^+ = \{c_{0,1}, c_{0,4}\}$, $\mathcal{C}^0 = \{c_{0,2}, c_{0,3}\}$ and $\mathcal{C}^- = \mathcal{E}^+ = \mathcal{E}^- = \mathcal{E}^0 = \emptyset$. Since $\mathcal{C}^+$ is not empty, we go to Case 3 of Algorithm 1. First we put the elements of $\mathcal{C}^0$ in $\mathcal{C}$: $c_{1,1} = c_{0,2}$ and $c_{1,2} = c_{0,3}$. Since $\mathcal{C}^-$ is empty, we do not have to transfer elements from $\mathcal{C}^-$ to $\mathcal{C}^+$. We set $c = c_{0,1}$. Since no group of inequalities has been processed completely yet, c satisfies the partial complementarity condition by definition. Therefore, we put it in $\mathcal{E}$: $e_{1,1} = c = c_{0,1}$. Finally, we combine $c = c_{0,1}$ and $c_{0,4}$ and put the result in $\mathcal{C}$:

$$c_{1,3} = \mathrm{res}(c_{0,4})c_{0,1} - \mathrm{res}(c_{0,1})c_{0,4} = c_{0,1} - c_{0,4}\ \ .$$

This leads to

$$c_{1,1} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},\ c_{1,2} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix},\ c_{1,3} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix},\ e_{1,1} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\ \ .$$

<u>$k = 2$</u>

We have

$$\mathrm{res}(c_{1,1}) = 2\,,\ \mathrm{res}(c_{1,2}) = -1\,,\ \mathrm{res}(c_{1,3}) = 0\,,\ \mathrm{res}(e_{1,1}) = -1\ \ .$$

So $\mathcal{C}^+ = \{c_{1,1}\}$, $\mathcal{C}^- = \{c_{1,2}\}$, $\mathcal{C}^0 = \{c_{1,3}\}$, $\mathcal{E}^- = \{e_{1,1}\}$ and $\mathcal{E}^+ = \mathcal{E}^0 = \emptyset$. Since $\mathcal{C}^+$ is not empty, we go again to Case 3. We put $c_{1,3}$ in $\mathcal{C}$: $c_{2,1} = c_{1,3}$ and we transfer $-c_{1,2}$ to $\mathcal{C}^+$. The generator $c = c_{1,1}$ satisfies the partial complementarity condition $(Pu)_1\,(Pu)_2 = 0$ since $P_{\{1,2\},.}c_{1,1} = \begin{bmatrix} 0 & 2 \end{bmatrix}^T$. Therefore, we put it in $\mathcal{E}$: $e_{2,1} = c_{1,1}$. We combine $c = c_{1,1}$ and $-c_{1,2}$ and we transfer the result to $\mathcal{C}$:

$$c_{2,2} = \mathrm{res}(-c_{1,2})c_{1,1} - \mathrm{res}(c_{1,1})(-c_{1,2}) = c_{1,1} + 2c_{1,2}\ \ .$$

The combination of $c = c_{1,1}$ and $e_{1,1}$ satisfies the partial complementarity condition and therefore we also put it to $\mathcal{E}$:

$$e_{2,2} = \mathrm{res}(c_{1,1})e_{1,1} - \mathrm{res}(e_{1,1})c_{1,1} = 2e_{1,1} + c_{1,1}\ \ .$$

This yields

$$c_{2,1} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix},\ c_{2,2} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix},\ e_{2,1} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},\ e_{2,2} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}\ \ .$$

$\underline{k=3}$

Using the same procedure as in the previous steps we find

$$c_{3,1} = c_{2,1} + 2c_{2,2}\ ,\ \ e_{3,1} = c_{2,1}\ ,\ \ e_{3,2} = 2e_{2,1} + 3c_{2,1}\ ,\ \ e_{3,3} = 2e_{2,2} + 3c_{2,1}\ \ .$$

This results in

$$c_{3,1} = \begin{bmatrix} 1 \\ 2 \\ 4 \\ -1 \end{bmatrix},\ e_{3,1} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix},\ e_{3,2} = \begin{bmatrix} 3 \\ 2 \\ 0 \\ -3 \end{bmatrix},\ e_{3,3} = \begin{bmatrix} 7 \\ 2 \\ 0 \\ -3 \end{bmatrix}\ .$$

We do not have to reject any of the extreme generators since they all satisfy the complementarity condition.

Since we did not encounter redundant inequalities, we have $P_{\text{nec}} = P$. Now we take the equality $Qu = 0$ into account:

$\underline{k=1}$

We have

$$\text{res}(c_{3,1}) = 0\ ,\ \text{res}(e_{3,1}) = -2\ ,\ \text{res}(e_{3,2}) = -4\ ,\ \text{res}(e_{3,3}) = 4\ \ .$$

So $\mathcal{C}^0 = \{c_{3,1}\}$, $\mathcal{E}^+ = \{e_{3,3}\}$, $\mathcal{E}^- = \{e_{3,1}, e_{3,2}\}$ and $\mathcal{C}^+ = \mathcal{C}^- = \mathcal{E}^0 = \emptyset$. Since $\mathcal{C}^+ = \mathcal{C}^- = \emptyset$, we go to Case 2 of Algorithm 2. All the elements of $\mathcal{C}$ stay in $\mathcal{C}$:

$$c_{4,1} = c_{3,1} = \begin{bmatrix} 1 & 2 & 4 & -1 \end{bmatrix}^T\ \ .$$

Now we have to determine which pairs of extreme generators are adjacent. The zero index sets of the extreme generators are given by

$$\mathcal{I}_0(e_{3,1}) = \{1,2\}\ ,\ \ \mathcal{I}_0(e_{3,2}) = \{1,3\}\ ,\ \ \mathcal{I}_0(e_{3,3}) = \{2,3\}\ \ .$$

If we consider Adjacency Test 1 then a necessary condition for two extreme generators to be adjacent is that their zero index sets have $n-t-2 = 4-1-2 = 1$ common element (Note that t is equal to 1 since at the beginning of this pass the minimal complete set of central generators $\mathcal{C}$ contained exactly one element). This means that all possible combinations of two different extreme generators pass Adjacency Test 1. Since we have not rejected any extreme generators, Adjacency Test 2 still yields a necessary and sufficient condition for adjacency. Since $\mathcal{I}_0(e_{3,3}) \cap \mathcal{I}_0(e_{3,1}) = \{2\} \not\subseteq \{1,3\} = \mathcal{I}_0(e_{3,2})$, the generators $e_{3,3}$ and $e_{3,1}$ are adjacent. If we combine them, we get a new extreme generator that satisfies the complementarity condition:

$$e_{4,1} = 4e_{3,1} + 2e_{3,3} = \begin{bmatrix} 18 & 4 & 0 & -10 \end{bmatrix}^T$$

with $Pe_{4,1} = \begin{bmatrix} 8 & 0 & 8 \end{bmatrix}^T$. For the combination

$$e = 4e_{3,2} + 4e_{3,3} = \begin{bmatrix} 40 & 16 & 0 & -24 \end{bmatrix}^T$$

of the adjacent generators $e_{3,3}$ and $e_{3,2}$, we have $Pe = \begin{bmatrix} 16 & 16 & 0 \end{bmatrix}^T$. Since e does not satisfy the complementarity condition, we discard it.

Now we have $\mathcal{C} = \left\{ \begin{bmatrix} 1 & 2 & 4 & -1 \end{bmatrix}^T \right\}$ and $\mathcal{E} = \left\{ \begin{bmatrix} 18 & 4 & 0 & -10 \end{bmatrix}^T \right\}$. Hence, every combination of the form

$$u = \lambda \begin{bmatrix} 1 \\ 2 \\ 4 \\ -1 \end{bmatrix} + \kappa \begin{bmatrix} 18 \\ 4 \\ 0 \\ -10 \end{bmatrix} \qquad \text{with } \lambda \in \mathbb{R} \text{ and } \kappa \geqslant 0$$

is a solution of the ELCP defined by (3.42) – (3.44). □

Since the determination of the maximal cross-complementary sets of extreme generators of an ELCP essentially reduces to the determination of the maximal cross-complementary sets of extreme generators of a GLCP, we demonstrate Algorithm 3 for a GLCP.

Example 3.6.2 : Determination of the maximal cross-complementary sets of extreme generators
Consider the following GLCP:

Given $Z = \begin{bmatrix} 1 & -1 & 0 & -1 & -1 \end{bmatrix}$, find $u \in \mathbb{R}^5$ such that

$$u_2 \, u_3 \, u_5 + u_3 \, u_4 = 0 \tag{3.45}$$

subject to $u \geqslant 0$ and $Zu = 0$.

The extreme generators of the solution set of this GLCP are

$$e_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \; e_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \; e_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \text{ and } e_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Now we use Algorithm 3 to determine which nonnegative combinations of these extreme generators are solutions of the GLCP.

- First we transform every generator into its binary equivalent. Since the generators are already binary, we can leave them as they are. The binary complementarity condition is given by

$$\Big((b_2 = 0) \vee (b_3 = 0) \vee (b_5 = 0) \Big) \wedge \Big((b_3 = 0) \vee (b_4 = 0) \Big) \tag{3.46}$$

where $b = \text{binary}(u)$.

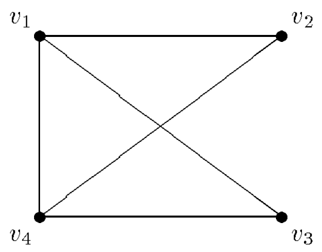


Figure 3.8: The cross-complementarity graph $\mathcal{G}_c$ for Example 3.6.2.

- Next we construct the cross-complementarity matrix, i.e. we determine which pairs of extreme generators are cross-complementary.
  The generators e_1 and e_2 are cross-complementary since
  $$e_1 \vee e_2 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \end{bmatrix}^T$$
  satisfies the binary complementarity condition (3.46).
  However, e_2 and e_3 are not cross-complementary since
  $$e_2 \vee e_3 = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \end{bmatrix}^T$$
  does not satisfy the binary complementarity condition.
  We check all other combinations of two different extreme generators and put the results in the cross-complementarity matrix $cross$.
  To make the subsequent steps easier to follow we have represented the cross-complementarity graph $\mathcal{G}_c$ that corresponds to $\{e_1, e_2, e_3, e_4\}$ in Figure 3.8. Recall that an edge between two vertices v_k and v_l of $\mathcal{G}_c$ indicates that the corresponding extreme generators e_k and e_l are cross-complementary.

- In Algorithm 3 we keep track of our progress by using lists of vertices to be investigated for each depth. These lists are stored in the matrix $vertices$: the dth row of $vertices$ contains the vertices for depth d. The column index for the first vertex of the list for depth d is $start(d)$, and the column index for the last vertex is $last(d)$. In our explanation the ordered set $\mathcal{L}_d$ will represent the list of vertices for depth d. After $start(depth)$ is incremented in the first step of the main loop, we have
  $$\mathcal{L}_d = \left(v_{vertices(d,start(d))}, v_{vertices(d,start(d)+1)}, \ldots, v_{vertices(d,last(d))} \right) .$$
  If the current depth is equal to d, then the set of extreme generators that we are investigating corresponds to the set obtained by taking the first vertex of each list $\mathcal{L}_i$ for $i = 1, 2, \ldots, d$.

- We start with a list of vertices for depth 1: $\mathcal{L}_1 = (v_1, v_2, v_3, v_4)$.
  Vertex v_1 is the first vertex in the list $\mathcal{L}_1$. Therefore, we look for other

vertices of $\mathcal{L}_1$ that are connected by an edge to v_1. Since the vertices v_2, v_3 and v_4 satisfy this condition, we get $\mathcal{L}_2 = (v_2, v_3, v_4)$.
The first vertices of $\mathcal{L}_1$ and $\mathcal{L}_2$ are v_1 and v_2 respectively, which means that the set of extreme generators that we are currently investigating is $\{e_1, e_2\}$. Now we try to extend this set. The only other vertex in $\mathcal{L}_2$ that is connected to both v_1 and v_2 is v_4. So we check whether the set $\{e_1, e_2, e_4\}$ is cross-complementary. This is not the case since

$$e_1 \vee e_2 \vee e_4 = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 \end{bmatrix}^T$$

does not satisfy the binary complementarity condition. So we do not find any vertices for depth 3, which means that the current set $\{e_1, e_2\}$ cannot be extended any more. Since Γ is empty, $\{e_1, e_2\}$ is a maximal cross-complementary set. Therefore, we put it in Γ. This yields $\Gamma = \big\{\{e_1, e_2\}\big\}$.

- Now we return to the previous point where a choice has been made: we remove vertex 2 from $\mathcal{L}_2$. This results in $\mathcal{L}_2 = (v_3, v_4)$.
  Now we are investigating the set $\{e_1, e_3\}$. Since v_4 is connected to both v_1 and v_3, we check whether

  $$e_1 \vee e_3 \vee e_4 = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 \end{bmatrix}^T$$

  satisfies the binary complementarity condition. Since this is the case, we add vertex v_4 to the list of depth 3: $\mathcal{L}_3 = (v_4)$. Since there is only one vertex in $\mathcal{L}_3$, we cannot extend the current set $\{e_1, e_3, e_4\}$. Since $\{e_1, e_3, e_4\}$ is not a subset of the set $\{e_1, e_2\} \in \Gamma$, we have again found a maximal cross-complementary set that should be added to Γ. This results in $\Gamma = \big\{\{e_1, e_2\}, \{e_1, e_3, e_4\}\big\}$.
- We return to previous point where a choice has been made: we go again to depth 2 and we remove v_3 from $\mathcal{L}_2$. Hence, $\mathcal{L}_2 = (v_4)$.
  The current set of extreme generators is $\{e_1, e_4\}$. Since there are no more vertices left for the next depth, the current set cannot be extended any further. However, since $\{e_1, e_4\}$ is a subset of the set $\{e_1, e_3, e_4\} \in \Gamma$, $\{e_1, e_4\}$ is not a maximal cross-complementary set and therefore we do not put it in Γ.
- Now we go back to depth 1, remove vertex v_1 from $\mathcal{L}_1$ and so on.

Finally we get

$$\Gamma = \big\{\{e_1, e_2\}, \{e_1, e_3, e_4\}, \{e_2, e_4\}\big\} \ .$$

This means that any combination of the form $u = \kappa_1 e_1 + \kappa_2 e_2$, $u = \kappa_1 e_1 + \kappa_3 e_3 + \kappa_4 e_4$ or $u = \kappa_2 e_2 + \kappa_4 e_4$ with κ_1, κ_2, κ_3, $\kappa_4 \geqslant 0$ is a solution of the given GLCP.
We see again that determining all maximal cross-complementary sets of extreme generators does not amount to determining all the maximal cliques of the cross-complementarity graph $\mathcal{G}_c$ since $\{v_1, v_2, v_4\}$ is a maximal clique of $\mathcal{G}_c$ but $\{e_1, e_2, e_4\}$ not a cross-complementary set. $\square$

3.7 Conclusion

In this chapter we have presented the Extended Linear Complementarity Problem (ELCP) and established a link between the ELCP and other linear complementarity problems. We have shown that the ELCP can be considered as a unifying framework for the Linear Complementarity Problem and its various generalizations. Furthermore, we have made a thorough study of the solution set of the general ELCP and developed an algorithm to find all solutions of an ELCP. Since our algorithm yields all solutions, it provides a geometrical insight in the solution set of an ELCP and other problems that can be reduced to an ELCP. On the other hand, this also leads to large computation times and storage space requirements if the number of variables and (in)equalities is large or if the ELCP has a complex solution set.

We have presented the results of some experiments that illustrate the influence of various parameters on the performance of our ELCP algorithm. We have also shown that the general ELCP with rational data is NP-hard (and NP-equivalent), which means that it is a computationally hard problem.

Although our algorithm yields all solutions of an ELCP, we are not always interested in obtaining all solutions of an ELCP. Therefore, it might be interesting to develop (heuristic) algorithms that yield only one solution. This will be a topic for further research. Furthermore, we could also investigate which subclasses of the ELCP can be solved by (pseudo-)polynomial time algorithms.

Chapter 4

Applications of the Extended Linear Complementarity Problem in the Max-Plus Algebra

In this chapter we show that the problem of finding all finite solutions of a system of multivariate max-algebraic polynomial equalities and inequalities can be transformed into an ELCP and vice versa. This enables us to find all solutions of a system of multivariate max-algebraic polynomial equalities and inequalities. Moreover, it also provides an insight in the geometrical structure of the solution set and in the computational complexity of this problem.

In this chapter and in the next chapters we shall show that many problems in the max-plus algebra, the symmetrized max-plus algebra and the max-min-plus algebra such as calculating max-algebraic matrix factorizations, performing max-algebraic state space transformations, constructing matrices with a given max-algebraic characteristic polynomial, determining partial state space realizations for max-linear time-invariant discrete event systems, and so on, can be reformulated as a system of multivariate max-algebraic polynomial equalities and inequalities and thus also as an ELCP.

This chapter is organized as follows. In Section 4.1 we show that the problem of solving a system of multivariate max-algebraic polynomial equalities and inequalities is equivalent to an ELCP and in Section 4.2 we discuss some max-algebraic problems that can be reformulated as an ELCP.

4.1 Systems of Multivariate Max-Algebraic Polynomial Equalities and Inequalities

In this next section we show that a system of multivariate max-algebraic polynomial equalities and inequalities can be transformed into an ELCP and vice versa.

4.1.1 Problem Formulation

Consider the following problem:

Given $p_1 + p_2$ integers $m_1, m_2, \ldots, m_{p_1+p_2} \in \mathbb{N}_0$ and real numbers a_{ki}, b_k and c_{kij} for $k = 1, 2, \ldots, p_1 + p_2$, $i = 1, 2, \ldots, m_k$ and $j = 1, 2, \ldots, n$, find $x \in \mathbb{R}^n$ such that

$$\bigoplus_{i=1}^{m_k} a_{ki} \otimes \bigotimes_{j=1}^{n} {x_j}^{\otimes^{c_{kij}}} = b_k \qquad \text{for } k = 1, 2, \ldots, p_1\,, \tag{4.1}$$

$$\bigoplus_{i=1}^{m_k} a_{ki} \otimes \bigotimes_{j=1}^{n} {x_j}^{\otimes^{c_{kij}}} \leqslant b_k \qquad \text{for } k = p_1 + 1, p_1 + 2, \ldots, p_1 + p_2\,, \tag{4.2}$$

or show that no such x exists.

We call (4.1) – (4.2) a *system of multivariate max-algebraic polynomial equalities and inequalities*. Note that the exponents may be negative or real. If x_j does not appear in the ith term of the kth equation then the exponent c_{kij} is equal to 0. In the next subsection we shall show that problem (4.1) – (4.2) can be transformed into an ELCP and vice versa.

4.1.2 A Connection between Systems of Multivariate Max-Algebraic Polynomial Equalities and Inequalities and Extended Linear Complementarity Problems

Theorem 4.1.1 *A system of multivariate max-algebraic polynomial equalities and inequalities is equivalent to an Extended Linear Complementarity Problem.*

Proof: First we show that a system of multivariate max-algebraic polynomial equalities and inequalities can be transformed into an ELCP.
Consider one equation of the form (4.1):

$$\bigoplus_{i=1}^{m_k} a_{ki} \otimes \bigotimes_{j=1}^{n} {x_j}^{\otimes^{c_{kij}}} = b_k$$

with $k \in \{1, 2, \ldots, p_1\}$. In conventional algebra this equation is equivalent to the system of linear inequalities

$$a_{ki} + \sum_{j=1}^{n} c_{kij} x_j \leqslant b_k \qquad \text{for } i = 1, 2, \ldots, m_k$$

where at least one inequality should hold with equality.
If we transfer the a_{ki}'s to the right-hand side and if we define $d_{ki} = b_k - a_{ki}$ for $i = 1, 2, \ldots, m_k$, we get the following system of a linear inequalities:

$$\sum_{j=1}^{n} c_{kij} x_j \leqslant d_{ki} \qquad \text{for } i = 1, 2, \ldots, m_k$$

where at least one inequality should hold with equality.
So (4.1) will lead to p_1 groups of linear inequalities where in each group at least one inequality should hold with equality.
Using the same reasoning equations of the form (4.2) can also be transformed into a system of linear inequalities, but without an extra condition.
If we define p_1+p_2 matrices C_1 , $C_2, \ldots, C_{p_1+p_2}$ and p_1+p_2 column vectors d_1, $d_2, \ldots, d_{p_1+p_2}$ such that $(C_k)_{ij} = c_{kij}$ and $(d_k)_i = d_{ki}$ for $k = 1, 2, \ldots, p_1+p_2$, $i = 1, 2, \ldots, m_k$ and $j = 1, 2, \ldots, n$ then our original problem is equivalent to p_1+p_2 groups of linear inequalities $C_k x \leqslant d_k$ where there has to be at least one inequality that holds with equality in each group $C_k x \leqslant d_k$ for $k = 1, 2, \ldots, p_1$.

Now we put all the C_k's in one large matrix $A = \begin{bmatrix} -C_1 \\ -C_2 \\ \vdots \\ -C_{p_1+p_2} \end{bmatrix}$ and all the d_k's in one large vector $c = \begin{bmatrix} -d_1 \\ -d_2 \\ \vdots \\ -d_{p_1+p_2} \end{bmatrix}$. We also define p_1 sets $\phi_1, \phi_2, \ldots, \phi_{p_1}$ such that $\phi_j = \{s_j+1, s_j+2, \ldots, s_j+m_j\}$ for $j = 1, 2, \ldots, p_1$ where $s_1 = 0$ and $s_{j+1} = s_j + m_j$ for $j = 1, 2, \ldots, p_1 - 1$. Problem (4.1)–(4.2) then results in the following ELCP:

Given A, c and $\phi_1, \phi_2, \ldots, \phi_{p_1}$, find $x \in \mathbb{R}^n$ such that

$$\sum_{j=1}^{p_1} \prod_{i \in \phi_j} (Ax - c)_i = 0$$

subject to $Ax \geqslant c$, or show that no such x exists.

Note that if any of the m_k's in (4.1) is equal to 1 we get a linear equality instead of a system of linear inequalities. These equalities can be removed from $Ax \geqslant c$ and from the complementarity condition and be put in $Bx = d$. Although this is not really necessary, it will certainly enhance the efficiency of the ELCP algorithm when we use it to solve the ELCP that corresponds to problem (4.1) – (4.2).

Now we prove that an ELCP can also be transformed into a system of multivariate max-algebraic polynomial equalities and inequalities.
Consider the following ELCP:

> Given $A \in \mathbb{R}^{p \times n}$, $B \in \mathbb{R}^{q \times n}$, $c \in \mathbb{R}^p$, $d \in \mathbb{R}^q$ and m subsets ϕ_1, ϕ_2, $\ldots$, ϕ_m of $\{1, 2, \ldots, p\}$, find $x \in \mathbb{R}^n$ such that
>
> $$\sum_{j=1}^{m} \prod_{i \in \phi_j} (Ax - c)_i = 0$$
>
> subject to $Ax \geqslant c$ and $Bx = d$.

First we define $\Phi = \bigcup_{j=1}^{m} \phi_j$ and $\Phi^c = \{1, 2, \ldots, p\} \setminus \Phi$. There are three cases that should be considered:

1. Groups of linear inequalities where at least one inequality should hold with equality:

$$\sum_{k=1}^{n} a_{ik} x_k \geqslant c_i \qquad \text{for all } i \in \Phi \tag{4.3}$$

with

$$\prod_{i \in \phi_j} (Ax - c)_i = 0 \qquad \text{for } j = 1, 2, \ldots, m\,, \tag{4.4}$$

where we have used the alternative formulation of the complementarity condition. Equation (4.3) is equivalent to

$$c_i + \sum_{k=1}^{n} (-a_{ik}) x_k \leqslant 0 \qquad \text{for all } i \in \Phi\,.$$

If we rewrite everything in max-algebraic notation and if we also take condition (4.4) into account, we get m multivariate max-algebraic polynomial equalities:

$$\bigoplus_{i \in \phi_j} c_i \otimes \bigotimes_{k=1}^{n} {x_k}^{\otimes^{(-a_{ik})}} = 0 \qquad \text{for } j = 1, 2, \ldots, m\,.$$

2. Linear equalities:

$$\sum_{k=1}^{n} b_{ik} x_k = d_i \qquad \text{for } i = 1, 2, \ldots, q\,.$$

These equations can be transformed into q multivariate max-algebraic polynomial equalities:

$$d_i \otimes \bigotimes_{k=1}^{n} x_k{}^{\otimes^{(-b_{ik})}} = 0 \qquad \text{for } i = 1, 2, \ldots, q\,.$$

3. The remaining linear inequalities:

$$\sum_{k=1}^{n} a_{ik} x_k \geqslant c_i \qquad \text{for all } i \in \Phi^{\mathrm{c}}$$

can be transformed into one multivariate max-algebraic polynomial inequality:

$$\bigoplus_{i \in \Phi^{\mathrm{c}}} c_i \otimes \bigotimes_{k=1}^{n} x_k{}^{\otimes^{(-a_{ik})}} \leqslant 0 \enspace .$$

So an ELCP can be transformed into a system of multivariate max-algebraic polynomial equalities and inequalities and vice versa. □

Since a system of multivariate max-algebraic polynomial equalities and inequalities can be transformed into an ELCP and vice versa, the solution set of a system of multivariate max-algebraic polynomial equalities and inequalities can be characterized by the following two propositions:

Proposition 4.1.2 *Let $\mathcal{X}^{\mathrm{cen}}$ be a minimal complete set of central generators of the ELCP that corresponds to a system of multivariate max-algebraic polynomial equalities and inequalities, let $\mathcal{X}^{\mathrm{ext}}$ be a minimal complete set of extreme generators of the ELCP, let $\mathcal{X}^{\mathrm{fin}}$ be a minimal complete set of finite points of the ELCP and let Λ be the set of ordered pairs of maximal cross-complementary subsets of $\mathcal{X}^{\mathrm{ext}}$ and $\mathcal{X}^{\mathrm{fin}}$. Then x is a (finite) solution of the system of multivariate max-algebraic polynomial equalities and inequalities if and only if there exists an ordered pair $\left(\mathcal{X}_s^{\mathrm{ext}}, \mathcal{X}_s^{\mathrm{fin}}\right) \in \Lambda$ such that*

$$x = \sum_{x_k^{\mathrm{c}} \in \mathcal{X}^{\mathrm{cen}}} \lambda_k x_k^{\mathrm{c}} + \sum_{x_k^{\mathrm{e}} \in \mathcal{X}_s^{\mathrm{ext}}} \kappa_k x_k^{\mathrm{e}} + \sum_{x_k^{\mathrm{f}} \in \mathcal{X}_s^{\mathrm{fin}}} \mu_k x_k^{\mathrm{f}} \tag{4.5}$$

with $\lambda_k \in \mathbb{R}$, $\kappa_k \geqslant 0$, $\mu_k \geqslant 0$ for all k and $\sum_k \mu_k = 1$.

Proposition 4.1.3 *In general the set of the (finite) solutions of a system of multivariate max-algebraic polynomial equalities and inequalities consists of the union of faces of a polyhedron.*

We have only considered finite coefficients and solutions with finite components in the formulation of problem (4.1) – (4.2). This was necessary to avoid problems arising from taking negative max-algebraic powers of ε. This assumption also allows us to transform problem (4.1) – (4.2) into an ELCP without causing problems arising from products like $0 \cdot \varepsilon$.
We could allow some of the a_{ki}'s to be equal to ε. However, if a_{ki} is equal to ε, then the corresponding term is also equal to ε, which means that it just disappears. Hence, the assumption that all the a_{ki}'s are finite is not restrictive.
Later on we shall show that the ELCP approach can also be used to solve problem (4.1) – (4.2) if some of the b_k's are equal to ε or if there (only) exist solutions for which some of the components are equal to ε. But first we consider the case where all the b_k's are finite and we show that if problem (4.1) – (4.2) has a solution then it also has a finite solution.

Proposition 4.1.4 *Let $\mathcal{S}$ be a system of multivariate max-algebraic polynomial equalities and inequalities with finite right-hand sides. If there exists a solution x of $\mathcal{S}$, then there also exists a solution $\tilde{x}$ of $\mathcal{S}$ with finite components.*

Proof: If all the components of x are finite, we set $\tilde{x} = x$ and then $\tilde{x}$ is a finite solution of $\mathcal{S}$.
From now on we assume that x has at least one component that is equal to ε. Suppose that $\mathcal{S}$ is defined by (4.1) – (4.2). Define $\Psi = \{\, j \,|\, x_j = \varepsilon \,\}$ and $\Psi^c = \{1, 2, \ldots, n\} \setminus \Psi$. Since negative max-algebraic powers of ε are not defined, x can only be a solution of $\mathcal{S}$ if $c_{kij} \geqslant 0$ for all k, i and all $j \in \Psi$.
Now we define $\tilde{x} \in \mathbb{R}^n$ such that

$$\tilde{x}_j = \begin{cases} x_j & \text{if } j \in \Psi^c \,, \\ M & \text{if } j \in \Psi \,, \end{cases}$$

where M is a real number the exact value of which will be chosen later on: we shall select the value of M such that $\tilde{x}$ will be a solution of $\mathcal{S}$.
Let us now determine under which conditions $\tilde{x}$ will be a solution of $\mathcal{S}$. Since $c_{kij} \geqslant 0$ for all k, i and all $j \in \Psi$, the left-hand sides of the system (4.1) – (4.2) can only increase if we replace x by $\tilde{x}$. Now we determine conditions on M such that the left-hand sides do not increase if we replace x by $\tilde{x}$. If we select M such that

$$a_{ki} \otimes \bigoplus_{j=1}^{n} \tilde{x}_j^{\otimes^{c_{kij}}} \leqslant b_k \tag{4.6}$$

for all k, i, then the left-hand sides of (4.1) – (4.2) will not change if we replace x by $\tilde{x}$ and then $\tilde{x}$ will be a solution of $\mathcal{S}$.

Consider arbitrary indices k and i. Since all the components of $\tilde{x}$ are finite, (4.6) can be rewritten as

$$a_{ki} + \sum_{j \in \Psi^c} c_{kij} \tilde{x}_j + \sum_{j \in \Psi} c_{kij} \tilde{x}_j \leqslant b_k \ ,$$

which is in its turn equivalent to

$$a_{ki} + \sum_{j \in \Psi^c} c_{kij} x_j + \sum_{j \in \Psi} c_{kij} M \leqslant b_k \ . \tag{4.7}$$

If $\sum\limits_{j \in \Psi} c_{kij}$ is equal to 0, then $c_{kij} = 0$ for all $j \in \Psi$ since $c_{kij} \geqslant 0$ for all $j \in \Psi$. So in that case we have $\bigoplus\limits_{j \in \Psi} \tilde{x}_j^{\otimes^{c_{kij}}} = 0 = \bigoplus\limits_{j \in \Psi} x_j{}^{\otimes^{c_{kij}}}$ and since x is a solution of $\mathcal{S}$, this means that condition (4.6) is satisfied.

From now on we assume that $\sum\limits_{j \in \Psi} c_{kij} \neq 0$. Hence, $\sum\limits_{j \in \Psi} c_{kij} > 0$. As a consequence, condition (4.7) can be rewritten as

$$M \leqslant \frac{b_k - a_{ki} - \sum\limits_{j \in \Psi^c} c_{kij} x_j}{\sum\limits_{j \in \Psi} c_{kij}} \tag{4.8}$$

The right-hand side of this expression is defined since b_k, a_{ki} and $\sum\limits_{j \in \Psi^c} c_{kij} x_j$ are finite and since $\sum\limits_{j \in \Psi} c_{kij} \neq 0$.

A sufficient condition for (4.6) to hold is that (4.8) is satisfied for all k, i for which $\sum\limits_{j \in \Psi} c_{kij} \neq 0$. If $\sum\limits_{j \in \Psi} c_{kij} = 0$ for all k, i then we may choose an arbitrary value for M, e.g. $M = 0$. So if we select M such that

$$M = \min \left(\left\{ \frac{b_k - a_{ki} - \sum\limits_{j \in \Psi^c} c_{kij} x_j}{\sum\limits_{j \in \Psi} c_{kij}} \;\middle|\; \sum_{j \in \Psi} c_{kij} \neq 0 \right\} \cup \{0\} \right) , \tag{4.9}$$

then $\tilde{x}$ is a solution of $\mathcal{S}$. Since the right-hand side of (4.9) is finite, M is finite. As a consequence, the components of $\tilde{x}$ are also finite. □

Let $\mathcal{S}$ be a system of multivariate max-algebraic polynomial equalities and inequalities with finite right-hand sides. From Proposition 4.1.4 it follows that if

$\mathcal{S}$ has a solution then we can use the ELCP approach to find a solution of $\mathcal{S}$. Now we present a procedure to reconstruct the solutions of $\mathcal{S}$ that have components that are equal to ε. We shall use the following proposition:

Proposition 4.1.5 *Let $\mathcal{S}$ be a system of multivariate max-algebraic polynomial equalities and inequalities with finite right-hand sides. If x is a solution of $\mathcal{S}$ that has components that are equal to ε, then there exists a solution $\tilde{x}$ of $\mathcal{S}$ with finite components and a vector $v \in \mathbb{R}^n$ such that every combination of the form $x(\eta) = \tilde{x} + \eta v$ with $\eta \in \mathbb{R}^+$ is also a solution of $\mathcal{S}$.*

Proof: Let $\tilde{x} \in \mathbb{R}^n$ be the finite solution of $\mathcal{S}$ obtained by applying the procedure of the proof of Proposition 4.1.4 and let $\Psi = \{\, j \,|\, x_j = \varepsilon \,\}$. Since the right-hand side of (4.9) is actually only an upper bound for M, we still have a solution of $\mathcal{S}$ if we replace every component of $\tilde{x}$ that is equal to M by an arbitrary real number that is less than M. So if we define $v \in \mathbb{R}^n$ such that

$$v_j = \begin{cases} -1 & \text{if } j \in \Psi\,, \\ 0 & \text{otherwise,} \end{cases}$$

then every combination of the form $x(\eta) = \tilde{x} + \eta v$ with $\eta \in \mathbb{R}^+$ is also a solution of $\mathcal{S}$. □

Although Proposition 4.1.5 states that every solution of $\mathcal{S}$ that has components that are equal to ε corresponds to the point at infinity of a ray that lies entirely in the solution set of $\mathcal{S}$, the reverse is not necessarily true as is shown by the following example.

Example 4.1.6 Consider the following problem:

$$\text{Find } x \in \mathbb{R}_\varepsilon^2 \text{ such that } x_1 \otimes x_2 = 0 \text{ and } x_2 \leqslant 0\,. \tag{4.10}$$

Obviously, $x(\eta) = \begin{bmatrix} \eta & -\eta \end{bmatrix}^T$ is a solution of this problem for every $\eta \in \mathbb{R}^+$. So if L is the ray defined by $L = \{x(\eta) \,|\, \eta \in \mathbb{R}^+\}$, then L is a subset of the solution set of problem (4.10). However, the point at infinity of L: $\begin{bmatrix} \infty & \varepsilon \end{bmatrix}^T$ clearly is not a solution of (4.10) since this point does not belong to $\mathbb{R}_\varepsilon^2$. This point is not even a solution in the max-min-plus algebra since we have $\infty \otimes \varepsilon = \varepsilon$ by definition (cf. Section 4.2.4).
Note that it is easy to verify that (4.10) can only have finite solutions. □

Let $\mathcal{S}$ be a system of multivariate max-algebraic polynomial equalities and inequalities of the form (4.1)–(4.2) with finite right-hand sides. Let $\Upsilon = \{\, j \,|\, c_{kij} \geqslant 0 \text{ for all } k, i \,\}$. Since every solution of $\mathcal{S}$ that has components that are equal to ε corresponds to the point at infinity of some ray that lies entirely in the solution set of $\mathcal{S}$, we can reconstruct the solutions of $\mathcal{S}$ that have components that are equal to ε by allowing some of the λ_k's or the κ_k's in (4.5) to become infinite. However, we have to take care that this does not cause any problems arising from taking negative max-algebraic powers of ε. So only components indexed by Υ are allowed to be equal to ε. Furthermore, the solutions should only have infinite components that are equal to $\varepsilon = -\infty$. Components that are equal to ∞ are not allowed since ∞ does not belong to $\mathbb{R}_\varepsilon$. Solutions obtained in this way will correspond to points at infinity of the polyhedron $\mathcal{P}$ defined by the system of linear equalities and inequalities of the ELCP that corresponds to $\mathcal{S}$. This limit technique will be illustrated in Example 4.1.9.

Remark 4.1.7 Let $\mathcal{S}$ be a system of multivariate max-algebraic polynomial equalities and inequalities of the form (4.1) – (4.2). Let $\Upsilon = \{\, j \,|\, c_{kij} \geqslant 0 \text{ for all } k, i \,\}$.
If some of the b_k's are equal to ε, then $\mathcal{S}$ cannot have finite solutions. However, we can still use the ELCP approach to solve $\mathcal{S}$ if we use the following procedure. We introduce a positive real number ξ and we transform every equation of the form $\bigoplus_i t_i = \varepsilon$ or $\bigoplus_i t_i \leqslant \varepsilon$ into $\bigoplus_i t_i \leqslant -\xi$. Now we have a system $\mathcal{S}(\xi)$ of multivariate max-algebraic polynomial equalities and inequalities with finite right-hand sides that can be solved using the ELCP approach. If we let ξ go to ∞ and if we see how the solution set of the intermediate ELCPs evolves, we obtain the solutions of $\mathcal{S}$.
Since the right-hand sides of the linear inequalities of the intermediate ELCPs depend linearly on ξ, the corresponding hyperplanes shift parallelly as ξ varies. This implies that if ξ is large enough the components of the finite points of the solution set of the intermediate ELCPs will depend affinely on ξ, i.e. if ξ is large enough then the ith component of any finite point $x(\xi)$ of $\mathcal{S}(\xi)$ can be written as $x_i(\xi) = a_i\xi + b_i$ for some $a_i, b_i \in \mathbb{R}$. Furthermore, if ξ is large enough then the solution set of all the intermediate ELCPs can be described by the same minimal complete sets of central and extreme generators. So in order to determine how the solution set of the intermediate ELCPs evolves as ξ tends to ∞, we only have to solve a *finite* number of intermediate ELCPs: we solve intermediate ELCPs for some values $\xi_1, \xi_2, \ldots, \xi_r$ of ξ until we notice that from a certain value of ξ on the minimal complete sets of central and extreme generators do not change any more and the components of the finite points depend affinely on ξ.
Note that we have to take care that in this way we do not create solutions with components that are equal to ∞ or solutions with components that are equal to ε but that are not indexed by Υ. Sometimes it is useful to normalize the representation of the solution set of the intermediate ELCPs in order to be able to see how the solution set evolves as ξ increases (See also Section 6.3.3).
Since the $\oplus$ operation hides small numbers from larger numbers, we could also use the following threshold procedure. First we select a positive real number ξ that is several orders of magnitude larger than $\dfrac{\alpha+\beta+\gamma}{\delta}$ where

$$\alpha = \max\{\, |a_{ki}| \,\big|\, a_{ki} \text{ is finite} \,\}$$

$$\beta = \max\{\, |b_k| \,\big|\, b_k \text{ is finite} \,\}$$

$$\gamma = \max\{\, |c_{kij}| \mid c_{kij} \text{ is finite} \,\}$$
$$\delta = \min\{\, c_{kij} \mid c_{kij} > 0 \,\} \ .$$

This heuristic rule for selecting ξ is based on expression (4.9).
Once we have found a solution x of $\mathcal{S}(\xi)$, we replace every negative component of x that has the same order of magnitude as ξ and that is not bounded from below by ε provided that in this way only components indexed by Υ are replaced by ε and provided that x has no positive components of the same order of magnitude as ξ. Note that positive components of the same order of magnitude as ξ would have to be replaced by ∞, but ∞ does not belong to $\mathbb{R}_\varepsilon$. If one uses this threshold technique, it is advisable to check whether the resulting solutions are truly solutions of $\mathcal{S}$ since it is possible that wrong results are obtained if ξ is not large enough.
In Example 4.2.1 we shall illustrate both the limit and the threshold technique that have been discussed in this remark. $\diamond$

Proposition 4.1.8 *Solving a general system of multivariate max-algebraic polynomial equalities and inequalities with rational data is an NP-hard problem.*

Proof: In the proof of Theorem 4.1.1 we have shown that a system of multivariate max-algebraic polynomial equalities and inequalities can be transformed into an ELCP. The time to perform this transformation is polynomial in the size of the problem. Furthermore, if the data of the system of multivariate max-algebraic polynomial equalities and inequalities are rational the data of the corresponding ELCP are also rational. By Theorem 3.5.1 the general ELCP with rational data is an NP-hard problem. Therefore, solving a system of multivariate max-algebraic polynomial equalities and inequalities with rational data is in general also an NP-hard problem. $\Box$

In a similar way we can also prove that solving a system of multivariate max-algebraic polynomial equalities and inequalities with rational data is in general an NP-equivalent problem.

4.1.3 A Worked Example

Now we give an example in which we show how the ELCP apporach can be used to find all solutions of a system of multivariate max-algebraic polynomial equalities and inequalities. Other examples can be found in [41, 47].

Example 4.1.9 Let $\mathcal{S}$ be the following system of multivariate max-algebraic polynomial equalities and inequalities:

$$7 \otimes {x_1}^{\otimes^4} \otimes {x_3}^{\otimes^2} \otimes {x_4}^{\otimes^{-2}} \otimes x_5 \ \oplus \ 6 \otimes {x_1}^{\otimes^{-3}} \otimes {x_2}^{\otimes^{-1}} \ \oplus$$
$${x_2}^{\otimes^{-2}} \otimes {x_3}^{\otimes^3} \otimes {x_4}^{\otimes^{-3}} \otimes x_5 \ = \ 4 \qquad (4.11)$$
$$3 \otimes {x_2}^{\otimes^2} \otimes {x_3}^{\otimes^2} \otimes {x_4}^{\otimes^{-2}} \otimes {x_5}^{\otimes^2} \ \oplus \ 5 \otimes x_1 \otimes {x_2}^{\otimes^3} \ = \ 3 \qquad (4.12)$$
$$1 \otimes {x_1}^{\otimes^{-2}} \otimes {x_3}^{\otimes^3} \otimes {x_4}^{\otimes^{-3}} \otimes x_5 \ \leqslant \ 5 \qquad (4.13)$$

with $x \in \mathbb{R}^5$.
Consider the first term of (4.11). In conventional algebra, this term is equal to

$$7 + 4x_1 + 2x_3 - 2x_4 + x_5 \ .$$

The other terms can be transformed in a similar way. Each term has to be less than or equal to 4 and at least one of the three terms has to be equal to 4. So we get a group of three inequalities in which at least one inequality should hold with equality. If we also take (4.12) and (4.13) into account, we get the following ELCP:

Given

$$A = \begin{bmatrix} -4 & 0 & -2 & 2 & -1 \\ 3 & 1 & 0 & 0 & 0 \\ 0 & 2 & -3 & 3 & -1 \\ 0 & -2 & -2 & 2 & -2 \\ -1 & -3 & 0 & 0 & 0 \\ 2 & 0 & -3 & 3 & -1 \end{bmatrix} \quad \text{and} \quad c = \begin{bmatrix} 3 \\ 2 \\ -4 \\ 0 \\ 2 \\ -4 \end{bmatrix} ,$$

find $x \in \mathbb{R}^5$ such that

$$(Ax - c)_1 \, (Ax - c)_2 \, (Ax - c)_3 \; + \; (Ax - c)_4 \, (Ax - c)_5 = 0$$

subject to $Ax \geqslant c$.

The ELCP algorithm of Section 3.4 yields the generators and the finite points of Table 4.1 and the pairs of maximal cross-complementary subsets of Table 4.2. Any finite solution of $\mathcal{S}$ can now be expressed as

$$x = \lambda_1 x_1^{\mathrm{c}} + \sum_{x_k^{\mathrm{e}} \in \mathcal{X}_s^{\mathrm{ext}}} \kappa_k x_k^{\mathrm{e}} + \sum_{x_k^{\mathrm{f}} \in \mathcal{X}_s^{\mathrm{fin}}} \mu_k x_k^{\mathrm{f}}$$

for some $s \in \{1, 2, \ldots, 5\}$ with $\lambda_1 \in \mathbb{R}$, $\kappa_k \geqslant 0$, $\mu_k \geqslant 0$ for all k and $\sum_k \mu_k = 1$.
Let us now construct a solution of $\mathcal{S}$ that has components that are equal to ε. The only variables for which all the exponents in (4.11) – (4.13) are nonnegative are x_3 and x_5. Hence, only the 3rd and the 5th component of the solutions of $\mathcal{S}$ are allowed to become equal to ε. Let L_1 be the ray defined by

$$\begin{aligned} L_1 &= \{ \, x_1^{\mathrm{f}} + \eta x_4^{\mathrm{e}} + \eta x_5^{\mathrm{e}} \mid \eta \in \mathbb{R}^+ \, \} \\ &= \left\{ \begin{bmatrix} 1 & -1 & 0 & 8 & 9 - 2\eta \end{bmatrix}^T \,\middle|\, \eta \in \mathbb{R}^+ \right\} \ . \end{aligned}$$

Since $\left(\{x_4^{\mathrm{e}}, x_5^{\mathrm{e}}\}, \{x_1^{\mathrm{f}}\} \right)$ is an ordered pair of cross-complementary subsets of $\mathcal{X}^{\mathrm{ext}}$ and $\mathcal{X}^{\mathrm{fin}}$, all the (finite) points of L_1 are solutions of $\mathcal{S}$. The point at infinity of L_1 is given by $v_1 = \begin{bmatrix} 1 & -1 & 0 & 8 & \varepsilon \end{bmatrix}^T$. Since only the 5th component of v_1 is equal to ε, v_1 is a solution of $\mathcal{S}$.

	$\mathcal{X}^{\rm cen}$	$\mathcal{X}^{\rm ext}$					$\mathcal{X}^{\rm fin}$	
	$x_1^{\rm c}$	$x_1^{\rm e}$	$x_2^{\rm e}$	$x_3^{\rm e}$	$x_4^{\rm e}$	$x_5^{\rm e}$	$x_1^{\rm f}$	$x_2^{\rm f}$
x_1	0	3	3	1	0	0	1	1
x_2	0	-1	-1	-3	0	0	-1	-1
x_3	1	0	0	0	0	0	0	0
x_4	1	13	-10	7	1	-1	8	-9
x_5	0	14	-32	10	1	-3	9	-25

Table 4.1: The generators and the finite points of the ELCP of Example 4.1.9.

s	$\mathcal{X}_s^{\rm ext}$	$\mathcal{X}_s^{\rm fin}$
1	$\{x_1^{\rm e}, x_2^{\rm e}\}$	$\{x_1^{\rm f}, x_2^{\rm f}\}$
2	$\{x_1^{\rm e}, x_3^{\rm e}\}$	$\{x_1^{\rm f}\}$
3	$\{x_2^{\rm e}, x_5^{\rm e}\}$	$\{x_2^{\rm f}\}$
4	$\{x_3^{\rm e}, x_4^{\rm e}\}$	$\{x_1^{\rm f}\}$
5	$\{x_4^{\rm e}, x_5^{\rm e}\}$	$\{x_1^{\rm f}, x_2^{\rm f}\}$

Table 4.2: The pairs of maximal cross-complementary subsets of the sets $\mathcal{X}^{\rm ext}$ and $\mathcal{X}^{\rm fin}$ of the ELCP of Example 4.1.9.

Now consider the ray $L_2 = \{ x_1^{\rm f} + \eta x_5^{\rm e} \,|\, \eta \in \mathbb{R}^+ \}$. All the (finite) points of L_2 are solutions of $\mathcal{S}$. However, the point at infinity of L_2, which is given by $v_2 = \begin{bmatrix} 1 & -1 & 0 & \varepsilon & \varepsilon \end{bmatrix}^T$ is not a solution of $\mathcal{S}$ since the 4th component of v_2 is equal to ε. □

4.2 Other Max-Algebraic Problems that Can Be Reformulated as an ELCP

In this section we treat some max-algebraic problems that can be reformulated as a system of multivariate max-algebraic polynomial equalities and inequalities. These problems can thus be solved using the ELCP approach (where we have to take Remark 4.1.7 into account). In general the solution set of the problems treated in this section consists of the union of faces of a polyhedron.

4.2.1 Max-Algebraic Matrix Factorizations

Consider the following problem:

> Given a matrix $T \in \mathbb{R}_\varepsilon^{m \times n}$ and an integer $l \in \mathbb{N}_0$, find $P \in \mathbb{R}_\varepsilon^{m \times l}$ and $Q \in \mathbb{R}_\varepsilon^{l \times n}$ such that $T = P \otimes Q$, or show that no such factorization exists.

So we have to find p_{ik} and q_{kj} for $i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, n$ and $k = 1, 2, \ldots, l$ such that

$$\bigoplus_{k=1}^{l} p_{ik} \otimes q_{kj} = t_{ij} \qquad \text{for } i = 1, 2, \ldots, m \text{ and } j = 1, 2, \ldots, n\,.$$

This can clearly be considered as a system of multivariate max-algebraic polynomial equations in the p_{ik}'s and the q_{kj}'s. Note that all the exponents are nonnegative.
The homogeneous ELCP that corresponds to this system of multivariate max-algebraic polynomial equations has $ml + nl + 1$ variables: the entries of P and Q and one extra variable α to make the ELCP homogeneous. The number of inequalities of the ELCP is equal to $mnl + 1$.
It is obvious that if we take l too small, the problem will not have any solutions. In fact, the smallest value of l for which the problem still has a solution corresponds to the max-algebraic Schein rank of T and is denoted by $\operatorname{rank}_{\oplus,\mathrm{Schein}}(T)$. We always have $\operatorname{rank}_\oplus(T) \leqslant \operatorname{rank}_{\oplus,\mathrm{Schein}}(T)$ [54].
The ELCP approach can also be used to compute the factorization of T as the max-algebraic product of a given number of matrices with specified sizes:

> Given a matrix $T \in \mathbb{R}_\varepsilon^{m \times n}$ and $k + 1$ integers $l_1, l_2, \ldots, l_{k+1} \in \mathbb{N}_0$ with $l_1 = m$ and $l_{k+1} = n$, find matrices $P_1, P_2, \ldots, P_k$ with $P_i \in \mathbb{R}_\varepsilon^{l_i \times l_{i+1}}$ for $i = 1, 2, \ldots, k$ such that $T = P_1 \otimes P_2 \otimes \ldots \otimes P_k$.

Furthermore, it is also possible to impose a certain structure on the composing matrices (e.g. triangular, diagonal, Hessenberg, ...).

Example 4.2.1 Consider the matrix

$$T = \begin{bmatrix} 9 & 7 \\ \varepsilon & 5 \\ 3 & -3 \end{bmatrix} .$$

Let us factorize this matrix as $T = P \otimes Q$ with $P \in \mathbb{R}_\varepsilon^{3 \times 2}$ and $Q \in \mathbb{R}_\varepsilon^{2 \times 2}$. We put all the variables in one large column vector x:

$$x = \begin{bmatrix} p_{11} & p_{12} & p_{21} & p_{22} & p_{31} & p_{32} & q_{11} & q_{12} & q_{21} & q_{22} \end{bmatrix}^T .$$

Since t_{21} is equal to ε, we apply the technique of Remark 4.1.7 and we replace the equation $p_{21} \otimes q_{11} \oplus p_{22} \otimes q_{21} = \varepsilon$ by $p_{21} \otimes q_{11} \oplus p_{22} \otimes q_{21} \leqslant -\xi$. As

	$\mathcal{X}^{\text{cen}}$		$\mathcal{X}^{\text{ext}}$								$\mathcal{X}^{\text{fin}}$	
	x_1^{c}	x_2^{c}	x_1^{e}	x_2^{e}	x_3^{e}	x_4^{e}	x_5^{e}	x_6^{e}	x_7^{e}	x_8^{e}	x_1^{f}	x_2^{f}
p_{11}	-1	0	0	0	0	0	0	0	0	0	0	0
p_{12}	0	-1	0	0	0	0	0	0	0	0	7	3
p_{21}	-1	0	-1	0	0	0	0	0	0	0	-1009	-2
p_{22}	0	-1	0	-1	0	0	0	0	0	0	5	-1006
p_{31}	-1	0	0	0	-1	0	0	0	0	0	-6	-10
p_{32}	0	-1	0	0	0	-1	0	0	0	0	-3	-3
q_{11}	1	0	0	0	0	0	-1	0	0	0	9	-998
q_{12}	1	0	0	0	0	0	0	-1	0	0	3	7
q_{21}	0	1	0	0	0	0	0	0	-1	0	-1005	6
q_{22}	0	1	0	0	0	0	0	0	0	-1	0	0

Table 4.3: The generators and the finite points of the ELCP of Example 4.2.1 for $\xi = 1000$.

s	$\mathcal{X}_s^{\text{ext}}$	$\mathcal{X}_s^{\text{fin}}$	s	$\mathcal{X}_s^{\text{ext}}$	$\mathcal{X}_s^{\text{fin}}$
1	$\{x_1^{\text{e}}, x_4^{\text{e}}, x_7^{\text{e}}\}$	$\{x_1^{\text{f}}\}$	3	$\{x_2^{\text{e}}, x_3^{\text{e}}, x_5^{\text{e}}\}$	$\{x_2^{\text{f}}\}$
2	$\{x_1^{\text{e}}, x_6^{\text{e}}, x_7^{\text{e}}\}$	$\{x_1^{\text{f}}\}$	4	$\{x_2^{\text{e}}, x_5^{\text{e}}, x_8^{\text{e}}\}$	$\{x_2^{\text{f}}\}$

Table 4.4: The pairs of maximal cross-complementary subsets of the sets $\mathcal{X}^{\text{ext}}$ and $\mathcal{X}^{\text{fin}}$ of Example 4.2.1.

explained in Remark 4.1.7 there are two methods to obtain the solutions of the ELCP that corresponds to $\xi = \infty$.

Let us first use the threshold technique. If we set $\xi = 1000$, the ELCP algorithm of Section 3.4 yields the generators and the finite points of Table 4.3 and the pairs of maximal cross-complementary subsets of Table 4.4. Since all the exponents in the system of multivariate max-algebraic polynomial equalities and inequalities that corresponds to $T = P \otimes Q$ are nonnegative, every entry of P and Q is allowed to become equal to ε. Consider x_1^{f}. The p_{21} and the q_{21} component of x_1^{f} are negative numbers of the same order of magnitude as ξ. Since $\left(\{x_1^{\text{e}}, x_7^{\text{e}}\}, \{x_1^{\text{f}}\}\right)$ is an ordered pair of cross-complementary subsets of $\mathcal{X}^{\text{ext}}$ and $\mathcal{X}^{\text{fin}}$, the p_{21} and the q_{21} component of x_1^{f} are not bounded from below. Furthermore, x_1^{f} has no positive components of the same order of magnitude as ξ. Therefore we replace the p_{21} and the q_{11} component of x_1^{f} by ε. This yields

	$\mathcal{X}^{\text{fin}}$	
	x_1^{f}	x_2^{f}
p_{11}	0	0
p_{12}	7	3
p_{21}	-1010	-2
p_{22}	5	-1007
p_{31}	-6	-10
p_{32}	-3	-3
q_{11}	9	-999
q_{12}	3	7
q_{21}	-1006	6
q_{22}	0	0

(a) $\xi = 1001$

	$\mathcal{X}^{\text{fin}}$	
	x_1^{f}	x_2^{f}
p_{11}	0	0
p_{12}	7	3
p_{21}	-1011	-2
p_{22}	5	-1008
p_{31}	-6	-10
p_{32}	-3	-3
q_{11}	9	-1000
q_{12}	3	7
q_{21}	-1007	6
q_{22}	0	0

(b) $\xi = 1002$

	$\mathcal{X}^{\text{fin}}$	
	x_1^{f}	x_2^{f}
p_{11}	0	0
p_{12}	7	3
p_{21}	-1012	-2
p_{22}	5	-1009
p_{31}	-6	-10
p_{32}	-3	-3
q_{11}	9	-1001
q_{12}	3	7
q_{21}	-1008	6
q_{22}	0	0

(c) $\xi = 1003$

Table 4.5: The finite points of the ELCP of Example 4.2.1 for $\xi = 1001, 1002, 1003$.

the following factorization of T:

$$T = \begin{bmatrix} 0 & 7 \\ \varepsilon & 5 \\ -6 & -3 \end{bmatrix} \otimes \begin{bmatrix} 9 & 3 \\ \varepsilon & 0 \end{bmatrix} . \tag{4.14}$$

We can also apply the same reasoning to x_2^{f}. For x_2^{f} we have to replace the p_{22} and the q_{11} component by ε. It is easy to verify that this also results in a valid factorization of T.

Alternatively, we can use the limit technique to compute max-algebraic matrix factorizations of T. If we use the ELCP algorithm of Section 3.4 to compute the solution set of the ELCP for some values of ξ that are greater than 1000, then we see that for any $\xi \geqslant 1000$ the central and the extreme generators are always the same as those of Table 4.3. In Table 4.5 we have listed the finite points of the solution set of the ELCP for some values of ξ that are greater than 1000. Since we know that the components of the finite points depend affinely on ξ if ξ is large enough, we conclude that the finite points are given by

$$\begin{aligned} x_1^{\text{f}}(\xi) &= \begin{bmatrix} 0 & 7 & -\xi-9 & 5 & -6 & -3 & 9 & 3 & -\xi-5 & 0 \end{bmatrix}^T \\ x_2^{\text{f}}(\xi) &= \begin{bmatrix} 0 & 3 & -2 & -\xi-6 & -10 & -3 & -\xi+2 & 7 & 6 & 0 \end{bmatrix}^T \end{aligned}$$

for any $\xi \geqslant 1000$. The pairs of maximal cross-complementary subsets are the same as those of Table 4.4 for any $\xi \geqslant 1000$ but with x_1^{f} and x_2^{f} replaced by

	$\tilde{\mathcal{X}}^{\text{cen}}$		$\tilde{\mathcal{X}}^{\text{ext}}$				$\tilde{\mathcal{X}}^{\text{fin}}$	
	$\tilde{x}_1^{\text{c}}$	$\tilde{x}_2^{\text{c}}$	$\tilde{x}_1^{\text{e}}$	$\tilde{x}_2^{\text{e}}$	$\tilde{x}_3^{\text{e}}$	$\tilde{x}_4^{\text{e}}$	$\tilde{x}_1^{\text{f}}$	$\tilde{x}_2^{\text{f}}$
p_{11}	-1	0	0	0	0	0	0	0
p_{12}	0	-1	0	0	0	0	7	3
p_{21}	-1	0	0	0	0	0	ε	-2
p_{22}	0	-1	0	0	0	0	5	ε
p_{31}	-1	0	-1	0	0	0	-6	-10
p_{32}	0	-1	0	-1	0	0	-3	-3
q_{11}	1	0	0	0	0	0	9	ε
q_{12}	1	0	0	0	-1	0	3	7
q_{21}	0	1	0	0	0	0	ε	6
q_{22}	0	1	0	0	0	-1	0	0

Table 4.6: The generators and the "finite" points of the ELCP of Example 4.2.1 for $\xi = \infty$.

$x_1^{\text{f}}(\xi)$ and $x_2^{\text{f}}(\xi)$ respectively. If we take the limit of $x_1^{\text{f}}(\xi)$ for ξ going ∞, we obtain again factorization (4.14).

Since the p_{21} component of x_1^{f} tends to ε as ξ tends to ∞, x_1^{e} becomes redundant when ξ goes to ∞. This also holds for x_2^{e}, x_5^{e} and x_7^{e}. So if we define $\tilde{x}_1^{\text{e}} = x_3^{\text{e}}$, $\tilde{x}_2^{\text{e}} = x_4^{\text{e}}$, $\tilde{x}_3^{\text{e}} = x_6^{\text{e}}$ and $\tilde{x}_4^{\text{e}} = x_8^{\text{e}}$, and $\tilde{x}_i^{\text{c}} = x_i^{\text{c}}$ and $\tilde{x}_i^{\text{f}} = \lim\limits_{\xi\to\infty} x_i^{\text{f}}(\xi)$ for $i = 1, 2$, then the set of solutions that corresponds to $\xi = \infty$ is described by the generators and the "finite" points of Table 4.6 and the set

$$\tilde{\Lambda} \;=\; \big\{ \big(\{\tilde{x}_2^{\text{e}}\}, \{\tilde{x}_1^{\text{f}}\}\big), \big(\{\tilde{x}_3^{\text{e}}\}, \{\tilde{x}_1^{\text{f}}\}\big), \big(\{\tilde{x}_1^{\text{e}}\}, \{\tilde{x}_2^{\text{f}}\}\big), \big(\{\tilde{x}_4^{\text{e}}\}, \{\tilde{x}_2^{\text{f}}\}\big) \big\}$$

of ordered pairs of maximal cross-complementary subsets of $\tilde{\mathcal{X}}^{\text{ext}}$ and $\tilde{\mathcal{X}}^{\text{fin}}$.

Any combination of the form

$$x \;=\; \lambda_1 \, \tilde{x}_1^{\text{c}} + \lambda_2 \, \tilde{x}_2^{\text{c}} + \kappa_k \, \tilde{x}_k^{\text{e}} + \tilde{x}_k^{\text{f}}$$

with $\big(\tilde{\mathcal{X}}_k^{\text{ext}}, \tilde{\mathcal{X}}_k^{\text{fin}}\big) \in \tilde{\Lambda}$ and with $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\kappa_k \geqslant 0$ corresponds to a factorization $P \otimes Q$ of T with $P \in \mathbb{R}_\varepsilon^{3\times 2}$ and $Q \in \mathbb{R}_\varepsilon^{2\times 2}$.

Now can use the same limit technique as the one used for systems of multivariate max-algebraic polynomial equalities and inequalities with finite right-hand sides to obtain other solutions that have components that are equal to ε. Consider e.g. the combination $x(\eta) = \eta \tilde{x}_2^{\text{e}} + \tilde{x}_1^{\text{f}}$ with $\eta \in \mathbb{R}^+$. If we take the limit of $x(\eta)$ for η going to ∞, we get the following factorization:

$$T = \begin{bmatrix} 0 & 7 \\ \varepsilon & 5 \\ -6 & \varepsilon \end{bmatrix} \otimes \begin{bmatrix} 9 & 3 \\ \varepsilon & 0 \end{bmatrix} \ . \qquad \square$$

4.2.2 Systems of Max-Linear Balances

In this subsection we address the following problem:

Given $A \in \mathbb{S}^{m \times n}$ and $b \in \mathbb{S}^m$, find $x \in (\mathbb{S}^\vee)^n$ such that $A \otimes x \nabla b$.

Now we show that this problem can be reformulated as a system of multivariate max-algebraic polynomial equalities and inequalities and that it can be solved using the ELCP approach.
First we assume that all the components of b are finite. The condition that the entries of x have to be signed leads to

$$x_i^\oplus \otimes x_i^\ominus = \varepsilon \qquad \text{for } i = 1, 2, \ldots, n\,. \tag{4.15}$$

If we extract the max-positive and the max-negative parts of A, b and x, then $Ax \nabla b$ can be rewritten as

$$(A^\oplus \ominus A^\ominus) \otimes (x^\oplus \ominus x^\ominus) \nabla b^\oplus \ominus b^\ominus .$$

If we write out the max-multiplications in this expression and if we use Proposition 2.3.6, we get

$$A^\oplus \otimes x^\oplus \oplus A^\ominus \otimes x^\ominus \oplus b^\ominus \nabla A^\oplus \otimes x^\ominus \oplus A^\ominus \otimes x^\oplus \oplus b^\oplus \ .$$

Both sides of this balance are signed. So by Proposition 2.3.7 we may replace the balance by an equality:

$$A^\oplus \otimes x^\oplus \oplus A^\ominus \otimes x^\ominus \oplus b^\ominus = A^\oplus \otimes x^\ominus \oplus A^\ominus \otimes x^\oplus \oplus b^\oplus \ . \tag{4.16}$$

Now we define a vector $p \in \mathbb{R}_\varepsilon^n$ such that

$$p = A^\oplus \otimes x^\oplus \oplus A^\ominus \otimes x^\ominus \oplus b^\ominus \ . \tag{4.17}$$

From (4.16) it follows that we also have

$$p = A^\oplus \otimes x^\ominus \oplus A^\ominus \otimes x^\oplus \oplus b^\oplus \ . \tag{4.18}$$

Since we have assumed that all the components of b are finite, the components of p are also finite and their max-algebraic inverses are defined. If we work out the max-multiplications in (4.17) and (4.18) and if we transfer the components of p to the other side, we get

$$\bigoplus_{j=1}^{n} a_{ij}^\oplus \otimes x_j^\oplus \otimes {p_i}^{\otimes^{-1}} \oplus \bigoplus_{j=1}^{n} a_{ij}^\ominus \otimes x_j^\ominus \otimes {p_i}^{\otimes^{-1}} \oplus b_i^\ominus \otimes {p_i}^{\otimes^{-1}} = 0 \tag{4.19}$$

$$\bigoplus_{j=1}^{n} a_{ij}^\oplus \otimes x_j^\ominus \otimes {p_i}^{\otimes^{-1}} \oplus \bigoplus_{j=1}^{n} a_{ij}^\ominus \otimes x_j^\oplus \otimes {p_i}^{\otimes^{-1}} \oplus b_i^\oplus \otimes {p_i}^{\otimes^{-1}} = 0 \tag{4.20}$$

for $i = 1, 2, \ldots, m$.
Clearly, equations (4.15) and (4.19) – (4.20) constitute a system of multivariate max-algebraic polynomial equalities. Using the technique explained in Section 4.1 and by taking Remark 4.1.7 into account, we can transform this system of multivariate max-algebraic polynomial equalities into an ELCP. The corresponding homogeneous ELCP will have $2n + m + 1$ variables and consist of $2m(2n+1) + n + 1 - 2n_\varepsilon$ inequalities where n_ε is the total number of entries of $A^\oplus$ and $A^\ominus$ that are equal to ε.

Note that if some of the components of b are not finite it is possible that some of the corresponding components of p are also not finite, which means that their max-algebraic inverses are not defined. In that case we have to use a technique that is similar to the one that has been discussed in Remark 4.1.7: we replace the infinite components of b by $(-\xi)^\bullet$ and then see how the solution set of the resulting ELCP evolves as ξ goes to ∞.
Once we have determined a solution of the ELCP that corresponds to $\xi = \infty$, we extract the $x_i^\oplus$'s and the $x_i^\ominus$'s and we set $x_i = x_i^\oplus \ominus x_i^\ominus$ for $i = 1, 2, \ldots, n$. Since the components of p are dummy variables that do not appear in the balance $A \otimes x \;\nabla\; b$, there will be no problems arising from taking negative max-algebraic powers of ε. Obviously, we still have to take care that we do not create solutions with components that are equal to ∞ when we use this technique.

The problem of finding a normalized solution a system of homogeneous max-linear balances:

$$\text{Given } A \in \mathbb{S}^{m \times n}, \text{ find } x \in (\mathbb{S}^\vee)^n \text{ such that } A \otimes x \;\nabla\; \mathcal{E}_{m\times 1} \text{ and } \|x\|_\oplus = 0 \,,$$

can also be solved using the ELCP approach.

Remark 4.2.2 If we only need one solution of a system of (homogeneous) max-linear balances, we normally do not have to use the ELCP technique since there exist more efficient algorithms to find one solution of a system of (homogeneous) max-linear balances (See [3, 54, 106]). $\diamond$

4.2.3 Other Problems in $\mathbb{R}_{\max}$ and $\mathbb{S}_{\max}$

In the next chapters we shall show that the following problems can also be reformulated as an ELCP or solved using the ELCP approach:

- performing max-algebraic state space transformations (See Section 6.2),
- determining partial or minimal state space realizations of the impulse response of a max-linear time-invariant DES (See Section 6.3),
- constructing matrices with a given max-algebraic characteristic polynomial (See Section 5.4),
- determining a singular value decomposition or a QR decomposition of a matrix in the symmetrized max-plus algebra (See Section 7.5).

4.2.4 Mixed Max-Min Problems

We can also use the technique of Section 4.1 to solve mixed max-min problems. Let us first introduce the max-min-plus algebra, which is an extension of the max-plus algebra. Let $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$. We still have $x \oplus y = \max(x, y)$ for all $x, y \in \overline{\mathbb{R}}$. The $\oplus'$ operation in $\overline{\mathbb{R}}$ is defined as follows: $x \oplus' y = \min(x, y)$ for all $x, y \in \overline{\mathbb{R}}$. We extend the definition of the $\otimes$ operation such that

$$\begin{aligned} &x \otimes y = x + y && \text{for all } x, y \in \overline{\mathbb{R}} \setminus \{-\infty\}\,, \\ &x \otimes (-\infty) = (-\infty) \otimes x = -\infty && \text{for all } x \in \overline{\mathbb{R}}\,. \end{aligned}$$

The resulting structure $(\overline{\mathbb{R}}, \oplus, \oplus', \otimes)$ is called the *max-min-plus algebra*. As for the order of evaluation of the max-algebraic operators, the $\oplus'$ operator has the same priority as the $\oplus$ operator. For more information about the max-min-plus algebra, the interested reader is referred to [33, 37, 38, 74, 119, 120, 121].
Consider the following problem:

> Given integers $m_k, m_{kl_1} \in \mathbb{N}_0$ for $k = 1, 2, \ldots, m$ and $l_1 = 1, 2, \ldots, m_k$ and real numbers $a_{kl_1l_2}$, b_k and $c_{kl_1l_2j}$ for $k = 1, 2, \ldots, m$, $l_1 = 1, 2, \ldots, m_k$, $l_2 = 1, 2, \ldots, m_{kl_1}$ and $j = 1, 2, \ldots, n$, find a vector $x \in \mathbb{R}^n$ that satisfies
>
> $$\bigoplus_{l_1=1}^{m_k}{}' \bigoplus_{l_2=1}^{m_{kl_1}} a_{kl_1l_2} \otimes \bigotimes_{j=1}^{n} x_j{}^{\otimes^{c_{kl_1l_2j}}} = b_k \qquad \text{for } k = 1, 2, \ldots, m\,, \tag{4.21}$$
>
> or show that no such vector exists.

Note that just like in the formulation of a system of multivariate max-algebraic polynomial equalities and inequalities in Section 4.1 we have only considered finite coefficients and solutions with finite components in the formulation of this problem.
Now we show that problem (4.21) can also be reformulated as an ELCP. If we define

$$t_{kl_1} = \bigoplus_{l_2=1}^{m_{kl_1}} a_{kl_1l_2} \otimes \bigotimes_{j=1}^{n} x_j{}^{\otimes^{c_{kl_1l_2j}}} \tag{4.22}$$

for $k = 1, 2, \ldots, m$ and $l_1 = 1, 2, \ldots, m_k$, we get

$$\bigoplus_{l_1=1}^{m_k}{}' t_{kl_1} = b_k \qquad \text{for } k = 1, 2, \ldots, m\,. \tag{4.23}$$

If we assume that the b_k's are finite, then the t_{kl_1}'s are also finite. Therefore, their max-algebraic inverses exist and (4.22) can be rewritten as

$$\bigoplus_{l_2=1}^{m_{kl_1}} a_{kl_1l_2} \otimes \bigotimes_{j=1}^{n} x_j{}^{\otimes^{c_{kl_1l_2j}}} \otimes t_{kl_1}{}^{\otimes^{-1}} = 0 \tag{4.24}$$

for $k = 1, 2, \ldots, m$ and $l_1 = 1, 2, \ldots, m_k$. Each equation of the form $\displaystyle\bigoplus_{l_1=1}^{m_k}{}' \, t_{kl_1} = b_k$ is equivalent to a system of linear inequalities of the form

$$t_{kl_1} \geqslant b_k \qquad \text{for } l_1 = 1, 2, \ldots, m_k$$

where at least one inequality should hold with equality. So (4.23) yields m groups of inequalities where in each group at least one inequality should hold with equality.

Equations of the form (4.24) are multivariate max-algebraic polynomial equations and can thus also be rewritten as groups of linear inequalities where in each group at least one inequality should hold with equality.
This means that the combined max-min problem (4.21) can be transformed into an ELCP.
We can also use this technique for systems of combined max-min equations of the form

$$\bigoplus_{l_1}{}' \bigoplus_{l_2} \bigoplus_{l_3}{}' \ldots \bigoplus_{l_q} a_{kl_1l_2\ldots l_q} \otimes \bigotimes_{j=1}^{n} x_i{}^{\otimes^{c_{kl_1l_2\ldots l_qj}}} = b_k \qquad \text{for } k = 1, 2, \ldots, m\,,$$

or for analogous equations but with $\oplus$ replaced by $\oplus'$ and vice versa or when some of the equalities are replaced by inequalities.

4.2.5 Max-Max and Max-Min Problems

In this section we consider systems of max-algebraic equations that also have multivariate max-algebraic polynomials (instead of constants) on the right-hand side. Since we are working in $\mathbb{R}_{\max}$, we cannot simply transfer terms from the right-hand side to the left-hand side as we would do in conventional algebra. However, these problems can also be transformed into an ELCP using a technique that is similar to the one used in Section 4.2.4.
Consider the following problem:

> Given integers $m_k, p_k \in \mathbb{N}_0$ for $k = 1, 2, \ldots, m$ and real numbers a_{ki}, b_{kij}, c_{kl} and d_{klj} for $k = 1, 2, \ldots, m$, $i = 1, 2, \ldots, m_k$, $j = 1, 2, \ldots, n$ and $l = 1, 2, \ldots, p_k$, find $x \in \mathbb{R}^n$ such that
>
> $$\bigoplus_{i=1}^{m_k} a_{ki} \otimes \bigotimes_{j=1}^{n} x_j{}^{\otimes^{b_{kij}}} = \bigoplus_{l=1}^{p_k} c_{kl} \otimes \bigotimes_{j=1}^{n} x_j{}^{\otimes^{d_{klj}}} \tag{4.25}$$
>
> for $k = 1, 2, \ldots, m$.

We define m dummy variables $t_1, t_2, \ldots, t_m$ such that

$$\bigoplus_{i=1}^{m_k} a_{ki} \otimes \bigotimes_{j=1}^{n} x_j{}^{\otimes^{b_{kij}}} = t_k \qquad \text{for } k = 1, 2, \ldots m\,.$$

Since the a_{ki}'s and the b_{kij}'s are finite and since the components of x are finite, the t_k's are also finite and therefore their max-algebraic inverses exist. So problem (4.25) is equivalent to

$$\bigoplus_{i=1}^{m_k} a_{ki} \otimes \bigotimes_{j=1}^{n} x_j{}^{\otimes^{b_{kij}}} \otimes t_k{}^{\otimes^{-1}} = 0 \qquad \text{for } k = 1, 2, \ldots, m\,,$$

$$\bigoplus_{l=1}^{p_k} c_{kl} \otimes \bigotimes_{j=1}^{n} x_j{}^{\otimes^{d_{klj}}} \otimes t_k{}^{\otimes^{-1}} = 0 \qquad \text{for } k = 1, 2, \ldots, m\,.$$

This is a system of multivariate max-algebraic polynomial equalities that can be transformed into an ELCP.

Using an analogous reasoning we can also transform problems that contain a mixture of equations of the following forms into an ELCP:

- $\displaystyle{\bigoplus_i}' \, l_i(x) = \bigoplus_i r_i(x)$
- $\displaystyle{\bigoplus_i}' \, l_i(x) = {\bigoplus_i}' \, r_i(x)$
- $\displaystyle{\bigoplus_i}' \, l_i(x) \leqslant \bigoplus_i r_i(x)$
- $\displaystyle{\bigoplus_i}' \, l_i(x) \geqslant \bigoplus_i r_i(x)$
- $\displaystyle\bigoplus_i l_i(x) \leqslant \bigoplus_i r_i(x)$
- $\displaystyle{\bigoplus_i}' \, l_i(x) \leqslant {\bigoplus_i}' \, r_i(x)$

where $l_i(x)$ and $r_i(x)$ are max-algebraic monomials of the form $a_i \otimes \displaystyle\bigotimes_{j=1}^{n} x_j{}^{\otimes^{b_{ij}}}$.

Remark 4.2.3 Other problems such as solving a system of max-linear equations, solving eigenvalue problems in the max-plus algebra, determining the roots of a max-algebraic characteristic polynomial and so on can also be transformed into an ELCP, but for these problems there are already efficient algorithms available, especially if we only want one solution (See e.g. [3, 9, 33, 54]). ◇

4.3 Conclusion

We have demonstrated that many problems in the max-plus algebra, the symmetrized max-plus algebra and the max-min-plus algebra such as solving a system of multivariate max-algebraic polynomial equalities and inequalities, calculating max-algebraic matrix factorizations, solving a system of max-linear balances, mixed max-min problems, max-max problems and max-min problems can be transformed into an ELCP. This gives us an insight in the geometrical structure of the solution set of these problems. This insight could be used to develop more efficient algorithms to solve these problems or to solve particular subclasses of the general problem of solving a system of multivariate max-algebraic polynomial equalities and inequalities.

Our ELCP algorithm yields all solutions of the ELCP that corresponds to a given max-algebraic problem. However, sometimes finding one solution suffices. Therefore, it might be interesting to develop (heuristic) algorithms that determine only one solution as we have done for the construction of matrices with a given MACP [43] (See also Section B.4).

It could also be interesting to make a more thorough study of the class of problems that can be reduced to solving a system of multivariate max-algebraic polynomial equalities and inequalities and to determine the computational complexity of the problems that belong to this class. We already know that in general the ELCP and the problem of solving a system of multivariate max-algebraic polynomial equalities and inequalities are NP-hard. However, it is still an open question whether the other problems that have been treated in this chapter are also NP-hard.

Chapter 5

The Max-Algebraic Characteristic Polynomial

In this chapter we determine necessary and for some cases also sufficient conditions for a max-algebraic polynomial with coefficients in $\mathbb{S}$ to be the max-algebraic characteristic polynomial of a matrix with elements in $\mathbb{R}_\varepsilon$. We also show that the problem of finding a matrix that has a given max-algebraic polynomial as its max-algebraic characteristic polynomial can be reformulated as an ELCP.

This chapter is organized as follows. In Section 5.1 we define the max-algebraic characteristic polynomial of a matrix and we give some definitions and properties. In Section 5.2 we derive necessary conditions for the coefficients of the max-algebraic characteristic polynomial of a matrix with entries in $\mathbb{R}_\varepsilon$. In Section 5.3 we give necessary and sufficient conditions for the coefficients of the max-algebraic characteristic polynomial of a matrix with entries in $\mathbb{R}_\varepsilon$ and with a dimension that is less than or equal to 4. Later on these results will be used to determine a lower bound for the minimal order of a state space description of a max-linear time-invariant DES (See Section 6.3.1). In Section 5.4 we show that the problem of constructing a matrix with a max-algebraic characteristic polynomial that is equal to a given max-algebraic polynomial can be reformulated as an ELCP.

5.1 Introduction

In this section we give the definitions and some properties of the max-algebraic characteristic equation and the max-algebraic characteristic polynomial.

Definition 5.1.1 (Max-algebraic characteristic equation) *Consider a matrix $A \in \mathbb{S}^{n \times n}$. The max-algebraic characteristic equation of A is given by* $\det_\oplus (A \ominus \lambda \otimes E_n) \,\nabla\, \varepsilon$.

Consider a matrix $A \in \mathbb{R}_\varepsilon^{n \times n}$. Let $\lambda \in \mathbb{R}_\varepsilon$ be a max-algebraic eigenvalue of A and let $v \in \mathbb{R}_\varepsilon^n$ be a corresponding max-algebraic eigenvector. Then we have $A \otimes v = \lambda \otimes v$ and thus also $A \otimes v \nabla \lambda \otimes v$, which can be rewritten as $(A \ominus \lambda \otimes E_n) \otimes v \nabla \mathcal{E}_{n\times 1}$ by Proposition 2.3.6. This means that the system of homogeneous max-linear balances $(A \ominus \lambda \otimes E_n) \otimes x \nabla \mathcal{E}_{n\times 1}$ has a non-trivial signed solution. Hence, $\det_\oplus (A \ominus \lambda \otimes E_n) \nabla \varepsilon$ by Theorem 2.3.15. So every max-algebraic eigenvalue of A is a "root" of the max-algebraic characteristic equation of A.

Consider $A \in \mathbb{S}^{n\times n}$. Since $\det_\oplus (A \ominus \lambda \otimes E_n) = (\ominus 0)^{\otimes^n} \otimes \det_\oplus (\lambda \otimes E_n \ominus A)$ and since we have $x \nabla \varepsilon \Leftrightarrow \ominus x \nabla \varepsilon$ for all $x \in \mathbb{S}$, the max-algebraic characteristic equation of A may also be represented by $\det_\oplus (\lambda \otimes E_n \ominus A) \nabla \varepsilon$.

Proposition 5.1.2 *Consider a matrix $A \in \mathbb{S}^{n\times n}$. If we write the formula $\det_\oplus (\lambda \otimes E_n \ominus A) \nabla \varepsilon$ out, we get $\lambda^{\otimes^n} \oplus \bigoplus_{k=1}^{n} a_k \otimes \lambda^{\otimes^{n-k}} \nabla \varepsilon$ with*

$$a_k = (\ominus 0)^{\otimes^k} \otimes \bigoplus_{\varphi \in \mathcal{C}_n^k} \det_\oplus A_{\varphi\varphi}$$

or

$$a_k = (\ominus 0)^{\otimes^k} \bigoplus_{\{i_1,\ldots,i_k\} \in \mathcal{C}_n^k} \ \bigoplus_{\sigma \in \mathcal{P}_k} \operatorname{sgn}_\oplus(\sigma) \otimes \bigotimes_{r=1}^{k} a_{i_r i_{\sigma(r)}} \tag{5.1}$$

for $k = 1, 2, \ldots, n$.

Proof: These formulas are the max-algebraic equivalent of similar formulas for the coefficients of the characteristic polynomial of a matrix in linear algebra. $\Box$

Let $A \in \mathbb{S}^{n\times n}$. The max-algebraic polynomial $\lambda^{\otimes^n} \oplus \bigoplus_{k=1}^{n} a_k \otimes \lambda^{\otimes^{n-k}}$ obtained by writing out $\det_\oplus (\lambda \otimes E_n \ominus A)$ is called the *max-algebraic characteristic polynomial* (MACP) of A. Since the coefficient of $\lambda^{\otimes^n}$ is equal to 0 (the identity element for $\otimes$), we say that this polynomial is a *monic* max-algebraic polynomial. From (5.1) it follows that $a_1 = \bigoplus_{i=1}^{n} a_{ii}$ and $a_n = (\ominus 0)^{\otimes^n} \otimes \det_\oplus A$.

Theorem 5.1.3 (Cayley-Hamilton) *In $\mathbb{S}_{\max}$ every square matrix satisfies its max-algebraic characteristic equation: if $A \in \mathbb{S}^{n\times n}$ and if $\lambda^{\otimes^n} \oplus \bigoplus_{k=1}^{n} a_k \otimes \lambda^{\otimes^{n-k}} \nabla \varepsilon$ is the max-algebraic characteristic equation of A, then we have*

$$A^{\otimes^n} \oplus \bigoplus_{k=1}^{n} a_k \otimes A^{\otimes^{n-k}} \nabla \mathcal{E}_{n\times n} \ .$$

Proof: See [109]. □

Let us illustrate these definitions and properties by an example:

Example 5.1.4 Consider again the matrix A of Example 2.2.6:

$$A = \begin{bmatrix} -2 & 1 & \varepsilon \\ 1 & 0 & 1 \\ \varepsilon & 0 & 2 \end{bmatrix} .$$

In Example 2.3.13 we have already calculated the max-algebraic determinant of A: $\det_\oplus A = \ominus 4$.
The coefficients of the MACP of A are given by

$$a_1 = \ominus a_{11} \ominus a_{22} \ominus a_{33} = \ominus(-2) \ominus 0 \ominus 2 = \ominus 2$$

$$\begin{aligned} a_2 &= \det\nolimits_\oplus A_{\{1,2\},\{1,2\}} \oplus \det\nolimits_\oplus A_{\{1,3\},\{1,3\}} \oplus \det\nolimits_\oplus A_{\{2,3\},\{2,3\}} \\ &= \det\nolimits_\oplus \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix} \oplus \det\nolimits_\oplus \begin{bmatrix} -2 & \varepsilon \\ \varepsilon & 2 \end{bmatrix} \oplus \det\nolimits_\oplus \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \\ &= -2 \ominus 2 \oplus 0 \ominus \varepsilon \oplus 2 \ominus 1 = 2^\bullet \end{aligned}$$

$$a_3 = \ominus \det\nolimits_\oplus A = 4 ,$$

and therefore the max-algebraic characteristic equation of A is given by

$$\lambda^{\otimes^3} \ominus 2 \otimes \lambda^{\otimes^2} \oplus 2^\bullet \otimes \lambda \oplus 4 \,\nabla\, \varepsilon .$$

The max-algebraic eigenvalue $\lambda = 2$ that has been calculated in Example 2.2.9 satisfies the max-algebraic characteristic equation of A since

$$2^{\otimes^3} \ominus 2 \otimes 2^{\otimes^2} \oplus 2^\bullet \otimes 2 \oplus 4 = 6 \ominus 6 \oplus 4^\bullet \oplus 4 = 6^\bullet \,\nabla\, \varepsilon .$$

Furthermore,

$$\begin{aligned} & A^{\otimes^3} \ominus 2 \otimes A^{\otimes^2} \oplus 2^\bullet \otimes A \oplus 4 \otimes E_3 \\ &= \begin{bmatrix} 2 & 3 & 4 \\ 3 & 3 & 5 \\ 3 & 4 & 6 \end{bmatrix} \oplus \begin{bmatrix} \ominus 4 & \ominus 3 & \ominus 4 \\ \ominus 3 & \ominus 4 & \ominus 5 \\ \ominus 3 & \ominus 4 & \ominus 6 \end{bmatrix} \oplus \\ &\qquad \begin{bmatrix} 0^\bullet & 3^\bullet & \varepsilon \\ 3^\bullet & 2^\bullet & 3^\bullet \\ \varepsilon & 2^\bullet & 4^\bullet \end{bmatrix} \oplus \begin{bmatrix} 4 & \varepsilon & \varepsilon \\ \varepsilon & 4 & \varepsilon \\ \varepsilon & \varepsilon & 4 \end{bmatrix} \\ &= \begin{bmatrix} 4^\bullet & 3^\bullet & 4^\bullet \\ 3^\bullet & 4^\bullet & 5^\bullet \\ 3^\bullet & 4^\bullet & 6^\bullet \end{bmatrix} \nabla\, \mathcal{E}_{3\times 3} . \end{aligned}$$

So A satisfies its max-algebraic characteristic equation. □

Proposition 5.1.5 *Every monic nth degree max-algebraic polynomial is the MACP of an n by n matrix with entries in* $\mathbb{S}$.

Proof: It is easy to verify that the monic max-algebraic polynomial

$$\lambda^{\otimes^n} \oplus a_1 \otimes \lambda^{\otimes^{n-1}} \oplus \ldots \oplus a_{n-1} \otimes \lambda \oplus a_n$$

is the MACP of the matrix

$$A = \begin{bmatrix} \varepsilon & 0 & \varepsilon & \ldots & \varepsilon \\ \varepsilon & \varepsilon & 0 & \ldots & \varepsilon \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \varepsilon & \varepsilon & \varepsilon & \ldots & 0 \\ \ominus a_n & \ominus a_{n-1} & \ominus a_{n-2} & \ldots & \ominus a_1 \end{bmatrix} . \qquad \square$$

In the next section we shall see that not every monic max-algebraic polynomial corresponds to the MACP of a matrix with entries in $\mathbb{R}_\varepsilon$.

5.2 Necessary Conditions for the Coefficients of the Max-Algebraic Characteristic Polynomial of a Matrix with Entries in $\mathbb{R}_\varepsilon$

The aim of this section is to derive necessary conditions for the coefficients of the MACP of a matrix with entries in $\mathbb{R}_\varepsilon$. Since some of the proofs of the propositions of this section and of the next section are rather tedious and not very instructive, we have put these proofs in an appendix (Appendix B).

Proposition 5.2.1 *If* $A \in \mathbb{R}_\varepsilon^{n\times n}$ *and if the MACP of* A *is given by* $\lambda^{\otimes^n} \oplus \bigoplus_{k=1}^{n} a_k \otimes \lambda^{\otimes^{n-k}}$, *then* $a_1 \in \mathbb{S}^{\ominus}$.

Proof: Since $a_1 = \ominus \bigoplus_{i=1}^{n} a_{ii}$ with $a_{ii} \in \mathbb{R}_\varepsilon$ for all i, we have $a_1 \in \mathbb{S}^{\ominus}$. $\square$

Lemma 5.2.2 *Let* $l \in \mathbb{N}_0$. *Any even permutation* $\sigma_{2l,\mathrm{even}}$ *of* $2l$ *elements can be decomposed into two even permutations* $\sigma_{2m+1,\mathrm{even}}$ *and* $\sigma_{2l-2m-1,\mathrm{even}}$ *of an odd number of elements with* $m \in \mathbb{N}$, *or into two odd permutations* $\sigma_{2p,\mathrm{odd}}$ *and* $\sigma_{2l-2p,\mathrm{odd}}$ *of an even number of elements with* $p \in \mathbb{N}_0$:

$$\sigma_{2l,\mathrm{even}} = \sigma_{2m+1,\mathrm{even}} \cup \sigma_{2l-2m-1,\mathrm{even}} \quad \textit{or} \quad \sigma_{2l,\mathrm{even}} = \sigma_{2p,\mathrm{odd}} \cup \sigma_{2l-2p,\mathrm{odd}} \ .$$

Any odd permutation $\sigma_{2l+1,\mathrm{odd}}$ *of* $2l+1$ *elements can be decomposed into an even permutation* $\sigma_{2q+1,\mathrm{even}}$ *of an odd number of elements and an odd permutation* $\sigma_{2l-2q,\mathrm{odd}}$ *of an even number of elements with* $q \in \mathbb{N}$:

$$\sigma_{2l+1,\mathrm{odd}} = \sigma_{2q+1,\mathrm{even}} \cup \sigma_{2l-2q,\mathrm{odd}} \ .$$

Proof: First we consider $\sigma_{2l,\text{even}}$. This is an even permutation of an even number of elements. Therefore, it follows from Lemma 2.1.2 that $\sigma_{2l,\text{even}}$ is not a cyclic permutation. Hence, it can be decomposed into elementary cycles. Suppose that there are c^{e} elementary cycles $\tau_1^{\text{e}}, \tau_2^{\text{e}}, \ldots, \tau_{c^{\text{e}}}^{\text{e}}$ with an even length, and c^{o} elementary cycles $\tau_1^{\text{o}}, \tau_2^{\text{o}}, \ldots, \tau_{c^{\text{o}}}^{\text{o}}$ with an odd length. Let n_i^{e} be the length of τ_i^{e} for $i = 1, 2, \ldots, c^{\text{e}}$ and let n_i^{o} be the length of τ_i^{o} for $i = 1, 2, \ldots, c^{\text{o}}$. Let $n_{\text{tot}}^{\text{e}} = \sum_{i=1}^{c^{\text{e}}} n_i^{\text{e}}$ and let $n_{\text{tot}}^{\text{o}} = \sum_{i=1}^{c^{\text{o}}} n_i^{\text{o}}$. Since the parity of $\sigma_{2l,\text{even}}$ is even, c^{e} should also be even by Lemma 2.1.2. Since $n_{\text{tot}}^{\text{e}}$ is always even and since $n_{\text{tot}}^{\text{e}} + n_{\text{tot}}^{\text{o}} = 2l$, $n_{\text{tot}}^{\text{o}}$ is even. Therefore, c^{o} is also even. Now we distinguish between two cases: $c^{\text{e}} = 0$ and $c^{\text{e}} \neq 0$.

If $c^{\text{e}} = 0$ then $c^{\text{o}} \neq 0$ since $2l \neq 0$. Take an arbitrary elementary cycle τ_j^{o} of odd length. We have $n_j^{\text{o}} = 2m + 1$ for some $m \in \mathbb{N}$. By Lemma 2.1.2 the parity of τ_j^{o} is even and therefore we represent τ_j^{o} by $\sigma_{2m+1,\text{even}}$. The other elementary cycles form a permutation with 0 cycles of even length. So they correspond to an even permutation of the remaining $2l-2m-1$ elements that will be denoted by $\sigma_{2l-2m-1,\text{even}}$.

If $c^{\text{e}} \neq 0$, we take one elementary cycle, say τ_k^{e}, of even length. We have $n_k^{\text{e}} = 2p$ for some $p \in \mathbb{N}_0$. Since the parity of τ_k^{e} is odd by Lemma 2.1.2, we represent τ_k^{e} by $\sigma_{2p,\text{odd}}$. The remaining elementary cycles constitute a permutation with an odd number $(c^{\text{e}} - 1)$ of cycles of even length: $\sigma_{2l-2p,\text{odd}}$.

Hence, $\sigma_{2l,\text{even}}$ can be decomposed as $\sigma_{2m+1,\text{even}} \cup \sigma_{2l-2m-1,\text{even}}$ or as $\sigma_{2p,\text{odd}} \cup \sigma_{2l-2p,\text{odd}}$ with $m \in \mathbb{N}$ and $p \in \mathbb{N}_0$.

Now we consider $\sigma_{2l+1,\text{odd}}$. This is an odd permutation of an odd number of elements. Hence, it is not a cyclic permutation and it can be decomposed into elementary cycles. We define c^{e}, c^{o}, $\tau_1^{\text{e}}, \tau_2^{\text{e}}, \ldots, \tau_{c^{\text{e}}}^{\text{e}}$, $\tau_1^{\text{o}}, \tau_2^{\text{o}}, \ldots, \tau_{c^{\text{o}}}^{\text{o}}$, $n_1^{\text{e}}, n_2^{\text{e}}, \ldots, n_{c^{\text{e}}}^{\text{e}}$, $n_1^{\text{o}}, n_2^{\text{o}}, \ldots, n_{c^{\text{o}}}^{\text{o}}$, $n_{\text{tot}}^{\text{e}}$ and $n_{\text{tot}}^{\text{o}}$ in the same way as in the first part of this proof. Since the parity of $\sigma_{2l+1,\text{odd}}$ is odd, c^{e} should also be odd. Since $n_{\text{tot}}^{\text{e}}$ is always even and since $n_{\text{tot}}^{\text{e}} + n_{\text{tot}}^{\text{o}} = 2l + 1$ is odd, $n_{\text{tot}}^{\text{o}}$ is odd. Hence, c^{o} is odd. This implies that $c^{\text{o}} \neq 0$. Let us take one elementary cycle τ_r^{o} of odd length. We have $n_r^{\text{o}} = 2q + 1$ for some $q \in \mathbb{N}$. Since τ_r^{o} is an even permutation of $2q + 1$ elements, we represent it by $\sigma_{2q+1,\text{even}}$. The other elementary cycles correspond to a permutation with c^{e} cycles of even length. Since c^{e} is odd, this permutation is an odd permutation of $2l - 2q$ elements: $\sigma_{2l-2q,\text{odd}}$.

Hence, $\sigma_{2l+1,\text{odd}} = \sigma_{2q+1,\text{even}} \cup \sigma_{2l-2q,\text{odd}}$. □

Now we extract the max-positive contributions to (5.1) and put them in a_k^{pos}; the max-negative contributions to (5.1) are collected in a_k^{neg}. This leads to

$$a_1^{\text{pos}} = \varepsilon \tag{5.2}$$

$$a_1^{\text{neg}} = \bigoplus_{i=1}^{n} a_{ii} \tag{5.3}$$

and

$$a_{2l}^{\text{pos}} = \bigoplus_{\{i_1,i_2,\ldots,i_{2l}\}\in\mathcal{C}_n^{2l}} \bigoplus_{\sigma\in\mathcal{P}_{2l,\text{even}}} \bigotimes_{r=1}^{2l} a_{i_r i_{\sigma(r)}} \tag{5.4}$$

$$a_{2l}^{\text{neg}} = \bigoplus_{\{i_1,i_2,\ldots,i_{2l}\}\in\mathcal{C}_n^{2l}} \bigoplus_{\sigma\in\mathcal{P}_{2l,\text{odd}}} \bigotimes_{r=1}^{2l} a_{i_r i_{\sigma(r)}} \tag{5.5}$$

$$a_{2l+1}^{\text{pos}} = \bigoplus_{\{i_1,i_2,\ldots,i_{2l+1}\}\in\mathcal{C}_n^{2l+1}} \bigoplus_{\sigma\in\mathcal{P}_{2l+1,\text{odd}}} \bigotimes_{r=1}^{2l+1} a_{i_r i_{\sigma(r)}} \tag{5.6}$$

$$a_{2l+1}^{\text{neg}} = \bigoplus_{\{i_1,i_2,\ldots,i_{2l+1}\}\in\mathcal{C}_n^{2l+1}} \bigoplus_{\sigma\in\mathcal{P}_{2l+1,\text{even}}} \bigotimes_{r=1}^{2l+1} a_{i_r i_{\sigma(r)}} \tag{5.7}$$

for $l = 1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor$ where the empty max-algebraic sum $\bigoplus_{\varphi\in\emptyset} \ldots$ is equal to ε by definition.

Proposition 5.2.3 *Let $A \in \mathbb{R}_\varepsilon^{n\times n}$ and let a_k^{pos} and a_k^{neg} be defined by (5.4)–(5.7) for $k = 2, 3, \ldots, n$. Then we have $a_k^{\text{pos}} \leqslant \bigoplus_{r=1}^{\lfloor \frac{k}{2} \rfloor} a_r^{\text{neg}} \otimes a_{k-r}^{\text{neg}}$ for $k = 2, 3, \ldots, n$.*

Proof: Let $l \in \left\{1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor\right\}$. First we consider (5.4). The terms of a_{2l}^{pos} are generated by even permutations of $2l$ elements. By Lemma 5.2.2 such permutations can be decomposed into two even permutations of an odd number of elements or into two odd permutations of an even number of elements. So if we consider all possible concatenations of two even permutations $\sigma_{2m+1,\text{even}}$ and $\sigma_{2l-2m-1,\text{even}}$ of an odd number of elements (which corresponds to $\bigoplus_{m=0}^{l-1} a_{2m+1}^{\text{neg}} \otimes a_{2l-2m-1}^{\text{neg}}$) or two odd permutations $\sigma_{2p,\text{odd}}$ and $\sigma_{2l-2p,\text{odd}}$ of an even number of elements (which corresponds to $\bigoplus_{p=1}^{l} a_{2p}^{\text{neg}} \otimes a_{2l-2p}^{\text{neg}}$), we are sure that we have included all the terms of a_{2l}^{pos}. In other words, $a_{2l}^{\text{pos}} \leqslant \bigoplus_{r=1}^{2l-1} a_r^{\text{neg}} \otimes a_{2l-r}^{\text{neg}}$. Since $(a_r^{\text{neg}} \otimes a_{2l-r}^{\text{neg}}) \oplus (a_{2l-r}^{\text{neg}} \otimes a_r^{\text{neg}}) = a_r^{\text{neg}} \otimes a_{2l-r}^{\text{neg}}$ for all r, we have

$$a_{2l}^{\text{pos}} \leqslant \bigoplus_{r=1}^{l} a_r^{\text{neg}} \otimes a_{2l-r}^{\text{neg}} \quad . \tag{5.8}$$

Now we consider (5.6). The terms of a_{2l+1}^{pos} are generated by odd permutations of $2l+1$ elements. Lemma 5.2.2 states that permutations of that sort can be decomposed into an odd permutation of an even number of elements and an even permutation of an odd number of elements. Using the same reasoning as for a_{2l}^{pos}, we find

$$a_{2l+1}^{\mathrm{pos}} \leqslant \bigoplus_{r=1}^{l} a_r^{\mathrm{neg}} \otimes a_{2l+1-r}^{\mathrm{neg}} \quad . \tag{5.9}$$

If we combine (5.8) and (5.9), we obtain $a_k^{\mathrm{pos}} \leqslant \bigoplus_{r=1}^{\lfloor \frac{k}{2} \rfloor} a_r^{\mathrm{neg}} \otimes a_{k-r}^{\mathrm{neg}}$ for all k. □

We do not have similar expressions for the a_k^{neg}'s since some of the generating permutations for the a_k^{neg}'s are cyclic permutations, and cyclic permutations cannot be decomposed into two or more elementary cycles.

When we calculate the coefficients of the MACP of a matrix, we normally use the simplification rules (2.9) and (2.10) during the calculation. Let $k \in \{1, 2, \ldots, n\}$. In general we only know the max-positive part $a_k^{\oplus}$ and the max-negative part $a_k^{\ominus}$ of a_k instead of a_k^{pos} and a_k^{neg}. Therefore, we now transform Proposition 5.2.3 into a property of $a_k^{\oplus}$ and $a_k^{\ominus}$. We can extract $a_k^{\oplus}$ and $a_k^{\ominus}$ from a_k^{pos} and a_k^{neg} as follows:

$$\begin{aligned}
&a_k^{\oplus} = a_k^{\mathrm{pos}} \ \text{ and } \ a_k^{\ominus} = \varepsilon && \text{if } a_k^{\mathrm{pos}} > a_k^{\mathrm{neg}} , \\
&a_k^{\oplus} = \varepsilon \quad\ \text{ and } \ a_k^{\ominus} = a_k^{\mathrm{neg}} && \text{if } a_k^{\mathrm{pos}} < a_k^{\mathrm{neg}} , \\
&a_k^{\oplus} = a_k^{\mathrm{pos}} \ \text{ and } \ a_k^{\ominus} = a_k^{\mathrm{neg}} && \text{if } a_k^{\mathrm{pos}} = a_k^{\mathrm{neg}} .
\end{aligned}$$

We have $a_k = a_k^{\mathrm{pos}} \ominus a_k^{\mathrm{neg}} = a_k^{\oplus} \ominus a_k^{\ominus}$, $a_k^{\mathrm{pos}} \leqslant a_k^{\oplus}$, $a_k^{\mathrm{neg}} \leqslant a_k^{\ominus}$ and $|a_k|_{\oplus} = a_k^{\mathrm{pos}} \oplus a_k^{\mathrm{neg}} = a_k^{\oplus} \oplus a_k^{\ominus}$ for all k. From Proposition 5.2.1 or from (5.2) it follows that we always have $a_1^{\oplus} = a_1^{\mathrm{pos}} = \varepsilon$ and $a_1^{\ominus} = a_1^{\mathrm{neg}}$.

Proposition 5.2.4 *If $A \in \mathbb{R}_{\varepsilon}^{n \times n}$ and if the MACP of A is given by $\lambda^{\otimes^n} \oplus \bigoplus_{k=1}^{n} a_k \otimes \lambda^{\otimes^{n-k}}$, then we have*

$$a_k^{\oplus} \ \leqslant \ \bigoplus_{r=1}^{\lfloor \frac{k}{2} \rfloor} (a_r^{\oplus} \ \oplus \ a_r^{\ominus}) \otimes (a_{k-r}^{\oplus} \ \oplus \ a_{k-r}^{\ominus}) \qquad \textit{for } k = 2, 3, \ldots, n \, .$$

Proof: Consider an arbitrary $k \in \{2, 3, \ldots, n\}$. Since $a_r^{\ominus} \leqslant |a_r|_{\oplus}$ for all r, Proposition 5.2.3 leads to

$$a_k^{\oplus} \ \leqslant \ a_k^{\mathrm{pos}} \ \leqslant \ \bigoplus_{r=1}^{\lfloor \frac{k}{2} \rfloor} |a_r|_{\oplus} \otimes |a_{k-r}|_{\oplus}$$

$$\leqslant \bigoplus_{r=1}^{\lfloor \frac{k}{2} \rfloor} (a_r^{\oplus} \ \oplus \ a_r^{\ominus}) \otimes (a_{k-r}^{\oplus} \ \oplus \ a_{k-r}^{\ominus}) \ . \qquad \square$$

We even have a more stringent property:

Proposition 5.2.5 *Let $A \in \mathbb{R}_\varepsilon^{n \times n}$ and let the MACP of A be given by $\lambda^{\otimes^n} \oplus \bigoplus_{k=1}^{n} a_k \otimes \lambda^{\otimes^{n-k}}$. For any $k \in \{1, 2, \ldots, n\}$ at least one of the following statements holds:*

$$(1) \quad a_k^{\oplus} \ \leqslant \ \bigoplus_{r=1}^{\lfloor \frac{k}{2} \rfloor} a_r^{\ominus} \otimes a_{k-r}^{\ominus}$$

$$(2) \quad a_k^{\oplus} \ < \ \bigoplus_{r=2}^{\lfloor \frac{k}{2} \rfloor} a_r^{\oplus} \otimes a_{k-r}^{\oplus}$$

$$(3) \quad a_k^{\oplus} \ < \ \bigoplus_{r=2}^{k-1} a_r^{\oplus} \otimes a_{k-r}^{\ominus} \ .$$

Proof: Consider an arbitrary $k \in \{2, 3, \ldots, n\}$. By Proposition 5.2.3 there exists an index $s \in \left\{1, 2, \ldots, \left\lfloor \dfrac{k}{2} \right\rfloor \right\}$ such that $a_k^{\oplus} \ \leqslant \ a_k^{\mathrm{pos}} \ \leqslant \ a_s^{\mathrm{neg}} \otimes a_{k-s}^{\mathrm{neg}}$. We have either $a_s^{\mathrm{neg}} = a_s^{\ominus}$ or $a_s^{\mathrm{neg}} < a_s^{\oplus}$, and either $a_{k-s}^{\mathrm{neg}} = a_{k-s}^{\ominus}$ or $a_{k-s}^{\mathrm{neg}} < a_{k-s}^{\oplus}$. This means that at least one of the following inequalities holds:

$$(1) \quad a_k^{\oplus} \ \leqslant \ a_s^{\ominus} \otimes a_{k-s}^{\ominus} \ \leqslant \ \bigoplus_{r=1}^{\lfloor \frac{k}{2} \rfloor} a_r^{\ominus} \otimes a_{k-r}^{\ominus}$$

$$(2) \quad a_k^{\oplus} \ < \ a_s^{\oplus} \otimes a_{k-s}^{\oplus} \ \leqslant \ \bigoplus_{r=2}^{\lfloor \frac{k}{2} \rfloor} a_r^{\oplus} \otimes a_{k-r}^{\oplus}$$

$$(3) \quad a_k^{\oplus} \ < \ a_s^{\ominus} \otimes a_{k-s}^{\oplus} \ \oplus \ a_s^{\oplus} \otimes a_{k-s}^{\ominus} \ \leqslant \ \bigoplus_{r=2}^{k-1} a_r^{\oplus} \otimes a_{k-r}^{\ominus} \ .$$

Note that in the max-algebraic sums of (2) and (3) we start from $r = 2$ instead of $r = 1$ since $a_1^{\oplus} = \varepsilon$. $\square$

Propositions 5.2.4 and 5.2.5 give necessary conditions for the coefficients of a max-algebraic polynomial to be the MACP of a matrix with entries in $\mathbb{R}_\varepsilon$. For max-algebraic polynomials of degree greater than or equal to 4, we have the following extra necessary conditions:

Proposition 5.2.6 *Let $A \in \mathbb{R}_\varepsilon^{n \times n}$ with $n \geqslant 4$ and let $\lambda^{\otimes^n} \oplus \bigoplus_{k=1}^{n} a_k \otimes \lambda^{\otimes^{n-k}}$ be the MACP of A. Then the coefficients of this max-algebraic polynomial always fall into exactly one of the following three cases:*

Case A: $a_4^\oplus \leqslant a_1^\ominus \otimes a_3^\ominus$ *or* $a_4^\oplus < a_1^\ominus \otimes a_3^\oplus$,

Case B: $a_4^\oplus > a_1^\ominus \otimes a_3^\ominus$ *and* $a_4^\oplus \geqslant a_1^\ominus \otimes a_3^\oplus$ *and* $a_4^\oplus \leqslant a_2^\ominus \otimes a_2^\ominus$ *and* $(a_1^\ominus = \varepsilon$ *or* $a_2^\oplus = \varepsilon$ *or* $a_4^\ominus = a_4^\oplus)$,

Case C: $a_4^\oplus > a_1^\ominus \otimes a_3^\ominus$ *and* $a_4^\oplus \geqslant a_1^\ominus \otimes a_3^\oplus$ *and* $a_4^\oplus \leqslant a_2^\ominus \otimes a_2^\ominus$ *and* $a_2^\oplus \neq \varepsilon$ *and* $a_4^\ominus = \varepsilon$.

Proof: See Section B.1, where also some additional necessary conditions for the coefficients of the MACP of a matrix with entries in $\mathbb{R}_\varepsilon$ can be found. □

5.3 Necessary and Sufficient Conditions for the Coefficients of the Max-Algebraic Characteristic Polynomial of a Matrix with Entries in $\mathbb{R}_\varepsilon$

Now we give some necessary and sufficient conditions for the coefficients of the MACP of a matrix with entries in $\mathbb{R}_\varepsilon$ and with a dimension that is less than or equal to 4. If we have a monic max-algebraic polynomial with a degree that is less than or equal to 4, then the results of this section will enable us to

1. check whether the given max-algebraic polynomial can be the MACP of a matrix with entries in $\mathbb{R}_\varepsilon$, and,
2. if the necessary and sufficient conditions are satisfied, construct a matrix with entries in $\mathbb{R}_\varepsilon$ such that its MACP is equal to the given max-algebraic polynomial.

For max-algebraic polynomials with a degree that is less than or equal to 4, we give an analytic description of the matrix we are looking for. For max-algebraic polynomials with a degree that is greater than 4 we have not yet found sufficient conditions. But in Section B.4 we shall state a conjecture based on the results of this section and then use this conjecture to develop a heuristic algorithm that will in most cases find a matrix with a MACP that is equal to a given max-algebraic polynomial.

In the next subsections we shall case by case determine necessary and sufficient conditions for

$$\lambda^{\otimes^n} \oplus \bigoplus_{k=1}^{n} a_k \otimes \lambda^{\otimes^{n-k}}$$

to be the MACP of a matrix with entries in $\mathbb{R}_\varepsilon$ and indicate how such a matrix can be found.
Define

$$\kappa_{i,j} = \begin{cases} \frac{a_j^{\oplus}}{a_i^{\ominus}} & \text{if } a_i^{\ominus} \neq \varepsilon\,, \\ \varepsilon & \text{if } a_i^{\ominus} = \varepsilon\,, \end{cases}$$

for $i = 1, 2, \ldots, n-1$ and $j = i+1, i+2, \ldots, n$.

5.3.1 The 1 by 1 Case

Proposition 5.3.1 *The only necessary and also sufficient condition for $\lambda \oplus a_1$ to be the MACP of a 1 by 1 matrix with entries in $\mathbb{R}_\varepsilon$ is $a_1^{\oplus} = \varepsilon$.*
The matrix $[a_1^{\ominus}]$ has the given max-algebraic polynomial as its MACP.

Proof: By Proposition 5.2.1 the condition $a_1^{\oplus} = \varepsilon$ is a necessary. And since the matrix $[a_1^{\ominus}]$ has $\lambda \oplus a_1$ as its MACP if $a_1^{\oplus} = \varepsilon$, the condition $a_1^{\oplus} = \varepsilon$ is also sufficient. □

5.3.2 The 2 by 2 Case

Proposition 5.3.2 *The necessary and also sufficient conditions for $\lambda^{\otimes^2} \oplus a_1 \otimes \lambda \oplus a_2$ to be the MACP of a 2 by 2 matrix with entries in $\mathbb{R}_\varepsilon$ are*

$$a_1^{\oplus} = \varepsilon \tag{5.10}$$
$$a_2^{\oplus} \leqslant a_1^{\ominus} \otimes a_1^{\ominus}\ . \tag{5.11}$$

The matrix $B = \begin{bmatrix} a_1^{\ominus} & a_2^{\ominus} \\ 0 & \kappa_{1,2} \end{bmatrix}$ has the given max-algebraic polynomial as its MACP.

Proof: By Propositions 5.2.1 and 5.2.4 the conditions $a_1^{\oplus} = \varepsilon$ and $a_2^{\oplus} \leqslant a_1^{\ominus} \otimes a_1^{\ominus}$ are necessary.
Now we show that the conditions (5.10) – (5.11) are sufficient by proving that if these conditions are satisfied then the matrix B has the given max-algebraic polynomial as its MACP.
If $a_2^{\oplus} \leqslant a_1^{\ominus} \otimes a_1^{\ominus}$ then we always have $\kappa_{1,2} \leqslant a_1^{\ominus}$ since

$$\kappa_{1,2} = \begin{cases} \frac{a_2^{\oplus}}{a_1^{\ominus}} \leqslant a_1^{\ominus} & \text{if } a_1^{\ominus} \neq \varepsilon\,, \\ \varepsilon \leqslant a_1^{\ominus} & \text{if } a_1^{\ominus} = \varepsilon\,. \end{cases}$$

If $a_2^{\oplus} \leqslant a_1^{\ominus} \otimes a_1^{\ominus}$ holds then $a_1^{\ominus} = \varepsilon$ implies that $a_2^{\oplus} = \varepsilon$ and thus also $\kappa_{1,2} = \varepsilon$. Now we prove that $a_1^{\ominus} \otimes \kappa_{1,2} = a_2^{\oplus}$. We have

$$a_1^{\ominus} \otimes \kappa_{1,2} = \begin{cases} a_1^{\ominus} \otimes \dfrac{a_2^{\oplus}}{a_1^{\ominus}} = a_2^{\oplus} & \text{if } a_1^{\ominus} \neq \varepsilon\,, \\ \varepsilon \otimes \varepsilon = \varepsilon = a_2^{\oplus} & \text{if } a_1^{\ominus} = \varepsilon\,. \end{cases}$$

If $\lambda^{\otimes^2} \oplus b_1 \otimes \lambda \oplus b_2$ is the MACP of B then we have

$$\begin{aligned} b_1 &= \ominus a_1^{\ominus} \ominus \kappa_{1,2} = \ominus a_1^{\ominus} = a_1 \\ b_2 &= a_1^{\ominus} \otimes \kappa_{1,2} \ominus a_2^{\ominus} = a_2^{\oplus} \ominus a_2^{\ominus} = a_2 \ . \end{aligned}$$

Hence, B has the given max-algebraic polynomial as its MACP. □

5.3.3 The 3 by 3 Case

Proposition 5.3.3 *The necessary and also sufficient conditions for* $\lambda^{\otimes^3} \oplus a_1 \otimes \lambda^{\otimes^2} \oplus a_2 \otimes \lambda \oplus a_3$ *to be the MACP of a 3 by 3 matrix with entries in* $\mathbb{R}_\varepsilon$ *are*

$$a_1^{\oplus} = \varepsilon \tag{5.12}$$

$$a_2^{\oplus} \leqslant a_1^{\ominus} \otimes a_1^{\ominus} \tag{5.13}$$

$$a_3^{\oplus} \leqslant a_1^{\ominus} \otimes a_2^{\ominus} \quad \textit{or} \quad a_3^{\oplus} < a_1^{\ominus} \otimes a_2^{\oplus} \ . \tag{5.14}$$

The matrix $B = \begin{bmatrix} a_1^{\ominus} & a_2^{\ominus} & a_3^{\ominus} \\ 0 & \kappa_{1,2} & \kappa_{1,3} \\ \varepsilon & 0 & \varepsilon \end{bmatrix}$ *has the given max-algebraic polynomial as its MACP.*

Proof: By Propositions 5.2.1 and 5.2.5 the conditions are necessary.
Assume that $\lambda^{\otimes^3} \oplus b_1 \otimes \lambda^{\otimes^2} \oplus b_2\lambda \oplus b_3$ is the MACP of B and that the conditions (5.12) – (5.14) are satisfied.
From the proof of Proposition 5.3.2 we already know that $\kappa_{1,2} \leqslant a_1^{\ominus}$ and $a_1^{\ominus} \otimes \kappa_{1,2} = a_2^{\oplus}$. Analogously we can prove that $a_1^{\ominus} \otimes \kappa_{1,3} = a_3^{\oplus}$ since if (5.14) holds and if $a_1^{\ominus} = \varepsilon$, we also have $a_3^{\oplus} = \varepsilon$.
Now we prove that $\kappa_{1,3} \leqslant a_2^{\ominus}$ if $a_2^{\ominus} \geqslant a_2^{\oplus}$ and that $\kappa_{1,3} < a_2^{\oplus}$ if $a_2^{\ominus} < a_2^{\oplus}$.
If $a_2^{\ominus} \geqslant a_2^{\oplus}$ then (5.14) implies that $a_3^{\oplus} \leqslant a_1^{\ominus} \otimes a_2^{\ominus}$. Hence,

$$\kappa_{1,3} = \begin{cases} \dfrac{a_3^{\oplus}}{a_1^{\ominus}} \leqslant a_2^{\ominus} & \text{if } a_1^{\ominus} \neq \varepsilon\,, \\ \varepsilon \leqslant a_2^{\ominus} & \text{if } a_1^{\ominus} = \varepsilon\,. \end{cases}$$

On the other hand, if $a_2^{\ominus} < a_2^{\oplus}$ then (5.14) implies that $a_3^{\oplus} < a_2^{\oplus} \otimes a_1^{\ominus}$. Hence, $a_1^{\ominus} \neq \varepsilon$ and $\kappa_{1,3} = \dfrac{a_3^{\oplus}}{a_1^{\ominus}} < a_2^{\oplus}$.

As a consequence, we always have $a_2^\oplus \ominus a_2^\ominus \ominus \kappa_{1,3} = a_2^\oplus \ominus a_2^\ominus$.
This leads to the following expressions for the coefficients of the MACP of B:

$$\begin{aligned}
b_1 &= \ominus a_1^\ominus \ominus \kappa_{1,2} = \ominus a_1^\ominus = a_1 \\
b_2 &= a_1^\ominus \otimes \kappa_{1,2} \ominus a_2^\ominus \ominus \kappa_{1,3} = a_2^\oplus \ominus a_2^\ominus \ominus \kappa_{1,3} = a_2^\oplus \ominus a_2^\ominus = a_2 \\
b_3 &= a_1^\ominus \otimes \kappa_{1,3} \ominus a_3^\ominus = a_3^\oplus \ominus a_3^\ominus = a_3 \ .
\end{aligned}$$

Hence, B has the given max-algebraic polynomial as its MACP. □

5.3.4 The 4 by 4 Case

Proposition 5.3.4 *Consider the max-algebraic polynomial*

$$\lambda^{\otimes^4} \oplus a_1 \otimes \lambda^{\otimes^3} \oplus a_2 \otimes \lambda^{\otimes^2} \oplus a_3 \otimes \lambda \oplus a_4 \ . \tag{5.15}$$

Assume that the coefficients of this max-algebraic polynomial satisfy the necessary conditions of Proposition 5.2.6. The (additional) necessary and sufficient conditions for (5.15) to be the MACP of a 4 by 4 matrix with entries in $\mathbb{R}_\varepsilon$ are:

$$a_1^\oplus = \varepsilon \tag{5.16}$$

$$a_2^\oplus \leqslant a_1^\ominus \otimes a_1^\ominus \tag{5.17}$$

$$a_3^\oplus \leqslant a_1^\ominus \otimes a_2^\ominus \ \text{ or } \ a_3^\oplus < a_1^\ominus \otimes a_2^\oplus \tag{5.18}$$

$$\text{for Case A: no extra conditions} \tag{5.19}$$

$$\text{for Case B: } a_1^\ominus \otimes a_4^\oplus \leqslant a_2^\ominus \otimes a_3^\oplus \ \text{ or } \ a_1^\ominus \otimes a_4^\oplus < a_2^\ominus \otimes a_3^\ominus \tag{5.20}$$

$$\text{for Case C: } a_1^\ominus \otimes a_3^\oplus = a_2^\ominus \otimes a_2^\oplus \ \text{ and } \ a_1^\ominus \otimes a_4^\oplus = a_2^\ominus \otimes a_3^\oplus \ . \tag{5.21}$$

If these necessary and sufficient conditions are satisfied, the following matrices have the given max-algebraic polynomial as their MACP:

$$B_{\mathrm{A}} = \begin{bmatrix} a_1^\ominus & a_2^\ominus & a_3^\ominus & a_4^\ominus \\ 0 & \kappa_{1,2} & \kappa_{1,3} & \kappa_{1,4} \\ \varepsilon & 0 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 0 & \varepsilon \end{bmatrix} \qquad \text{for Case } A\,,$$

$$B_{\mathrm{B}} = \begin{bmatrix} a_1^\ominus & a_2^\ominus & a_3^\ominus & a_4^\ominus \\ 0 & \kappa_{1,2} & \kappa_{1,3} & \varepsilon \\ \varepsilon & 0 & \varepsilon & \kappa_{2,4} \\ \varepsilon & \varepsilon & 0 & \varepsilon \end{bmatrix} \qquad \text{for Case } B\,,$$

$$B_{\mathrm{C}} = \begin{bmatrix} a_1^\ominus & a_2^\ominus & \varepsilon & \varepsilon \\ 0 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 0 & \kappa_{2,3} & \kappa_{2,4} \\ \varepsilon & \varepsilon & 0 & \varepsilon \end{bmatrix} \qquad \text{for Case } C.$$

Proof: See Section B.2. □

Let us now illustrate the properties of this and the previous section with some examples.

Example 5.3.5 Consider the monic max-algebraic polynomial $\lambda^{\otimes^4} \ominus 3 \otimes \lambda^{\otimes^3} \ominus 4 \otimes \lambda^{\otimes^2} \ominus 8 \otimes \lambda \oplus 12$.
We have

$$\begin{aligned} a_4^{\oplus} &= 12 \ > \ 11 = 3 \otimes 8 = a_1^{\ominus} \otimes a_3^{\ominus} \\ a_4^{\oplus} &= 12 \ \geqslant \ \varepsilon = 3 \otimes \varepsilon = a_1^{\ominus} \otimes a_3^{\oplus} \\ a_4^{\oplus} &= 12 \ > \ 8 = 4 \otimes 4 = a_2^{\ominus} \otimes a_2^{\ominus} \ . \end{aligned}$$

Since the coefficients do not satisfy the conditions of one of the three possible cases of Proposition 5.2.6, the given max-algebraic polynomial cannot be the MACP of a matrix with entries in $\mathbb{R}_\varepsilon$. □

Example 5.3.6 Consider $\lambda^{\otimes^4} \ominus 1 \otimes \lambda^{\otimes^3} \ominus 5 \otimes \lambda^{\otimes^2} \oplus 6 \otimes \lambda \oplus 8^{\bullet}$.
We have

$$\begin{aligned} a_4^{\oplus} &= 8 \ > \ \varepsilon = 1 \otimes \varepsilon = a_1^{\ominus} \otimes a_3^{\ominus} \\ a_4^{\oplus} &= 8 \ \geqslant \ 7 = 1 \otimes 6 = a_1^{\ominus} \otimes a_3^{\oplus} \\ a_4^{\oplus} &= 8 \ \leqslant \ 10 = 5 \otimes 5 = a_2^{\ominus} \otimes a_2^{\ominus} \\ a_2^{\oplus} &= \varepsilon \ . \end{aligned}$$

Hence, we are in Case B. The necessary and sufficient conditions of Proposition 5.3.4 for Case B are fulfilled since

$$\begin{aligned} a_1^{\oplus} \ &= \ \varepsilon \\ a_2^{\oplus} = \varepsilon \ &\leqslant \ 2 = 1 \otimes 1 = a_1^{\ominus} \otimes a_1^{\ominus} \\ a_3^{\oplus} = 6 \ &\leqslant \ 6 = 1 \otimes 5 = a_1^{\ominus} \otimes a_2^{\ominus} \\ a_1^{\ominus} \otimes a_4^{\oplus} = 1 \otimes 8 = 9 \ &\leqslant \ 11 = 5 \otimes 6 = a_2^{\ominus} \otimes a_3^{\oplus} \ . \end{aligned}$$

The matrix

$$B = \begin{bmatrix} 1 & 5 & \varepsilon & 8 \\ 0 & \varepsilon & 5 & \varepsilon \\ \varepsilon & 0 & \varepsilon & 3 \\ \varepsilon & \varepsilon & 0 & \varepsilon \end{bmatrix}$$

has the given max-algebraic polynomial as its MACP. □

5.4 Construction of Matrices with a Given Max-Algebraic Characteristic Polynomial

Consider the following problem:

Given a monic max-algebraic polynomial with coefficients in $\mathbb{S}$:

$$\lambda^{\otimes^n} \oplus \bigotimes_{k=1}^{n} a_k \otimes \lambda^{\otimes^{n-k}} \quad , \tag{5.22}$$

find a matrix $B \in \mathbb{R}_\varepsilon^{n \times n}$ such that the MACP of B is equal to the given max-algebraic polynomial.

Let $B \in \mathbb{R}_\varepsilon^{n \times n}$ and let $\lambda^{\otimes^n} \oplus \bigoplus_{k=1}^{n} b_k \otimes \lambda^{\otimes^{n-k}}$ be the MACP of B. Note that we always have $b_1 \in \mathbb{S}^{\ominus}$. So if $a_1 \notin \mathbb{S}^{\ominus}$ then (5.22) cannot be the MACP of a matrix with entries in $\mathbb{R}_\varepsilon$.
Consider the formulas (5.2) – (5.7) for the max-positive contribution b_k^{pos} and the max-negative contribution b_k^{neg} to b_k for $k = 1, 2, \ldots, n$. The expressions for the b_k^{pos}'s and the b_k^{neg}'s are max-algebraic sums of max-algebraic products of entries of B. So these expressions are in fact multivariate max-algebraic polynomial expressions. We have to find the entries of the matrix B such that $b_k^{\mathrm{pos}} \ominus b_k^{\mathrm{neg}} = a_k$ for $k = 1, 2, \ldots, n$.
For each $k \in \{1, 2, \ldots, n\}$ there are three possible cases:

1. if $a_k \in \mathbb{S}^{\oplus}$, we should have $\begin{cases} b_k^{\mathrm{pos}} = |a_k|_\oplus \\ b_k^{\mathrm{neg}} < |a_k|_\oplus \end{cases}$,
2. if $a_k \in \mathbb{S}^{\ominus}$, we should have $\begin{cases} b_k^{\mathrm{pos}} < |a_k|_\oplus \\ b_k^{\mathrm{neg}} = |a_k|_\oplus \end{cases}$,
3. if $a_k \in \mathbb{S}^{\bullet}$, we should have $\begin{cases} b_k^{\mathrm{pos}} = |a_k|_\oplus \\ b_k^{\mathrm{neg}} = |a_k|_\oplus \end{cases}$.

Note that it is always possible to transform the strict inequalities into non-strict inequalities by subtracting a small positive real number δ from the right-hand side. This leads to a combination of multivariate polynomial equalities and inequalities in the max-plus algebra with the entries of B as unknowns.

Lemma 5.4.1 *Let*

$$\lambda^{\otimes^n} \oplus \bigotimes_{k=1}^{n} a_k \otimes \lambda^{\otimes^{n-k}} \quad , \tag{5.23}$$

be a monic max-algebraic polynomial with coefficients in $\mathbb{S}$*. If the coefficients* $a_1,\ a_2,\ \ldots,\ a_n$ *are finite and if there exists a matrix* $B \in \mathbb{R}_\varepsilon^{n \times n}$ *such that* *the MACP of B is equal to (5.23), there also exists a matrix $\tilde{B} \in \mathbb{R}_\varepsilon^{n \times n}$ with finite entries such that the MACP of $\tilde{B}$ is equal to the given max-algebraic polynomial.*

Proof: We have already shown that if (5.23) is the MACP of B the vector x obtained by putting all the entries of B in a large column vector is a solution of a system of multivariate max-algebraic polynomial equalities and *strict* inequalities with the a_k's as right-hand sides. Assume that the MACP of B is given by $\lambda^{\otimes^n} \oplus \bigoplus_{k=1}^{n} b_k \otimes \lambda^{\otimes^{n-k}}$. Let b_k^{pos} and b_k^{neg} be the value of respectively the max-positive and the max-negative contribution to b_k obtained by evaluating the formulas (5.2) – (5.7) for $k = 1, 2, \ldots, n$. Define

$$\begin{aligned} \mathcal{D} = &\left\{ \left| a_k \right|_\oplus - b_k^{\mathrm{neg}} \,\middle|\, k \in \{1, 2, \ldots, n\},\ a_k \in \mathbb{S}^\oplus \text{ and } b_k^{\mathrm{neg}} \neq \varepsilon \right\} \cup \\ &\left\{ \left| a_k \right|_\oplus - b_k^{\mathrm{pos}} \,\middle|\, k \in \{1, 2, \ldots, n\},\ a_k \in \mathbb{S}^\ominus \text{ and } b_k^{\mathrm{pos}} \neq \varepsilon \right\} \cup \left\{ 1 \right\} \end{aligned}$$

and $d = \min \mathcal{D}$. Note that $d > 0$. Now we have

$$\begin{array}{llll} b_k^{\mathrm{pos}} = \left| a_k \right|_\oplus & \text{and} & b_k^{\mathrm{neg}} \leqslant \left| a_k \right|_\oplus - d & \text{if } a_k \in \mathbb{S}^\oplus , \\ b_k^{\mathrm{pos}} \leqslant \left| a_k \right|_\oplus - d & \text{and} & b_k^{\mathrm{neg}} = \left| a_k \right|_\oplus & \text{if } a_k \in \mathbb{S}^\ominus , \\ b_k^{\mathrm{pos}} = \left| a_k \right|_\oplus & \text{and} & b_k^{\mathrm{neg}} = \left| a_k \right|_\oplus & \text{if } a_k \in \mathbb{S}^\bullet , \end{array}$$

for every $k \in \{1, 2, \ldots, n\}$.
So the vector x is also a solution of a system $\mathcal{S}$ of multivariate max-algebraic polynomial equalities and *non-strict* inequalities. Since d and the coefficients $a_1, a_2, \ldots, a_n$ are finite, the right-hand sides of $\mathcal{S}$ are also finite. From Proposition 4.1.4 it follows that $\mathcal{S}$ also has a solution $\tilde{x}$ with finite components. Therefore, there exists a matrix $\tilde{B} \in \mathbb{R}_\varepsilon^{n \times n}$ with finite entries and with a MACP that is equal to (5.23). □

Hence, the problem of finding a matrix B with a given MACP can also be reformulated as an ELCP and solved with the ELCP algorithm of Section 3.4. If we do not find a solution then our estimate of δ was too large. In that case we have to decrease δ and repeat the procedure.
If the given max-algebraic polynomial has degree n, the resulting homogeneous ELCP has $n^2 + 1$ variables: the entries of the matrix B and an extra variable α to make the ELCP homogeneous (cf. Section 3.2.2). The number of inequalities of the ELCP will grow very rapidly as the degree of the given max-algebraic polynomial grows: in a straightforward implementation (without removal of redundant inequalities) the number $N(n)$ of inequalities of the ELCP that corresponds to a max-algebraic polynomial of degree n is given by

$$N(n) = n + 1 + \sum_{k=2}^{n} \frac{n!}{(n-k)!} \quad .$$

We have $e(n! - n) \leqslant N(n) \leqslant en! + 1$. Since in general the execution time of the ELCP algorithm of Section 3.4 depends more or less exponentially on the number of variables and polynomially on the number of inequalities, this means that the ELCP approach in connection with our ELCP algorithm cannot be used in practice to construct a matrix with a given MACP.

In Section B.4 we shall present a heuristic algorithm to find a matrix with a given MACP. Note however that this heuristic algorithm will not always yield a result, even if one exists. Since not every monic nth degree max-algebraic polynomial with coefficients in $\mathbb{S}$ will be the MACP of a matrix with entries in $\mathbb{R}_\varepsilon$, it is advisable to determine whether a solution can exist before starting the algorithm, i.e. we should first check whether the coefficients of the given max-algebraic polynomial satisfy the necessary conditions of Sections 5.2 and 5.3.

5.5 Conclusions

In this chapter we have derived some necessary conditions for the coefficients of the max-algebraic characteristic polynomial of a matrix with entries in $\mathbb{R}_\varepsilon$. For matrices with a dimension that is less than or equal to 4 we have also derived necessary and sufficient conditions for the coefficients of the max-algebraic characteristic polynomial. So if we have a max-algebraic polynomial with a degree that is less than or equal to 4 then we can check whether this max-algebraic polynomial can be the max-algebraic characteristic polynomial of a matrix with entries in $\mathbb{R}_\varepsilon$ and construct such a matrix, if it exists. For square matrices with a dimension that is greater than 4 we have not yet found necessary and sufficient conditions for the coefficients of the max-algebraic characteristic polynomial.

Furthermore, we have also shown that the problem of constructing a matrix with a given max-algebraic characteristic polynomial can be reformulated as an ELCP. However, since the size of the resulting ELCP is enormous even for max-algebraic polynomials with a small degree, this approach is not workable in practice.

Chapter 6

State Space Transformations and State Space Realization for Max-Linear Time-Invariant Discrete Event Systems

After introducing some definitions in Section 6.1, we discuss state space transformations for max-linear time-invariant discrete event systems in Section 6.2. We also show how the ELCP can be used to compute these state space transformations. In order to analyze systems it is advantageous to have a compact description. Therefore, and also because it is one of the fundamental open problems in max-linear system theory for discrete event systems, we address the minimal state space realization problem for max-linear time-invariant discrete event systems. The aim of the minimal state space realization problem for a max-linear time-invariant discrete event system is to find a max-algebraic state space model of minimal size of the impulse response of the system. In Section 6.3 we present a procedure to determine the minimal system order of a max-linear time-invariant discrete event system given its impulse response. We show how the ELCP can be used to solve the problem of finding all fixed order partial state space realizations and all minimal state space realizations of a given impulse response. In Section 6.4 we illustrate the techniques of Sections 6.2 and 6.3 with some worked examples.

6.1 Introduction

In this section we present some definitions and theorems in connection with state space models and impulse responses of max-linear time-invariant DESs.

Consider a max-linear time-invariant DES that can be described by the following nth order state space model:

$$x(k+1) = A \otimes x(k) \;\oplus\; B \otimes u(k) \tag{6.1}$$

$$y(k) = C \otimes x(k) \tag{6.2}$$

with $A \in \mathbb{R}_\varepsilon^{n\times n}$, $B \in \mathbb{R}_\varepsilon^{n\times m}$ and $C \in \mathbb{R}_\varepsilon^{l\times n}$. We shall characterize a model of this form by the triple (A, B, C) of system matrices. If the initial condition $x(0)$ is available then we characterize the model by the 4-tuple $(A, B, C, x(0))$. A system with one input and one output is called a single input single output (SISO) system. A system with more than one input and more than one output is called a multiple input multiple output (MIMO) system.

Definition 6.1.1 (Equivalent state space realizations) *We say that the triples (A, B, C) and $(\tilde{A}, \tilde{B}, \tilde{C})$ are equivalent if the corresponding state space models have the same impulse response, i.e. if*

$$C \otimes A^{\otimes^k} \otimes B \;=\; \tilde{C} \otimes \tilde{A}^{\otimes^k} \otimes \tilde{B} \qquad \textit{for all } k \in \mathbb{N}\,.$$

Two 4-tuples $(A, B, C, x(0))$ and $(\tilde{A}, \tilde{B}, \tilde{C}, x(0))$ are called equivalent if the corresponding state space models have the same input-output behavior for the given initial condition, i.e. if

$$C \otimes A^{\otimes^k} \otimes B \;=\; \tilde{C} \otimes \tilde{A}^{\otimes^k} \otimes \tilde{B} \quad \textit{and} \quad C \otimes A^{\otimes^k} \otimes x(0) \;=\; \tilde{C} \otimes \tilde{A}^{\otimes^k} \otimes \tilde{x}(0)$$

for all $k \in \mathbb{N}$.

Note that if $(A, B, C, x(0))$ and $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{x}(0))$ are two state space realizations of a max-linear time-invariant DES then the matrices A and $\tilde{A}$ do not necessarily have the same size.

Definition 6.1.2 (Ultimately geometric impulse response)
Let $\{G_k\}_{k=0}^{\infty}$ be the impulse response of a max-linear time-invariant DES. If

$$\exists k_0 \in \mathbb{N},\, \exists c \in \mathbb{N}_0,\, \exists \lambda \in \mathbb{R}_\varepsilon \textit{ such that } \forall k \geqslant k_0 \,:\, G_{k+c} = \lambda^{\otimes^c} \otimes G_k \;, \tag{6.3}$$

then we say that the impulse response $\{G_k\}_{k=0}^{\infty}$ is ultimately geometric.

The term "ultimately geometric" was introduced by Gaubert in [56, 58]. Note that "geometric" has to be understood in the max-algebraic sense: the Markov parameters are max-multiplied by a constant factor.
If $G = \{G_k\}_{k=0}^{\infty}$ is an ultimately geometric sequence then the smallest possible c for which (6.3) holds is called the *period* of G.

Suppose that we have a DES that can be characterized by a triple (A, B, C). A sufficient but not necessary condition for the impulse response of this DES to be ultimately geometric is that A is irreducible (cf. Theorem 2.2.8). This will e.g. be the case for a DES without separate independent subsystems and with a cyclic behavior or with feedback from the output to the input (such as e.g. a flexible production system in which the parts are carried around on a limited number of pallets that circulate in the system [20]). As was shown in Example 2.4.1 ultimately geometric behavior can also occur if the system matrix A is not irreducible.

In general, the impulse response of a max-linear time-invariant DES can be characterized by the following theorem:

Theorem 6.1.3 *If $\{G_k\}_{k=0}^{\infty}$ is the impulse response of a max-linear time-invariant DES with m inputs and l outputs then*

$$\forall i \in \{1, 2, \ldots, l\}, \forall j \in \{1, 2, \ldots, m\}, \exists c \in \mathbb{N}_0,$$
$$\exists \lambda_1, \lambda_2, \ldots, \lambda_c \in \mathbb{R}_\varepsilon, \exists k_0 \in \mathbb{N} \textit{ such that } \forall k \geqslant k_0 :$$
$$(G_{kc+c+s-1})_{ij} = \lambda_s{}^{\otimes^c} \otimes (G_{kc+s-1})_{ij} \textit{ for } s = 1, 2, \ldots, c\,. \tag{6.4}$$

Proof: This is a direct consequence of e.g. Corollary 1.1.9 of [54, p. 166] or of Proposition 1.2.2 of [56] (See also Section C.1). □

If a sequence $G = \{G_k\}_{k=0}^{\infty}$ exhibits a behavior of the form (6.4) then we say that the sequence G is *ultimately periodic*.

Proposition 6.1.4 *A sequence $G = \{G_k\}_{k=0}^{\infty}$ with $G_k \in \mathbb{R}_\varepsilon^{l \times m}$ for all k is the impulse response of a max-linear time-invariant DES if and only if it is an ultimately periodic sequence.*

Proof: A proof of this proposition for SISO systems can be found in e.g. [3, 54, 54].
For MIMO systems the "only if" part corresponds to Theorem 6.1.3. Therefore, we only have to prove the "if" part. Consider arbitrary indices $i \in \{1, 2, \ldots, l\}$ and $j \in \{1, 2, \ldots, m\}$. Since G is ultimately periodic, the sequence $g_{ij} = \{(G_k)_{ij}\}_{k=0}^{\infty}$ is also ultimately periodic. From the first part of this proof, it follows that g_{ij} is the impulse response of a max-linear time-invariant SISO DES. Let (A_{ij}, B_{ij}, C_{ij}) be a realization of this system. If we repeat this reasoning for all pairs of indices (i, j) with $i \in \{1, 2, \ldots, l\}$ and $j \in \{1, 2, \ldots, m\}$ and if define block matrices A, B, C such that

$$A = \begin{bmatrix} A_{11} & \varepsilon & \ldots & \varepsilon & \varepsilon & \ldots & \varepsilon \\ \varepsilon & A_{12} & \ldots & \varepsilon & \varepsilon & \ldots & \varepsilon \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ \varepsilon & \varepsilon & \ldots & A_{1m} & \varepsilon & \ldots & \varepsilon \\ \varepsilon & \varepsilon & \ldots & \varepsilon & A_{21} & \ldots & \varepsilon \\ \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots \\ \varepsilon & \varepsilon & \ldots & \varepsilon & \varepsilon & \ldots & A_{lm} \end{bmatrix}$$

$$B = \begin{bmatrix} B_{11} & \varepsilon & \dots & \varepsilon \\ \varepsilon & B_{12} & \dots & \varepsilon \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon & \varepsilon & \dots & B_{1m} \\ B_{21} & \varepsilon & \dots & \varepsilon \\ \varepsilon & B_{22} & \dots & \varepsilon \\ \vdots & \vdots & & \vdots \\ \varepsilon & \varepsilon & \dots & B_{lm} \end{bmatrix}$$

$$C = \begin{bmatrix} C_{11} & C_{12} & \dots & C_{1m} & \varepsilon & \varepsilon & \dots & \varepsilon \\ \varepsilon & \varepsilon & \dots & \varepsilon & C_{21} & C_{22} & \dots & \varepsilon \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \varepsilon & \varepsilon & \dots & \varepsilon & \varepsilon & \varepsilon & \dots & C_{lm} \end{bmatrix} ,$$

then (A, B, C) is a realization of G. □

If $G = \{G_k\}_{k=0}^{\infty}$ is the impulse response of time-invariant max-linear DES then we call the (semi-infinite) block Hankel matrix

$$H(G) \stackrel{\text{def}}{=} \begin{bmatrix} G_0 & G_1 & G_2 & \dots \\ G_1 & G_2 & G_3 & \dots \\ G_2 & G_3 & G_4 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

the (block) Hankel matrix that corresponds to the impulse response G. This matrix will play an important role in the procedures to determine lower and upper bounds for the minimal system order (See Section 6.3.1).

Definition 6.1.5 (Max-algebraic weak column rank) *Let $A \in \mathbb{R}_\varepsilon^{m \times n}$. If $A \neq \mathcal{E}_{m \times n}$ then the max-algebraic weak column rank of A is defined by*

$$\begin{aligned} \operatorname{rank}_{\oplus,\mathrm{wc}}(A) = \min \Big\{ \#I \ \Big| \ & I \subseteq \{1, 2, \dots, n\} \text{ and } \forall k \in \{1, 2, \dots, n\}, \\ & \exists l \in \mathbb{N}_0, \exists i_1, i_2, \dots, i_l \in I, \exists \alpha_1, \alpha_2, \dots, \alpha_l \in \mathbb{R}_\varepsilon \\ & \text{such that } A_{.,k} = \bigoplus_{j=1}^{l} \alpha_j A_{.,i_j} \Big\} . \end{aligned}$$

If $A = \mathcal{E}_{m \times n}$ then we have $\operatorname{rank}_{\oplus,\mathrm{wc}}(A) = 0$.

A more formal definition of the max-algebraic weak column rank of a matrix can be found in e.g. [54, 56]. Efficient methods to compute the max-algebraic weak column rank of a matrix are described in [33, 35, 54].
If $A \in \mathbb{R}_\varepsilon^{m \times n}$ then we have $\operatorname{rank}_{\oplus}(A) \leqslant \operatorname{rank}_{\oplus,\mathrm{wc}}(A)$.

Example 6.1.6 Consider the matrix

$$A = \begin{bmatrix} 0 & 2 & \varepsilon & 0 \\ 7 & 9 & 3 & 7 \\ \varepsilon & 5 & 6 & 9 \\ 2 & 4 & 0 & 3 \end{bmatrix} .$$

It is obvious that the first column of A cannot be written as a max-linear combination of the other columns of A. This also holds for the third column of A. However, the second and the fourth column of A are max-linear combinations of the first and the third column:

$$A_{.,2} = 2 \otimes A_{.,1} \oplus (-1) \otimes A_{.,3} \text{ and } A_{.,4} = A_{.,1} \oplus 3 \otimes A_{.,3} .$$

This implies that $I = \{1, 3\}$ is a minimal set of column indices such that every column of A can be written as a max-linear combination of the columns that are indexed by I. Hence, $\mathrm{rank}_{\oplus,\mathrm{wc}}(A) = 2$. □

6.2 Transformation of State Space Models

In this section we present some theorems and propositions in connection with max-algebraic state space transformations. We shall again encounter these theorems and propositions when we look at the set of all the equivalent state space realizations of a given impulse response in Example 6.4.1.

If the 4-tuples $(A, B, C, \mathcal{E})$ and $(\tilde{A}, \tilde{B}, \tilde{C}, \mathcal{E})$ are equivalent, then the triples (A, B, C) and $(\tilde{A}, \tilde{B}, \tilde{C})$ are also equivalent. This means that all the theorems and propositions on equivalent 4-tuples of this section can also be reformulated for equivalent triples.

Proposition 6.2.1 (Max-algebraic similarity transformation) *Let $T \in \mathbb{R}_\varepsilon^{n \times n}$ be max-invertible. If $(A, B, C, x(0))$ is an nth order state space realization of a max-linear time-invariant DES then $(T \otimes A \otimes T^{\otimes^{-1}}, T \otimes B, C \otimes T^{\otimes^{-1}}, T \otimes x(0))$ is an equivalent realization.*

Proof: Consider an arbitrary $k \in \mathbb{N}$. We have

$$\begin{aligned} \left(C \otimes T^{\otimes^{-1}}\right) \otimes \left(T \otimes A \otimes T^{\otimes^{-1}}\right)^{\otimes^k} \otimes \left(T \otimes B\right) \\ = C \otimes T^{\otimes^{-1}} \otimes T \otimes A^{\otimes^k} \otimes T^{\otimes^{-1}} \otimes T \otimes B \\ = C \otimes A^{\otimes^k} \otimes B . \end{aligned}$$

Using a similar reasoning we also find

$$\left(C \otimes T^{\otimes^{-1}}\right) \otimes \left(T \otimes A \otimes T^{\otimes^{-1}}\right)^{\otimes^k} \otimes \left(T \otimes x(0)\right) = C \otimes A^{\otimes^k} \otimes x(0) .$$

Hence, $(A, B, C, x(0))$ and $(T \otimes A \otimes T^{\otimes^{-1}}, T \otimes B, C \otimes T^{\otimes^{-1}}, T \otimes x(0))$ are equivalent realizations. □

Corollary 6.2.2 *If $(A, B, C, x(0))$ is a state space realization of a max-linear time-invariant DES then the 4-tuple $(A,\, \alpha \otimes B,\, (-\alpha) \otimes C,\, \alpha \otimes x(0))$ with $\alpha \in \mathbb{R}$ is an equivalent realization.*

Proof: Apply Proposition 6.2.1 with $T = \alpha \otimes E_n$ and thus $T^{\otimes^{-1}} = \alpha^{\otimes^{-1}} \otimes E_n = (-\alpha) \otimes E_n$. □

The transformation of Proposition 6.2.1 is the max-algebraic equivalent of the similarity transformation of conventional linear algebra and linear system theory. We can give the following interpretation to max-algebraic similarity transformations. Assume that we have an nth order state space model of the form (6.1)–(6.2) with a state space vector x. Let $T \in \mathbb{R}_\varepsilon^{n \times n}$ be max-invertible. If we apply a max-algebraic similarity transformation with T to the given state space model, then the state space vector $\tilde{x}$ of the resulting state space model satisfies $\tilde{x} = T \otimes x$. By Proposition 2.2.1 the matrix T can be factorized as $T = D \otimes P$ with D a max-algebraic diagonal matrix with finite diagonal entries and P a max-algebraic permutation matrix. If we define $\hat{x} = P \otimes x$ then we have $\tilde{x} = D \otimes \hat{x}$. Right max-multiplication of x by the permutation matrix P corresponds to a permutation of the components of x. In conventional algebra we have $\tilde{x}_i = \hat{x}_i + d_{ii}$ for $i = 1, 2, \ldots, n$. So the D matrix corresponds to a translation of the origin of the state space. Hence, a max-algebraic similarity transformation corresponds to a permutation of the coordinates followed by a translation of the origin of the state space.

Since the class of max-invertible matrices is rather limited, max-algebraic similarity transformations have a limited scope. For linear time-invariant systems all the minimal state space realizations are related by similarity transformations [86]. However, in Example 6.4.1 we shall show that in general two arbitrary minimal state space realizations of a given max-linear time-invariant DES are not always related by max-algebraic similarity transformations.

The class of invertible matrices in $\mathbb{S}_{\max}$ can also be characterized as the set of all the matrices that can be written as the max-algebraic product of a max-algebraic diagonal matrix with finite diagonal entries and a max-algebraic permutation matrix. So if we transfer the problem to $\mathbb{S}_{\max}$, we are not much better off either. Furthermore, an approach based on $\mathbb{S}_{\max}$ has two other major drawbacks. First of all we get balances instead of equalities in $\mathbb{S}_{\max}$. Moreover, it is not trivial to find a similarity transformation such that the resulting system matrices will have entries that belong to $\mathbb{R}_\varepsilon$. This means that in general we cannot transfer the results back to $\mathbb{R}_{\max}$.

Therefore, we now present a method to perform state space transformations that is entirely based on $\mathbb{R}_{\max}$. A part of this approach — the L-transformations — was hinted at but not proved in [109]. We extend it such that the dimension of the state space vector can change. We also add another type of transformations: the M-transformations.

Theorem 6.2.3 (L-transformation) *Let the triple $(A, B, C, x(0))$ be an nth order state space realization of a max-linear time-invariant DES. Let $L \in \mathbb{R}_\varepsilon^{p\times n}$ be a common factor of A and C such that $A = \hat{A} \otimes L$ and $C = \hat{C} \otimes L$. Then the 4-tuple $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{x}(0))$ with*

$$\tilde{A} = L \otimes \hat{A}, \quad \tilde{B} = L \otimes B, \quad \tilde{C} = \hat{C} \quad \text{and} \quad \tilde{x}(0) = L \otimes x(0) \tag{6.5}$$

is an equivalent realization.

Proof: Consider an arbitrary $k \in \mathbb{N}$. We have

$$\begin{aligned} \tilde{C} \otimes \tilde{A}^{\otimes^k} \otimes \tilde{B} &= \hat{C} \otimes \left(L \otimes \hat{A} \right)^{\otimes^k} \otimes L \otimes B \\ &= \hat{C} \otimes L \otimes \left(\hat{A} \otimes L \right)^{\otimes^k} \otimes B \\ &= C \otimes A^{\otimes^k} \otimes B \ . \end{aligned}$$

Analogously, we find $\tilde{C} \otimes \tilde{A}^{\otimes^k} \otimes \tilde{x(0)} = C \otimes A^{\otimes^k} \otimes x(0)$.
This means that $(A, B, C, x(0))$ and $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{x}(0))$ are equivalent state space realizations. □

We can also use the dual of this theorem:

Theorem 6.2.4 (M-transformation) *Let the triple $(A, B, C, x(0))$ be an nth order state space realization of a max-linear time-invariant DES. Let $M \in \mathbb{R}_\varepsilon^{n\times p}$ be a common factor of A, B and $x(0)$ such that $A = M \otimes \hat{A}$, $B = M \otimes \hat{B}$ and $x(0) = M \otimes \hat{x}(0)$. Then the 4-tuple $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{x}(0))$ with*

$$\tilde{A} = \hat{A} \otimes M, \quad \tilde{B} = \hat{B}, \quad \tilde{C} = C \otimes M \quad \text{and} \quad \tilde{x}(0) = \hat{x}(0)$$

is an equivalent realization.

If we are considering triples instead of 4-tuples — i.e. if we are considering equivalent realizations of an impulse response — we do not have to include the condition $x(0) = M \otimes \hat{x}(0) = M \otimes \tilde{x}(0)$ when we determine M since we always have $\mathcal{E}_{n\times 1} = M \otimes \mathcal{E}_{p\times 1}$.
Let $(A, B, C, x(0))$ be an nth order state space realization of a max-linear time-invariant DES. To obtain another state space realization of the given system, we try to find a factorization

$$\begin{bmatrix} A \\ C \end{bmatrix} = \begin{bmatrix} \hat{A} \\ \hat{C} \end{bmatrix} \otimes L \qquad \text{or} \qquad \begin{bmatrix} A & B & x(0) \end{bmatrix} = M \otimes \begin{bmatrix} \hat{A} & \hat{B} & \hat{x}(0) \end{bmatrix}$$

with $L \in \mathbb{R}_\varepsilon^{p\times n}$ or $M \in \mathbb{R}_\varepsilon^{n\times p}$. These matrix factorizations can be considered as systems of multivariate max-algebraic equalities with the entries of $\hat{A}$, $\hat{C}$ and

L, or $\hat{A}$, $\hat{B}$, $\hat{x}(0)$ and M as unknowns (cf. Section 4.2.1). This means that we can use the ELCP approach to determine $\hat{A}$, $\hat{C}$ and L or $\hat{A}$, $\hat{B}$, $\hat{x}(0)$ and M.

Even if $p = n$, the matrices L and M are not necessarily max-invertible. So in general L-transformations and M-transformations are not max-algebraic similarity transformations. However, L-transformations and M-transformations can be considered as a generalization of max-algebraic similarity transformations since if L is max-invertible then (6.5) results in

$$\tilde{A} = L \otimes A \otimes L^{\otimes^{-1}}, \quad \tilde{B} = L \otimes B, \quad \tilde{C} = C \otimes L^{\otimes^{-1}} \text{ and } \tilde{x}(0) = L \otimes x(0) \ ,$$

and since an analogous result holds for M-transformations if M is max-invertible.

If $p = n$ then L or M will be square and then the 4-tuple $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{x}(0))$ will also be an nth order realization. If we take a rectangular L or M matrix, we can change the dimension of the state space vector and get a pth order state space model. It is obvious that p always has to be greater than or equal to the minimal system order since otherwise it is impossible to find a common factor of A and C or of A, B and $x(0)$.

Suppose that we can transform the 4-tuple $(A_1, B_1, C_1, x_1(0))$ into the 4-tuple $(A_2, B_2, C_2, x_2(0))$ with an L-transformation. Then it follows from (6.5) that

$$\begin{bmatrix} A_2 & B_2 & x_2(0) \end{bmatrix} = L \otimes \begin{bmatrix} \hat{A} & \hat{B} & \hat{x}(0) \end{bmatrix}$$

where $\hat{B} = B_1$ and $\hat{x}(0) = x_1(0)$. Furthermore,

$$A_1 = \hat{A} \otimes L \quad \text{and} \quad C_1 = \hat{C} \otimes L = C_2 \otimes L \ .$$

This implies that we can go back from $(A_2, B_2, C_2, x_2(0))$ to $(A_1, B_1, C_1, x_1(0))$ by an M-transformation with $M = L$. Hence, L-transformations and M-transformations can be considered as inverse transformations. However, in Example 6.4.1 we shall show that L-transformations and M-transformations in general do not yield the entire set of all equivalent state space realizations in one step.

In the next section we demonstrate how the set of all the minimal state space realizations of a given impulse response can be determined.

6.3 The Minimal State Space Realization Problem

Consider a max-linear time-invariant DES with m inputs and l outputs that can be described by an nth order state space model of the form (6.1)–(6.2). Suppose that the system matrices A, B and C of this system are unknown, and that we only know the impulse response $\{G_k\}_{k=0}^{\infty}$. How can we construct A, B and C from the sequence $\{G_k\}_{k=0}^{\infty}$? This process is called *state space realization*.

If we make the dimension of A minimal, then the dimension of A is equal to the *minimal system order* and the triple (A, B, C) is a *minimal state space realization* of $\{G_k\}_{k=0}^{\infty}$. Note that in general a minimal state space realization is not unique since Corollary 6.2.2 implies that any non-trivial impulse response has infinitely many equivalent (minimal) state space realizations.

There are several reasons why we study the minimal state space realization problem for max-linear time-invariant DESs. First of all this problem is the max-algebraic equivalent of one of the most elementary problems in linear system theory. Note however that for max-linear time-invariant DESs this problem is far more difficult to solve than for linear time-invariant systems. Apart from further enhancing the system theory for max-linear time-invariant DESs the solution of the minimal realization problem can also be seen as the first step towards identification of DESs. Finally the technique presented in this section can also be used to reduce the order of existing state space models.

The minimal state space realization problem for max-linear time-invariant DESs has been studied by many authors and for some specific cases the problem has been solved [35, 36, 54, 56, 115, 116, 122, 133, 146, 147, 148]. Related results can be found in [151, 152].

If certain conditions are satisfied then there exist efficient algorithms to compute a minimal state space realization, e.g.

- if the impulse response $g = \{g_k\}_{k=0}^{\infty}$ of a SISO DES exhibits a "uniformly up-terrace" behavior [146, 147, 148], i.e. if g consists of M subsequences with lengths $n_1, n_2, \ldots, n_M$ and increments $c_1, c_2, \ldots, c_M$ respectively such that

$$g_{k+1} = g_k + c_i \qquad \text{for } i = 1, 2, \ldots, M \text{ and } k = t_i, \ldots, t_i + n_i - 1 \ ,$$

  with $n_M = +\infty$, $t_1 = 0$ and $t_{i+1} = t_i + n_i$ and $c_{i+1} > c_i$ for $i = 1, 2, \ldots, M - 1$.

- if the impulse response $\{g_k\}_{k=0}^{\infty}$ of a SISO DES has finite Markov parameters and exhibits a "strictly convex" transient behavior and an ultimately geometric behavior with period 1 [36], i.e. if there exists an integer $k_0 \in \mathbb{N}$ and a real number λ such that

$$\begin{aligned} g_{k+1} - g_k &> g_k - g_{k-1} && \text{for } k = 0, 1, \ldots, k_0 \,, \\ g_{k+1} &= \lambda \otimes g_k && \text{for } k = k_0, k_0 + 1, \ldots \ . \end{aligned}$$

The methods given in [35, 54, 56, 133] do not always yield a minimal realization. In [115, 116, 122] Olsder uses a transformation from the max-plus algebra to a ring of sums of exponentials with conventional addition and multiplication as basic operations (This transformation is related to the mapping that we shall present in Section 7.2). Next he solves the realization problem in conventional algebra and transforms the results back to the max-plus algebra. However, it is not always obvious how and whether a realization can be constructed that can be mapped back to a realization with entries in $\mathbb{R}_\varepsilon$ (instead of $\mathbb{S}$ (cf. Section 7.2)). Note that this problem is related to the case of max-algebraic similarity transformations with entries that belong to $\mathbb{S}$: in that case it is also not trivial to obtain a realization with entries that belong to $\mathbb{R}_\varepsilon$ (See Section 6.2).

In this section we shall present a method that will always result in a minimal state space realization. If we use the ELCP algorithm of Section 3.4 to solve the resulting ELCP, we can — at least theoretically — compute *all* the minimal state space realizations of a given impulse response. Moreover, our method works for both SISO and MIMO systems. However, the major drawback of our method is that at this moment there are no efficient, polynomial time algorithms available to solve some of the subproblems encountered in this approach.

Let $G = \{G_k\}_{k=0}^{\infty}$ be an ultimately periodic sequence with $G_k \in \mathbb{R}_\varepsilon^{l \times m}$ for all k. So by Proposition 6.1.4 G is the impulse response of a max-linear time-invariant DES. We shall construct minimal state space realizations of G in three steps. First we determine a lower bound for the minimal system order; next we determine all the minimal state space realizations of a finite subsequence of G (i.e. we solve the partial realization problem) and finally we construct all the minimal state space realizations of the full sequence.

6.3.1 Determination of the Minimal System Order

We shall use the following lemma the proof of which is trivial:

Lemma 6.3.1 *Let $A \in \mathbb{R}_\varepsilon^{n \times n}$, $B \in \mathbb{R}_\varepsilon^{n \times m}$ and $C \in \mathbb{R}_\varepsilon^{l \times n}$. If A satisfies an equation of the form $\displaystyle\bigoplus_{i=0}^{n} a_i \otimes A^{\otimes^{n-i}} \nabla \mathcal{E}_{n \times n}$ (e.g. its max-algebraic characteristic equation) then the Markov parameters of the DES that corresponds to the triple (A, B, C) satisfy $\displaystyle\bigoplus_{i=0}^{n} a_i \otimes G_{k+n-i} \nabla \mathcal{E}_{l \times m}$ for all $k \in \mathbb{N}$.*

We could use the following theorem [54, 56] to determine a lower bound for the minimal system order:

Theorem 6.3.2 *If G is the impulse response of a time-invariant max-linear DES, then the max-algebraic minor rank of $H(G)$ is a lower bound for the minimal system order.*

It is still an open question whether it is possible to develop efficient algorithms to determine the max-algebraic minor rank of a matrix.

Therefore, we now present an alternative method to obtain a lower bound for the minimal system order. Parts of this method have also been used by other authors in [35, 133] and in a slightly different form in [116, 122]. Our main contribution is that we enhance the method by including our results on the necessary and/or sufficient conditions of Sections 5.2 and 5.3 for the coefficients of the MACP of a matrix with entries in $\mathbb{R}_\varepsilon$.

First we develop this approach for SISO systems. So we have to solve the following problem:

> Given the impulse response $g = \{g_k\}_{k=0}^{\infty}$ of a max-linear time-variant SISO DES, what is the minimal dimension of the system matrix A over the set of all state space realizations (A,B,C) of the given impulse response?

We shall use the following proposition:

Proposition 6.3.3 *If g is the impulse response of a SISO max-linear time-invariant DES with minimal system order n and if $H = H(g)$, then we have $\det_\oplus H_{\alpha\beta} \ \nabla \ \varepsilon$ for any $\alpha \subseteq \mathbb{N}_0$ with $\#\alpha > n$ and for any $\beta = \{j+1, j+2, \ldots, j+(\#\alpha)\}$ with $j \in \mathbb{N}$.*

Proof: Let (A, B, C) be an nth order state space realization of g. Since A is a matrix with entries in $\mathbb{R}_\varepsilon$, there also exists a signed version of the max-algebraic characteristic equation of A in which all the coefficients are signed (See [116] and Section A.1). The Cayley-Hamilton theorem also holds for this signed version of the max-algebraic characteristic equation. This means that the columns of H also satisfy an expression of the form

$$\bigoplus_{i=0}^{n} b_i \otimes H_{.,k+n-i} \ \nabla \ \mathcal{E}_{\infty\times 1} \qquad \text{for all } k \in \mathbb{N}_0$$

where the coefficients $b_1, b_2, \ldots, b_n$ are signed and where $b_0 = 0$. Hence, every system of max-linear balances of the form $H_{\alpha\beta} \otimes v \ \nabla \ \mathcal{E}_{\infty\times 1}$ with $\alpha \subseteq \mathbb{N}_0$ and $\#\alpha > n$ and with $\beta = \{j+1, j+2, \ldots, j+(\#\alpha)\}$ and $j \in \mathbb{N}$ has at least one non-trivial signed solution, namely $\begin{bmatrix} b_n & b_{n-1} & \ldots & b_0 & \varepsilon & \ldots & \varepsilon \end{bmatrix}^T$. Hence, it follows from Theorem 2.3.15 that $\det_\oplus H_{\alpha\beta} \nabla \varepsilon$. □

Note that Proposition 6.3.3 is in fact a direct consequence of Theorem 6.3.2. However, we have given a separate proof for this proposition since the reasoning given in this proof will be used in the alternative procedure to obtain a lower bound for the minimal system order.

Let M be a p by q matrix. If $\alpha \subseteq \{1, 2, \ldots, p\}$ and if $\beta = \{j+1, j+2, \ldots, j+l\}$ for some $j \in \mathbb{N}$ and some $l \in \mathbb{N}_0$ with $j \leqslant q-1$ and $l \leqslant q-j$, then $M_{\alpha\beta}$ is called a *consecutive column submatrix* of M. So Proposition 6.3.3 states that the dimension of the largest square consecutive column submatrix of $H(g)$ that has a non-balanced max-algebraic determinant is less than or equal to the minimal system order. We represent this dimension by $\text{rank}_{\oplus,\text{cc}}(H(g))$:

Definition 6.3.4 (Max-algebraic consecutive column rank)
Let $P \in \mathbb{S}^{m\times n}$. The max-algebraic consecutive column rank of P, denoted by

$\operatorname{rank}_{\oplus,\mathrm{cc}}(P)$, *is the dimension of the largest square consecutive column submatrix of P that has a max-algebraic determinant that is not balanced:*

$$\begin{aligned}\operatorname{rank}_{\oplus,\mathrm{cc}}(P) = \max\Big\{ \#\gamma \,\Big|\, \gamma \subseteq \mathbb{N}_0,\ 1 \leqslant \#\gamma \leqslant \min(m,n) \text{ with} \\ \exists j \in \{0,1,\ldots,n-(\#\gamma)\} \text{ such that } \det{}_\oplus P_{\gamma\delta} \not\nabla \varepsilon \\ \text{where } \delta = \{j+1, j+2, \ldots, j+(\#\gamma)\}\Big\}\end{aligned}$$

where $\max \emptyset$ *is equal to* 0 *by definition.*

Let $P \in \mathbb{S}^{m\times n}$. We could also define the max-algebraic consecutive row rank of P, $\operatorname{rank}_{\oplus,\mathrm{cr}}(P)$. However, in this section we only determine ranks of Hankel matrices, which are symmetric, and therefore we only need the max-algebraic consecutive column rank: if $P = P^T$ then $\operatorname{rank}_{\oplus,\mathrm{cc}}(P) = \operatorname{rank}_{\oplus,\mathrm{cr}}(P)$. Note that in general we have $\operatorname{rank}_{\oplus,\mathrm{cc}}(P) \neq \operatorname{rank}_{\oplus,\mathrm{cr}}(P)$ and $\operatorname{rank}_{\oplus,\mathrm{cc}}(P)$, $\operatorname{rank}_{\oplus,\mathrm{cr}}(P) \leqslant \operatorname{rank}_{\oplus}(P)$.

Let $H = H(g)$. Suppose that the max-algebraic characteristic equation of the unknown system matrix A is given by $\displaystyle\bigoplus_{i=0}^{n} a_i \otimes \lambda^{\otimes^{n-i}} \nabla\, \varepsilon$. As a direct consequence of Theorem 5.1.3 and Lemma 6.3.1, we have

$$\bigoplus_{i=0}^{n} a_i \otimes H_{.,k+n-i} \nabla\, \mathcal{E}_{\infty\times 1} \qquad \text{for all } k \in \mathbb{N}_0\,. \tag{6.6}$$

To determine a lower bound for the minimal system order we try to find a relation of the form (6.6) among the columns of H with a minimal number of terms. This number of terms will be a first estimate of the lower bound for the minimal system order. Since we know that the entries of the system matrix A belong to $\mathbb{R}_\varepsilon$, we look for coefficients that correspond to a matrix with entries in $\mathbb{R}_\varepsilon$. So the a_i's should satisfy the necessary and/or sufficient conditions of Sections 5.2 and 5.3.

This leads to the following procedure to determine a lower bound r for the minimal system order of the max-linear time-invariant SISO DES that corresponds to a given ultimately periodic sequence $g = \{g_k\}_{k=0}^{\infty}$:

First we construct a p by q Hankel matrix

$$\tilde{H} = \begin{bmatrix} g_0 & g_1 & \cdots & g_{q-1} \\ g_1 & g_2 & \cdots & g_q \\ \vdots & \vdots & \ddots & \vdots \\ g_{p-1} & g_p & \cdots & g_{p+q-2} \end{bmatrix}$$

with p and q large enough: $p, q \gg n$, where n is the real (but unknown) minimal system order. If we do not take p and q large enough, the lower

bound that we shall obtain will still be less than or equal to the minimal system order. Note that $\tilde{H}$ is a submatrix of $H(g)$.
Now we try to find r and $a_0, a_1, \ldots, a_r$ such that the columns of $\tilde{H}$ satisfy an expression of the form

$$\bigoplus_{i=0}^{r} a_i \otimes \tilde{H}_{.,k+r-i} \nabla \mathcal{E}_{\infty \times 1} \qquad \text{for } k = 1, 2 \ldots, q-r\,. \tag{6.7}$$

We start with r equal to $\operatorname{rank}_{\oplus,\mathrm{cc}}(\tilde{H})$. Let $\tilde{H}_{\gamma\delta}$ be an r by r consecutive column submatrix of $\tilde{H}$ such that $\det_{\oplus} \tilde{H}_{\gamma\delta} \not\nabla \varepsilon$. Assume that $\delta = \{j+1, j+2, \ldots, j+r\}$ with $j+r < q$. Let $i \in \{1, 2, \ldots, p\} \setminus \gamma$ and define $\alpha = \gamma \cup \{i\}$ and $\beta = \delta \cup \{j+r+1\}$. We have $\det_{\oplus} \tilde{H}_{\alpha\beta} \nabla \varepsilon$. So by Theorem 2.3.15 the system of homogeneous max-linear balances

$$\tilde{H}_{\alpha\beta} \otimes v \nabla \mathcal{E}_{r+1 \times 1} \tag{6.8}$$

has a (signed) solution. We look for a solution $v = \begin{bmatrix} a_r & a_{r-1} & \ldots & a_0 \end{bmatrix}^T$ that corresponds to the max-algebraic characteristic equation of a matrix with entries in $\mathbb{R}_\varepsilon$. An algorithm to compute a non-trivial signed solution of a system of homogeneous max-linear balances can be found in [54]. Note that we could also use the ELCP approach to determine all (finite) solutions of the system of homogeneous max-linear balances (6.8) (cf. Section 4.2.2).
Once we have found a solution of (6.8), we normalize the a_0 component to 0 and then we check if the necessary and/or sufficient conditions of Sections 5.2 and 5.3 for the coefficients of the MACP of a matrix with entries in $\mathbb{R}_\varepsilon$ are satisfied. Note that v should not necessarily be a signed solution: a signed solution would correspond to the signed version of the max-algebraic characteristic equation.
If we cannot find any solution of (6.8) that satisfies the necessary and/or sufficient conditions of Sections 5.2 and 5.3, we augment r and repeat the procedure. Note that even if the necessary conditions are satisfied we do not necessarily have coefficients that correspond to a matrix with entries in $\mathbb{R}_\varepsilon$.
We continue until we get a relation of the form (6.7) among the columns of $\tilde{H}$.
Since g is ultimately periodic, it corresponds to a max-linear time-invariant DES. Since we have assumed that $p, q \gg n$, this means that we can always find a relation of the form (6.7) by gradually augmenting r. The r that results from this procedure is a lower bound for the minimal system order, since it corresponds to the smallest possible number of terms in a relation of the form (6.7) among the columns of $\tilde{H}$.

The efficiency of this approach depends on many factors:

- In general, we have $\operatorname{rank}_{\oplus,\mathrm{cc}}(\tilde{H}) \leqslant \operatorname{rank}_{\oplus}(\tilde{H})$. However, this does not necessarily mean that the lower bound of Theorem 6.3.2 is better than the lower bound obtained by the procedure presented above, since it is possible that there does not exist a relation of the form (6.7) with $r = \operatorname{rank}_{\oplus}(\tilde{H})$ terms among the columns of $\tilde{H}$.

- At present, we have to use minor inspection to compute $\operatorname{rank}_{\oplus}(\tilde{H})$ and $\operatorname{rank}_{\oplus,\mathrm{cc}}(\tilde{H})$. This implies that with the current algorithms it takes much more time to compute $\operatorname{rank}_{\oplus}(\tilde{H})$ than to compute $\operatorname{rank}_{\oplus,\mathrm{cc}}(\tilde{H})$ since there are $\binom{r}{p}\binom{r}{q}$ minors of size r in $\tilde{H}$ compared to $\binom{r}{p}(q+1-r)$ consecutive column submatrices of size r.

- Finally, we have to remark that the algorithm of [54] for solving a system of homogeneous max-linear balances does not yield all solutions. So if the solution provided by this algorithm does not satisfy the necessary conditions for the coefficients of the MACP of a matrix with entries in $\mathbb{R}_{\varepsilon}$, we do not know for sure whether there exists a solution that satisfies these necessary conditions.
  However, (6.8) can be transformed into a system of max-algebraic polynomial equalities and inequalities (cf. Section 4.2.2). Hence, we could use the ELCP algorithm to find all finite solutions of (6.8) but then it is possible that from a computational point of view the alternative procedure to determine a lower bound for the minimal system order is not attractive any more. Therefore, there certainly is a need for efficient algorithms to find all solutions of a system of homogeneous max-linear balances.

These remarks clearly illustrate the need for efficient, polynomial time algorithms to solve some basic numerical problems in $\mathbb{R}_{\max}$ and $\mathbb{S}_{\max}$ such as computing the max-algebraic minor rank of a matrix or determining all solutions of a system of homogeneous max-linear balances. The answer to the question as to which of these two problems can be solved most efficiently determines which of the two methods to determine a lower bound for the minimal system order should be preferred.
Furthermore, the results of Chapter 5 should be extended such that necessary and sufficient conditions for the MACP of a matrix of any size with entries in $\mathbb{R}_{\varepsilon}$ are obtained.

If the given impulse response is ultimately geometric, we can use the following theorem [35, 54, 56] to determine an upper bound for the minimal system order:

Theorem 6.3.5 *Let g be the impulse response of a max-linear time-invariant SISO DES with $g \neq \{\varepsilon\}_{k=0}^{\infty}$. If g is ultimately geometric then the max-algebraic weak column rank of $H(g)$ is an upper bound for the minimal system order.*

A comparison of the upper bound of this theorem and the lower bound of Theorem 6.3.2, and some illustrative examples can be found in [54, 56]. In [54, 56] Gaubert has also given a generalization of Theorem 6.3.5 for impulse responses that are not ultimately geometric (See Section C.3).

If s is the upper bound for the minimal system order obtained by using Theorem 6.3.5 or its generalization, then there exist efficient methods to construct an sth order state space realization of the given impulse response [35, 54, 56].

The alternative procedure to determine a lower bounded for the minimal system order of a max-linear time-invariant SISO DES based on the max-algebraic characteristic equation can be extended to the MIMO case. Then we have to find a relation of the form

$$\bigoplus_{i=0}^{r} a_i \otimes G_{k+r-i} \ \nabla \ \mathcal{E}_{l\times m} \qquad \text{for all } k \in \mathbb{N}$$

with a minimal number of terms and with $a_0 = 0$ and where the coefficients $a_1, a_2, \ldots, a_r$ satisfy the necessary and/or sufficient conditions of Sections 5.2 and 5.3 for the coefficients of the MACP of a matrix with entries in $\mathbb{R}_\varepsilon$.

6.3.2 The Partial State Realization Problem

Now we consider the *partial realization problem*: we try to determine a realization that fits the first N Markov parameters of the sequence $G = \{G_k\}_{k=0}^{\infty}$ for some $N \in \mathbb{N}_0$. Suppose that r is a lower bound for the minimal system order (obtained e.g. by using one of the techniques discussed in Section 6.3.1). We have to find $A \in \mathbb{R}_\varepsilon^{r\times r}$, $B \in \mathbb{R}_\varepsilon^{r\times m}$ and $C \in \mathbb{R}_\varepsilon^{l\times r}$ such that

$$C \otimes A^{\otimes^k} \otimes B = G_k \qquad \text{for } k = 0, 1, \ldots, N-1\,. \tag{6.9}$$

If we write out the equations of the form (6.9), we get

$$\bigoplus_{p=1}^{r} c_{ip} \otimes b_{pj} \ = \ (G_0)_{ij}$$

for $i = 1, 2, \ldots, l$ and $j = 1, 2, \ldots, m$, and

$$\bigoplus_{p=1}^{r} \bigoplus_{q=1}^{r} c_{ip} \otimes (A^{\otimes^k})_{pq} \otimes b_{qj} \ = \ (G_k)_{ij} \tag{6.10}$$

for $i = 1, 2, \ldots, l,\ j = 1, 2, \ldots, m$ and $k = 1, 2, \ldots, N-1$.
Since

$$(A^{\otimes^k})_{pq} \ = \ \bigoplus_{i_1=1}^{r} \bigoplus_{i_2=1}^{r} \ldots \bigoplus_{i_{k-1}=1}^{r} a_{pi_1} \otimes a_{i_1 i_2} \otimes \ldots \otimes a_{i_{k-1} q} \ ,$$

equation (6.10) can be rewritten as

$$\bigoplus_{p=1}^{r} \bigoplus_{q=1}^{r} \bigoplus_{s=1}^{r^{k-1}} c_{ip} \otimes \bigotimes_{u=1}^{r} \bigotimes_{v=1}^{r} a_{uv}{}^{\otimes^{\gamma_{kpqsuv}}} \otimes b_{qj} \ = \ (G_k)_{ij} \tag{6.11}$$

where γ_{kpqsuv} is the number of times that a_{uv} appears in the sth term of $(A^{\otimes^k})_{pq}$. Note that if a_{uv} does not appear in that term we have $\gamma_{kpqsuv} = 0$ since $a^{\otimes^0} = 0 \cdot a = 0$, the identity element for $\otimes$. If we use the fact that $x \oplus x = x$ and $x \otimes y \leqslant x \otimes x \oplus y \otimes y$ for all $x, y \in \mathbb{R}_\varepsilon$, we can remove many redundant terms. Suppose that after removing the redundant terms, there are w_{kij} terms left in (6.11). Note that $w_{kij} \leqslant r^{k+1}$.
If we put the entries of A, B and C in one large vector x of length $r(r+m+l)$, we have to solve a system of multivariate max-algebraic polynomial equations of the following form:

$$\bigoplus_{p=1}^{r} \bigotimes_{q=1}^{r(r+m+l)} x_q^{\otimes^{\delta_{0ijpq}}} = (G_0)_{ij} \tag{6.12}$$

$$\bigoplus_{p=1}^{w_{kij}} \bigotimes_{q=1}^{r(r+m+l)} x_q^{\otimes^{\delta_{kijpq}}} = (G_k)_{ij} \tag{6.13}$$

for $i = 1, 2, \ldots, l$, $j = 1, 2, \ldots, m$ and $k = 1, 2, \ldots, N-1$. If all the impulse response matrices have finite entries then it follows from Proposition 4.1.4 that (6.12) – (6.13) always has a finite solution. This solution can be found using the ELCP approach. If some impulse response matrices have entries that are equal to ε, we can also use the ELCP approach to solve the system of max-algebraic polynomial equalities (6.12) – (6.13) if we apply the threshold or the limit procedure described in Remark 4.1.7. Note that all the exponents in (6.12) – (6.13) are nonnegative.
Once we have found a solution of (6.12) – (6.13), we extract the entries of the system matrices A, B and C from x. This results in a partial realization of the given impulse response. If we do not get any solutions, then r is less than the minimal system order, i.e. it is not possible to describe the given impulse response with an rth order state space model. In that case we augment r and we repeat the above procedure until we finally get a solution.

Now we can characterize the set of all finite partial state space realizations of a given impulse response:

Proposition 6.3.6 *Let $r \in \mathbb{N}$. In general the set of all finite rth order partial state space realizations of the impulse response of a max-linear time-invariant DES with finite Markov parameters corresponds to the union of faces of a polyhedron in the x space, where x is the vector obtained by putting the components of the system matrices in one large column vector.*

The set of all the rth order state space realizations of the first N Markov parameters of G will be denoted by $\mathcal{R}_r(G, N)$. So

$$\mathcal{R}_r(G,N) = \Big\{ (A,B,C) \;\Big|\; A \in \mathbb{R}_\varepsilon^{r \times r}, B \in \mathbb{R}_\varepsilon^{r \times m}, C \in \mathbb{R}_\varepsilon^{l \times r} \text{ and } C \otimes A^{\otimes^k} \otimes B = G_k \text{ for } k = 0, 1, \ldots, N-1 \Big\} .$$

6.3.3 Realizations of the Entire Impulse Response

Let r be a lower bound for the minimal system order and suppose that $\mathcal{R}_r(G,N)$ is nonempty for all $N \in \mathbb{N}_0$. If we want to find all rth order state space realizations of G, we have to determine $\lim_{N\to\infty} \mathcal{R}_r(G,N)$. When N becomes larger and larger, there are two possible situations that can occur:

(1) there exists an index $N_0 \in \mathbb{N}_0$ such that $\mathcal{R}_r(G,N) = \mathcal{R}_r(G,N_0)$ for all $N \geqslant N_0$;

(2) the sequence $\{\mathcal{R}_r(G,N)\}_{N=1}^{\infty}$ does not become stationary after a finite number of terms.

The first situation typically occurs if G is ultimately geometric. Unfortunately, it is not obvious how N_0 can be determined without explicitly computing the terms of the sequence $\{\mathcal{R}_r(G,N)\}_{N=1}^{\infty}$. Therefore, we start with an arbitrary integer $\tilde{N} \in \mathbb{N}$ and we construct the sequence $\mathcal{R}_r(G,\tilde{N})$, $\mathcal{R}_r(G,\tilde{N}+1)$, $\mathcal{R}_r(G,\tilde{N}+2)$, ... and we check whether this sequence becomes stationary from a certain index $\tilde{N}_0$ on. It is obvious that we have to take our estimate of $\tilde{N}$ large enough. In practice it appears that we should at least include the transient behavior and the first cycles of the geometric behavior.

Case (2) occurs if G is not ultimately geometric since then G cannot be realized by a triple (A,B,C) with an irreducible A matrix. So A has to contain entries that are equal to ε. Since $\mathcal{R}_r(G,N)$ always contains finite elements for any $N \in \mathbb{N}$ (under the assumption that all the entries of the Markov parameters are finite), the sequence $\{\mathcal{R}_r(G,N)\}_{N=1}^{\infty}$ cannot reach its limit after a finite numbers of terms in this case. However, we can still use the ELCP approach by applying a limit procedure and by observing how $\mathcal{R}_r(G,N)$ evolves as N goes to ∞. In the limit some of the entries of the system matrix A will become equal to ε. In order to be able to determine to evolution of $\mathcal{R}_r(G,N)$ as N goes to ∞ it is advisable to perform certain normalizations and to sort the extreme generators and the finite points lexicographically[1] before listing them.

Lemma 6.3.7 *Consider an arbitrary minimal realization (A,B,C) of the impulse response G of a max-linear time-invariant DES. If $L^*(G)$ is the set of the smallest possible values for the λ_s's in (6.4) and if $\max L^*(G) \neq \varepsilon$, then $\max L^*(G)$ is equal to the largest max-algebraic eigenvalue of A.*

Proof: See Section C.1. □

Lemma 6.3.8 *Consider $A \in \mathbb{R}_\varepsilon^{n\times n}$. If λ is the largest max-algebraic eigenvalue of A and if $\lambda \neq \varepsilon$ then there exists a max-invertible matrix $T \in \mathbb{R}_\varepsilon^{n\times n}$ such that $\left\| T \otimes A \otimes T^{\otimes^{-1}} \right\|_\oplus = \lambda$.*

Proof: See Section C.2. □

[1] A vector $x \in \mathbb{R}^n$ is lexicographically greater than or equal to a vector $y \in \mathbb{R}^n$ if and only if the first non-zero component of $x - y$ is greater than or equal to 0.

Proposition 6.3.9 *Let G be the impulse response of a max-linear time-invariant DES, and let $L^*(G)$ be the set of the smallest possible values for the λ_s's in (6.4). Let (A, B, C) be a minimal state space realization of G and let n be the minimal system order. If $\lambda = \max L^*(G)$ and if $\lambda \neq \varepsilon$, then there exists a max-algebraic similarity transformation that transforms (A, B, C) in an equivalent state space realization $(\tilde{A}, \tilde{B}, \tilde{C})$ of G with $\|\tilde{A}\|_\oplus = \lambda$, $\|\tilde{B}\|_\oplus = 0$ and $\tilde{a}_{11} \geqslant \tilde{a}_{22} \geqslant \ldots \geqslant \tilde{a}_{nn}$.*

Proof: Since (A, B, C) is a minimal state space realization of G, the maximal max-algebraic eigenvalue of A is equal to λ by Lemma 6.3.7. Furthermore, since $\lambda \neq \varepsilon$, B has at least one finite entry. Hence, $\|B\|_\oplus$ is finite. By Lemma 6.3.8 there exists a max-invertible matrix T such that $\left\|T \otimes A \otimes T^{\otimes^{-1}}\right\|_\oplus = \lambda$. If we define $\hat{A} = T \otimes A \otimes T^{\otimes^{-1}}$, $\hat{B} = T \otimes B$ and $\hat{C} = C \otimes T^{\otimes^{-1}}$, then $(\hat{A}, \hat{B}, \hat{C})$ is a realization of G by Proposition 6.2.1. We have $\|\hat{A}\|_\oplus = \lambda$. Furthermore, $\|\hat{B}\|_\oplus$ is finite.
Now we define $\alpha = \|\hat{B}\|_\oplus$, $\bar{A} = \hat{A}$, $\bar{B} = (-\alpha) \otimes \hat{B}$ and $\bar{C} = \alpha \otimes \hat{C}$. By Corollary 6.2.2 $(\bar{A}, \bar{B}, \bar{C})$ is also a realization of G. Moreover, $\|\bar{A}\|_\oplus = \lambda$ and $\|\bar{B}\|_\oplus = \|(-\alpha) \otimes \hat{B}\|_\oplus = (-\alpha) \otimes \|\hat{B}\|_\oplus = (-\alpha) \otimes \alpha = 0$.
Finally, we reorder the diagonal entries of $\hat{A}$ as follows. Let σ be a permutation of $\{1, 2, \ldots, n\}$ such that $\bar{a}_{\sigma(i)\sigma(i)} \geqslant \bar{a}_{\sigma(i+1),\sigma(i+1)}$ for $i = 1, 2, \ldots, n-1$. Now we define a max-algebraic permutation matrix $P \in \mathbb{R}_\varepsilon^{n \times n}$ such that $p_{ij} = 0$ if $j = \sigma(i)$ and $p_{ij} = \varepsilon$ if $j \neq \sigma(i)$. If we define $\tilde{A} = P \otimes \bar{A} \otimes P^{\otimes^{-1}}$ then the diagonal entries of $\tilde{A}$ are sorted in descending order. If we also define $\tilde{B} = P \otimes \bar{B}$ and $\tilde{C} = \bar{C} \otimes P^{\otimes^{-1}}$, then $(\tilde{A}, \tilde{B}, \tilde{C})$ is a realization of G by Proposition 6.2.1. Since the transformation from $\bar{A}$ to $\tilde{A}$ and from $\bar{B}$ to $\tilde{B}$ corresponds to a permutation of the rows and the columns of $\bar{A}$ and a permutation of the rows of $\bar{B}$, we have $\|\tilde{A}\|_\oplus = \|\bar{A}\|_\oplus = \lambda$ and $\|\tilde{B}\|_\oplus = \|\bar{B}\|_\oplus = 0$. □

So by applying max-algebraic similarity transformations we can always bring a minimal state space realization into a normalized form. Therefore, we may always add the following extra constraints to the system of multivariate max-algebraic polynomial equalities (6.12) – (6.13) if we are computing minimal state space realizations:

$$\bigoplus_{i=1}^{r} \bigoplus_{j=1}^{r} a_{ij} = \lambda \tag{6.14}$$

$$\bigoplus_{i=1}^{r} \bigoplus_{j=1}^{m} b_{ij} = 0 \tag{6.15}$$

$$a_{ii} \geqslant a_{i+1,i+1} \qquad \text{for } i = 1, 2, \ldots, r-1\,, \tag{6.16}$$

where $\lambda = \max L^*(G) \neq \varepsilon$ (Note that λ can only be equal to ε if $G_k = \mathcal{E}_{l \times m}$ for all k).

So instead of determining the evolution of $\mathcal{R}_r(G,N)$, we determine the evolution of

$$\mathcal{R}_r^{\text{nor}}(G,N) \stackrel{\text{def}}{=} \Big\{(A,B,C) \in \mathcal{R}_r(G,N) \,\Big|\, \|A\|_\oplus = \lambda,\ \|B\|_\oplus = 0 \text{ and }$$
$$a_{11} \geqslant a_{22} \geqslant \ldots \geqslant a_{rr}\Big\} \ .$$

Once we have determined $\mathcal{R}_r^{\text{nor}}(G) \stackrel{\text{def}}{=} \lim_{N\to\infty} \mathcal{R}_r^{\text{nor}}(G,N)$, we can reconstruct the elements of the set $\mathcal{R}_r(G) \stackrel{\text{def}}{=} \lim_{N\to\infty} \mathcal{R}_r(G,N)$ by applying max-algebraic similarity transformations to the elements of $\mathcal{R}_r^{\text{nor}}(G)$:

$$\mathcal{R}_r(G) = \Bigg\{ \left(T \otimes A \otimes T^{\otimes^{-1}}, T \otimes B, C \otimes T^{\otimes^{-1}}\right) \Bigg| (A,B,C) \in \mathcal{R}_r^{\text{nor}}(G) \text{ and }$$
$$T \in \mathbb{R}_\varepsilon^{r\times r} \text{ is max-invertible} \Bigg\} \ .$$

This procedure will be illustrated in Example 6.4.3.

If the above procedure does not yield a realization of the complete impulse response, we have to augment r and repeat the procedure of Sections 6.3.2 and 6.3.3 until we finally get a solution.

Remark 6.3.10 If we already have a state space realization of the given impulse response, we could try to use L-transformations or M-transformations with the lower bound r as the number of rows of L or the number of columns of M to get a minimal realization. If we do not get any solutions, we could augment r and repeat the procedure until we would finally get a solution. However, in Example 6.4.1 we shall show that it is not always possible to obtain a minimal state space realization in this way: if we use the procedure described in this remark then it is possible that the system order of the final solution is greater than the minimal system order. $\diamond$

6.3.4 Computational Complexity and Algorithmic Aspects

As already indicated before, the execution time and the storage space requirements of the ELCP algorithm depend on the number of variables and inequalities. For the minimal realization problem the number of variables and inequalities grows rapidly when the minimal system order increases or when the number of Markov parameters that should be considered increases. Therefore, the ELCP algorithm in its present form is not well suited to solve the minimal state space realization problem for DESs with a large minimal system order or with a long and complex transient behavior. Moreover, we are not always interested in finding all minimal realizations.

Note that we can also apply the normalization technique of Section 6.3.3 if we compute partial realizations of an ultimately geometric impulse response.

This will reduce the solution set of the resulting ELCP and it will in general result in faster execution times for our ELCP algorithm (provided that the inequalities that correspond to the extra conditions (6.14) – (6.15) are processed before the inequalities that correspond to the system (6.12) – (6.13)).

Sometimes it is possible to determine the system matrices in two steps. First we construct the system matrix A starting from the coefficients a_1, a_2, $\ldots$, a_r that result from the procedure to determine a lower bound r for the minimal system order. If r is less than or equal to 4, we can use the propositions of Section 5.3 to construct A. Otherwise, we can use the heuristic algorithm of Section B.4 to construct A. Once we have determined A, we can determine the matrices B and C by solving a system of max-algebraic polynomial equalities that is similar to the system (6.12) – (6.13). However, since the entries of A are known, the number of variables and inequalities of the corresponding ELCP will be considerably smaller than the ELCP that would result from the "direct" approach in which the matrices A, B and C are determined simultaneously. As a consequence, solving the ELCP for B and C will require far less CPU time than solving the full ELCP that would result from the direct approach. However, we shall show in Example 6.4.2 that for some matrices A that have a MACP that is equal to $\lambda^{\otimes^r} \oplus \bigoplus_{i=1}^{r} a_i \otimes \lambda^{\otimes^{r-i}}$ we cannot find matrices B and C such that the triple (A, B, C) is a (partial) state space realization of the given impulse response, even if r is equal to the minimal system order.

Since the method to solve the ELCP is an iterative process where in each step a new (in)equality is taken into account, we can make use of the special structure of our problem to speed up the algorithm. To each Markov parameter there corresponds a group of linear inequalities of the homogeneous ELCP. After each group we can test whether the impulse responses of the solutions of the ELCP that corresponds to that group and all the previous groups match the desired impulse response. If this is the case we do not have to take the other groups of inequalities into account, since they will automatically be satisfied. This means that we can start with a small number of Markov parameters and gradually take more and more groups of inequalities into account. Note that we do not have to start all over again for each new group since we can continue with the central and the extreme generators of the solution set of the homogeneous ELCP that corresponds to the whole of all the previous groups of inequalities.

Since we have one group of inequalities for each Markov parameter that we take into consideration and since the computational complexity increases as the number of inequalities grows, it is important to use as few Markov parameters as possible. But if we take too few Markov parameters, we can get solutions with an impulse response that does not coincide entirely with the desired impulse response. In Example 6.4.2 we shall show that in order to obtain a realization of the first N Markov parameters it is not always necessary to solve the full system of multivariate max-algebraic polynomial equalities that corresponds to the sequence G_0, G_1, $\ldots$, G_{N-1}: sometimes it is sufficient to consider the reduced system obtained by removing all the multivariate max-algebraic polynomial equalities that correspond to one or more Markov parameters of the sequence $G_0, G_1, \ldots, G_{N-1}$. However, it is still an open question how we can determine a minimal set of Markov parameters such that any minimal state space realization of this minimal set is also a realization of the entire impulse response.

6.4 Worked Examples

In this section we illustrate the procedure of the previous section with some examples. For the first example we also give an interpretation of the solution set of the minimal state space realization problem in terms of the theorems on state space transformations of Section 6.2. The second and the third example will show that our ELCP technique can be used to find minimal realizations in cases that are not covered by the authors cited in Section 6.3.

Example 6.4.1 We consider again the max-linear time-invariant DES of Example 1.2.1. This system can be realized by the triple (A, B, C) with

$$A = \begin{bmatrix} 5 & \varepsilon & \varepsilon \\ \varepsilon & 6 & \varepsilon \\ 11 & 12 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix} \text{ and } C = \begin{bmatrix} \varepsilon & \varepsilon & 3 \end{bmatrix} .$$

We are going to construct all equivalent minimal state space realizations of this DES starting from its impulse response, which is given by

$$g = \{g_k\}_{k=0}^{\infty} = 11, 16, 21, 27, 33, 39, 45, 51, 57, 63, 69, 75 \ldots \ .$$

This impulse response is ultimately geometric since $g_{k+1} = 6 \otimes g_k$ for all $k \geqslant 2$. First we determine a lower bound for the minimal system order. Let $H = H(g)$. Consider the following 8 by 8 submatrix of H:

$$\tilde{H} = H_{\{1,2,\ldots,8\},\{1,2,\ldots,8\}} = \begin{bmatrix} 11 & 16 & 21 & 27 & 33 & 39 & 45 & 51 \\ 16 & 21 & 27 & 33 & 39 & 45 & 51 & 57 \\ 21 & 27 & 33 & 39 & 45 & 51 & 57 & 63 \\ 27 & 33 & 39 & 45 & 51 & 57 & 63 & 69 \\ 33 & 39 & 45 & 51 & 57 & 63 & 69 & 75 \\ 39 & 45 & 51 & 57 & 63 & 69 & 75 & 81 \\ 45 & 51 & 57 & 63 & 69 & 75 & 81 & 87 \\ 51 & 57 & 63 & 69 & 75 & 81 & 87 & 93 \end{bmatrix} .$$

The max-algebraic minor rank of $\tilde{H}$ is equal to 2.
The max-algebraic consecutive column rank of $\tilde{H}$ is also equal to 2. The max-algebraic determinant of $\tilde{H}_{\{1,3\},\{1,2\}} = \begin{bmatrix} 11 & 16 \\ 21 & 27 \end{bmatrix}$ is not balanced. We add

one row and one column and then we look for a solution of the system of max-linear balances

$$\tilde{H}_{\{1,2,3\},\{1,2,3\}} \otimes v = \begin{bmatrix} 11 & 16 & 21 \\ 16 & 21 & 27 \\ 21 & 27 & 33 \end{bmatrix} \otimes \begin{bmatrix} a_2 \\ a_1 \\ a_0 \end{bmatrix} \nabla \, \mathcal{E}_{3\times 1} \; .$$

If we use the algorithm of [54] to solve this balance and if we normalize the a_0 component of the resulting solution, we obtain $a_0 = 0$, $a_1 = \ominus 6$, $a_2 = 11$. This solution satisfies the necessary and sufficient conditions of Proposition 5.3.2 for the coefficients of the MACP of a 2 by 2 matrix with elements in $\mathbb{R}_\varepsilon$ since $a_1^\oplus = \varepsilon$ and $a_2^\oplus = 11 \leqslant 12 = 6 \otimes 6 = a_1^\ominus \otimes a_1^\ominus$. This solution also corresponds to a stable relation among the columns of $\tilde{H}$:

$$\tilde{H}_{.,k+2} \; \oplus \; 11 \otimes \tilde{H}_{.,k} = 6 \otimes \tilde{H}_{.,k+1} \qquad \text{for } k = 1, 2, \ldots, 6 \, .$$

It is easy to verify that this relation also holds for all the columns of the full matrix H.
So the minimal system order is greater than or equal to 2.
Let us now compute the max-algebraic weak column rank of H in order to obtain an upper bound for the minimal system order. It is impossible to find a number $\alpha \in \mathbb{R}_\varepsilon$ and an index $i \in \mathbb{N}_0 \setminus \{1\}$ such that $H_{.,1} = \alpha \otimes H_{.,i}$. It is also impossible to find a number $\beta \in \mathbb{R}_\varepsilon$ and an index $j \in \mathbb{N}_0 \setminus \{3\}$ such that $H_{.,3} = \beta \otimes H_{.,j}$. This implies that $\operatorname{rank}_{\oplus,\mathrm{wc}}(H)$ is greater than 1. We can express every column of H as a max-linear combination of $H_{.,1}$ and $H_{.,3}$: $H_{.,2} = 5 \otimes H_{.,1} \oplus (-6) \otimes H_{.,3}$ and $H_{.,k} = (6(k-3)) \otimes H_{.,3}$ for all $k \geqslant 4$. Hence, $\operatorname{rank}_{\oplus,\mathrm{wc}}(H) = 2$. So the minimal system order is less than or equal to 2 by Theorem 6.3.5.
This implies that the minimal system order is equal to 2.

Now we consider the sequence $\{\mathcal{R}_2(g,N)\}_{N=1}^{\infty}$. If we use the ELCP algorithm of Section 3.4 to solve the system of max-algebraic polynomial equalities that corresponds to $\mathcal{R}_2(g,N)$, we get the generators and the finite points of Table 6.1 and the pairs of maximal cross-complementary subsets of Table 6.2 for any $N \geqslant 5$. Hence, $\mathcal{R}_2(g) = \mathcal{R}_2(g,N) = \mathcal{R}_2(g,5)$ for all $N \geqslant 5$. This means that the generators and the finite points of Table 6.1 and the pairs of maximal cross-complementary subsets of Table 6.2 also describe the set of all 2nd order state space realizations of g.
If we consider the set $\mathcal{R}_2(g,N)$ with $N < 5$, some combinations of the generators and the finite points only yield a partial realization of the given impulse response: they only fit the first N Markov parameters.
If we remove all the inequalities that correspond to one or more of the Markov parameters of the sequence g_0, g_1, g_2, g_3, g_4 from the ELCP for $N = 5$, some combinations of the generators and the finite points of the solution set of the resulting ELCP do not result in a state space realization of the given impulse response. Hence, $\{g_0, g_1, g_2, g_3, g_4\}$ is a minimal set of Markov parameters such that any minimal state space realization of this set is also a realization of g.

	$\mathcal{X}^{\text{cen}}$		$\mathcal{X}^{\text{ext}}$						$\mathcal{X}^{\text{fin}}$	
	x_1^{c}	x_2^{c}	x_1^{e}	x_2^{e}	x_3^{e}	x_4^{e}	x_5^{e}	x_6^{e}	x_1^{f}	x_2^{f}
a_{11}	0	0	0	0	0	0	0	0	6	5
a_{12}	1	0	−1	0	0	0	0	0	10	10
a_{21}	−1	0	0	−1	0	0	0	0	0	0
a_{22}	0	0	0	0	0	0	0	0	5	6
b_1	0	1	0	0	−1	0	0	0	0	0
b_2	−1	1	0	0	0	−1	0	0	−4	−6
c_1	0	−1	0	0	0	0	−1	0	9	11
c_2	1	−1	0	0	0	0	0	−1	15	15

Table 6.1: The generators and the finite points that correspond to the set $\mathcal{R}_2(g, N)$ of Example 6.4.1 for $N \geqslant 5$.

s	$\mathcal{X}_s^{\text{ext}}$	$\mathcal{X}_s^{\text{fin}}$
1	$\{x_1^{\text{e}}, x_2^{\text{e}}\}$	$\{x_1^{\text{f}}\}$
2	$\{x_1^{\text{e}}, x_2^{\text{e}}\}$	$\{x_2^{\text{f}}\}$
3	$\{x_1^{\text{e}}, x_4^{\text{e}}\}$	$\{x_2^{\text{f}}\}$
4	$\{x_1^{\text{e}}, x_5^{\text{e}}\}$	$\{x_1^{\text{f}}\}$

s	$\mathcal{X}_s^{\text{ext}}$	$\mathcal{X}_s^{\text{fin}}$
5	$\{x_2^{\text{e}}, x_3^{\text{e}}\}$	$\{x_1^{\text{f}}\}$
6	$\{x_2^{\text{e}}, x_6^{\text{e}}\}$	$\{x_2^{\text{f}}\}$
7	$\{x_3^{\text{e}}, x_5^{\text{e}}\}$	$\{x_1^{\text{f}}\}$
8	$\{x_4^{\text{e}}, x_6^{\text{e}}\}$	$\{x_2^{\text{f}}\}$

Table 6.2: The pairs of maximal cross-complementary subsets of the sets $\mathcal{X}^{\text{ext}}$ and $\mathcal{X}^{\text{fin}}$ of Example 6.4.1 for $N \geqslant 5$.

Any finite minimal realization of g can now be expressed as

$$x = \begin{bmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \\ b_1 \\ b_2 \\ c_1 \\ c_2 \end{bmatrix} = \lambda_1 x_1^{\text{c}} + \lambda_2 x_2^{\text{c}} + \kappa_1 x_{i_1}^{\text{e}} + \kappa_2 x_{i_2}^{\text{e}} + x_{j_1}^{\text{f}} \tag{6.17}$$

for some $s \in \{1, 2, \ldots, 8\}$, $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\kappa_1, \kappa_2 \geqslant 0$ with $x_{i_1}^{\text{e}}, x_{i_2}^{\text{e}} \in \mathcal{X}_s^{\text{ext}}$ and $x_{j_1}^{\text{f}} \in \mathcal{X}_s^{\text{fin}}$. Expression (6.17) shows that the set of all the finite minimal state space realizations of g corresponds to the union of 8 faces of a polyhedron in the x space.

A state space realization for which some of the entries of A, B or C are infinite can be obtained by allowing some of the coefficients in (6.17) to become infinite (if we take care that in this way we do not introduce entries that are equal to ∞). Since all the exponents of the system of multivariate max-algebraic polynomial equalities that defines the set $\mathcal{R}_2(g)$ are nonnegative, there will never be problems arising from taking negative max-algebraic powers of ε. The combination $\eta x_1^{\mathrm{e}} + \eta x_2^{\mathrm{e}} + x_1^{\mathrm{f}}$ with $\eta \in \mathbb{R}^+$ of the extreme generators and the finite point of the ordered pair $(\mathcal{X}_1^{\mathrm{ext}}, \mathcal{X}_1^{\mathrm{fin}})$ corresponds to

$$A = \begin{bmatrix} 6 & 10-\eta \\ -\eta & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ -4 \end{bmatrix} \text{ and } C = \begin{bmatrix} 9 & 15 \end{bmatrix} .$$

If we take the limit for η going to ∞, we get

$$A = \begin{bmatrix} 6 & \varepsilon \\ \varepsilon & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ -4 \end{bmatrix} \text{ and } C = \begin{bmatrix} 9 & 15 \end{bmatrix} ,$$

which also is a realization of g.

Let us now give an interpretation of the central generators and the finite points of Table 6.1 in terms of the theorems on state space transformations of Section 6.2.

Generator x_1^{c} corresponds to a max-algebraic similarity transformation with $T = \begin{bmatrix} 0 & \varepsilon \\ \varepsilon & -1 \end{bmatrix}$. Generator x_2^{c} corresponds to the equivalence of Corollary 6.2.2 or to a max-algebraic similarity transformation with $T = \begin{bmatrix} 1 & \varepsilon \\ \varepsilon & 1 \end{bmatrix}$. The triple that corresponds to x_2^{f} can be obtained from the triple that corresponds to x_1^{f} by a max-algebraic similarity transformation with $T = \begin{bmatrix} \varepsilon & 4 \\ -6 & \varepsilon \end{bmatrix}$. The set of combinations of the central generators and one of the finite points of the solution set of the ELCP:

$$\mathcal{S} = \left\{ x \,\middle|\, x = \lambda_1 x_1^{\mathrm{c}} + \lambda_2 x_2^{\mathrm{c}} + x_1^{\mathrm{f}} \text{ or } x = \lambda_1 x_1^{\mathrm{c}} + \lambda_2 x_2^{\mathrm{c}} + x_2^{\mathrm{f}} \text{ with } \lambda_1, \lambda_2 \in \mathbb{R} \right\} ,$$

corresponds to a full class of 2nd order state space realizations of g that are linked by max-algebraic similarity transformations. But in this way we cannot construct the entire set of all minimal state space realizations since e.g. the combination $x_1^{\mathrm{f}} + x_1^{\mathrm{e}}$ does not belong to $\mathcal{S}$.

Let (A_i, B_i, C_i) be the realization that corresponds to $x_1^{\mathrm{f}} + x_i^{\mathrm{e}}$ for $i = 1, 2, 5$. The triple (A_1, B_1, C_1) can be obtained from the triple that corresponds to x_1^{f} by an L-transformation with e.g. $L = \begin{bmatrix} 0 & 4 \\ -6 & 0 \end{bmatrix}$, $\hat{A} = \begin{bmatrix} 6 & 9 \\ 0 & 5 \end{bmatrix}$ and $\hat{C} = \begin{bmatrix} 9 & 15 \end{bmatrix}$.

The triple (A_1, B_1, C_1) can be obtained from (A_2, B_2, C_2) by an L-transformation, and (A_2, B_2, C_2) can be obtained from (A_5, B_5, C_5) by an M-transformation, but it is not possible to transform (A_5, B_5, C_5) into (A_1, B_1, C_1) with an L-transformation or an M-transformation. So starting from an arbitrary realization, we cannot get the set of all equivalent minimal state space realizations

in one step by applying L-transformations or M-transformations.
Furthermore, it is also impossible to find an L-transformation or an M-transformation that transforms the original 3rd order state space model into a 2nd order model. □

The next example will show that the two-step approach to find a minimal state space realization of a given impulse response does not always yield a solution, even if one exists. This example will also show that it is not always necessary to solve the full system of multivariate max-algebraic polynomial equalities that corresponds to the first, say N, Markov parameters of the impulse response $\{G_k\}_{k=0}^{\infty}$ of a max-linear time-invariant DES in order to obtain a minimal state space realization of the sequence $G_0, G_1, \ldots, G_{N-1}$.

Example 6.4.2 Consider a DES that can be described by a max-linear time-invariant state space model with system matrices

$$A = \begin{bmatrix} 2 & \varepsilon & \varepsilon \\ \varepsilon & 1 & 3 \\ 0 & 3 & \varepsilon \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ \varepsilon \\ 0 \end{bmatrix} \text{ and } C = \begin{bmatrix} 2 & \varepsilon & 2 \end{bmatrix} . \tag{6.18}$$

The impulse response of this system is:

$$g = \{g_k\}_{k=0}^{\infty} = 3,\ 5,\ 8,\ 9,\ 14,\ 15,\ 20,\ 21,\ 26,\ 27,\ 32,\ 33,\ \ldots\ .$$

Since $g_{k+2} = 6 \otimes g_k = 3^{\otimes^2} \otimes g_k$ for all $k \geqslant 2$, g is an ultimately geometric impulse response with period 2. Since there are two different alternating increments in the ultimately geometric behavior: 1 and 5, we cannot use the techniques of [36, 146, 147, 148] to compute a minimal state space realization of g.
Let $H = H(g)$ and let

$$\tilde{H} = H_{\{1,2,\ldots,8\},\{1,2,\ldots,8\}} = \begin{bmatrix} 3 & 5 & 8 & 9 & 14 & 15 & 20 & 21 \\ 5 & 8 & 9 & 14 & 15 & 20 & 21 & 26 \\ 8 & 9 & 14 & 15 & 20 & 21 & 26 & 27 \\ 9 & 14 & 15 & 20 & 21 & 26 & 27 & 32 \\ 14 & 15 & 20 & 21 & 26 & 27 & 32 & 33 \\ 15 & 20 & 21 & 26 & 27 & 32 & 33 & 38 \\ 20 & 21 & 26 & 27 & 32 & 33 & 38 & 39 \\ 21 & 26 & 27 & 32 & 33 & 38 & 39 & 44 \end{bmatrix} .$$

We have $\text{rank}_{\oplus}(\tilde{H}) = \text{rank}_{\oplus,\text{cc}}(\tilde{H}) = 3$. The max-algebraic determinant of the 3 by 3 submatrix $\tilde{H}_{\{1,2,3\},\{1,2,3\}}$ of $\tilde{H}$ is not balanced. If we use the algorithm of [54] to compute a solution of the system of max-linear homogeneous balances

$$\tilde{H}_{\{1,2,3,4\},\{1,2,3,4\}} \otimes \begin{bmatrix} a_3 \\ a_2 \\ a_1 \\ a_0 \end{bmatrix} \nabla\ \mathcal{E}_{4\times 1}$$

and if we normalize the a_0 component of the result, we obtain

$$a_0 = 0,\ \ a_1 = \ominus 2,\ \ a_2 = \ominus 6 \text{ and } a_3 = 8\ . \tag{6.19}$$

This solution satisfies the necessary and sufficient conditions for the coefficients of the MACP of a 3 by 3 matrix with elements in $\mathbb{R}_\varepsilon$ (cf. Proposition 5.3.3) since

$$\begin{aligned} a_1^{\oplus} &= \varepsilon \\ a_2^{\oplus} = \varepsilon \leqslant 4 &= 2 \otimes 2 = a_1^{\ominus} \otimes a_1^{\ominus} \\ a_3^{\oplus} = 8 \leqslant 8 &= 2 \otimes 6 = a_1^{\ominus} \otimes a_2^{\ominus}\ . \end{aligned}$$

This solution also corresponds to a stable relation among the columns of $\tilde{H}$:

$$\tilde{H}_{.,k+3} \ \oplus\ 8 \otimes \tilde{H}_{.,k} = 2 \otimes \tilde{H}_{.,k+2} \ \oplus\ 6 \otimes \tilde{H}_{.,k+1}$$

for $k = 1, 2, \ldots, 5$. This relation also holds for all the columns of H.
We have $\operatorname{rank}_{\oplus,\mathrm{wc}}(H) = 4$. So the upper bound of Theorem 6.3.5 is not tight for this example.

Let us also use the normalizations (6.14)–(6.16) in this example. Since $g_{k+2} = 6 \otimes g_k = 3^{\otimes^2} \otimes g_k$ for all $k \geqslant 2$, the set $L^*(g)$ of the smallest possible values for the λ_s's in (6.4) is given by $L^*(g) = \{3\}$. Since it follows from Lemma 6.3.7 that the largest max-algebraic eigenvalue of the A matrix of any minimal state space realization of g will be equal to 3, we have

$$\begin{aligned} \mathcal{R}_3^{\mathrm{nor}}(g, N) = \Big\{\, (A, B, C) \in \mathcal{R}_3(g, N) \,\Big|\, \|A\|_{\oplus} = 3,\ \ \|B\|_{\oplus} = 0 \text{ and} \\ a_{11} \geqslant a_{22} \geqslant a_{33} \Big\}\ . \end{aligned}$$

Now we consider the sequence $\{\mathcal{R}_3^{\mathrm{nor}}(g, N)\}_{N=1}^{\infty}$. If we use the ELCP algorithm of Section 3.4 to solve the system of max-algebraic polynomial equalities that corresponds to $\mathcal{R}_3^{\mathrm{nor}}(g, N)$, we get the extreme generators and the finite points of Table 6.3 and the pairs of maximal cross-complementary subsets of Table 6.4 for any $N \geqslant 6$. There are no central generators. For $N < 6$ some solutions only fit the first N Markov parameters. If we solve the reduced ELCP that only consists of the inequalities that correspond to subsequence g_0, g_1, g_2, g_3, g_5 of g or to the subsequence g_0, g_1, g_2, g_4, g_5, we obtain the same solutions as for the sequence g_0, g_1, $\ldots$, g_{N-1} with $N \geqslant 6$. If we consider an arbitrary subset S of $S_1 = \{g_0, g_1, g_2, g_3, g_5\}$ or $S_2 = \{g_0, g_1, g_2, g_4, g_5\}$, then S does not always result in a realization of the entire impulse response g. Hence, S_1 and S_2 are minimal sets of Markov parameters that are needed in order to get a minimal state space realization of the entire impulse response g.
This shows that it is not always necessary to consider the entire sequence G_0,

	$\mathcal{X}^{\text{ext}}$											
	x_1^{e}	x_2^{e}	x_3^{e}	x_4^{e}	x_5^{e}	x_6^{e}	x_7^{e}	x_8^{e}	x_9^{e}	x_{10}^{e}	x_{11}^{e}	x_{12}^{e}
a_{11}	0	0	0	0	0	0	0	0	0	0	0	0
a_{12}	-1	-1	0	0	0	0	0	0	0	0	0	0
a_{13}	-1	0	-1	0	0	0	0	0	0	0	0	0
a_{21}	0	0	0	-1	-1	0	0	0	0	0	0	0
a_{22}	0	0	0	0	0	-1	0	0	0	0	0	0
a_{23}	0	0	0	0	0	0	0	0	0	0	0	0
a_{31}	0	0	0	-1	0	0	-1	0	0	0	0	0
a_{32}	0	0	0	0	0	0	0	0	0	0	0	0
a_{33}	0	0	0	0	0	-1	0	-1	0	0	0	0
b_1	-1	0	0	0	0	0	0	0	0	0	0	0
b_2	0	0	0	-1	0	0	0	0	-1	0	0	0
b_3	0	0	0	-1	0	0	0	0	0	-1	0	0
c_1	1	0	0	0	0	0	0	0	0	0	0	0
c_2	0	0	0	1	0	0	0	0	0	0	-1	0
c_3	0	0	0	1	0	0	0	0	0	0	0	-1

	$\mathcal{X}^{\text{fin}}$									
	x_1^{f}	x_2^{f}	x_3^{f}	x_4^{f}	x_5^{f}	x_6^{f}	x_7^{f}	x_8^{f}	x_9^{f}	x_{10}^{f}
a_{11}	2	2	2	2	2	2	2	2	2	2
a_{12}	3	3	3	2	1	0	0	-1	-1	-2
a_{13}	3	3	2	3	0	1	-1	0	-2	-1
a_{21}	-1	-2	0	-1	2	1	3	2	3	3
a_{22}	1	1	1	1	1	1	1	1	1	1
a_{23}	3	3	3	3	3	3	3	3	3	3
a_{31}	-2	-1	-1	0	1	2	2	3	3	3
a_{32}	3	3	3	3	3	3	3	3	3	3
a_{33}	1	1	1	1	1	1	1	1	1	1
b_1	0	0	0	0	0	0	-1	-1	-2	-2
b_2	-5	-3	-4	-2	-2	0	-2	0	-2	0
b_3	-3	-5	-2	-4	0	-2	0	-2	0	-2
c_1	3	3	3	3	3	3	4	4	5	5
c_2	3	5	2	4	0	2	0	2	0	2
c_3	5	3	4	2	2	0	2	0	2	0

Table 6.3: The generators and the finite points of the set $\mathcal{R}_3^{\text{nor}}(g, N)$ of Example 6.4.2 for $N \geqslant 6$.

s	$\mathcal{X}_s^{\mathrm{ext}}$	$\mathcal{X}_s^{\mathrm{fin}}$
1	$\{x_1^{\mathrm{e}}, x_2^{\mathrm{e}}, x_3^{\mathrm{e}}, x_5^{\mathrm{e}}, x_6^{\mathrm{e}}, x_7^{\mathrm{e}}, x_8^{\mathrm{e}}, x_9^{\mathrm{e}}\}$	$\{x_5^{\mathrm{f}}, x_7^{\mathrm{f}}, x_9^{\mathrm{f}}\}$
2	$\{x_1^{\mathrm{e}}, x_2^{\mathrm{e}}, x_3^{\mathrm{e}}, x_5^{\mathrm{e}}, x_6^{\mathrm{e}}, x_7^{\mathrm{e}}, x_8^{\mathrm{e}}, x_{10}^{\mathrm{e}}\}$	$\{x_6^{\mathrm{f}}, x_8^{\mathrm{f}}, x_{10}^{\mathrm{f}}\}$
3	$\{x_1^{\mathrm{e}}, x_2^{\mathrm{e}}, x_3^{\mathrm{e}}, x_5^{\mathrm{e}}, x_6^{\mathrm{e}}, x_7^{\mathrm{e}}, x_8^{\mathrm{e}}, x_{11}^{\mathrm{e}}\}$	$\{x_5^{\mathrm{f}}, x_7^{\mathrm{f}}, x_9^{\mathrm{f}}\}$
4	$\{x_1^{\mathrm{e}}, x_2^{\mathrm{e}}, x_3^{\mathrm{e}}, x_5^{\mathrm{e}}, x_6^{\mathrm{e}}, x_7^{\mathrm{e}}, x_8^{\mathrm{e}}, x_{12}^{\mathrm{e}}\}$	$\{x_6^{\mathrm{f}}, x_8^{\mathrm{f}}, x_{10}^{\mathrm{f}}\}$
5	$\{x_1^{\mathrm{e}}, x_2^{\mathrm{e}}, x_3^{\mathrm{e}}, x_5^{\mathrm{e}}, x_7^{\mathrm{e}}, x_8^{\mathrm{e}}, x_9^{\mathrm{e}}, x_{11}^{\mathrm{e}}\}$	$\{x_5^{\mathrm{f}}, x_7^{\mathrm{f}}, x_9^{\mathrm{f}}\}$
6	$\{x_1^{\mathrm{e}}, x_2^{\mathrm{e}}, x_3^{\mathrm{e}}, x_5^{\mathrm{e}}, x_7^{\mathrm{e}}, x_8^{\mathrm{e}}, x_{10}^{\mathrm{e}}, x_{12}^{\mathrm{e}}\}$	$\{x_6^{\mathrm{f}}, x_8^{\mathrm{f}}, x_{10}^{\mathrm{f}}\}$
7	$\{x_1^{\mathrm{e}}, x_2^{\mathrm{e}}, x_5^{\mathrm{e}}, x_6^{\mathrm{e}}, x_7^{\mathrm{e}}, x_8^{\mathrm{e}}, x_9^{\mathrm{e}}, x_{11}^{\mathrm{e}}\}$	$\{x_5^{\mathrm{f}}, x_7^{\mathrm{f}}, x_9^{\mathrm{f}}\}$
8	$\{x_1^{\mathrm{e}}, x_2^{\mathrm{e}}, x_5^{\mathrm{e}}, x_6^{\mathrm{e}}, x_7^{\mathrm{e}}, x_8^{\mathrm{e}}, x_{10}^{\mathrm{e}}, x_{12}^{\mathrm{e}}\}$	$\{x_6^{\mathrm{f}}, x_8^{\mathrm{f}}, x_{10}^{\mathrm{f}}\}$
9	$\{x_1^{\mathrm{e}}, x_3^{\mathrm{e}}, x_5^{\mathrm{e}}, x_6^{\mathrm{e}}, x_7^{\mathrm{e}}, x_8^{\mathrm{e}}, x_9^{\mathrm{e}}, x_{11}^{\mathrm{e}}\}$	$\{x_5^{\mathrm{f}}, x_7^{\mathrm{f}}, x_9^{\mathrm{f}}\}$
10	$\{x_1^{\mathrm{e}}, x_3^{\mathrm{e}}, x_5^{\mathrm{e}}, x_6^{\mathrm{e}}, x_7^{\mathrm{e}}, x_8^{\mathrm{e}}, x_{10}^{\mathrm{e}}, x_{12}^{\mathrm{e}}\}$	$\{x_6^{\mathrm{f}}, x_8^{\mathrm{f}}, x_{10}^{\mathrm{f}}\}$
11	$\{x_2^{\mathrm{e}}, x_3^{\mathrm{e}}, x_4^{\mathrm{e}}, x_5^{\mathrm{e}}, x_6^{\mathrm{e}}, x_7^{\mathrm{e}}, x_8^{\mathrm{e}}, x_9^{\mathrm{e}}\}$	$\{x_1^{\mathrm{f}}, x_3^{\mathrm{f}}, x_5^{\mathrm{f}}\}$
12	$\{x_2^{\mathrm{e}}, x_3^{\mathrm{e}}, x_4^{\mathrm{e}}, x_5^{\mathrm{e}}, x_6^{\mathrm{e}}, x_7^{\mathrm{e}}, x_8^{\mathrm{e}}, x_{10}^{\mathrm{e}}\}$	$\{x_2^{\mathrm{f}}, x_4^{\mathrm{f}}, x_6^{\mathrm{f}}\}$
13	$\{x_2^{\mathrm{e}}, x_3^{\mathrm{e}}, x_4^{\mathrm{e}}, x_5^{\mathrm{e}}, x_6^{\mathrm{e}}, x_7^{\mathrm{e}}, x_8^{\mathrm{e}}, x_{11}^{\mathrm{e}}\}$	$\{x_1^{\mathrm{f}}, x_3^{\mathrm{f}}, x_5^{\mathrm{f}}\}$
14	$\{x_2^{\mathrm{e}}, x_3^{\mathrm{e}}, x_4^{\mathrm{e}}, x_5^{\mathrm{e}}, x_6^{\mathrm{e}}, x_7^{\mathrm{e}}, x_8^{\mathrm{e}}, x_{12}^{\mathrm{e}}\}$	$\{x_2^{\mathrm{f}}, x_4^{\mathrm{f}}, x_6^{\mathrm{f}}\}$
15	$\{x_2^{\mathrm{e}}, x_3^{\mathrm{e}}, x_4^{\mathrm{e}}, x_5^{\mathrm{e}}, x_6^{\mathrm{e}}, x_8^{\mathrm{e}}, x_9^{\mathrm{e}}, x_{11}^{\mathrm{e}}\}$	$\{x_1^{\mathrm{f}}, x_3^{\mathrm{f}}, x_5^{\mathrm{f}}\}$
16	$\{x_2^{\mathrm{e}}, x_3^{\mathrm{e}}, x_4^{\mathrm{e}}, x_5^{\mathrm{e}}, x_6^{\mathrm{e}}, x_8^{\mathrm{e}}, x_{10}^{\mathrm{e}}, x_{12}^{\mathrm{e}}\}$	$\{x_2^{\mathrm{f}}, x_4^{\mathrm{f}}, x_6^{\mathrm{f}}\}$
17	$\{x_2^{\mathrm{e}}, x_3^{\mathrm{e}}, x_4^{\mathrm{e}}, x_5^{\mathrm{e}}, x_7^{\mathrm{e}}, x_8^{\mathrm{e}}, x_9^{\mathrm{e}}, x_{11}^{\mathrm{e}}\}$	$\{x_1^{\mathrm{f}}, x_3^{\mathrm{f}}, x_5^{\mathrm{f}}\}$
18	$\{x_2^{\mathrm{e}}, x_3^{\mathrm{e}}, x_4^{\mathrm{e}}, x_5^{\mathrm{e}}, x_7^{\mathrm{e}}, x_8^{\mathrm{e}}, x_{10}^{\mathrm{e}}, x_{12}^{\mathrm{e}}\}$	$\{x_2^{\mathrm{f}}, x_4^{\mathrm{f}}, x_6^{\mathrm{f}}\}$
19	$\{x_2^{\mathrm{e}}, x_3^{\mathrm{e}}, x_4^{\mathrm{e}}, x_6^{\mathrm{e}}, x_7^{\mathrm{e}}, x_8^{\mathrm{e}}, x_9^{\mathrm{e}}, x_{11}^{\mathrm{e}}\}$	$\{x_1^{\mathrm{f}}, x_3^{\mathrm{f}}, x_5^{\mathrm{f}}\}$
20	$\{x_2^{\mathrm{e}}, x_3^{\mathrm{e}}, x_4^{\mathrm{e}}, x_6^{\mathrm{e}}, x_7^{\mathrm{e}}, x_8^{\mathrm{e}}, x_{10}^{\mathrm{e}}, x_{12}^{\mathrm{e}}\}$	$\{x_2^{\mathrm{f}}, x_4^{\mathrm{f}}, x_6^{\mathrm{f}}\}$
21	$\{x_2^{\mathrm{e}}, x_3^{\mathrm{e}}, x_5^{\mathrm{e}}, x_6^{\mathrm{e}}, x_8^{\mathrm{e}}, x_9^{\mathrm{e}}, x_{11}^{\mathrm{e}}\}$	$\{x_5^{\mathrm{f}}, x_7^{\mathrm{f}}, x_9^{\mathrm{f}}\}$
22	$\{x_2^{\mathrm{e}}, x_3^{\mathrm{e}}, x_5^{\mathrm{e}}, x_6^{\mathrm{e}}, x_8^{\mathrm{e}}, x_{10}^{\mathrm{e}}, x_{12}^{\mathrm{e}}\}$	$\{x_6^{\mathrm{f}}, x_8^{\mathrm{f}}\}$
23	$\{x_2^{\mathrm{e}}, x_3^{\mathrm{e}}, x_6^{\mathrm{e}}, x_7^{\mathrm{e}}, x_8^{\mathrm{e}}, x_9^{\mathrm{e}}, x_{11}^{\mathrm{e}}\}$	$\{x_5^{\mathrm{f}}, x_7^{\mathrm{f}}\}$
24	$\{x_2^{\mathrm{e}}, x_3^{\mathrm{e}}, x_6^{\mathrm{e}}, x_7^{\mathrm{e}}, x_8^{\mathrm{e}}, x_{10}^{\mathrm{e}}, x_{12}^{\mathrm{e}}\}$	$\{x_6^{\mathrm{f}}, x_8^{\mathrm{f}}, x_{10}^{\mathrm{f}}\}$
25	$\{x_2^{\mathrm{e}}, x_5^{\mathrm{e}}, x_6^{\mathrm{e}}, x_7^{\mathrm{e}}, x_8^{\mathrm{e}}, x_9^{\mathrm{e}}, x_{11}^{\mathrm{e}}\}$	$\{x_1^{\mathrm{f}}, x_3^{\mathrm{f}}, x_5^{\mathrm{f}}\}$
26	$\{x_2^{\mathrm{e}}, x_5^{\mathrm{e}}, x_6^{\mathrm{e}}, x_7^{\mathrm{e}}, x_8^{\mathrm{e}}, x_{10}^{\mathrm{e}}, x_{12}^{\mathrm{e}}\}$	$\{x_4^{\mathrm{f}}, x_6^{\mathrm{f}}\}$
27	$\{x_3^{\mathrm{e}}, x_5^{\mathrm{e}}, x_6^{\mathrm{e}}, x_7^{\mathrm{e}}, x_8^{\mathrm{e}}, x_9^{\mathrm{e}}, x_{11}^{\mathrm{e}}\}$	$\{x_3^{\mathrm{f}}, x_5^{\mathrm{f}}\}$
28	$\{x_3^{\mathrm{e}}, x_5^{\mathrm{e}}, x_6^{\mathrm{e}}, x_7^{\mathrm{e}}, x_8^{\mathrm{e}}, x_{10}^{\mathrm{e}}, x_{12}^{\mathrm{e}}\}$	$\{x_2^{\mathrm{f}}, x_4^{\mathrm{f}}, x_6^{\mathrm{f}}\}$

Table 6.4: The pairs of maximal cross-complementary subsets of the sets $\mathcal{X}^{\mathrm{ext}}$ and $\mathcal{X}^{\mathrm{fin}}$ of Example 6.4.2 for $N \geqslant 6$.

$G_1, \ldots, G_{N-1}$ to find all minimal state space realizations of (the first N Markov parameters of) a given impulse response $\{G_k\}_{k=0}^{\infty}$.
The system matrices A, B and C of (6.18) can be obtained by performing a max-algebraic similarity transformation with $T = 1 \otimes E_3$ on the realization $(\tilde{A}, \tilde{B}, \tilde{C})$ that corresponds to the point at infinity of the ray $L_1 = \{\, \eta x_2^{\mathrm{e}} + \eta x_3^{\mathrm{e}} + x_4^{\mathrm{e}} + \eta x_5^{\mathrm{e}} + \eta x_8^{\mathrm{e}} + \eta x_9^{\mathrm{e}} + \eta x_{11}^{\mathrm{e}} + x_5^{\mathrm{f}} \mid \eta \in \mathbb{R}^+ \,\}$ generated by extreme generators and a finite point of the ordered pair $\left(\mathcal{X}_{15}^{\mathrm{ext}}, \mathcal{X}_{15}^{\mathrm{fin}}\right)$. We have

$$\tilde{A} = \lim_{\eta \to \infty} \begin{bmatrix} 2 & 1-\eta & -\eta \\ 1-\eta & 1 & 3 \\ 0 & 3 & 1-\eta \end{bmatrix} = A$$

$$\tilde{B} = \lim_{\eta \to \infty} \begin{bmatrix} 0 \\ -3-\eta \\ -1 \end{bmatrix} = (-1) \otimes B$$

$$\tilde{C} = \begin{bmatrix} 3 & 1-\eta & 3 \end{bmatrix} = 1 \otimes C \ .$$

Now we try to use the two-step method to construct a minimal realization (A_1, B_1, C_1) of g. First we construct a matrix A_1 such that the coefficients of its MACP are given by (6.19). From Proposition 5.3.3 it follows that we can take e.g.

$$A_1 = \begin{bmatrix} 2 & 6 & \varepsilon \\ 0 & \varepsilon & 6 \\ \varepsilon & 0 & \varepsilon \end{bmatrix} \ .$$

Since $A_1{}^{\otimes^2} = \begin{bmatrix} 6 & 8 & 12 \\ 2 & 6 & \varepsilon \\ 0 & \varepsilon & 6 \end{bmatrix}$, the condition $C_1 \otimes A_1{}^{\otimes^2} \otimes B_1 = g_2 = 8$ implies that $C_1 \otimes B_1 \leqslant 2$. But then we cannot have $C_1 \otimes B_1 = g_1 = 3$. Therefore, it is impossible to find matrices B_1 and C_1 such that $(A_1, B_1, C_1) \in \mathcal{R}_3(g, N)$ if $N \geqslant 3$.
Now consider the matrix

$$A_2 = \begin{bmatrix} 2 & \varepsilon & \varepsilon \\ 0 & \varepsilon & 6 \\ \varepsilon & 0 & \varepsilon \end{bmatrix} \ .$$

The coefficients of the MACP of A_2 are also given by (6.19). However, for this matrix it is possible to find matrices B_2 and C_2 such that $(A_2, B_2, C_2) \in \mathcal{R}_3(g)$. The solution set of the ELCP that corresponds to the system of max-algebraic equalities $C_2 \otimes A_2{}^{\otimes^k} \otimes B_2 = g_k$ for $k = 0, 1, \ldots, N-1$ can be described by the generators and the finite points of Table 6.5 and the pairs of maximal cross-complementary subsets of Table 6.6 for any $N \geqslant 6$.

Generator $\hat{x}_1^{\mathrm{f}}$ yields $B_2 = \begin{bmatrix} 6 \\ 5 \\ 0 \end{bmatrix}$ and $C_2 = \begin{bmatrix} -3 & -3 & -2 \end{bmatrix}$. The triple

	$\hat{\mathcal{X}}^{\text{cen}}$	$\hat{\mathcal{X}}^{\text{ext}}$					$\hat{\mathcal{X}}^{\text{fin}}$	
	$\hat{x}_1^{\text{c}}$	$\hat{x}_1^{\text{e}}$	$\hat{x}_2^{\text{e}}$	$\hat{x}_3^{\text{e}}$	$\hat{x}_4^{\text{e}}$	$\hat{x}_5^{\text{e}}$	$\hat{x}_1^{\text{f}}$	$\hat{x}_2^{\text{f}}$
b_1	1	−1	0	0	0	0	6	5
b_2	1	0	−1	0	0	0	5	1
b_3	1	0	0	−1	0	0	0	0
c_1	−1	1	0	0	0	0	−3	−2
c_2	−1	0	0	0	−1	0	−3	−3
c_3	−1	0	0	0	0	−1	−2	2

Table 6.5: The generators and the finite points that correspond to the set of all finite matrices B_2 and C_2 of Example 6.4.2 such that $(A_2, B_2, C_2) \in \mathcal{R}_3(g, N)$ for $N \geqslant 6$.

s	$\hat{\mathcal{X}}_s^{\text{ext}}$	$\hat{\mathcal{X}}_s^{\text{fin}}$
1	$\{\hat{x}_1^{\text{e}}, \hat{x}_2^{\text{e}}\}$	$\{\hat{x}_2^{\text{f}}\}$
2	$\{\hat{x}_1^{\text{e}}, \hat{x}_3^{\text{e}}\}$	$\{\hat{x}_1^{\text{f}}\}$
3	$\{\hat{x}_1^{\text{e}}, \hat{x}_4^{\text{e}}\}$	$\{\hat{x}_2^{\text{f}}\}$

s	$\hat{\mathcal{X}}_s^{\text{ext}}$	$\hat{\mathcal{X}}_s^{\text{fin}}$
4	$\{\hat{x}_1^{\text{e}}, \hat{x}_5^{\text{e}}\}$	$\{\hat{x}_1^{\text{f}}\}$
5	$\{\hat{x}_2^{\text{e}}, \hat{x}_4^{\text{e}}\}$	$\{\hat{x}_2^{\text{f}}\}$
6	$\{\hat{x}_3^{\text{e}}, \hat{x}_5^{\text{e}}\}$	$\{\hat{x}_1^{\text{f}}\}$

Table 6.6: The pairs of maximal cross-complementary subsets of the sets $\hat{\mathcal{X}}^{\text{ext}}$ and $\hat{\mathcal{X}}^{\text{fin}}$ of Example 6.4.2.

(A_2, B_2, C_2) can be brought into a normalized form by a performing a max-algebraic similarity transformation with $T = \begin{bmatrix} -8 & \varepsilon & \varepsilon \\ \varepsilon & -5 & \varepsilon \\ \varepsilon & \varepsilon & -2 \end{bmatrix}$ (cf. Proposition 6.3.9). This yields the triple $(\tilde{A}_2, \tilde{B}_2, \tilde{C}_2) \in \mathcal{R}_3^{\text{nor}}(g)$ with

$$\tilde{A}_2 = \begin{bmatrix} 2 & \varepsilon & \varepsilon \\ 3 & \varepsilon & 3 \\ \varepsilon & 3 & \varepsilon \end{bmatrix}, \tilde{B}_2 = \begin{bmatrix} -2 \\ 0 \\ -2 \end{bmatrix} \text{ and } \tilde{C}_2 = \begin{bmatrix} 5 & 2 & 0 \end{bmatrix} .$$

It is easy to verify that the triple $(\tilde{A}_2, \tilde{B}_2, \tilde{C}_2)$ corresponds to the point at infinity of the ray $L_2 = \{\, \eta x_2^{\text{e}} + \eta x_3^{\text{e}} + \eta x_6^{\text{e}} + \eta x_7^{\text{e}} + x_{10}^{\text{f}} \,|\, \eta \in \mathbb{R}^+ \}$ generated by extreme generators and a finite point of the ordered pair $\left(\mathcal{X}_2^{\text{ext}}, \mathcal{X}_2^{\text{fin}}\right)$. □

Example 6.4.3 Here we consider the sequence

$$g = \{g_k\}_{k=0}^{\infty} = 0,\, 1,\, 0,\, 3,\, 0,\, 5,\, 0,\, 7,\, 0,\, 9,\, 0,\, 11,\, 0,\, 13,\, 0,\, 15,\, \ldots$$

of an example of [54, 56]. This sequence is not ultimately geometric, but it does exhibit an ultimately periodic behavior since we have

$$g_{2k+2} = 0^{\otimes^2} \otimes g_{2k} \quad \text{and} \quad g_{2k+3} = 1^{\otimes^2} \otimes g_{2k+1} \qquad \text{for all } k \in \mathbb{N}\,.$$

So it follows from Proposition 6.1.4 that there exists a max-linear time-invariant SISO DES that has g as its impulse response.
Let $H = H(g)$. Both the max-algebraic minor rank and the max-algebraic consecutive column rank of $\tilde{H}$ are equal to 3. If we use the procedure that has been described in Section 6.3.1, we find e.g. the following stable relation among the columns of H

$$H_{.,k+3} \,\oplus\, 2 \otimes H_{.,k} \,=\, H_{.,k+2} \,\oplus\, 2 \otimes H_{.,k+1} \qquad \text{for all } k \in \mathbb{N}_0\,.$$

The coefficients of this relation satisfy the necessary and sufficient conditions for the coefficients of the MACP of a 3 by 3 matrix with entries in $\mathbb{R}_\varepsilon$. Therefore, we conclude that the minimal system order is greater than or equal to 3.
Since g is not ultimately geometric, we cannot apply Theorem 6.3.5. However, if we use the generalization of this theorem by Gaubert, we find that the minimal system order is less than or equal to 3 (See Example C.3.2).
This implies that the minimal system order is equal to 3.

Let us now use the ELCP approach to construct the set of all 3rd order state space realizations of g. Since g is not ultimately geometric, it cannot be realized by a triple (A, B, C) for which all the entries of A are finite. Therefore, we shall determine how the set $\mathcal{R}_3^{\mathrm{nor}}(g, N)$ evolves as N goes to ∞. The set $L^*(g)$ of the smallest possible values for the λ_s's in (6.4) is given by $L^*(g) = \{0, 1\}$. Since $\max L^*(g) = 1$, the largest max-algebraic eigenvalue of the system matrix A of any minimal state space realization (A, B, C) of g is equal to 1 by Lemma 6.3.7. This means that $\mathcal{R}_3^{\mathrm{nor}}(g, N) = \left\{\, (A, B, C) \in \mathcal{R}_3(g, N) \;\middle|\; \|A\|_\oplus = 1,\ \|B\|_\oplus = 0 \text{ and } a_{11} \geqslant a_{22} \geqslant a_{33} \,\right\}$.
If we use the ELCP algorithm of Section 3.4 to solve the ELCP that corresponds to $\mathcal{R}_3^{\mathrm{nor}}(g, N)$, we get the extreme generators and the finite points of Table 6.7 and the pairs of maximal cross-complementary subsets of Table 6.8 for any $N \geqslant 5$. There are no central generators. We have $\mathcal{R}_3^{\mathrm{nor}}(g, 2l+1) = \mathcal{R}_3^{\mathrm{nor}}(g, 2l+2)$ for all $l \geqslant 2$. Note that the components of the finite points of $\mathcal{R}_3^{\mathrm{nor}}(g, N)$ depend affinely on l where $l = \left\lfloor \dfrac{N-1}{2} \right\rfloor$.
If l goes to ∞ then only $x_5^{\mathrm{f}}(l)$ and $x_6^{\mathrm{f}}(l)$ have components that are bounded from above. Define $\tilde{x}_1^{\mathrm{f}} = \lim\limits_{l \to \infty} x_5^{\mathrm{f}}(l)$ and $\tilde{x}_2^{\mathrm{f}} = \lim\limits_{l \to \infty} x_6^{\mathrm{f}}(l)$. Note that all the extreme generators except for $\tilde{x}_1^{\mathrm{e}} \stackrel{\mathrm{def}}{=} x_1^{\mathrm{e}}$ and $\tilde{x}_2^{\mathrm{e}} \stackrel{\mathrm{def}}{=} x_4^{\mathrm{e}}$ become redundant when l goes to ∞. As a consequence, the set $\mathcal{R}_3^{\mathrm{nor}}(g) = \lim\limits_{N \to \infty} \mathcal{R}_3^{\mathrm{nor}}(g, N)$ corresponds to the extreme generators and the "finite" points of Table 6.9 and the set

$$\tilde{\Lambda} \,=\, \left\{ \left(\{\tilde{x}_1^{\mathrm{e}}\}, \{\tilde{x}_1^{\mathrm{f}}\}\right), \left(\{\tilde{x}_1^{\mathrm{e}}\}, \{\tilde{x}_2^{\mathrm{f}}\}\right), \left(\{\tilde{x}_2^{\mathrm{e}}\}, \{\tilde{x}_1^{\mathrm{f}}\}\right), \left(\{\tilde{x}_2^{\mathrm{e}}\}, \{\tilde{x}_2^{\mathrm{f}}\}\right) \right\}$$

	$\mathcal{X}^{\mathrm{ext}}(l)$											
	x_1^{e}	x_2^{e}	x_3^{e}	x_4^{e}	x_5^{e}	x_6^{e}	x_7^{e}	x_8^{e}	x_9^{e}	x_{10}^{e}	x_{11}^{e}	x_{12}^{e}
a_{11}	0	0	0	0	0	0	0	0	0	0	0	0
a_{12}	-1	-1	0	0	0	0	0	0	0	0	0	0
a_{13}	-1	0	-1	0	0	0	0	0	0	0	0	0
a_{21}	0	0	0	-1	-1	0	0	0	0	0	0	0
a_{22}	0	0	0	0	0	-1	0	0	0	0	0	0
a_{23}	0	0	0	0	0	0	0	0	0	0	0	0
a_{31}	0	0	0	-1	0	0	-1	0	0	0	0	0
a_{32}	0	0	0	0	0	0	0	0	0	0	0	0
a_{33}	0	0	0	0	0	-1	0	-1	0	0	0	0
b_1	-1	0	0	0	0	0	0	0	0	0	0	0
b_2	0	0	0	-1	0	0	0	0	-1	0	0	0
b_3	0	0	0	-1	0	0	0	0	0	-1	0	0
c_1	1	0	0	0	0	0	0	0	0	0	0	0
c_2	0	0	0	1	0	0	0	0	0	0	-1	0
c_3	0	0	0	1	0	0	0	0	0	0	0	-1

	$\mathcal{X}^{\mathrm{fin}}(l)$									
	$x_1^{\mathrm{f}}(l)$	$x_2^{\mathrm{f}}(l)$	$x_3^{\mathrm{f}}(l)$	$x_4^{\mathrm{f}}(l)$	$x_5^{\mathrm{f}}(l)$	$x_6^{\mathrm{f}}(l)$	$x_7^{\mathrm{f}}(l)$	$x_8^{\mathrm{f}}(l)$	$x_9^{\mathrm{f}}(l)$	$x_{10}^{\mathrm{f}}(l)$
a_{11}	0	0	0	0	0	0	0	0	0	0
a_{12}	1	1	1	0	$2-2l$	$1-2l$	$3-4l$	$2-4l$	$2-4l$	$1-4l$
a_{13}	1	1	0	1	$1-2l$	$2-2l$	$2-4l$	$3-4l$	$1-4l$	$2-4l$
a_{21}	$2-4l$	$1-4l$	$2-4l$	$3-4l$	$1-2l$	$2-2l$	0	1	1	1
a_{22}	$1-2l$	$1-2l$	$1-2l$	$1-2l$	$1-2l$	$1-2l$	$1-2l$	$1-2l$	$1-2l$	$1-2l$
a_{23}	1	1	1	1	1	1	1	1	1	1
a_{31}	$1-4l$	$2-4l$	$3-4l$	$2-4l$	$2-2l$	$1-2l$	1	0	1	1
a_{32}	1	1	1	1	1	1	1	1	1	1
a_{33}	$1-2l$	$1-2l$	$1-2l$	$1-2l$	$1-2l$	$1-2l$	$1-2l$	$1-2l$	$1-2l$	$1-2l$
b_1	0	0	0	0	0	0	$1-2l$	$1-2l$	$-2l$	$-2l$
b_2	$-4l$	$-2l$	$1-2l$	$1-4l$	0	$-2l$	0	$-2l$	0	$-2l$
b_3	$-2l$	$-4l$	$1-4l$	$1-2l$	$-2l$	0	$-2l$	0	$-2l$	0
c_1	0	0	0	0	0	0	$2l-1$	$2l-1$	$2l$	$2l$
c_2	$2l$	0	-1	$2l-1$	$-2l$	0	$-2l$	0	$-2l$	0
c_3	0	$2l$	$2l-1$	-1	0	$-2l$	0	$-2l$	0	$-2l$

Table 6.7: The generators and the finite points of the sets $\mathcal{R}_3^{\mathrm{nor}}(g, 2l+1)$ and $\mathcal{R}_3^{\mathrm{nor}}(g, 2l+2)$ of Example 6.4.3 for $l \geqslant 2$.

s	$\mathcal{X}_s^{\text{ext}}(l)$	$\mathcal{X}_s^{\text{fin}}(l)$
1	$\{x_1^{\text{e}}, x_2^{\text{e}}, x_3^{\text{e}}, x_5^{\text{e}}, x_6^{\text{e}}, x_7^{\text{e}}, x_8^{\text{e}}, x_9^{\text{e}}, x_{12}^{\text{e}}\}$	$\{x_6^{\text{f}}(l), x_8^{\text{f}}(l), x_{10}^{\text{f}}(l)\}$
2	$\{x_1^{\text{e}}, x_2^{\text{e}}, x_3^{\text{e}}, x_5^{\text{e}}, x_6^{\text{e}}, x_7^{\text{e}}, x_8^{\text{e}}, x_{10}^{\text{e}}, x_{11}^{\text{e}}\}$	$\{x_5^{\text{f}}(l), x_7^{\text{f}}(l), x_9^{\text{f}}(l)\}$
3	$\{x_2^{\text{e}}, x_3^{\text{e}}, x_4^{\text{e}}, x_5^{\text{e}}, x_6^{\text{e}}, x_7^{\text{e}}, x_8^{\text{e}}, x_9^{\text{e}}, x_{12}^{\text{e}}\}$	$\{x_1^{\text{f}}(l), x_4^{\text{f}}(l), x_6^{\text{f}}(l)\}$
4	$\{x_2^{\text{e}}, x_3^{\text{e}}, x_4^{\text{e}}, x_5^{\text{e}}, x_6^{\text{e}}, x_7^{\text{e}}, x_8^{\text{e}}, x_{10}^{\text{e}}, x_{11}^{\text{e}}\}$	$\{x_2^{\text{f}}(l), x_3^{\text{f}}(l), x_5^{\text{f}}(l)\}$

Table 6.8: The pairs of maximal cross-complementary subsets of the sets $\mathcal{X}^{\text{ext}}(l)$ and $\mathcal{X}^{\text{fin}}(l)$ of Example 6.4.3 for $l \geqslant 2$.

	$\tilde{\mathcal{X}}^{\text{ext}}$		$\tilde{\mathcal{X}}^{\text{fin}}$	
	$\tilde{x}_1^{\text{e}}$	$\tilde{x}_2^{\text{e}}$	$\tilde{x}_1^{\text{f}}$	$\tilde{x}_2^{\text{f}}$
a_{11}	0	0	0	0
a_{12}	-1	0	ε	ε
a_{13}	-1	0	ε	ε
a_{21}	0	-1	ε	ε
a_{22}	0	0	ε	ε
a_{23}	0	0	1	1
a_{31}	0	-1	ε	ε
a_{32}	0	0	1	1
a_{33}	0	0	ε	ε
b_1	-1	0	0	0
b_2	0	-1	0	ε
b_3	0	-1	ε	0
c_1	1	0	0	0
c_2	0	1	ε	0
c_3	0	1	0	ε

Table 6.9: The generators and the "finite" points of the set $\mathcal{R}_3^{\text{nor}}(g)$ of Example 6.4.3.

of ordered pairs of maximal cross-complementary subsets of $\tilde{\mathcal{X}}^{\text{ext}}$ and $\tilde{\mathcal{X}}^{\text{fin}}$. The elements of $\mathcal{R}_3(g)$ can now be reconstructed from $\mathcal{R}_3^{\text{nor}}(g)$ by applying max-algebraic similarity transformations to the elements of $\mathcal{R}_3^{\text{nor}}(g)$.

In Example C.3.2 we shall use the technique of [54, 56] to construct a 3rd order state space realization (A_1, B_1, C_1) of the given impulse response. Note however that since this technique is based on (an extension of) Theorem 6.3.5, it will only result in a *minimal* state space realization of the given impulse response if the upper bound given by this theorem is equal to the minimal system order, which is not always the case (See e.g. Example 6.4.2). We have

$$A_1 = \begin{bmatrix} 0 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 2 \\ \varepsilon & 0 & \varepsilon \end{bmatrix}, \ B_1 = \begin{bmatrix} 0 \\ 0 \\ \varepsilon \end{bmatrix} \text{ and } C_1 = \begin{bmatrix} 0 & \varepsilon & 1 \end{bmatrix} .$$

The triple (A_1, B_1, C_1) can be brought into a normalized form by performing a max-algebraic similarity transformation with $T = \begin{bmatrix} 0 & \varepsilon & \varepsilon \\ \varepsilon & 0 & \varepsilon \\ \varepsilon & \varepsilon & 1 \end{bmatrix}$ (cf. Proposition 6.3.9). This results in the triple $(\tilde{A}_1, \tilde{B}_1, \tilde{C}_1) \in \mathcal{R}_3^{\mathrm{nor}}(g)$ with

$$\tilde{A}_1 = \begin{bmatrix} 0 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 1 \\ \varepsilon & 1 & \varepsilon \end{bmatrix}, \ \tilde{B}_1 = \begin{bmatrix} 0 \\ 0 \\ \varepsilon \end{bmatrix} \text{ and } \tilde{C}_1 = \begin{bmatrix} 0 & \varepsilon & 0 \end{bmatrix} .$$

Note that the triple $(\tilde{A}_1, \tilde{B}_1, \tilde{C}_1)$ corresponds to the "finite" point $\tilde{x}_1^{\mathrm{f}}$. □

For other examples with SISO systems the interested reader is referred to [43, 44, 45]. In [46] we have given an example with a MIMO system.

6.5 Conclusion

In this chapter we have used the fact that a system of multivariate max-algebraic polynomial equations can be transformed into an ELCP to perform state space transformations for max-linear time-invariant discrete event systems. Next we have presented a method to solve the minimal state space realization problem for max-linear time-invariant discrete event systems. This method consists of three major steps. First we determine a lower bound for the minimal system order. Then we construct (minimal) state space realizations of a finite subsequence of the impulse response of the discrete event system and finally we construct minimal state space realizations of the entire impulse response. We have shown that we can use the ELCP to compute all fixed order *partial* state space realizations of a given impulse response. We can also use the ELCP to determine all *minimal* state space realizations of a given impulse response. These procedures have been illustrated with some worked examples.

We have also briefly indicated how the constructive proofs of the propositions of Section 5.3 on the coefficients of the max-algebraic characteristic polynomial of a matrix with entries in $\mathbb{R}_\varepsilon$ or the heuristic algorithm of Section B.4 can sometimes be used to find a minimal state space realization of a given impulse response with a two-step method. In this method we only consider one solution for the system matrix A. As a consequence, this approach results in much smaller ELCPs than the direct approach in which the system matrices A, B and C are determined simultaneously. Therefore, the two-step approach allows us to tackle larger problems than the direct approach. However, the two-step approach does not always yield a solution, even if there exists one.

We have to admit that using the ELCP approach to solve the minimal state space realization problem for max-linear time-invariant discrete event systems is not a very elegant method. However, by showing that the partial state space realization problem can be reformulated as an ELCP we have gained an insight in the geometrical structure of the set of all fixed order state space realizations of a given impulse response. We hope that this insight will lead to the development of more efficient algorithms to solve the partial or the minimal state space realization problem for max-linear time-invariant discrete event systems or to solve special cases of these problems. One of the main characteristics of the ELCP algorithm of Section 3.4 is that it finds all solutions. Since this also leads to large computation times and storage space requirements if the number of variables and (in)equalities is large, it would be useful to develop methods that find only one solution.

It is still an open question how the smallest possible N such that any minimal state space realization of the first N Markov parameters is also a realization of the entire impulse response can be determined. Furthermore, we do not yet know how to determine a minimal set of Markov parameters such that any minimal state space realization of this set is also a realization of the entire impulse response.

Chapter 7

The Singular Value Decomposition and the QR Decomposition in the Symmetrized Max-Plus Algebra

In [126] Olsder and Roos have used asymptotic equivalences to show that every matrix has at least one max-algebraic eigenvalue and to prove a max-algebraic version of Cramer's rule and of the Cayley-Hamilton theorem. We shall extend and formalize their technique and use it to define the singular value decomposition and the QR decomposition in the symmetrized max-plus algebra.

After introducing some new concepts and definitions in Section 7.1, we establish a link between a ring of real functions (with conventional addition and multiplication as basic operations) and the symmetrized max-plus algebra in Section 7.2. We also introduce a further extension of the max-plus algebra that will correspond to a ring of complex functions. In Section 7.3 we use the link between the ring of real functions and the symmetrized max-plus algebra to define the singular value decomposition and the QR decomposition of a matrix in the symmetrized max-plus algebra and to prove the existence of these decompositions. In Section 7.4 we study some properties of the max-algebraic singular value decomposition. Next we discuss a possible application of the max-algebraic singular value decomposition in connection with the identification problem for max-linear time-invariant discrete event systems. In Section 7.5 we show that the problem of finding all max-algebraic singular value decompositions or all max-algebraic QR decompositions of a given matrix can also be solved using the ELCP approach.

7.1 Introduction

In this section we give some definitions that will be needed in the next sections.
If $x, y \in \mathbb{R}_\varepsilon^n$ then $x^T \otimes y$ is called the max-algebraic inner product of x and y.
A function $f : D \to \mathbb{R}$ with $D \subseteq \mathbb{R}$ is called a real function. Likewise, we say that a function $\tilde{F} : D \to \mathbb{R}^{m \times n}$ with $D \subseteq \mathbb{R}$ is a real m by n matrix-valued function.
The 2-norm of the vector a is defined by $\|a\|_2 = \sqrt{a^T a}$.
A matrix $A \in \mathbb{R}^{n \times n}$ is called orthogonal if $A^T A = I_n$.
Consider a matrix $A \in \mathbb{R}^{m \times n}$. The Frobenius norm of A is given by

$$\|A\|_\mathrm{F} = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}^2} \quad .$$

The 2-norm of A is defined by $\|A\|_2 = \max\limits_{\|x\|_2 = 1} \|Ax\|_2$. We have

$$\frac{1}{\sqrt{n}} \|A\|_\mathrm{F} \leqslant \|A\|_2 \leqslant \|A\|_\mathrm{F} \quad . \tag{7.1}$$

If $U \in \mathbb{R}^{m \times m}$ and $V \in \mathbb{R}^{n \times n}$ are orthogonal matrices then we have $\|UA\|_\mathrm{F} = \|AV\|_\mathrm{F} = \|A\|_\mathrm{F}$ and $\operatorname{rank} A = \operatorname{rank}(UA) = \operatorname{rank}(AV)$.
Now we briefly discuss two basic matrix factorizations from linear algebra: the QR decomposition and the singular value decomposition. These decompositions are used in many linear algebra algorithms (See e.g. [65] and the references cited therein) and in many contemporary algorithms for the identification of linear systems (See e.g. [96, 98, 138, 139, 140, 142] and the references cited therein). The proofs of the theorems and the properties given below can be found in e.g. [65, 82].

Theorem 7.1.1 (QR decomposition) *If $A \in \mathbb{R}^{m \times n}$ then there exist an orthogonal matrix $Q \in \mathbb{R}^{m \times m}$ and an upper triangular matrix $R \in \mathbb{R}^{m \times n}$ such that*

$$A = QR \quad . \tag{7.2}$$

Factorization (7.2) is called the QR decomposition (QRD) of A.
Let $A \in \mathbb{R}^{m \times n}$. If QR is a QRD of A, then $\|A\|_\mathrm{F} = \|R\|_\mathrm{F}$ since Q is an orthogonal matrix.

Theorem 7.1.2 (Singular Value Decomposition) *Let $A \in \mathbb{R}^{m \times n}$ and let $r = \min(m, n)$. Then there exists a diagonal matrix $\Sigma \in \mathbb{R}^{m \times n}$ and two orthogonal matrices $U \in \mathbb{R}^{m \times m}$ and $V \in \mathbb{R}^{n \times n}$ such that*

$$A = U \Sigma V^T \tag{7.3}$$

with $\sigma_1 \geqslant \sigma_2 \geqslant \ldots \geqslant \sigma_r \geqslant 0$ where $\sigma_i = (\Sigma)_{ii}$ for $i = 1, 2, \ldots, r$.

Factorization (7.3) is called the singular value decomposition (SVD) of A. The diagonal entries of Σ are called the singular values of A. The columns of U are called the left singular vectors of A and the columns of V are called the right singular vectors of A. In the remainder of this chapter and in Appendix D σ_i will always represent the ith diagonal element of the matrix Σ, and u_i will always represent the ith column of the matrix U. Likewise, v_i will represent the ith column of V. We shall also use this notation for the max-algebraic SVD.
The singular values of a matrix $A \in \mathbb{R}^{m\times n}$ are unique. Singular vectors corresponding to simple singular values are also uniquely determined (apart from the sign). If two or more singular values coincide, only the linear subspace generated by the corresponding singular vectors is well determined: any choice of sets $\{u_1, u_2, \ldots, u_m\}$ and $\{v_1, v_2, \ldots, v_n\}$ of orthonormal basis vectors such that $A^T u_i = \sigma_i v_i$ and $Av_i = \sigma_i u_i$ for $i = 1, 2, \ldots, \min(m, n)$ yields valid sets of singular vectors. If σ_1 is the largest singular value of A then $\sigma_1 = \|A\|_2$. The number of non-zero singular values of A is equal to the rank of A.

Definition 7.1.3 (Analytic function) *Let f be a real function and let $\alpha \in \mathbb{R}$ be an interior point of* $\operatorname{dom} f$. *Then f is analytic in α if the Taylor series of f with center α exists and if there is a neighborhood of α where this Taylor series converges to f.*
A real function f is analytic in an interval $[\alpha, \beta] \subseteq \operatorname{dom} f$ if it is analytic in every point of that interval.
A real matrix-valued function $\tilde{F}$ is analytic in $[\alpha, \beta] \subseteq \operatorname{dom} \tilde{F}$ if all its entries are analytic in $[\alpha, \beta]$.

If a real function f is analytic in $[\alpha, \beta] \subseteq \operatorname{dom} f$ then it is also continuous on $[\alpha, \beta]$.

Definition 7.1.4 (Asymptotic equivalence) *Let f and g be real functions and let $\alpha \in \mathbb{R} \cup \{\infty\}$ be an accumulation point of* $\operatorname{dom} f$ *and* $\operatorname{dom} g$. *We say that f is asymptotically equivalent to g in the neighborhood of α, denoted by $f(x) \sim g(x)$, $x \to \alpha$, if $\displaystyle\lim_{x\to\alpha} \frac{f(x)}{g(x)} = 1$.*
If $\beta \in \mathbb{R}$ is an accumulation point of $\operatorname{dom} f$ *and if there exists a real number $\delta > 0$ such that $\forall x \in (\beta - \delta, \beta + \delta) \setminus \{\beta\} : f(x) = 0$ then $f(x) \sim 0$, $x \to \beta$.*
If ∞ is an accumulation point of $\operatorname{dom} f$, *we say that $f(x) \sim 0$, $x \to \infty$ if there exists a real number K such that $\forall x \geqslant K : f(x) = 0$.*
Let $\tilde{F}$ and $\tilde{G}$ be real m by n matrix-valued functions and let $\alpha \in \mathbb{R} \cup \{\infty\}$ be an accumulation point of $\operatorname{dom} \tilde{F}$ *and* $\operatorname{dom} \tilde{G}$. *Then $\tilde{F}(x) \sim \tilde{G}(x)$, $x \to \alpha$ if $\tilde{f}_{ij}(x) \sim \tilde{g}_{ij}(x)$, $x \to \alpha$ for $i = 1, 2, \ldots, m$ and j=1,2, ... ,n.*

The main difference with the conventional definition of asymptotic equivalence is that Definition 7.1.4 also allows us to say that a function is asymptotically equivalent to 0.

7.2 A Link between Conventional Algebra and the Symmetrized Max-Plus Algebra

In [126] Olsder and Roos have used a kind of link between conventional algebra and the max-plus algebra based on asymptotic equivalences to show that every matrix has at least one max-algebraic eigenvalue and to prove a max-algebraic version of Cramer's rule and of the Cayley-Hamilton theorem. In this section we extend and formalize this link. We also introduce the max-complex numbers, which yields a further extension of the max-plus algebra.

In this chapter we shall frequently encounter functions that are asymptotically equivalent to an exponential of the form νe^{xs} for $s \to \infty$. Since we want to allow exponents that are equal to ε, we set $e^{\varepsilon s}$ equal to 0 for all positive real values of s by definition. We also define the following classes of functions:

$$\mathcal{R}_{\mathrm{e}}^{+} = \Big\{ f : \mathbb{R}_0^+ \to \mathbb{R}^+ \,\Big|\, f(s) = \sum_{i=0}^{n} \mu_i e^{x_i s} \text{ with } n \in \mathbb{N}, \\ \mu_i \in \mathbb{R}_0^+ \text{ and } x_i \in \mathbb{R}_\varepsilon \text{ for all } i \Big\}$$

$$\mathcal{R}_{\mathrm{e}} = \Big\{ f : \mathbb{R}_0^+ \to \mathbb{R} \,\Big|\, f(s) = \sum_{i=0}^{n} \nu_i e^{x_i s} \text{ with } n \in \mathbb{N}, \\ \nu_i \in \mathbb{R}_0 \text{ and } x_i \in \mathbb{R}_\varepsilon \text{ for all } i \Big\}$$

$$\mathcal{C}_{\mathrm{e}} = \big\{ f + gj \,\big|\, f, g \in \mathcal{R}_{\mathrm{e}} \big\}$$

where j is the imaginary unit ($j^2 = -1$). It is easy to verify that $(\mathcal{R}_{\mathrm{e}}, +, \cdot)$ and $(\mathcal{C}_{\mathrm{e}}, +, \cdot)$ are rings.
For all $x, y, z \in \mathbb{R}_\varepsilon$ we have

$$x \oplus y = z \quad \Leftrightarrow \quad e^{xs} + e^{ys} \sim c e^{zs} \;,\; s \to \infty$$

$$x \otimes y = z \quad \Leftrightarrow \quad e^{xs} \cdot e^{ys} = e^{zs} \qquad \text{for all } s \in \mathbb{R}_0^+$$

where $c = 1$ if $x \neq y$ and $c = 2$ if $x = y$. We shall extend this link between $(\mathcal{R}_{\mathrm{e}}^{+}, +, \cdot)$ and $\mathbb{R}_{\max}$ that has already been used in [115, 116, 122, 125, 126] — and under a slightly different form in [34] — to $\mathbb{S}_{\max}$. If $x \in \mathbb{R}_\varepsilon$ then $\mathcal{F}(x, \cdot)$ is a function with domain of definition $\mathbb{R}_0^+$ and with

$$\mathcal{F}(x, s) = \mu e^{xs}$$

$$\mathcal{F}(\ominus x, s) = -\mu e^{xs}$$

$$\mathcal{F}(x^{\bullet}, s) = \nu e^{xs}$$

for all $s \in \mathbb{R}_0^+$, where μ is an arbitrary positive real number or parameter, and where ν is an arbitrary real number or parameter different from 0. Note that $\mathcal{F}(\varepsilon, \cdot) = 0$ since we have $e^{\varepsilon s} = 0$ for all $s \in \mathbb{R}_0^+$ by definition.
To reverse the mapping $\mathcal{F}(x, \cdot)$ we have to take $\lim\limits_{s\to\infty} \dfrac{\log(\,|\mathcal{F}(x,s)|\,)}{s}$ and adapt the max-algebraic sign depending on the sign of the coefficient of the exponential. So if f is a real function, if $x \in \mathbb{R}_\varepsilon$ and if μ is a positive real number or if μ is a parameter that can only take on positive real values then we have

$$f(s) \sim \mu e^{xs}\ ,\ s \to \infty \quad \Rightarrow \quad \mathcal{R}(f) = x$$

$$f(s) \sim -\mu e^{xs}\ ,\ s \to \infty \quad \Rightarrow \quad \mathcal{R}(f) = \ominus x$$

where $\mathcal{R}$ is the reverse mapping of $\mathcal{F}$. If ν is a parameter that can take on any non-zero real value then we have

$$f(s) \sim \nu e^{xs}\ ,\ s \to \infty \quad \Rightarrow \quad \mathcal{R}(f) = x^\bullet\ .$$

Note that the reverse mapping always yields a signed result if the coefficient of e^{xs} is a number (and not a parameter).
For all $a, b, c \in \mathbb{S}$ we have

$$a \oplus b = c \quad \Rightarrow \quad \mathcal{F}(a,s) + \mathcal{F}(b,s) \sim \mathcal{F}(c,s)\ ,\ s \to \infty \tag{7.4}$$

$$\mathcal{F}(a,s) + \mathcal{F}(b,s) \sim \mathcal{F}(c,s)\ ,\ s \to \infty \quad \Rightarrow \quad a \oplus b \mathrel{\nabla} c \tag{7.5}$$

$$a \otimes b = c \quad \Leftrightarrow \quad \mathcal{F}(a,s) \cdot \mathcal{F}(b,s) = \mathcal{F}(c,s) \qquad \text{for all } s \in \mathbb{R}_0^+ \tag{7.6}$$

for an appropriate choice of the μ and the ν coefficients in $\mathcal{F}(c,s)$ in (7.4) and in (7.6) from the left to the right. The balance in (7.5) results from the fact that we can have cancelation of equal terms with opposite sign in $(\mathcal{R}_\mathrm{e}^+, +, \cdot)$ whereas this is in general not possible in the symmetrized max-plus algebra since $\forall a \in \mathbb{S} \setminus \{\varepsilon\} : a \ominus a \neq \varepsilon$. So we have the following correspondences:

$$(\mathcal{R}_\mathrm{e}^+, +, \cdot) \quad \leftrightarrow \quad (\mathbb{R}_\varepsilon, \oplus, \otimes) = \mathbb{R}_{\max}$$

$$(\mathcal{R}_\mathrm{e}, +, \cdot) \quad \leftrightarrow \quad (\mathbb{S}, \oplus, \otimes) = \mathbb{S}_{\max}\ .$$

Now we extend the mapping $\mathcal{F}$ to matrices as follows. If $A \in \mathbb{S}^{m\times n}$ then $\tilde{A} = \mathcal{F}(A, \cdot)$ is a real m by n matrix-valued function with domain of definition $\mathbb{R}_0^+$ and with $\tilde{a}_{ij}(s) = \mathcal{F}(a_{ij}, s)$ for all i, j for some choice of the μ and the ν coefficients. Note that the mapping is performed entrywise — it is not a matrix exponential! The reverse mapping $\mathcal{R}$ is extended to matrices in a similar way: if $\tilde{A}$ is a real matrix-valued function with entries that are asymptotically equivalent to an exponential in the neighborhood of ∞, then $(\mathcal{R}(\tilde{A}))_{ij} = \mathcal{R}(\tilde{a}_{ij})$ for all i, j. If A, B and C are matrices with entries in $\mathbb{S}$, we have

$$A \oplus B = C \quad \Rightarrow \quad \mathcal{F}(A,s) + \mathcal{F}(B,s) \sim \mathcal{F}(C,s)\ ,\ s \to \infty \tag{7.7}$$

$$\mathcal{F}(A,s)+\mathcal{F}(B,s) \;\sim\; \mathcal{F}(C,s)\;,\; s\to\infty \quad\Rightarrow\quad A\oplus B \;\nabla\; C \tag{7.8}$$

$$A\otimes B = C \quad\Rightarrow\quad \mathcal{F}(A,s)\cdot\mathcal{F}(B,s) \;\sim\; \mathcal{F}(C,s)\;,\; s\to\infty \tag{7.9}$$

$$\mathcal{F}(A,s)\cdot\mathcal{F}(B,s) \;\sim\; \mathcal{F}(C,s)\;,\; s\to\infty \quad\Rightarrow\quad A\otimes B \;\nabla\; C \tag{7.10}$$

for an appropriate choice of the μ's and the ν's in $\mathcal{F}(C,s)$ in (7.7) and (7.9).

Example 7.2.1 Let $A = \begin{bmatrix} 0 & \varepsilon \\ \ominus 1 & \ominus 2 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 1 \\ 2^{\bullet} & \ominus 0 \end{bmatrix}$. Hence, $A\otimes B = \begin{bmatrix} -3 & 1 \\ 4^{\bullet} & 2^{\bullet} \end{bmatrix}$. In general we have $\mathcal{F}(A,s) = \begin{bmatrix} \mu_1 & 0 \\ -\mu_2\, e^{s} & -\mu_3\, e^{2s} \end{bmatrix}$, $\mathcal{F}(B,s) = \begin{bmatrix} \mu_4\, e^{-3s} & \mu_5\, e^{s} \\ \nu_1\, e^{2s} & -\mu_6 \end{bmatrix}$ and $\mathcal{F}(A\otimes B,s) = \begin{bmatrix} \mu_7\, e^{-3s} & \mu_8\, e^{s} \\ \nu_2\, e^{4s} & \nu_3\, e^{2s} \end{bmatrix}$ for all $s\in\mathbb{R}_0^+$ with $\mu_1,\mu_2,\ldots,\mu_8\in\mathbb{R}_0^+$ and $\nu_1,\nu_2,\nu_3\in\mathbb{R}_0$. Furthermore,

$$\mathcal{F}(A,s)\cdot\mathcal{F}(B,s) = \begin{bmatrix} \mu_1\,\mu_4\, e^{-3s} & \mu_1\,\mu_5\, e^{s} \\ -\mu_2\,\mu_4\, e^{-2s} \;-\; \nu_1\,\mu_3\, e^{4s} & (-\mu_2\,\mu_5 \;+\; \mu_3\,\mu_6)\, e^{2s} \end{bmatrix}$$

for all $s\in\mathbb{R}_0^+$.
If we take

$$\mu_7 = \mu_1\,\mu_4\;,\; \mu_8 = \mu_1\,\mu_5\;,\; \nu_2 = -\nu_1\,\mu_3 \quad\text{and}\quad \nu_3 = -\mu_2\,\mu_5 \;+\; \mu_3\,\mu_6\;,$$

then we have $\mathcal{F}(A,s)\cdot\mathcal{F}(B,s) \;\sim\; \mathcal{F}(A\otimes B,s)\;,\; s\to\infty$.
If we take $\mu_i = 1$ for $i = 1,2,\ldots,6$ and $\nu_1 = 1$, we get

$$\mathcal{F}(A,s)\cdot\mathcal{F}(B,s) \;\sim\; \begin{bmatrix} e^{-3s} & e^{s} \\ -e^{4s} & 0 \end{bmatrix} \stackrel{\text{def}}{=} \tilde{C}(s)\;,\; s\to\infty\;.$$

The reverse mapping results in $C = \mathcal{R}(\tilde{C}) = \begin{bmatrix} -3 & 1 \\ \ominus 4 & \varepsilon \end{bmatrix}$. Note that $A\otimes B \;\nabla\; C$.
Taking $\mu_i = i$ for $i = 1,2,\ldots,6$ and $\nu_1 = -1$ leads to

$$\mathcal{F}(A,s)\cdot\mathcal{F}(B,s) \;\sim\; \begin{bmatrix} 4e^{-3s} & 5e^{s} \\ 3e^{4s} & 8e^{2s} \end{bmatrix} \stackrel{\text{def}}{=} \tilde{D}(s)\;,\; s\to\infty\;.$$

The reverse mapping results in $D = \mathcal{R}(\tilde{D}) = \begin{bmatrix} -3 & 1 \\ 4 & 2 \end{bmatrix}$ and again we have $A\otimes B \;\nabla\; D$. □

We can extend the link between $(\mathcal{R}_{\mathrm{e}},+,\cdot)$ and $\mathbb{S}_{\max}$ even further by introducing the "max-complex" numbers. First we define $\bar{k}$ such that $\bar{k}\otimes\bar{k} = \ominus 0$. This yields $\mathbb{T} = \{\, a\oplus b\otimes\bar{k} \mid a,b\in\mathbb{S}\,\}$, the set of the *max-complex* numbers. The set $\mathbb{S}\subset\mathbb{T}$ is the set of the max-real numbers and $\mathbb{R}_{\varepsilon}\subset\mathbb{S}\subset\mathbb{T}$ is the set consisting of ε and the max-positive max-real numbers. Using a method that is analogous to the method that is used to construct $\mathbb{C}$ from $\mathbb{R}$ we get the following calculation rules:

$$(a \oplus b \otimes \bar{k}) \;\oplus\; (c \oplus d \otimes \bar{k}) = (a \oplus c) \;\oplus\; (b \oplus d) \otimes \bar{k}$$

$$(a \oplus b \otimes \bar{k}) \;\otimes\; (c \oplus d \otimes \bar{k}) = (a \otimes c \ominus b \otimes d) \;\oplus\; (a \otimes d \oplus b \otimes c) \otimes \bar{k}$$

where a, b, c and $d \in \mathbb{S}$. This results in the structure $\mathbb{T}_{\max} = (\mathbb{T}, \oplus, \otimes)$. It is easy to verify that $\mathbb{T}_{\max}$ is a commutative dioid.
If $a, b \in \mathbb{S}$ and if f and g are real functions that are asymptotically equivalent to an exponential in the neighborhood of ∞, we define

$$\mathcal{F}(a \oplus b \otimes \bar{k}, \cdot) = \mathcal{F}(a, \cdot) + \mathcal{F}(b, \cdot) j$$

$$\mathcal{R}(f + g\, j) = \mathcal{R}(f) \oplus \mathcal{R}(g) \otimes \bar{k}$$

where j is the imaginary unit. This leads to the following correspondence:

$$(\mathcal{C}_{\mathrm{e}}, +, \cdot) \quad \leftrightarrow \quad (\mathbb{T}, \oplus, \otimes) \;=\; \mathbb{T}_{\max} \ .$$

We shall not further elaborate this correspondence between $(\mathcal{C}_{\mathrm{e}}, +, \cdot)$ and $\mathbb{T}_{\max}$ since it will not be needed in the remainder of this thesis. Note however that we can use $\mathbb{T}_{\max}$ to define max-algebraic analogues of matrix decompositions from linear algebra for real or complex matrices (such as the eigenvalue decomposition or the Jordan decomposition).

7.3 Existence Proof for the Singular Value Decomposition in the Symmetrized Max-Plus algebra

In [42] we have used the mapping from $(\mathcal{R}_{\mathrm{e}}, +, \cdot)$ to $\mathbb{S}_{\max}$ and the reverse mapping to prove the existence of a kind of singular value decomposition in $\mathbb{S}_{\max}$. In this section we present an alternative proof for the existence theorem of the max-algebraic SVD. The major advantage of the new proof technique that will be developed in this section over the one of [42] is that it can easily be extended to prove the existence of many other matrix decompositions in the symmetrized max-plus algebra such as e.g. the max-algebraic QR decomposition.
In this section we first introduce a class $\mathcal{S}_{\mathrm{e}}$ of functions that can be written as a sum or a series of exponentials if the argument is large enough. This class of functions is closed under elementary operations such as additions, multiplications, subtractions, divisions, square roots and absolute values. We prove that for a matrix with entries in $\mathcal{S}_{\mathrm{e}}$ there exists a QR decomposition with entries that also belong to $\mathcal{S}_{\mathrm{e}}$. Next we use these results to prove the existence of

max-algebraic analogues of the SVD and the QRD. We conclude this section with a worked example.

The entries of the matrices that are used in the alternative existence proof for the max-algebraic SVD that will be presented in this section are sums or series of exponentials. Therefore, we first study some properties of this kind of functions.

Definition 7.3.1 (Sum or series of exponentials) *Let $\mathcal{S}_{\mathrm{e}}$ be the set of real functions that are analytic and that can be written as a (possibly infinite, but absolutely convergent) sum of exponentials in a neighborhood of ∞:*

$$\mathcal{S}_{\mathrm{e}} = \Big\{ f : A \to \mathbb{R} \;\Big|\; A \subseteq \mathbb{R},\ \exists K \in \mathbb{R}_0^+ \text{ such that } [K, \infty) \subseteq A \text{ and}$$

$$f \text{ is analytic in } [K,\infty) \text{ and either}$$

$$\forall x \geqslant K : f(x) = \sum_{i=0}^{n} \alpha_i e^{a_i x} \tag{7.11}$$

$$\text{with } n \in \mathbb{N},\ \alpha_i \in \mathbb{R}_0,\ a_i \in \mathbb{R}_\varepsilon \text{ for all } i \text{ and } a_0 > a_1 > \ldots > a_n \text{ ; or}$$

$$\forall x \geqslant K : f(x) = \sum_{i=0}^{\infty} \alpha_i e^{a_i x} \tag{7.12}$$

$$\text{with } \alpha_i \in \mathbb{R}_0,\ a_i \in \mathbb{R},\ a_i > a_{i+1},\ \lim_{i\to\infty} a_i = \varepsilon \text{ for all } i \text{ and}$$

$$\text{where the series converges absolutely for every } x \geqslant K \Big\} \ .$$

If $f \in \mathcal{S}_{\mathrm{e}}$ then the largest exponent in the sum or the series of exponentials that corresponds to f is called the *dominant exponent* of f.
Recall that by definition we have $e^{\varepsilon s} = 0$ for all $s \in \mathbb{R}_0^+$. Since we allow exponents that are equal to $\varepsilon = -\infty$ in the definition of $\mathcal{S}_{\mathrm{e}}$, the zero function also belongs to $\mathcal{S}_{\mathrm{e}}$. Since we require that the sequence of the exponents that appear in (7.11) or (7.12) is decreasing and since the coefficients cannot be equal to 0, any sum of exponentials of the form (7.11) or (7.12) that corresponds to the zero function consists of exactly one term: e.g. $1 \cdot e^{\varepsilon x}$.
If $f \in \mathcal{S}_{\mathrm{e}}$ is a series of the form (7.12) then the set $\{a_i \,|\, i = 0, 1, \ldots, \infty\}$ has no finite accumulation point since the sequence $\{a_i\}_{i=0}^{\infty}$ is decreasing and unbounded from below. Note that series of the form (7.12) are related to (generalized) Dirichlet series [100].
The behavior of the functions in $\mathcal{S}_{\mathrm{e}}$ in the neighborhood of ∞ is given by the following property:

Lemma 7.3.2 *Every function $f \in \mathcal{S}_{\mathrm{e}}$ is asymptotically equivalent to an exponential in the neighborhood of ∞:*

$$f \in \mathcal{S}_{\mathrm{e}} \ \Rightarrow\ f(x) \sim \alpha_0 e^{a_0 x} \ , \ x \to \infty$$

for some $\alpha_0 \in \mathbb{R}_0$ *and some* $a_0 \in \mathbb{R}_\varepsilon$.

Proof: See Section D.1. □

The set $\mathcal{S}_\mathrm{e}$ is closed under some basic operations:

Proposition 7.3.3 *If* f *and* g *belong to* $\mathcal{S}_\mathrm{e}$ *then* ρf, $f+g$, $f-g$, fg, f^l *and* $|f|$ *also belong to* $\mathcal{S}_\mathrm{e}$ *for any* $\rho \in \mathbb{R}$ *and any* $l \in \mathbb{N}$.
Furthermore, if there exists a real number P *such that* $f(x) \neq 0$ *for all* $x \geqslant P$ *then the functions* $\dfrac{1}{f}$ *and* $\dfrac{g}{f}$ *restricted to* $[P, \infty)$ *also belong to* $\mathcal{S}_\mathrm{e}$.
If there exists a real number Q *such that* $f(x) > 0$ *for all* $x \geqslant Q$ *then the function* $\sqrt{f}$ *restricted to* $[Q, \infty)$ *also belongs to* $\mathcal{S}_\mathrm{e}$.

Proof: See Section D.2. □

Let $\tilde{A}$ and $\tilde{R}$ be real m by n matrix-valued functions and let $\tilde{Q}$ be a real m by m matrix-valued function. Suppose that these matrix-valued functions are defined in $J \subseteq \mathbb{R}$. If $\tilde{Q}(s)\,\tilde{R}(s) = \tilde{A}(s)$, $\tilde{Q}^T(s)\,\tilde{Q}(s) = I_m$ and $\tilde{R}(s)$ is an upper triangular matrix for all $s \in J$ then we say that $\tilde{Q}\tilde{R}$ is a *path of QR decompositions* of $\tilde{A}$ on J. A path of SVDs is defined in a similar way.
Note that if $\tilde{Q}\tilde{R}$ is a path of QR decompositions of $\tilde{A}$ on J then we have $\|\tilde{R}(s)\|_\mathrm{F} = \|\tilde{A}(s)\|_\mathrm{F}$ for all $s \in J$.

Proposition 7.3.4 *If* $\tilde{A} \in \mathcal{S}_\mathrm{e}^{m \times n}$ *then there exists a path of QR decompositions* $\tilde{Q}\tilde{R}$ *of* $\tilde{A}$ *for which the entries of* $\tilde{Q}$ *and* $\tilde{R}$ *belong to* $\mathcal{S}_\mathrm{e}$.

Proof: Let $A \in \mathbb{R}^{m \times n}$. To compute the QR decomposition QR of A we can use an algorithm based on Givens rotations (See e.g. [65]). A Givens rotation is characterized by an n by n matrix of the form

$$G(n,i,k,c,s) = \begin{array}{cl} \left[\begin{array}{cccccccc} 1 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & & \vdots & & \vdots \\ 0 & 0 & \dots & c & \dots & s & \dots & 0 \\ \vdots & \vdots & & \vdots & \ddots & \vdots & & \vdots \\ 0 & 0 & \dots & -s & \dots & c & \dots & 0 \\ \vdots & \vdots & & \vdots & & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \end{array}\right] & \begin{array}{l} \\ \\ \\ \leftarrow \text{row } i \\ \\ \leftarrow \text{row } k \\ \\ \\ \end{array} \\ \begin{array}{cccccccc} & & & \uparrow & & \uparrow & & \\ & & & \text{column} & & \text{column} & & \\ & & & i & & k & & \end{array} & \end{array} \tag{7.13}$$

with $i, k \in \{1, 2, \dots, n\}$ and where $c = \cos(\theta)$ and $s = \sin(\theta)$ for some θ. So if $G = G(n,i,k,c,s)$ then we have $G_{ii} = G_{kk} = c$, $G_{ik} = -s$, $G_{ki} = s$; all the

other entries of G are the same as those of the identity matrix I_n. Consider an arbitrary $\theta \in \mathbb{R}$ and let $c = \cos(\theta)$ and $s = \sin(\theta)$. It is easy to verify that $G(n, i, k, c, s)$ is an orthogonal matrix and that left multiplication of a vector $x \in \mathbb{R}^n$ by $G^T(n, i, k, c, s)$ corresponds to a counterclockwise rotation of θ radians in the x_i-x_k plane.
In the Givens QR algorithm we apply Givens rotations to A such that the entries of the strictly lower triangular part of A are zeroed column by column until we finally obtain an upper triangular matrix. We now give the Givens QR algorithm in its most elementary form (i.e. without paying attention to efficiency and without the refinements necessary to avoid overflow and to guarantee numerical stability). We use the same notation as the one that has been used in Section 3.4 to describe the ELCP algorithm.

The Givens QR algorithm

Input: m, n, $A \in \mathbb{R}^{m \times n}$

$R \leftarrow A$
$Q \leftarrow I_m$
for $j = 1, 2, \ldots, n$ **do**
  for $i = m, m-1, \ldots, j+1$ **do**
    $\delta \leftarrow \sqrt{r_{i-1,j}^2 + r_{ij}^2}$
    if $\delta \neq 0$ **then**
      $c \leftarrow \dfrac{r_{i-1,j}}{\delta}$
      $s \leftarrow \dfrac{-r_{ij}}{\delta}$
      $Q \leftarrow Q\, G(m, i-1, i, c, s)$
      $R \leftarrow G^T(m, i-1, i, c, s)\, R$
    end if
  end for
end for

Output: Q, R

The operations used in this algorithm are additions, multiplications, subtractions, divisions and square roots.
So if we apply this algorithm to a matrix-valued function $\tilde{A}$ with entries in $\mathcal{S}_{\mathrm{e}}$ then the entries of the resulting matrix-valued functions $\tilde{Q}$ and $\tilde{R}$ will also belong to $\mathcal{S}_{\mathrm{e}}$ by Proposition 7.3.3. $\Box$

Now we give an alternative proof for the existence theorem of the max-algebraic SVD:

Theorem 7.3.5 (Existence of the SVD in $\mathbb{S}_{\max}$) *Let $A \in \mathbb{S}^{m \times n}$ and let $r = \min(m, n)$. Then there exist a max-algebraic diagonal matrix $\Sigma \in \mathbb{R}_\varepsilon^{m \times n}$*

and matrices $U \in (\mathbb{S}^\vee)^{m \times m}$ and $V \in (\mathbb{S}^\vee)^{n \times n}$ such that

$$A \mathrel{\nabla} U \otimes \Sigma \otimes V^T \tag{7.14}$$

with $U^T \otimes U \mathrel{\nabla} E_m$, $V^T \otimes V \mathrel{\nabla} E_n$ and $\|A\|_\oplus \geqslant \sigma_1 \geqslant \sigma_2 \geqslant \ldots \geqslant \sigma_r$ where $\sigma_i = (\Sigma)_{ii}$ for $i = 1, 2, \ldots, r$.
Every decomposition of the form (7.14) that satisfies the above conditions is called a max-algebraic singular value decomposition of A.

Proof: If $A \in \mathbb{S}^{m \times n}$ has entries that are not signed, we can always define a matrix $\hat{A} \in (\mathbb{S}^\vee)^{m \times n}$ such that

$$\hat{a}_{ij} = \begin{cases} a_{ij} & \text{if } a_{ij} \text{ is signed,} \\ |a_{ij}|_\oplus & \text{if } a_{ij} \text{ is not signed,} \end{cases}$$

for all i, j. Since $|\hat{a}_{ij}|_\oplus = |a_{ij}|_\oplus$ for all i, j, we have $\|\hat{A}\|_\oplus = \|A\|_\oplus$. Moreover, we have $\forall a, b \in \mathbb{S} : a \mathrel{\nabla} b \Rightarrow a^\bullet \mathrel{\nabla} b$, which means that if $\hat{A} \mathrel{\nabla} U \otimes \Sigma \otimes V^T$ then also $A \mathrel{\nabla} U \otimes \Sigma \otimes V^T$. Therefore, it is sufficient to prove this theorem for signed matrices A.

So from now on we assume that A is signed. Define $c = \|A\|_\oplus$.
If $c = \varepsilon$ then $A = \mathcal{E}_{m \times n}$. If we take $U = E_m$, $\Sigma = \mathcal{E}_{m \times n}$ and $V = E_n$, we have $A = U \otimes \Sigma \otimes V^T$, $U^T \otimes U = E_m$, $V^T \otimes V = E_n$ and $\sigma_1 = \sigma_2 = \ldots = \sigma_r = \varepsilon = \|A\|_\oplus$. So $U \otimes \Sigma \otimes V^T$ is a max-algebraic SVD of A.
From now on we assume that $c \neq \varepsilon$. We may assume without loss of generality that $m \geqslant n$: if $m < n$, we can apply the subsequent reasoning to A^T since $A \mathrel{\nabla} U \otimes \Sigma \otimes V^T$ if and only if $A^T \mathrel{\nabla} V \otimes \Sigma^T \otimes U^T$. So $U \otimes \Sigma \otimes V^T$ is a max-algebraic SVD of A if and only if $V \otimes \Sigma^T \otimes U^T$ is a max-algebraic SVD of A^T.
Now we distinguish between two different situations depending on whether or not all the a_{ij}'s have a finite max-absolute value. In Remark 7.3.6 we shall explain why this distinction is necessary. Note that is proof is rather long: it will end on p. 209.

Case 1: All the a_{ij}'s have a finite max-absolute value.

First we construct $\tilde{A} = \mathcal{F}(A, \cdot)$. Hence, $\tilde{a}_{ij}(s) = \gamma_{ij} e^{c_{ij} s}$ for all $s \in \mathbb{R}_0^+$ and for all i, j with $\gamma_{ij} \in \mathbb{R}_0$ and $c_{ij} = |a_{ij}|_\oplus \in \mathbb{R}_\varepsilon$ for all i, j. Note that the entries of $\tilde{A}$ belong to $\mathcal{S}_e$.
We are going to use Kogbetliantz's SVD algorithm [93] to construct a path of SVDs $\tilde{U}\tilde{\Sigma}\tilde{V}^T$ of $\tilde{A}$. In the next paragraphs we shall describe Kogbetliantz's SVD algorithm for matrices with real entries. This algorithm can be considered as an extension of Jacobi's method for the computation of the eigenvalue decomposition of a real symmetric matrix. The explanation below is mainly based on [15] and [78].

The off-norm of a matrix $M \in \mathbb{R}^{m\times n}$ is defined by

$$\|M\|_{\text{off}} = \sqrt{\sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} m_{ij}^2}$$

where the empty sum is equal to 0 by definition (So if M is a 1 by 1 matrix then we have $\|M\|_{\text{off}} = 0$).
Let $B \in \mathbb{R}^{m\times n}$ with $m \geqslant n$. Before applying Kogbetliantz's SVD algorithm we compute a QRD of B:

$$B = Q \begin{bmatrix} R \\ O_{(m-n)\times n} \end{bmatrix}$$

where R is an n by n upper triangular matrix.
Now we apply Kogbetliantz's SVD algorithm to R. In this algorithm a sequence of matrices is generated as follows:

$$\begin{aligned} &U_0 = I_n\,, && V_0 = I_n\,, && S_0 = R\,, \\ &U_k = U_{k-1} G_k\,, && V_k = V_{k-1} H_k\,, && S_k = G_k^T S_{k-1} H_k \qquad \text{for } k = 1, 2, 3, \ldots \end{aligned}$$

such that $\|S_k\|_{\text{off}}$ decreases monotonously as k increases. So S_k tends more and more to a diagonal matrix as the iteration process progresses. The absolute values of the diagonal entries of S_k will converge to the singular values of R as k goes to ∞.
The matrices G_k and H_k are Givens matrices of the form (7.13) and their parameters are chosen such that $(S_k)_{i_k j_k} = (S_k)_{j_k i_k} = 0$ for some ordered pair of indices (i_k, j_k). As a result we have

$$\|S_k\|_{\text{off}}^2 = \|S_{k-1}\|_{\text{off}}^2 - (S_{k-1})_{i_k j_k}^2 - (S_{k-1})_{j_k i_k}^2 \ .$$

Since the matrices G_k and H_k are orthogonal for all $k \in \mathbb{N}_0$, we have

$$\|S_k\|_{\text{F}} = \|R\|_{\text{F}}\,, \quad R = U_k S_k V_k^T\,, \quad U_k^T U_k = I_n \ \text{ and } \ V_k^T V_k = I_n \qquad (7.15)$$

for all $k \in \mathbb{N}$.
We use the row-cyclic version of Kogbetliantz's SVD algorithm: in each cycle the indices i_k and j_k are chosen such that the entries in the strictly upper triangular part of the S_k's are selected row by row. This yields the following sequence for the ordered pairs of indices (i_k, j_k):

$$(1,2) \rightarrow (1,3) \rightarrow \ldots \rightarrow (1,n) \rightarrow (2,3) \rightarrow (2,4) \rightarrow \ldots \rightarrow (n-1,n) \ .$$

One full cycle $(1,2) \rightarrow \ldots \rightarrow (n-1,n)$ is called a *sweep*. Note that a sweep corresponds to $N = \dfrac{(n-1)n}{2}$ iterations. Sweeps are repeated until S_k

becomes diagonal. If we have an upper triangular matrix at the beginning of a sweep then we shall have a lower triangular matrix after the sweep and vice versa. In order to keep the description of the algorithm short and simple we shall transpose the matrix S and switch U and V after each sweep. This ensures that S is always upper triangular at the beginning of a sweep.
For triangular matrices the row-cyclic Kogbetliantz algorithm as it will be given below is globally convergent [50, 78]. Furthermore, for triangular matrices the convergence of this algorithm is quadratic if k is large enough [4, 14, 76, 77, 127]:

$$\exists K \in \mathbb{N} \text{ such that } \forall k \geqslant K : \|S_{k+N}\|_{\text{off}} \leqslant \gamma \, \|S_k\|_{\text{off}}^2 \tag{7.16}$$

for some constant γ that does not depend on k, under the assumption that diagonal entries that correspond to the same singular value or that are affiliated with the same cluster of singular values occupy successive positions on the diagonal. This assumption is not restrictive since we can always reorder the diagonal entries of S_k by inserting an extra step in which we select a permutation matrix $\hat{P} \in \mathbb{R}^{n \times n}$ such that the diagonal entries of $S_{k+1} = \hat{P}^T S_k \hat{P}$ exhibit the required ordering. Note that $\|S_{k+1}\|_{\text{F}} = \|S_k\|_{\text{F}}$. If we define $U_{k+1} = U_k \hat{P}$ and $V_{k+1} = V_k \hat{P}$ then U_{k+1} and V_{k+1} are orthogonal since $\hat{P}^T \hat{P} = I_n$. We also have

$$U_{k+1} S_{k+1} V_{k+1}^T \;=\; \left(U_k \hat{P} \right) \, \left(\hat{P}^T S_k \hat{P} \right) \, \left(\hat{P}^T V_k^T \right) \;=\; U_k S_k V_k^T \;=\; R \;.$$

Furthermore, once the diagonal entries have the required ordering, they hold it provided that k is sufficiently large [76].
If we define $S = \lim\limits_{k \to \infty} S_k$, $U = \lim\limits_{k \to \infty} U_k$ and $V = \lim\limits_{k \to \infty} V_k$ then S is a diagonal matrix, U and V are orthogonal matrices and $USV^T = R$.
We make all the diagonal entries of S nonnegative by multiplying S with an n by n diagonal matrix D with $d_{ii} = 1$ if $s_{ii} \geqslant 0$ and $d_{ii} = -1$ if $s_{ii} < 0$ for $i = 1, 2, \ldots, n$. Next we construct a permutation matrix P such that the diagonal entries of $P^T SDP$ are arranged in descending order. If we define $U_R = UP$, $S_R = P^T SDP$ and $V_R = VD^{-1}P$, then U_R and V_R are orthogonal, the diagonal entries of S_R are nonnegative and ordered and

$$U_R S_R V_R^T = (UP) \left(P^T SDP \right) \left(P^T D^{-1} V^T \right) = USV^T = R \;.$$

Hence, $U_R S_R V_R^T$ is an SVD of R. If we define

$$U_B = Q \begin{bmatrix} U_R & O_{n \times (m-n)} \\ O_{(m-n) \times n} & I_{m-n} \end{bmatrix}, \;\; S_B = \begin{bmatrix} S_R \\ O_{(m-n) \times n} \end{bmatrix} \text{ and } \; V_B = V_R \;,$$

then $U_B S_B V_B^T$ is an SVD of B.
We now give the row-cyclic Kogbetliantz algorithm for upper triangular matrices (without paying attention to efficiency and without the refinements

necessary to avoid overflow and to guarantee numerical stability). The inputs of the algorithm are an upper triangular matrix $R \in \mathbb{R}^{n \times n}$ and a tolerance $\xi > 0$. The algorithm then computes orthogonal vectors U and V and an almost diagonal matrix $S = U^T RV$ such that $\|S\|_{\text{off}} \leqslant \xi \, \|R\|_{\text{F}}$.

The row-cyclic Kogbetliantz algorithm

Input: n, $R \in \mathbb{R}^{n \times n}$, ξ

$S_0 \leftarrow R$
$U_0 \leftarrow I_n$
$V_0 \leftarrow I_n$
$k \leftarrow 0$
$\eta \leftarrow \xi \, \|R\|_{\text{F}}$
$upper \leftarrow 1$
while $\|S_k\|_{\text{off}} > \eta$ **do**
  for $i = 1, 2, \ldots, n-1$ **do**
    for $j = i+1, i+2, \ldots, n$ **do**
      if $(S_k)_{ij} \neq 0$ **then**
        if $(S_k)_{jj} = 0$ **then**
          $\tau \leftarrow \dfrac{(S_k)_{ii}}{(S_k)_{ij}}$
          if $|\tau| < 1$ **then**
            $t_\psi \leftarrow \tau$
            $c_\phi \leftarrow 0$
            $s_\phi \leftarrow 1$
          else
            $t_\psi \leftarrow -\dfrac{1}{\tau}$
            $c_\phi \leftarrow 1$
            $s_\phi \leftarrow 0$
          end if
        else
          $\delta \leftarrow (S_k)_{jj}^2 - (S_k)_{ii}^2 + (S_k)_{ij}^2$
          if $\delta = 0$ **then**
            $t_\psi \leftarrow 1$
          else
            $t_{2\psi} \leftarrow \dfrac{2\,(S_k)_{ii}\,(S_k)_{ij}}{\delta}$
            $t_\psi \leftarrow \dfrac{t_{2\psi}}{1 + \sqrt{1 + {t_{2\psi}}^2}}$
          end if

$$t_\phi \leftarrow \frac{(S_k)_{ij} + (S_k)_{ii}\, t_\psi}{(S_k)_{jj}}$$
$$c_\phi \leftarrow \frac{1}{\sqrt{1+t_\phi^2}}$$
$$s_\phi \leftarrow c_\phi t_\phi$$

end if

$$c_\psi \leftarrow \frac{1}{\sqrt{1+t_\psi^2}}$$
$$s_\psi \leftarrow c_\psi t_\psi$$
$$S_{k+1} \leftarrow G^T(n,i,j,c_\phi,s_\phi)\, S_k\, G(n,i,j,c_\psi,s_\psi)$$
$$U_{k+1} \leftarrow U_k\, G(n,i,j,c_\phi,s_\phi)$$
$$V_{k+1} \leftarrow V_k\, G(n,i,j,c_\psi,s_\psi)$$

else

$S_{k+1} \leftarrow S_k$
$U_{k+1} \leftarrow U_k$
$V_{k+1} \leftarrow V_k$

end if

$k \leftarrow k+1$

end for

end for

$S_k \leftarrow S_k^T$
$T \leftarrow V_k$
$V_k \leftarrow U_k$
$U_k \leftarrow T$
$upper \leftarrow 1 - upper$

end while

if $upper = 1$ **then**

$S \leftarrow S_k$
$U \leftarrow U_k$
$V \leftarrow V_k$

else

$S \leftarrow S_k^T$
$U \leftarrow V_k$
$V \leftarrow U_k$

end if

Output: U, S, V

Now we use the row-cyclic Kogbetliantz algorithm to construct a path of SVDs $\tilde{U}\tilde{\Sigma}\tilde{V}^T$ of $\tilde{A}$. In order to prove convergence of the row-cyclic Kog-

betliantz algorithm when applied to matrices with entries in $\mathcal{S}_{\rm e}$, we need a matrix-valued function for which all the terms of all the entries have negative exponents. Therefore, we define a matrix-valued function $\tilde{B}$ with $\operatorname{dom}\tilde{B} = \mathbb{R}_0^+$ such that $\tilde{B}(s) = e^{-(c+1)s}\tilde{A}(s)$ for all $s \in \mathbb{R}_0^+$. Hence, $\tilde{b}_{ij}(s) = \gamma_{ij}e^{-b_{ij}s}$ for all $s \in \mathbb{R}_0^+$ and for all i,j with $b_{ij} = c + 1 - c_{ij} > 0$ for all i,j. Obviously, the entries of $\tilde{B}$ are in $\mathcal{S}_{\rm e}$. Note that the dominant exponent of $\|\tilde{B}\|_{\rm F}$ is negative.
Let $J \subseteq \mathbb{R}_0^+$. If $\tilde{U}\tilde{S}\tilde{V}^T$ is a path of SVDs of $\tilde{B}$ on J and if $\tilde{\Sigma}$ is a real m by n matrix-valued function with $\operatorname{dom}\tilde{\Sigma} = J$ such that $\tilde{\Sigma}(s) = e^{(c+1)s}\tilde{S}(s)$ for all $s \in J$ then $\tilde{U}\tilde{\Sigma}\tilde{V}^T$ is a path of SVDs of $\tilde{A}$ on J.
In order to determine a path of SVDs of $\tilde{B}$, we first compute a path of QRDs of $\tilde{B}$ on $\mathbb{R}_0^+$:

$$\tilde{B} = \tilde{Q}\begin{bmatrix} \tilde{R} \\ O_{(m-n)\times n} \end{bmatrix}$$

where $\tilde{R}$ is an n by n upper triangular matrix-valued function. By Proposition 7.3.4 the entries of $\tilde{Q}$ and $\tilde{R}$ belong to $\mathcal{S}_{\rm e}$.
Now we use the row-cyclic Kogbetliantz algorithm to compute a path of SVDs of $\tilde{R}$. The operations used in this algorithm are additions, multiplications, subtractions, divisions, square roots and absolute values. So if we apply this algorithm to a matrix with entries in $\mathcal{S}_{\rm e}$, the entries of all the matrices generated during the iteration process also belong to $\mathcal{S}_{\rm e}$ by Proposition 7.3.3.
If f, g and h belong to $\mathcal{S}_{\rm e}$ then they are asymptotically equivalent to an exponential in the neighborhood of ∞ by Lemma 7.3.2. So if L is large enough, then $f(L) \geqslant 0$ and $g(L) \geqslant h(L)$ imply that $f(s) \geqslant 0$ and $g(s) \geqslant h(s)$ for all $s \in [L,\infty)$. This is one of the reasons that Kogbetliantz's SVD algorithm and other algorithms from linear algebra also work for matrices with entries that belong to $\mathcal{S}_{\rm e}$ instead of $\mathbb{R}$.
Let $\tilde{S}_k$, $\tilde{U}_k$ and $\tilde{V}_k$ be the matrix-valued functions that are computed in the kth pass through the main loop of the algorithm. The dominant exponents of all the entries of $\tilde{R}$ are negative since $\|\tilde{R}\|_{\rm F} = \|\tilde{B}\|_{\rm F}$. Since $\|\tilde{S}_k\|_{\rm F} = \|\tilde{R}\|_{\rm F}$ by (7.15), the dominant exponents of the entries of $\tilde{S}_k$ are also negative. Therefore it follows from (7.16) that the largest dominant exponent of the off-diagonal entries of $\tilde{S}_k$ approximately (at least) doubles each N passes provided that k is large enough. So the dominant exponents of the off-diagonal entries of $\tilde{S}_k$ become more and more negative as the iteration progresses. Furthermore, since the Frobenius norm of $\tilde{S}_k$ stays constant during the iteration, the exponents of the updates $(\tilde{S}_k)_{ii} - (\tilde{S}_{k-1})_{ii}$ of the diagonal entries also approximately double each N passes. If we assume that the terms of the sums or the series that correspond to the diagonal entries of $\tilde{S}_k$ are ordered such that the exponents form a decreasing sequence, more and more successive terms of the sums or the series that correspond to the diagonal entries of $\tilde{S}_k$ stay constant as the iteration process progresses. This

also holds for the sums or the series that correspond to the entries of $\tilde{U}_k$ and $\tilde{V}_k$.

In theory the row-cyclic Kogbetliantz algorithm should run forever in order to produce a path of exact SVDs of $\tilde{B}$. However, since we are mainly interested in the asymptotic behavior of the singular values and the entries of the singular vectors of $\tilde{B}$, we may stop the iteration process as soon as the dominant exponents of the matrix-valued functions obtained at the end of consecutive sweeps do not change any more. Furthermore, we are not really interested in a path of exact SVDs of $\tilde{B}$. Assume that we have performed p sweeps of the row-cyclic Kogbetliantz algorithm. Let $\tilde{\Delta}_p$ be the diagonal matrix-valued function obtained by removing the off-diagonal entries of $\tilde{S}_{pN}$ after the pth sweep, making all diagonal entries nonnegative and arranging them in descending order, and adding $m-n$ zero rows (cf. the transformations that were used to go from S to S_B on p. 203). Let $\tilde{X}_p$ and $\tilde{Y}_p$ be the matrix-valued functions obtained by applying the corresponding transformations to $\tilde{U}_{pN}$ and $\tilde{V}_{pN}$ respectively. If we define a matrix-valued function $\tilde{C}_p = \tilde{X}_p \tilde{\Delta}_p \tilde{Y}_p^T$, we have a path of *exact* SVDs of $\tilde{C}_p$ on some interval $[L, \infty)$. This means that we may stop the iteration process as soon as

$$\tilde{b}_{ij}(s) \sim (\tilde{C}_p(s))_{ij} \ , \ s \rightarrow \infty \qquad \text{for all } i, j \, . \tag{7.17}$$

Let $\tilde{U}\tilde{S}\tilde{V}^T$ be a path of approximate SVDs of $\tilde{B}$ on some interval $[L, \infty)$ that was obtained by the procedure given above. Since we have performed a *finite* number of elementary operations on the entries of $\tilde{B}$, the entries of $\tilde{U}$, $\tilde{S}$ and $\tilde{V}$ belong to $\mathcal{S}_e$.

If we define a matrix-valued function $\tilde{\Sigma}$ with $\operatorname{dom}\tilde{\Sigma} = [L, \infty)$ such that $\tilde{\Sigma}(s) = e^{(c+1)s}\tilde{S}(s)$ for all $s \in [L, \infty)$, then we have

$$\tilde{A}(s) \sim \tilde{U}(s)\, \tilde{\Sigma}(s)\, \tilde{V}^T(s) \ , \ s \rightarrow \infty \tag{7.18}$$

$$\tilde{U}^T(s)\, \tilde{U}(s) = I_m \qquad \text{for all } s \geqslant L \tag{7.19}$$

$$\tilde{V}^T(s)\, \tilde{V}(s) = I_n \qquad \text{for all } s \geqslant L \, . \tag{7.20}$$

The diagonal entries of $\tilde{\Sigma}$ and the entries of $\tilde{U}$ and $\tilde{V}$ belong to $\mathcal{S}_e$ and are thus asymptotically equivalent to an exponential in the neighborhood of ∞ by Lemma 7.3.2.

Now we apply the reverse mapping $\mathcal{R}$ in order to obtain a max-algebraic SVD of A. Since we have used numbers instead of parameters for the coefficients of the exponentials in $\mathcal{F}(A, \cdot)$, the coefficients of the exponentials in the singular values and the entries of the singular vectors are also numbers. Therefore, the reverse mapping only yields signed results. If we define

$$\Sigma = \mathcal{R}(\tilde{\Sigma})\, , \ U = \mathcal{R}(\tilde{U})\, , \ V = \mathcal{R}(\tilde{V}) \ \text{ and } \ \sigma_i = (\Sigma)_{ii} = \mathcal{R}(\tilde{\sigma}_i) \text{ for all } i \, ,$$

then Σ is a max-algebraic diagonal matrix and U and V have signed entries. If we apply the reverse mapping $\mathcal{R}$ to (7.18) – (7.20), we get

$$A \nabla U \otimes \Sigma \otimes V^T, \quad U^T \otimes U \nabla E_m \text{ and } V^T \otimes V \nabla E_n \ .$$

The $\tilde{\sigma}_i$'s are nonnegative in $[L, \infty)$ and therefore we have $\sigma_i \in \mathbb{R}_\varepsilon$ for all i. Since the $\tilde{\sigma}_i$'s are ordered in $[L, \infty)$, their dominant exponentials are also ordered. Hence, $\sigma_1 \geqslant \sigma_2 \geqslant \ldots \geqslant \sigma_r$.
We have $\|\tilde{A}(s)\|_\mathrm{F} \sim \gamma e^{cs}$, $s \rightarrow \infty$ for some $\gamma > 0$ since $c = \|A\|_\oplus$ is the largest exponent that appears in the entries of $\tilde{A}$. Hence, $\mathcal{R}\left(\|\tilde{A}\|_\mathrm{F}\right) = c = \|A\|_\oplus$. By (7.1) we have $\dfrac{1}{\sqrt{n}}\|\tilde{A}\|_\mathrm{F} \leqslant \|\tilde{A}\|_2 \leqslant \|\tilde{A}\|_\mathrm{F}$ in $[L, \infty)$. Since $\tilde{\sigma}_1(s) \sim \|\tilde{A}(s)\|_2$, $s \rightarrow \infty$ and since the mapping $\mathcal{R}$ preserves the order, this leads to $\|A\|_\oplus \leqslant \sigma_1 \leqslant \|A\|_\oplus$ and consequently, $\sigma_1 = \|A\|_\oplus$.

Case 2: Not all the a_{ij}'s have a finite max-absolute value.
First we construct a sequence $\{A_l\}_{l=1}^\infty$ of m by n matrices such that

$$(A_l)_{ij} = \begin{cases} a_{ij} & \text{if } |a_{ij}|_\oplus \neq \varepsilon \,, \\ \|A\|_\oplus - l & \text{if } |a_{ij}|_\oplus = \varepsilon \,, \end{cases}$$

for all i, j. So the entries of the matrices A_l are finite and $\|A\|_\oplus = \|A_l\|_\oplus$ for all $l \in \mathbb{N}_0$. Furthermore, $\lim\limits_{l \rightarrow \infty} A_l = A$.
Now we construct the sequence $\{\tilde{A}_l\}_{l=1}^\infty$ with $\tilde{A}_l = \mathcal{F}(A_l, \cdot)$ for $l = 1, 2, 3, \ldots$ where we take the same coefficients γ_{ij} for all the $\tilde{A}_l$'s. We compute a path of approximate SVDs $\tilde{U}_l \, \tilde{\Sigma}_l \, \tilde{V}_l^T$ of each $\tilde{A}_l$ using the method of Case 1 of this proof.
In general, it is possible that for some of the entries of the $\tilde{U}_l$'s and the $\tilde{V}_l$'s the sequence of the dominant exponents and the sequence of the corresponding coefficients have more than one accumulation point (since if two or more singular values coincide the corresponding left and right singular vectors are not uniquely defined). However, since we use a fixed computation scheme (the row-cyclic Kogbetliantz algorithm), all the sequences will have exactly one accumulation point. So some of the dominant exponents will reach a finite limit as l goes to ∞, while the other dominant exponents will tend to $-\infty$. If we take the reverse mapping $\mathcal{R}$, we get a sequence of max-algebraic SVDs $\{U_l \otimes \Sigma_l \otimes V_l^T\}_{l=1}^\infty$ where some of the entries, viz. those that correspond to dominant exponents that tend to $-\infty$, tend to ε as l goes to ∞.
Note that $(\Sigma_l)_{ii} \leqslant (\Sigma_l)_{11} = \|A\|_\oplus$ for $i = 1, 2, \ldots, n$ and for all $l \in \mathbb{N}_0$.
If we define

$$U = \lim_{l \rightarrow \infty} U_l, \quad \Sigma = \lim_{l \rightarrow \infty} \Sigma_l \text{ and } V = \lim_{l \rightarrow \infty} V_l$$

then we have

$$A \nabla U \otimes \Sigma \otimes V^T, \quad U^T \otimes U \nabla E_m \text{ and } V^T \otimes V \nabla E_n \ .$$

Since the diagonal entries of all the Σ_l's are max-positive or max-zero, ordered and less than or equal to $\|A\|_{\oplus}$, the diagonal entries of Σ are also max-positive or max-zero, ordered and less than or equal to $\|A\|_{\oplus}$. Hence, $U \otimes \Sigma \otimes V^T$ is a max-algebraic SVD of A. $\square$

Remark 7.3.6 Now we explain why we have distinguished between two different cases in the proof of Theorem 7.3.5.
If there exist indices i and j such that $a_{ij} = \varepsilon$ then $\tilde{b}_{ij}(s) = 0$ for all $s \in \mathbb{R}_0^+$, which means then we cannot guarantee that condition (7.17) will be satisfied after a finite number of sweeps. This is why we make a distinction between the case where all the entries of A are finite and the case where at least one entry of A is equal to ε.
Let us now show that this argument does not hold for the singular values. If $\tilde{\Psi}$ is a real matrix-valued function that is analytic in some interval $J \subseteq \mathbb{R}$ then the rank of $\tilde{\Psi}$ is constant in J except in some isolated points where the rank drops [64]. If the rank of $\tilde{\Psi}(s)$ is equal to ρ for all $s \in J$ except for some isolated points then we say that the *generic rank* of $\tilde{\Psi}$ in J is equal to ρ. The entries of all the matrix-valued functions created in the row-cyclic Kogbetliantz algorithm when applied to $\tilde{A}$ are real and analytic in some interval $[L^*, \infty)$. Furthermore, for a fixed value of s the matrices $\tilde{A}(s)$, $\tilde{B}(s)$, $\tilde{R}(s)$, $\tilde{S}_1(s)$, $\tilde{S}_2(s)$, $\ldots$ all have the same rank since $\tilde{B}(s) = e^{(c+1)s}\tilde{A}(s)$ and since the matrices $\tilde{B}(s)$, $\tilde{R}(s)$, $\tilde{S}_1(s)$, $\tilde{S}_2(s)$, $\ldots$ are related by orthogonal transformations. So if ρ is the generic rank of $\tilde{A}$ in $[L^*, \infty)$ then the generic rank of $\tilde{B}$, $\tilde{R}$, $\tilde{S}_1$, $\tilde{S}_2$, $\ldots$ in $[L^*, \infty)$ is also equal to ρ. If $\rho < n$ then the $n - \rho$ smallest singular values of $\tilde{R}$ will be identically zero in $[L^*, \infty)$. However, since $\tilde{R}$, $\tilde{S}_N$, $\tilde{S}_{2N}$, $\ldots$ are upper triangular matrices, they have at least $n - \rho$ diagonal entries that are identically zero in $[L^*, \infty)$ since otherwise their generic rank would be greater than ρ. In fact this also holds for $\tilde{S}_1$, $\tilde{S}_2$, $\ldots$ since these matrix-valued functions are hierarchically triangular, i.e. block triangular such that the diagonal blocks are again block triangular, etc. [78]. Furthermore, we have already said that if k is large enough, diagonal entries do not change their affiliation any more, i.e. if a diagonal entry corresponds to a specific singular value in the kth pass through the main loop of the algorithm then it will also correspond to that singular value in all the next passes. Since the diagonal entries of $\tilde{S}_k$ are asymptotically equivalent to an exponential in the neighborhood of ∞, this means that at least $n - \rho$ diagonal entries (with a fixed position) of $\tilde{S}_k$, $\tilde{S}_{k+1}$, $\ldots$ will be identically zero in some interval $[L, \infty) \subseteq [L^*, \infty)$ if k is large enough. Hence, we do not have to take special precautions if $\tilde{A}$ has singular values that are identically zero in the neighborhood of ∞ since convergence to these singular values in a finite number of iteration steps is guaranteed.
For inner products of two different columns of $\tilde{U}$ there are no problems either: these inner products are equal to 0 by construction since the matrix-valued function $\tilde{U}_k$ is orthogonal on $[L, \infty)$ for all $k \in \mathbb{N}$.
This also holds for inner products of two different columns of $\tilde{V}$. $\diamond$

Remark 7.3.7 In Section 7.4 we shall explain why the condition $\sigma_1 \leqslant \|A\|_\oplus$ is needed in the definition of the max-algebraic SVD (See Example 7.4.1). Note that there does not appear a similar condition in the definition of the SVD in linear algebra (cf. Theorem 7.1.2).
In the proof of Theorem 7.3.5 we have in fact proved that $\sigma_1 = \|A\|_\oplus$. However, in Section 7.4 we shall make clear why we have used the condition $\sigma_1 \leqslant \|A\|_\oplus$ instead in the formulation of Theorem 7.3.5.
When we shall formulate the existence theorem for the max-algebraic QRD, we shall also use a similar condition. $\diamond$

The alternative proof technique that has been used in this section leads to a proof that is longer than that of [42]. However, since it essentially consists in applying an algorithm from linear algebra to a matrix with entries in $\mathcal{S}_{\mathrm{e}}$, the proof technique of this section has the advantage that it can also be used to prove the existence of many other max-algebraic matrix decompositions fairly easily. Let us illustrate this by showing the existence of a max-algebraic analogue of the QRD.

Theorem 7.3.8 (Max-algebraic QR decomposition) *If $A \in \mathbb{S}^{m\times n}$ then there exist a matrix $Q \in (\mathbb{S}^\vee)^{m\times m}$ and a max-algebraic upper triangular matrix $R \in (\mathbb{S}^\vee)^{m\times n}$ such that*

$$A \,\nabla\, Q \otimes R \tag{7.21}$$

with $Q^T \otimes Q \,\nabla\, E_m$ and $\|R\|_\oplus \leqslant \|A\|_\oplus$.
Every decomposition of the form (7.21) that satisfies the above conditions is called a max-algebraic QR decomposition of A.

Proof: This is a direct consequence of Proposition 7.3.4. $\Box$

Furthermore, the proof technique of this section can easily be adapted to prove the existence of a max-algebraic eigenvalue decomposition for symmetric matrices (by using the Jacobi algorithm for the computation of the eigenvalue decomposition of a real symmetric matrix), a max-algebraic LU decomposition, a max-algebraic Schur decomposition, a max-algebraic Hessenberg decomposition and so on (See e.g. [65] for a definition of these decompositions in linear algebra).

Now we give an example of the computation of a max-algebraic SVD of a matrix using the mapping $\mathcal{F}$. Another example of this technique can be found in [42]. An example of the computation of a max-algebraic QRD will be given in Section D.4.

Example 7.3.9 Consider

$$A = \begin{bmatrix} \ominus 0 & 4 \\ 1 & \ominus 5 \end{bmatrix} .$$

We shall compute a max-algebraic SVD of A using the mapping $\mathcal{F}$. We define $\tilde{A} = \mathcal{F}(A, \cdot)$ where we set all the coefficients of the exponentials equal to 1. So

$$\tilde{A}(s) = \begin{bmatrix} -1 & e^{4s} \\ e^{s} & -e^{5s} \end{bmatrix} \qquad \text{for all } s \in \mathbb{R}_0^+ .$$

Since $\tilde{A}$ is a 2 by 2 matrix-valued function, we can compute a path of SVDs $\tilde{U}\tilde{\Sigma}\tilde{V}^T$ of $\tilde{A}$ analytically, e.g. via the eigenvalue decomposition of $\tilde{A}^T\tilde{A}$ (See e.g. [65, 135]). This yields

$$\tilde{U}(s) = \begin{bmatrix} \dfrac{-1}{\sqrt{e^{2s}+1}} & \dfrac{e^{s}}{\sqrt{e^{2s}+1}} \\ \dfrac{e^{s}}{\sqrt{e^{2s}+1}} & \dfrac{1}{\sqrt{e^{2s}+1}} \end{bmatrix} \sim \begin{bmatrix} -e^{-s} & 1 \\ 1 & e^{-s} \end{bmatrix}, \ s \to \infty$$

$$\tilde{\Sigma}(s) = \begin{bmatrix} \sqrt{e^{10s}+e^{8s}+e^{2s}+1} & 0 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} e^{5s} & 0 \\ 0 & 0 \end{bmatrix}, \ s \to \infty$$

$$\tilde{V}(s) = \begin{bmatrix} \dfrac{1}{\sqrt{e^{8s}+1}} & \dfrac{e^{4s}}{\sqrt{e^{8s}+1}} \\ \dfrac{-e^{4s}}{\sqrt{e^{8s}+1}} & \dfrac{1}{\sqrt{e^{8s}+1}} \end{bmatrix} \sim \begin{bmatrix} e^{-4s} & 1 \\ -1 & e^{-4s} \end{bmatrix}, \ s \to \infty \ .$$

If we apply the reverse mapping $\mathcal{R}$, we get a max-algebraic SVD $U \otimes \Sigma \otimes V^T$ of A with

$$U = \mathcal{R}(\tilde{U}) = \begin{bmatrix} \ominus(-1) & 0 \\ 0 & -1 \end{bmatrix}, \quad \Sigma = \mathcal{R}(\tilde{\Sigma}) = \begin{bmatrix} 5 & \varepsilon \\ \varepsilon & \varepsilon \end{bmatrix} \text{ and}$$

$$V = \mathcal{R}(\tilde{V}) = \begin{bmatrix} -4 & 0 \\ \ominus 0 & -4 \end{bmatrix} .$$

We have

$$U \otimes \Sigma \otimes V^T = \begin{bmatrix} \ominus 0 & 4 \\ 1 & \ominus 5 \end{bmatrix} = A$$

$$U^T \otimes U = \begin{bmatrix} 0 & (-1)^{\bullet} \\ (-1)^{\bullet} & 0 \end{bmatrix} \nabla E_2$$

$$V^T \otimes V = \begin{bmatrix} 0 & (-4)^{\bullet} \\ (-4)^{\bullet} & 0 \end{bmatrix} \nabla E_2 \ .$$

Note that $\sigma_1 = 5 = \|A\|_{\oplus}$.
We have $U \otimes U^T = U^T \otimes U \nabla E_2$, $V \otimes V^T = V^T \otimes V \nabla E_2$, $\det_{\oplus} U = \ominus 0 \not\nabla \varepsilon$ and $\det_{\oplus} V = 0 \not\nabla \varepsilon$ (cf. Section D.5). □

7.4 Properties of the Max-Algebraic SVD

In this section we derive some properties of the max-algebraic SVD and we indicate how the max-algebraic SVD might be used in the identification of max-linear time-invariant DESs.

First we show by an example that the condition $\sigma_1 \leqslant \|A\|_\oplus$ in the definition of the max-algebraic SVD is necessary in order to obtain singular values that are bounded from above.

Example 7.4.1 Consider

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} .$$

If we define

$$U = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \ominus 0 & \ominus 0 & 0 \\ 0 & 0 & 0 & \ominus 0 \\ 0 & 0 & \ominus 0 & 0 \end{bmatrix} , \quad \Sigma = \begin{bmatrix} \sigma & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \sigma & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \sigma & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \sigma \end{bmatrix}$$

and

$$V = \begin{bmatrix} 0 & \ominus 0 & \ominus 0 & \ominus 0 \\ 0 & 0 & \ominus 0 & \ominus 0 \\ \ominus 0 & 0 & \ominus 0 & \ominus 0 \\ 0 & \ominus 0 & 0 & \ominus 0 \end{bmatrix} ,$$

then we have

$$U^T \otimes U = V^T \otimes V = \begin{bmatrix} 0 & 0^\bullet & 0^\bullet & 0^\bullet \\ 0^\bullet & 0 & 0^\bullet & 0^\bullet \\ 0^\bullet & 0^\bullet & 0 & 0^\bullet \\ 0^\bullet & 0^\bullet & 0^\bullet & 0 \end{bmatrix} \nabla E_4$$

and

$$U \otimes \Sigma \otimes V^T = \begin{bmatrix} \sigma^\bullet & \sigma^\bullet & \sigma^\bullet & \sigma^\bullet \\ \sigma^\bullet & \sigma^\bullet & \sigma^\bullet & \sigma^\bullet \\ \sigma^\bullet & \sigma^\bullet & \sigma^\bullet & \sigma^\bullet \\ \sigma^\bullet & \sigma^\bullet & \sigma^\bullet & \sigma^\bullet \end{bmatrix} , \tag{7.22}$$

which means that $U \otimes \Sigma \otimes V^T \nabla A$ for every $\sigma \geqslant 0$.
So if the condition $\sigma_1 \leqslant \|A\|_\oplus$ would not have been included in the definition of the max-algebraic SVD, (7.22) would be a max-algebraic SVD of A for every $\sigma \geqslant 0$. □

Likewise, the condition $\|R\|_\oplus \leqslant \|A\|_\oplus$ in Theorem 7.3.8 is necessary to bound the components of R from above:

Example 7.4.2 Consider

$$A = \begin{bmatrix} \ominus 0 & 0 & 0 \\ 0 & \ominus 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} .$$

Without the condition $\|R\|_\oplus \leqslant \|A\|_\oplus$ every max-algebraic product of the form

$$Q \otimes R = \begin{bmatrix} \ominus 0 & 0 & 0 \\ 0 & \ominus 0 & 0 \\ 0 & 0 & \ominus 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & \varepsilon & \rho \\ \varepsilon & 0 & \rho \\ \varepsilon & \varepsilon & \rho \end{bmatrix} = \begin{bmatrix} \ominus 0 & 0 & \rho^\bullet \\ 0 & \ominus 0 & \rho^\bullet \\ 0 & 0 & \rho^\bullet \end{bmatrix}$$

with $\rho \geqslant 0$ would have been a max-algebraic QRD of A. □

If $A \in \mathbb{S}^{m\times n}$ and if $U \otimes \Sigma \otimes V^T$ is a max-algebraic SVD of A then U is a square matrix with signed entries that satisfies $U^T \otimes U \ \nabla \ E_m$. We shall now prove some properties of this kind of matrices.

Proposition 7.4.3 *Consider $U \in (\mathbb{S}^\vee)^{m\times m}$. If $U^T \otimes U \ \nabla \ E_m$ then we have $\|U_{.,i}\|_\oplus = 0$ for $i = 1, 2, \ldots, m$.*

Proof: Let $i \in \{1, 2, \ldots, m\}$. Since $U^T \otimes U \ \nabla \ E_m$, we have $(U^T \otimes U)_{ii} \ \nabla \ 0$. Hence,

$$\bigoplus_{k=1}^{m} {u_{ki}}^{\otimes^2} \ \nabla \ 0 \ . \tag{7.23}$$

Consider an arbitrary index $k \in \{1, 2, \ldots, n\}$. We have

$$\begin{aligned} {u_{ki}}^{\otimes^2} &= (u_{ki}^{\oplus} \ominus u_{ki}^{\ominus})^{\otimes^2} \\ &= (u_{ki}^{\oplus})^{\otimes^2} \ominus u_{ki}^{\oplus} \otimes u_{ki}^{\ominus} \ominus u_{ki}^{\ominus} \otimes u_{ki}^{\oplus} \oplus (u_{ki}^{\ominus})^{\otimes^2} \\ &= (u_{ki}^{\oplus})^{\otimes^2} \oplus (u_{ki}^{\ominus})^{\otimes^2} \end{aligned}$$

since the entries of U are signed and thus $u_{ki}^{\oplus} = \varepsilon$ or $u_{ki}^{\ominus} = \varepsilon$, which implies that $u_{ki}^{\oplus} \otimes u_{ki}^{\ominus} = \varepsilon$.

So ${u_{ki}}^{\otimes^2}$ is signed for all k, which means that both sides of the balance (7.23) are signed. Therefore, it follows from Proposition 2.3.3 that

$$\bigoplus_{k=1}^{m} \left((u_{ki}^{\oplus})^{\otimes^2} \oplus (u_{ki}^{\ominus})^{\otimes^2} \right) = \bigoplus_{k=1}^{m} {u_{ki}}^{\otimes^2} = 0 \ .$$

Since $(x \oplus y)^{\otimes^2} = x^{\otimes^2} \oplus y^{\otimes^2}$ for all $x, y \in \mathbb{R}_\varepsilon$, this results in

$$\bigoplus_{k=1}^{m} (u_{ki}^{\oplus} \oplus u_{ki}^{\ominus})^{\otimes^2} = 0 \ . \tag{7.24}$$

If $x \in \mathbb{R}_\varepsilon$ then $x^{\otimes^2}$ is equal to $2x$ in conventional algebra. Therefore, (7.24) is equivalent to $\bigoplus_{k=1}^{m} (u_{ki}^{\oplus} \oplus u_{ki}^{\ominus}) = 0$, and this results in $\|U_{.,i}\|_\oplus = 0$. □

Corollary 7.4.4 *Consider $U \in (\mathbb{S}^\vee)^{m \times m}$. If $U \otimes U^T \ \nabla \ E_m$ then we have $|u_{ij}|_\oplus \leqslant 0$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, m$.*

Now we explain why we really need the symmetrized max-plus algebra $\mathbb{S}_{\max}$ to define the max-algebraic singular value decomposition: we shall show that the class of matrices with entries in $\mathbb{R}_\varepsilon$ that have max-algebraic SVD in which U and V have only entries in $\mathbb{R}_\varepsilon$ is rather limited. Let $A \in \mathbb{R}_\varepsilon^{m \times n}$ and let $U \otimes \Sigma \otimes V^T$ be a max-algebraic SVD of A with $U \in \mathbb{R}_\varepsilon^{m \times m}$ and $V \in \mathbb{R}_\varepsilon^{n \times n}$. Since the entries of U are signed, it follows from Proposition 2.3.7 that the balance $U^T \otimes U \ \nabla \ E_m$ results in $U^T \otimes U = E_m$. It is easy to verify that this is only possible if every column and every row of U contains exactly one entry that is equal to 0 and if all the other entries of U are equal to ε. Hence, U is max-algebraic permutation matrix. This also holds for V. So $U = P_1$ and $V = P_2$ where P_1 and P_2 are max-algebraic permutation matrices. As a consequence, we have $A = U \otimes \Sigma \otimes V^T = P_1 \otimes \Sigma \otimes P_2^T$. So A has to be a permuted max-algebraic diagonal matrix.
So only permuted max-algebraic diagonal matrices with entries in $\mathbb{R}_\varepsilon$ have a max-algebraic SVD with entries in $\mathbb{R}_\varepsilon$. This could be compared with the class of real matrices in linear algebra that have an SVD with only nonnegative entries: using an analogous reasoning one can prove that this class coincides with the set of the real permuted diagonal matrices. Furthermore, it is obvious that each SVD in $\mathbb{R}_{\max}$ is also an SVD in $\mathbb{S}_{\max}$.

Proposition 7.4.5 *Let $A \in \mathbb{S}^{m \times n}$. There always exists a max-algebraic SVD $U \otimes \Sigma \otimes V^T$ of A for which $\sigma_1 = \|A\|_\oplus$.*

Proof: This has already been proved in the proof of Theorem 7.3.5. □

Theorem 7.3.5 tells us that the max-algebraic singular values of a matrix A are bounded from above by $\|A\|_\oplus$. Furthermore, by Proposition 7.4.5 there always exists a max-algebraic SVD for which σ_1 is equal to this upper bound. The following proposition tells us when the upper bound for σ_1 is tight for all the max-algebraic SVDs of A:

Proposition 7.4.6 *Consider $A \in \mathbb{S}^{m \times n}$. If there is at least one signed entry in A that is equal to $\|A\|_\oplus$ in max-absolute value then $\sigma_1 = \|A\|_\oplus$ for every max-algebraic SVD of A.*

Proof: Consider an arbitrary max-algebraic SVD $U \otimes \Sigma \otimes V^T$ of A. So $A \,\nabla\, U \otimes \Sigma \otimes V^T$. If we extract the max-positive and the max-negative part of each matrix that appears in this balance, we get

$$A^{\oplus} \ominus A^{\ominus} \;\nabla\; (U^{\oplus} \ominus U^{\ominus}) \otimes \Sigma \otimes (V^{\oplus} \ominus V^{\ominus})^T \enspace .$$

Using Proposition 2.3.6 this balance can be rewritten as

$$\begin{aligned} A^{\oplus} \;\oplus\; U^{\oplus} \otimes \Sigma \otimes (V^{\ominus})^T \;\oplus\; U^{\ominus} \otimes \Sigma \otimes (V^{\oplus})^T \;\nabla \\ A^{\ominus} \;\oplus\; U^{\oplus} \otimes \Sigma \otimes (V^{\oplus})^T \;\oplus\; U^{\ominus} \otimes \Sigma \otimes (V^{\ominus})^T \enspace . \end{aligned} \tag{7.25}$$

Both sides of this balance are signed. So by Proposition 2.3.7 we may replace the balance by an equality. Let $r = \min(m, n)$ and let a_{pq} be the signed entry of A for which $|a_{pq}|_{\oplus} = \|A\|_{\oplus}$. Now we consider the equality that corresponds to the pth row and the qth column of (7.25):

$$\begin{aligned} a_{pq}^{\oplus} \;\oplus\; \bigoplus_{k=1}^{r} u_{pk}^{\oplus} \otimes \sigma_k \otimes v_{qk}^{\ominus} \;\oplus\; \bigoplus_{k=1}^{r} u_{pk}^{\ominus} \otimes \sigma_k \otimes v_{qk}^{\oplus} \;= \\ a_{pq}^{\ominus} \;\oplus\; \bigoplus_{k=1}^{r} u_{pk}^{\oplus} \otimes \sigma_k \otimes v_{qk}^{\oplus} \;\oplus\; \bigoplus_{k=1}^{r} u_{pk}^{\ominus} \otimes \sigma_k \otimes v_{qk}^{\ominus} \enspace . \end{aligned} \tag{7.26}$$

First we assume that $a_{pq} \in \mathbb{S}^{\oplus}$. Hence, $a_{pq}^{\ominus} = \varepsilon$. The entries of U and V are less than or equal to 0 in max-absolute value by Corollary 7.4.4. Hence,

$$u_{pk}^{\oplus},\, u_{pk}^{\ominus},\, v_{qk}^{\oplus},\, v_{qk}^{\ominus} \;\leqslant\; 0 \qquad \text{for } k = 1, 2, \ldots, m \tag{7.27}$$

and thus

$$u_{pk}^{\oplus} \otimes \sigma_k \otimes v_{qk}^{\ominus} \leqslant \sigma_k \leqslant \|A\|_{\oplus} \quad \text{and} \quad u_{pk}^{\ominus} \otimes \sigma_k \otimes v_{qk}^{\oplus} \leqslant \sigma_k \leqslant \|A\|_{\oplus}$$

for $k = 1, 2, \ldots, m$. So the left-hand side of (7.26) is equal to $a_{pq}^{\oplus} = \|A\|_{\oplus}$, which means that there has to exist an index $l \in \{1, 2, \ldots, r\}$ such that

$$u_{pl}^{\oplus} \otimes \sigma_l \otimes v_{ql}^{\oplus} \;=\; a_{pq}^{\oplus} \quad \text{ or } \quad u_{pl}^{\ominus} \otimes \sigma_l \otimes v_{ql}^{\ominus} \;=\; a_{pq}^{\oplus} \enspace .$$

From (7.27) it follows that this is only possible if $\sigma_l \geqslant a_{pq}^{\oplus} = \|A\|_{\oplus}$. Since $\|A\|_{\oplus} \geqslant \sigma_1 \geqslant \sigma_l$, this means that $\sigma_1 = \sigma_l = \|A\|_{\oplus}$.
If $a_{pq} \in \mathbb{S}^{\ominus}$, an analogous reasoning also leads to the conclusion that $\sigma_1 = \|A\|_{\oplus}$. □

Note that the condition of Proposition 7.4.6 is always satisfied if all the entries of the matrix A are signed. For a matrix A that does not satisfy the condition of Proposition 7.4.6 it is possible that there exists a max-algebraic SVD for which the largest singular value is less than $\|A\|_{\oplus}$ as is shown by the following example:

Example 7.4.7 Consider $A = \left[\, 1^\bullet \,\right]$. Then $0 \otimes \sigma \otimes 0$ is a max-algebraic SVD of A for every $\sigma \in \mathbb{R}_\varepsilon$ with $\sigma \leqslant 1 = \|A\|_\oplus$ since $0 \otimes \sigma \otimes 0 = \sigma \ \nabla \ 1^\bullet$ if $\sigma \leqslant 1$. □

So in contrast to the singular values in linear algebra the max-algebraic singular values are not always unique. This leads to the definition of a *maximal* max-algebraic SVD — where all the singular values are as large as possible — and a *minimal* max-algebraic SVD — where all the singular values are as small as possible. The maximal max-algebraic SVD of the matrix A of Example 7.4.7 is given by $0 \otimes 1 \otimes 0$ and the minimal max-algebraic SVD is given by $0 \otimes \varepsilon \otimes 0$.

Proposition 7.4.8 *Let $A \in \mathbb{S}^{m \times n}$. If $U \otimes \Sigma_{\max} \otimes V^T$ is a maximal max-algebraic SVD of A, then we have $\sigma_{\max,1} \stackrel{\mathrm{def}}{=} (\Sigma_{\max})_{11} = \|A\|_\oplus$*

Proof: The definition of the max-algebraic SVD yields an upper bound for $\sigma_{\max,1}$: $\sigma_{\max,1} \leqslant \|A\|_\oplus$ and Proposition 7.4.5 tells us that this upper bound is tight. □

Recall that if $C \in \mathbb{R}^{m \times n}$ with $\operatorname{rank}(C) = \rho$ and if $U \Sigma V^T$ is a (conventional) SVD of C then we have $\sigma_1, \sigma_2, \ldots, \sigma_\rho \neq 0$ and $\sigma_{\rho+1} = \sigma_{\rho+2} = \ldots = \sigma_r = 0$ where $r = \min(m,n)$. By analogy we could define a rank based on the max-algebraic SVD:

Definition 7.4.9 (Max-algebraic SVD rank) *Let $A \in \mathbb{S}^{m \times n}$ and let $r = \min(m,n)$. The max-algebraic SVD rank of A is defined by*

$$\begin{aligned} \operatorname{rank}_{\oplus,\mathrm{SVD}}(A) = \min \Big\{ \rho \;\Big|\; & U \otimes \Sigma \otimes V^T \text{ is a max-algebraic SVD of } A \\ & \text{with } \sigma_1, \sigma_2, \ldots, \sigma_\rho \neq \varepsilon \text{ and} \\ & \sigma_{\rho+1} = \sigma_{\rho+2} = \ldots = \sigma_r = \varepsilon \Big\} \end{aligned} \tag{7.28}$$

where $\sigma_i = (\Sigma)_{ii}$ for $i = 1, 2, \ldots, r$.

So the max-algebraic SVD rank of $A \in \mathbb{S}^{m \times n}$ is equal to the least number of non-ε max-algebraic singular values over the set of all the max-algebraic SVDs of A. Furthermore, if $\rho_A = \operatorname{rank}_{\oplus,\mathrm{SVD}}(A)$ and if the minimum in (7.28) is reached for the max-algebraic SVD $\hat{U} \otimes \hat{\Sigma} \otimes \hat{V}^T$ of A then we have

$$A \ \nabla \ \bigoplus_{i=1}^{\rho_A} \hat{\sigma}_i \otimes \hat{u}_i \otimes \hat{v}_i^T \tag{7.29}$$

where the empty max-algebraic sum $\displaystyle\bigoplus_{i=1}^{0} \ldots$ is equal to $\mathcal{E}_{m \times n}$ by definition. The max-algebraic sum (7.29) contains the least number of terms over all the decompositions $A \ \nabla \ \displaystyle\bigoplus_{i=1}^{\rho} \sigma_i \otimes u_i \otimes v_i^T$ that correspond to a max-algebraic SVD

$U \otimes \Sigma \otimes V^T$ of A. This explains why we have used the condition $\sigma_1 \leqslant \|A\|_\oplus$ instead of $\sigma_1 = \|A\|_\oplus$ in Theorem 7.3.5. If the condition $\sigma_1 = \|A\|_\oplus$ would have been used then the matrix A of Example 7.4.7 would have only one max-algebraic SVD: $1^\bullet \;\nabla\; 0 \otimes 1 \otimes 0$ with $\sigma_1 = 1 \neq \varepsilon$, and then the max-algebraic SVD rank of A would be equal to 1. However, if we use the condition $\sigma_1 \leqslant \|A\|_\oplus$ in the definition of the max-algebraic SVD then $0 \otimes \varepsilon \otimes 0$ is a minimal max-algebraic SVD of A and then we have $\operatorname{rank}_{\oplus,\mathrm{SVD}}(A) = 0$, and this is in accordance with the fact that A can be written as a max-algebraic sum of the form (7.29) with 0 terms: $A = 1^\bullet \;\nabla\; \varepsilon = \displaystyle\bigoplus_{i=1}^{0} \sigma_i \otimes u_i \otimes v_i^T$.

Consider a matrix $C \in \mathbb{R}^{m \times n}$. If $U \Sigma V^T$ is a (conventional) SVD of C and if $\hat{\rho} \in \mathbb{N}$ with $\hat{\rho} \leqslant \operatorname{rank}(C)$, then it can be shown (See e.g. [65]) that the matrix $R_{\hat{\rho}} = \displaystyle\sum_{i=1}^{\hat{\rho}} \sigma_i u_i v_i^T$ is the best rank-$\hat{\rho}$ approximation of A in the sense that $\displaystyle\min_{\operatorname{rank}(R)=\hat{\rho}} \|C - R\|_2 = \|C - R_{\hat{\rho}}\|_2$. By analogy we could also define a "best rank-$\hat{\rho}$ approximation" based on the max-algebraic SVD.
Let us now indicate how this approximation could be used in the identification of max-linear time-invariant DESs.
Suppose that we have a max-linear time-invariant DES with m inputs and l outputs. Let $N \in \mathbb{N}$ with $N \gg n$ where n is the (unknown) minimal system order of the given system. Let $\{y_i(k)\}_{k=1}^{2N+1}$ be the output sequence that is *measured* if we apply a unit impulse to the ith input for $i = 1, 2, \ldots, m$. Define $\hat{G}_{k-1} = \begin{bmatrix} y_1(k) & y_2(k) & \ldots & y_m(k) \end{bmatrix}^T$ for $k = 1, 2, \ldots, 2N+1$ and let

$$\hat{H} = \begin{bmatrix} \hat{G}_0 & \hat{G}_1 & \ldots & \hat{G}_N \\ \hat{G}_1 & \hat{G}_2 & \ldots & \hat{G}_{N+1} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{G}_N & \hat{G}_{N+1} & \ldots & \hat{G}_{2N} \end{bmatrix} .$$

If there is no noise present then $\operatorname{rank}_\oplus(\hat{H})$ is a lower bound for the minimal system order (cf. Theorem 6.3.2). But in the presence of noise $\hat{H}$ will almost always be of full rank. If $U \otimes \Sigma \otimes V^T$ is a max-algebraic SVD of $\hat{H}$, then we have

$$\hat{H} \;\nabla\; \bigoplus_{i=1}^{\rho} \sigma_i \otimes u_i \otimes v_i^T \tag{7.30}$$

with $\rho = (N+1)\min(m, l)$. If we do not consider all the terms of the right-hand side of (7.30), i.e. if we stop adding terms in (7.30) as soon as the matrix A is approximated "accurately" enough, say if $\rho = \hat{\rho}$, we could use $\hat{\rho}$ as an estimate of the minimal system order of the given system. Next we could try to find a state space realization (A, B, C) of order $\hat{\rho}$ for which the "difference" between the sequence $\{C \otimes A^{\otimes^k} \otimes B\}_{k=0}^{2N}$ and the sequence $\{\hat{G}_k\}_{k=0}^{2N}$ is minimal. This is of course mainly an intuitive reasoning and there are still many open questions left such as: When is $\hat{H}$ approximated "accurately" enough? How do we characterize this "accuracy"? How "good" is the estimate of the minimal system order? What is the best way to characterize the "difference" between the sequence $\{C \otimes A^{\otimes^k} \otimes B\}_{k=0}^{2N}$ and the sequence $\{\hat{G}_k\}_{k=0}^{2N}$? and so on.

7.5 The Max-Algebraic SVD and the ELCP

In this section we show that the problem of determining a max-algebraic SVD of a given matrix can be reformulated as a system of multivariate max-algebraic polynomial equalities and inequalities that can be solved using the ELCP approach. This result also holds for the max-algebraic QRD. For small-sized matrices this allows us to compute all the max-algebraic SVDs or QRDs of the given matrix with the ELCP algorithm of Section 3.4. The proof technique that will be developed in this section can also be used to prove that many other max-algebraic matrix decompositions in the symmetrized max-plus algebra can also be computed using the ELCP approach.

Proposition 7.5.1 *If $A \in \mathbb{S}^{m \times n}$ is a matrix with finite entries, i.e. if $|a_{ij}|_{\oplus} \neq \varepsilon$ for all i, j, then there exists a max-algebraic SVD of A for which all the singular values and all the components of the singular vectors are finite.*

Proof: See Section D.3. □

Proposition 7.5.2 *If $A \in \mathbb{S}^{m \times n}$ is a matrix with finite entries, i.e. if $|a_{ij}|_{\oplus} \neq \varepsilon$ for all i, j, then there exists a max-algebraic QR decomposition $Q \otimes R$ of A for which all the entries of Q and all the entries of the upper triangular part of R are finite.*

Proof: Use a proof that is similar to that of Proposition 7.5.1. □

We have already shown that the mapping $\mathcal{F}$ can be used to compute the max-algebraic SVD of a matrix A. We could compute a path of SVDs of $\tilde{A} = \mathcal{F}(A, \cdot)$ analytically via the eigenvalue decomposition of $\tilde{A}^T\tilde{A}$ (cf. Example 7.3.9). However, symbolic calculation of the eigenvalues and eigenvectors of $\tilde{A}^T\tilde{A}$ is not always possible (especially if the size of the matrix $\tilde{A}^T\tilde{A}$ is greater than 4 since by Abel's theorem on algebraic equations there do not exist general formulas that express the roots of a polynomial of degree 5 or higher in terms of the coefficients of the polynomial by means of radicals (See e.g. [6, 11])). Furthermore, applying the row-cyclic Kogbetliantz algorithm to $\tilde{A}$ as we have done in the proof of Theorem 7.3.5 is too complicated and arduous in practice. Alternatively, we could compute constant SVDs of $\tilde{A}$ in a set of discrete points and then use interpolation to obtain a path of SVDs $\tilde{U}\tilde{\Sigma}\tilde{V}^T$ of $\tilde{A}$. However, max-algebraic singular values and components of the max-algebraic singular vectors that are asymptotically equivalent to an exponential of the form γe^{cs} with $c < 0$ in the neighborhood of ∞ will become almost 0 even for relatively small s. Numerically they are then equal to 0 and they will be mapped to ε instead of c by the reverse mapping $\mathcal{R}$. Especially in such cases, the ELCP approach could be considered as an alternative method to compute the max-algebraic SVD of the given matrix. Note however that if we use the ELCP algorithm of Section 3.4, this approach is only feasible for small-sized matrices.

Let us now show that the problem of finding a max-algebraic SVD of a matrix can be reformulated as an ELCP.
The original problem is:

> Given $A \in \mathbb{S}^{m \times n}$, find a max-algebraic diagonal matrix $\Sigma \in \mathbb{R}_\varepsilon^{m \times n}$ and matrices $U \in (\mathbb{S}^\vee)^{m \times m}$ and $V \in (\mathbb{S}^\vee)^{n \times n}$ such that
>
> $$A \nabla U \otimes \Sigma \otimes V^T \tag{7.31}$$
>
> $$U^T \otimes U \nabla E_m \tag{7.32}$$
>
> $$V^T \otimes V \nabla E_n \tag{7.33}$$
>
> with $\|A\|_\oplus \geqslant \sigma_1 \geqslant \sigma_2 \geqslant \ldots \geqslant \sigma_r$, where $r = \min(m, n)$ and $\sigma_i = (\Sigma)_{ii}$ for $i = 1, 2, \ldots, r$.

We shall show that the above conditions can be transformed into a system of multivariate max-algebraic polynomial equalities and inequalities.
First we assume that all the entries of A are finite. Then it follows from Proposition 7.5.1 that there exists a max-algebraic SVD of A with finite singular values and finite singular vectors.
Now we write down the equations that will yield a max-algebraic SVD $U \otimes \Sigma \otimes V^T$ of A with finite singular values and finite singular vectors.
First of all, we want the entries of U and V to be signed:

$$u_{ij}^\oplus \otimes u_{ij}^\ominus = \varepsilon \qquad \text{for } i = 1, 2, \ldots, m \text{ and } j = 1, 2, \ldots, m\,, \tag{7.34}$$

$$v_{ij}^\oplus \otimes v_{ij}^\ominus = \varepsilon \qquad \text{for } i = 1, 2, \ldots, n \text{ and } j = 1, 2, \ldots, n\,. \tag{7.35}$$

If we extract the max-positive and the max-negative parts of each matrix, (7.31) – (7.33) result in

$$A^\oplus \ominus A^\ominus \nabla (U^\oplus \ominus U^\ominus) \otimes \Sigma \otimes (V^\oplus \ominus V^\ominus)^T \tag{7.36}$$

$$(U^\oplus \ominus U^\ominus)^T \otimes (U^\oplus \ominus U^\ominus) \nabla E_m \tag{7.37}$$

$$(V^\oplus \ominus V^\ominus)^T \otimes (V^\oplus \ominus V^\ominus) \nabla E_n \tag{7.38}$$

or

$$A^{\oplus} \;\oplus\; U^{\oplus} \otimes \Sigma \otimes (V^{\ominus})^T \;\oplus\; U^{\ominus} \otimes \Sigma \otimes (V^{\oplus})^T \;\nabla\; A^{\ominus} \;\oplus\; U^{\oplus} \otimes \Sigma \otimes (V^{\oplus})^T \;\oplus\; U^{\ominus} \otimes \Sigma \otimes (V^{\ominus})^T$$

$$(U^{\oplus})^T \otimes U^{\oplus} \;\oplus\; (U^{\ominus})^T \otimes U^{\ominus} \;\nabla\; E_m \;\oplus\; (U^{\oplus})^T \otimes U^{\ominus} \;\oplus\; (U^{\ominus})^T \otimes U^{\oplus}$$

$$(V^{\oplus})^T \otimes V^{\oplus} \;\oplus\; (V^{\ominus})^T \otimes V^{\ominus} \;\nabla\; E_n \;\oplus\; (V^{\oplus})^T \otimes V^{\ominus} \;\oplus\; (V^{\ominus})^T \otimes V^{\oplus}$$

by Proposition 2.3.6. Both sides of all the balances are now signed. So by Proposition 2.3.7 we may replace the balances by equalities. We define three matrices $T \in \mathbb{R}_\varepsilon^{m\times n}$, $P \in \mathbb{R}_\varepsilon^{m\times m}$ and $Q \in \mathbb{R}_\varepsilon^{n\times n}$ such that

$$T = A^{\oplus} \;\oplus\; U^{\oplus} \otimes \Sigma \otimes (V^{\ominus})^T \;\oplus\; U^{\ominus} \otimes \Sigma \otimes (V^{\oplus})^T \tag{7.39}$$

$$P = (U^{\oplus})^T \otimes U^{\oplus} \;\oplus\; (U^{\ominus})^T \otimes U^{\ominus} \tag{7.40}$$

$$Q = (V^{\oplus})^T \otimes V^{\oplus} \;\oplus\; (V^{\ominus})^T \otimes V^{\ominus} \;, \tag{7.41}$$

and thus also

$$T = A^{\ominus} \;\oplus\; U^{\oplus} \otimes \Sigma \otimes (V^{\oplus})^T \;\oplus\; U^{\ominus} \otimes \Sigma \otimes (V^{\ominus})^T \tag{7.42}$$

$$P = E_m \;\oplus\; (U^{\oplus})^T \otimes U^{\ominus} \;\oplus\; (U^{\ominus})^T \otimes U^{\oplus} \tag{7.43}$$

$$Q = E_n \;\oplus\; (V^{\oplus})^T \otimes V^{\ominus} \;\oplus\; (V^{\ominus})^T \otimes V^{\oplus} \;. \tag{7.44}$$

Note that P and Q are symmetric matrices. Since the max-algebraic singular values are finite, their max-algebraic inverses are defined. Since the entries of A, U and V are finite, the entries of T, P and Q are also finite. So their max-algebraic inverses are also defined.

If we write out the max-algebraic matrix multiplications in (7.39) and (7.42) and if we transfer the entries of T to the opposite side, we get

$$a_{ij}^{\oplus} \otimes t_{ij}{}^{\otimes^{-1}} \;\oplus\; \bigoplus_{k=1}^{r} u_{ik}^{\oplus} \otimes \sigma_k \otimes v_{jk}^{\ominus} \otimes t_{ij}{}^{\otimes^{-1}} \;\oplus\; \bigoplus_{k=1}^{r} u_{ik}^{\ominus} \otimes \sigma_k \otimes v_{jk}^{\oplus} \otimes t_{ij}{}^{\otimes^{-1}} = 0 \tag{7.45}$$

$$a_{ij}^{\ominus} \otimes t_{ij}{}^{\otimes^{-1}} \;\oplus\; \bigoplus_{k=1}^{r} u_{ik}^{\oplus} \otimes \sigma_k \otimes v_{jk}^{\oplus} \otimes t_{ij}{}^{\otimes^{-1}} \;\oplus \tag{7.46}$$

$$\bigoplus_{k=1}^{r} u_{ik}^{\ominus} \otimes \sigma_k \otimes v_{jk}^{\ominus} \otimes t_{ij}{}^{\otimes^{-1}} = 0 \tag{7.47}$$

for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.
Since P is symmetric and since $(E_m)_{ij} = 0$ if $i = j$ and $(E_m)_{ij} = \varepsilon$ if $i \neq j$, (7.40) and (7.43) lead to

$$\bigoplus_{k=1}^{m} u_{ki}^{\oplus} \otimes u_{kj}^{\oplus} \otimes p_{ij}{}^{\otimes^{-1}} \oplus \bigoplus_{k=1}^{m} u_{ki}^{\ominus} \otimes u_{kj}^{\ominus} \otimes p_{ij}{}^{\otimes^{-1}} = 0 \tag{7.48}$$

$$\bigoplus_{k=1}^{m} u_{ki}^{\oplus} \otimes u_{kj}^{\ominus} \otimes p_{ij}{}^{\otimes^{-1}} \oplus \bigoplus_{k=1}^{m} u_{ki}^{\ominus} \otimes u_{kj}^{\oplus} \otimes p_{ij}{}^{\otimes^{-1}} = 0 \tag{7.49}$$

for $i = 1, 2, \ldots, m$ and $j = i+1, i+2, \ldots, m$. Furthermore,

$$\bigoplus_{k=1}^{m} u_{ki}^{\oplus} \otimes u_{ki}^{\oplus} \oplus \bigoplus_{k=1}^{m} u_{ki}^{\ominus} \otimes u_{ki}^{\ominus} = 0 \oplus \bigoplus_{k=1}^{m} u_{ki}^{\oplus} \otimes u_{ki}^{\ominus} \oplus \bigoplus_{k=1}^{m} u_{ki}^{\ominus} \otimes u_{ki}^{\oplus} = p_{ii}$$

for $i = 1, 2, \ldots, m$, or equivalently

$$\bigoplus_{k=1}^{m} (u_{ki}^{\oplus})^{\otimes^2} \oplus \bigoplus_{k=1}^{m} (u_{ki}^{\ominus})^{\otimes^2} = 0 = p_{ii} \qquad \text{for } i = 1, 2, \ldots, m$$

since the entries of U are signed.
If $x \in \mathbb{R}_\varepsilon$ then $x^{\otimes^2}$ is equal to $2x$ in conventional algebra. Hence,

$$\bigoplus_{k=1}^{m} u_{ki}^{\oplus} \oplus \bigoplus_{k=1}^{m} u_{ki}^{\ominus} = 0 \qquad \text{for } i = 1, 2, \ldots, m\,. \tag{7.50}$$

Note that $p_{ii} = 0$ for $i = 1, 2, \ldots, m$.
Analogously we obtain

$$\bigoplus_{k=1}^{n} v_{ki}^{\oplus} \otimes v_{kj}^{\oplus} \otimes q_{ij}{}^{\otimes^{-1}} \oplus \bigoplus_{k=1}^{n} v_{ki}^{\ominus} \otimes v_{kj}^{\ominus} \otimes q_{ij}{}^{\otimes^{-1}} = 0 \tag{7.51}$$

$$\bigoplus_{k=1}^{n} v_{ki}^{\oplus} \otimes v_{kj}^{\ominus} \otimes q_{ij}{}^{\otimes^{-1}} \oplus \bigoplus_{k=1}^{n} v_{ki}^{\ominus} \otimes v_{kj}^{\oplus} \otimes q_{ij}{}^{\otimes^{-1}} = 0 \tag{7.52}$$

for $i = 1, 2, \ldots, n$ and $j = i+1, i+2, \ldots, n$;

$$\bigoplus_{k=1}^{n} v_{ki}^{\oplus} \oplus \bigoplus_{k=1}^{n} v_{ki}^{\ominus} = 0 \qquad \text{for } i = 1, 2, \ldots, n \tag{7.53}$$

and $q_{ii} = 0$ for $i = 1, 2, \ldots, n$.
The condition $\sigma_1 \leqslant \|A\|_\oplus$ can be rewritten as

$$\|A\|_\oplus \otimes \sigma_1{}^{\otimes^{-1}} \geqslant 0 \ . \tag{7.54}$$

Finally, we order the max-algebraic singular values by requiring that $\sigma_i \geqslant \sigma_{i+1}$ for $i = 1, 2, \ldots, r-1$ or equivalently

$$\sigma_i \otimes (\sigma_{i+1})^{\otimes^{-1}} \geqslant 0 \qquad \text{for } i = 1, 2, \ldots, r-1 . \tag{7.55}$$

Expressions (7.34) – (7.35) and (7.45) – (7.55) constitute a system of multivariate max-algebraic polynomial equalities and inequalities. By using the technique explained in Section 4.1 and by taking Remark 4.1.7 into account we can transform them into an ELCP. So we can use the ELCP algorithm to find all the solutions of the system of multivariate max-algebraic polynomial equalities and inequalities (7.34) – (7.35), (7.45) – (7.55).

The resulting homogeneous ELCP has $\frac{5}{2}(m^2+n^2) - \frac{1}{2}(m+n) + mn + r + 1$ variables: the max-positive and the max-negative parts of the entries of U and V, the diagonal entries of Σ, the entries of T, the entries of the upper triangular part of P and Q and an extra variable α to make the ELCP homogeneous. The number of inequalities of this ELCP is equal to $2mn(2r+1) + 2m^3 + 2n^3 + m^2 + n^2 + r + 1 - n_\varepsilon$ where n_ε is the total number of entries of $A^\oplus$ and $A^\ominus$ that are equal to ε. So if A is an n by n matrix and if all the entries of A are finite and signed (which implies that $n_\varepsilon = n^2$), we have an ELCP with $6n^2 + 1$ variables and $8n^3 + 3n^2 + n + 1$ inequalities.
Recall that any arbitrary solution of an ELCP corresponds to the sum of a linear combination of the central generators, a nonnegative combination of cross-complementary extreme generators and a convex combination of cross-complementary finite points that are also cross-complementary with these extreme generators. Since the max-algebraic singular values are bounded from above by $\|A\|_\oplus$ and since the max-absolute values of the components of the max-algebraic singular vectors are bounded from above by 0, there are no central generators in the solution set of the ELCP that corresponds to the system of multivariate max-algebraic polynomial equalities and inequalities (7.34) – (7.35), (7.45) – (7.55). Furthermore, since all the components of the solutions are bounded from above, the components of the extreme generators are less than or equal to 0. Therefore, the finite points that result from the ELCP algorithm always correspond to a maximal max-algebraic SVD of A.
Since every matrix with finite entries has at least one max-algebraic SVD with finite singular values and finite singular vectors by Proposition 7.5.1, the solution set of the ELCP that corresponds to the system (7.34) – (7.35), (7.45) – (7.55) cannot be empty.

Remark 7.5.3 As explained in Remark 4.1.7 equations of the form $x_i^\oplus \otimes x_i^\ominus = \varepsilon$ will be replaced by equations of the form

$$x_i^\boxplus \otimes x_i^\boxminus \leqslant -\xi \tag{7.56}$$

where ξ is a (large) positive number and where $x_i^\boxplus$ and $x_i^\boxminus$ are "finite approximations" of $x_i^\oplus$ and $x_i^\ominus$ respectively. Note that we cannot use $x_i^\oplus$ and $x_i^\ominus$ any

more in (7.56) since if x_i is signed, we always have $x_i^{\oplus} \otimes x_i^{\ominus} = \varepsilon$, which means that $x_i^{\oplus} = \varepsilon$ or $x_i^{\ominus} = \varepsilon$. However, we still have $x_i = x_i^{\boxplus} \ominus x_i^{\boxminus}$ provided that ξ is large enough.
As explained in Remark 4.1.7 we can use a limit or a threshold technique to obtain the solutions of the original system of multivariate max-algebraic polynomial equalities and inequalities (7.34) – (7.35), (7.45) – (7.55). Since the solutions of systems of multivariate max-algebraic polynomial equalities and inequalities that arise from max-algebraic SVDs will always be bounded from above, there will be no solutions with positive components of the same order of magnitude as ξ if ξ is large enough. So we shall never create solutions with components that are equal to ∞ if we use the limit or the threshold technique. $\diamond$

Remark 7.5.4 If some of the entries of A are not finite, we can use the same technique as in Section 4.2.2 and replace the infinite components of A by $(-\xi)^{\bullet}$ and then see how the solution set of the resulting ELCP evolves as ξ tends to ∞. Since the entries of U, V and Σ are bounded from above, this procedure will never result in solutions with components that are equal to ∞. Furthermore, since the entries of T, P and Q are dummy variables that do not appear in the conditions $A \nabla U \otimes \Sigma \otimes V^T$, $U^T \otimes U \nabla E_m$, $V^T \otimes V \nabla E_n$ and $\|A\|_{\oplus} \geqslant \sigma_1 \geqslant \sigma_2 \geqslant \ldots \geqslant \sigma_r$, there cannot be any problems arising from taking negative max-algebraic powers of ε. $\diamond$

The time and the memory space needed to solve an ELCP with the algorithm described in Section 3.4 increases rapidly as the number of variables and inequalities increases. Therefore, it is advantageous to reduce the number of variables and inequalities as much as possible.
Let $A \in \mathbb{S}^{m \times n}$. If there is a signed entry in A that is equal to $\|A\|_{\oplus}$ in max-absolute value then $\sigma_1 = \|A\|_{\oplus}$ by Proposition 7.4.6. So in that case we do not have to consider σ_1 as an unknown.
Let $U \otimes \Sigma \otimes V^T$ be a max-algebraic SVD of A, let $i \in \{1, 2, \ldots, m\}$ and let $r = \min(m, n)$. Let us now see what happens if we replace u_i (the ith column of U) by $\ominus u_i$. Obviously, we still have $U^T \otimes U \nabla E_m$. The balance $A \nabla U \otimes \Sigma \otimes V^T$ can be rewritten as

$$A \nabla \bigoplus_{i=1}^{r} \sigma_i \otimes u_i \otimes v_i^T \quad . \tag{7.57}$$

If we also replace v_i by $\ominus v_i$ if $i \leqslant r$ then (7.57) still holds. So we still have a max-algebraic SVD of A. This means that we can reduce the number of variables and inequalities of the ELCP that is used to determine a max-algebraic SVD of A by requiring that the diagonal entries of U (or V, depending on which one has the largest dimension) are max-positive or max-zero: $u_{ii}^{\ominus} = \varepsilon$ for $i = 1, 2, \ldots, m$.

It is obvious that the max-algebraic QRD of a matrix $A \in \mathbb{S}^{m \times n}$ can also be computed using the ELCP technique. It is easy to verify that in this case we can also reduce the number of variables by requiring that the diagonal entries of Q belong to $\mathbb{S}^{\oplus}$.

Now we use the ELCP approach to compute all the max-algebraic SVDs of the matrix of Example 7.3.9. An example of the use of the ELCP approach to compute all the max-algebraic QRDs of a given matrix can be found in Section D.4.

Example 7.5.5 Consider

$$A = \begin{bmatrix} \ominus 0 & 4 \\ 1 & \ominus 5 \end{bmatrix} .$$

Note that $A_{.,1}$ and $A_{.,2}$ are max-linearly dependent since $A_{.,2} = \ominus 4 \otimes A_{.,1}$.
In Example 7.3.9 we have used the mapping $\mathcal{F}$ to compute a max-algebraic SVD of A. Now we use the transformation into a system of multivariate max-algebraic polynomial equalities and inequalities to compute all the max-algebraic SVDs of A. We reduce the number of variables by requiring that the diagonal entries of U belong to $\mathbb{S}^{\oplus}$.
We apply the technique of Remark 7.5.3 and we introduce new variables $u_{ij}^{\boxplus}$ and $u_{ij}^{\boxminus}$ for $i = 1, 2$ and $j = 1, 2$ with $i \neq j$ such that $u_{ij}^{\boxplus} \otimes u_{ij}^{\boxminus} \leqslant -\xi$ for all i, j with $i \neq j$ where ξ is a (large) positive number. In a similar way we also define $v_{ij}^{\boxplus}$ and $v_{ij}^{\boxminus}$ for $i = 1, 2$ and $j = 1, 2$. Since u_{11}, $u_{22} \in \mathbb{S}^{\oplus}$, we have $u_{11}^{\oplus} = u_{11}$, $u_{22}^{\oplus} = u_{22}$ and $u_{11}^{\ominus} = u_{22}^{\ominus} = \varepsilon$. Therefore, we do not have to replace $u_{11}^{\oplus}$, $u_{11}^{\ominus}$, $u_{22}^{\oplus}$ and $u_{22}^{\ominus}$ by new variables. We put all the variables in one large column vector x:

$$x = \begin{bmatrix} \sigma_1 & \sigma_2 & u_{11} & u_{12}^{\boxplus} & u_{21}^{\boxplus} & u_{22} & u_{12}^{\boxminus} & u_{21}^{\boxminus} & v_{11}^{\boxplus} & v_{12}^{\boxplus} & v_{21}^{\boxplus} & v_{22}^{\boxplus} \end{bmatrix.$$
$$\begin{matrix} v_{11}^{\boxminus} & v_{12}^{\boxminus} & v_{21}^{\boxminus} & v_{22}^{\boxminus} & t_{11} & t_{12} & t_{21} & t_{22} & p_{12} & q_{12} \end{matrix} \,]^T \ .$$

Note that p_{11}, p_{22}, q_{11}, q_{22}, p_{21} and q_{21} are not considered as unknowns since we already know that $p_{11} = p_{22} = q_{11} = q_{22} = 0$ and $p_{21} = p_{12}$ and $q_{21} = q_{12}$.
If we set ξ equal to 1000, the ELCP algorithm of Section 3.4 yields the generators and the finite points of Table 7.1 and the pairs of maximal cross-complementary subsets of Table 7.2. There are no central generators.
Let us now use the threshold technique to recover the solutions of the original system of multivariate max-algebraic polynomial equalities and inequalities $\mathcal{S}(\infty)$ that describes that set of the max-algebraic SVDs $U \otimes \Sigma \otimes V^T$ of A for which the diagonal entries of U belong to $\mathbb{S}^{\oplus}$. Consider the generator x_1^{f}. The components $u_{21}^{\boxplus}$, $u_{12}^{\boxminus}$, $v_{11}^{\boxplus}$, $v_{12}^{\boxminus}$, $v_{21}^{\boxminus}$ and $v_{22}^{\boxminus}$ are negative numbers of the same order of magnitude as ξ. These components are not bounded from below since the extreme generators x_2^{e}, x_3^{e}, x_4^{e}, x_7^{e}, x_8^{e}, x_9^{e} appear in the same ordered pair of maximal cross-complementary subsets $(\mathcal{X}_1^{\mathrm{ext}}, \mathcal{X}_1^{\mathrm{fin}})$ as x_1^{f}. There are no positive components of the same order of magnitude as ξ in x_1^{f} (This was to be expected since the max-algebraic singular values are bounded from above by $\|A\|_{\oplus} = 5$ and since the max-absolute values of the components of the max-algebraic singular vectors are bounded from above by 0). Moreover, the exponents of $u_{21}^{\boxplus}$,

	$\mathcal{X}^{\text{ext}}$									$\mathcal{X}^{\text{fin}}$	
	x_1^{e}	x_2^{e}	x_3^{e}	x_4^{e}	x_5^{e}	x_6^{e}	x_7^{e}	x_8^{e}	x_9^{e}	x_1^{f}	x_2^{f}
σ_1	0	0	0	0	0	0	0	0	0	5	5
σ_2	-1	0	0	0	0	0	0	0	0	0	0
u_{11}	0	0	0	0	0	0	0	0	0	-1	-1
$u_{12}^{\boxplus}$	0	0	0	0	0	0	0	0	0	0	0
$u_{21}^{\boxplus}$	0	-1	0	0	0	0	0	0	0	-1000	-1000
u_{22}	0	0	0	0	0	0	0	0	0	-1	-1
$u_{12}^{\boxminus}$	0	0	-1	0	0	0	0	0	0	-1000	-1000
$u_{21}^{\boxminus}$	0	0	0	0	0	0	0	0	0	0	0
$v_{11}^{\boxplus}$	0	0	0	-1	0	0	0	0	0	-996	-996
$v_{12}^{\boxplus}$	0	0	0	0	-1	0	0	0	0	0	-1000
$v_{21}^{\boxplus}$	0	0	0	0	0	0	0	0	0	0	0
$v_{22}^{\boxplus}$	0	0	0	0	0	-1	0	0	0	-4	-996
$v_{11}^{\boxminus}$	0	0	0	0	0	0	0	0	0	-4	-4
$v_{12}^{\boxminus}$	0	0	0	0	0	0	-1	0	0	-1000	0
$v_{21}^{\boxminus}$	0	0	0	0	0	0	0	-1	0	-1000	-1000
$v_{22}^{\boxminus}$	0	0	0	0	0	0	0	0	-1	-996	-4
t_{11}	0	0	0	0	0	0	0	0	0	0	0
t_{12}	0	0	0	0	0	0	0	0	0	4	4
t_{21}	0	0	0	0	0	0	0	0	0	1	1
t_{22}	0	0	0	0	0	0	0	0	0	5	5
p_{12}	0	0	0	0	0	0	0	0	0	-1	-1
q_{12}	0	0	0	0	0	0	0	0	0	-4	-4

Table 7.1: The generators and the finite points of the ELCP of Example 7.5.5 for $\xi = 1000$.

s	$\mathcal{X}_s^{\text{ext}}$	$\mathcal{X}_s^{\text{fin}}$
1	$\{x_1^{\text{e}}, x_2^{\text{e}}, x_3^{\text{e}}, x_4^{\text{e}}, x_7^{\text{e}}, x_8^{\text{e}}, x_9^{\text{e}}\}$	$\{x_1^{\text{f}}\}$
2	$\{x_1^{\text{e}}, x_2^{\text{e}}, x_3^{\text{e}}, x_4^{\text{e}}, x_5^{\text{e}}, x_6^{\text{e}}, x_8^{\text{e}}\}$	$\{x_2^{\text{f}}\}$

Table 7.2: The pairs of maximal cross-complementary subsets of the sets $\mathcal{X}^{\text{ext}}$ and $\mathcal{X}^{\text{fin}}$ of Example 7.5.5 for $\xi = 1000$.

$u_{12}^{\boxminus}$, $v_{11}^{\boxplus}$, $v_{12}^{\boxminus}$, $v_{21}^{\boxminus}$ and $v_{22}^{\boxminus}$ in the system $\mathcal{S}(\infty)$ are nonnegative. Therefore, we replace these entries by ε (cf. Remark 4.1.7). This results in

$$U_1 = \begin{bmatrix} -1 & 0 \\ \ominus 0 & -1 \end{bmatrix}, \ \Sigma_1 = \begin{bmatrix} 5 & \varepsilon \\ \varepsilon & 0 \end{bmatrix} \text{ and } V_1 = \begin{bmatrix} \ominus(-4) & 0 \\ 0 & -4 \end{bmatrix} . \qquad (7.58)$$

Note that we can also obtain this solution by using the fact that

$$u_{ij} = u_{ij}^{\boxplus} \ominus u_{ij}^{\boxminus} = \begin{cases} u_{ij}^{\boxplus} & \text{if } u_{ij}^{\boxplus} > u_{ij}^{\boxminus} , \\ u_{ij}^{\boxminus} & \text{if } u_{ij}^{\boxminus} > u_{ij}^{\boxplus} , \end{cases}$$

for all i, j if ξ is large enough, and an analogous expression for the v_{ij}'s. Since

$$U_1 \otimes \Sigma_1 \otimes V_1^T = \begin{bmatrix} 0^{\bullet} & 4 \\ 1 & \ominus 5 \end{bmatrix} \nabla A$$

$$U_1^T \otimes U_1 = \begin{bmatrix} 0 & (-1)^{\bullet} \\ (-1)^{\bullet} & 0 \end{bmatrix} \nabla E_2$$

$$V_1^T \otimes V_1 = \begin{bmatrix} 0 & (-4)^{\bullet} \\ (-4)^{\bullet} & 0 \end{bmatrix} \nabla E_2 ,$$

$U_1 \otimes \Sigma_1 \otimes V_1^T$ really is a max-algebraic SVD of A. So the value that we have chosen for ξ was large enough.

Since the extreme generator x_1^{e} belongs to the set $\mathcal{X}_1^{\mathrm{ext}}$, we may replace $(\Sigma_1)_{22}$ by any nonpositive real number or by ε. This implies that

$$U \otimes \Sigma \otimes V^T = \begin{bmatrix} -1 & 0 \\ \ominus 0 & -1 \end{bmatrix} \otimes \begin{bmatrix} 5 & \varepsilon \\ \varepsilon & \sigma \end{bmatrix} \otimes \begin{bmatrix} \ominus(-4) & 0 \\ 0 & -4 \end{bmatrix}^T \qquad (7.59)$$

is a max-algebraic SVD of A for every $\sigma \in \mathbb{R}_\varepsilon$ with $\sigma \leqslant 0$.

The decomposition $U_2 \otimes \Sigma_2 \otimes V_2^T$ that corresponds to x_2^{f} can be obtained from decomposition (7.58) by replacing $(V_1)_{.,2}$ by $\ominus(V_1)_{.,2}$. Note that we may also replace $(\Sigma_2)_{22}$ by any nonpositive real number or by ε.

The set of all the max-algebraic SVDs of A can be obtained from $U_1 \otimes \Sigma_1 \otimes V_1^T$ and $U_2 \otimes \Sigma_2 \otimes V_2^T$ by replacing σ_2 by a nonpositive real number or by ε; by replacing the left singular vector u_1 by $\ominus u_1$ and the right singular vector v_1 by $\ominus v_1$; by replacing the left singular vector u_2 by $\ominus u_2$ and the right singular vector v_2 by $\ominus v_2$; or by a combination of these replacements.

Note that $\sigma_1 = 5 = \|A\|_{\oplus}$ for all the max-algebraic SVDs of A (cf. Proposition 7.4.6).

Taking $\sigma = \varepsilon$ in (7.59) yields a minimal max-algebraic SVD of A. Since $\sigma_1 = 5$ and $\sigma_2 = \varepsilon$ for all the minimal max-algebraic SVDs of A, we have $\mathrm{rank}_{\oplus,\mathrm{SVD}}(A) = 1$.

If we set $\sigma = \sigma_{\max,2} = 0$ in (7.59), we obtain again (7.58) which is thus a maximal max-algebraic SVD of A. For decomposition (7.59) we have

$$\sigma_1 \otimes u_1 \otimes v_1^T = \begin{bmatrix} \ominus 0 & 4 \\ 1 & \ominus 5 \end{bmatrix}$$

$$\sigma_{\max,2} \otimes u_2 \otimes v_2^T = \begin{bmatrix} 0 & -4 \\ -1 & -5 \end{bmatrix} .$$

Note that the max-absolute value of each entry of $\sigma_{\max,2} \otimes u_2 \otimes v_2^T$ is less than or equal to the max-absolute value of the corresponding entry of $\sigma_1 \otimes u_1 \otimes v_1^T$. The solution of Example 7.3.9 can be obtained by taking $\sigma = \varepsilon$ in (7.59) and by replacing u_1 by $\ominus u_1$ and v_1 by $\ominus v_1$. □

7.6 Conclusion

First we have established a link between the ring $(\mathcal{R}_e, +, \cdot)$ and the symmetrized max-plus algebra. We have used this link to introduce the max-complex structure $\mathbb{T}_{\max}$, which can be considered as a further extension of the max-plus algebra. Next we have introduced the class $\mathcal{S}_e$ of functions that are analytic and that can be written as a sum or a series of exponentials in a neighborhood of ∞. This class is closed under basic operations such as additions, subtractions, multiplications, divisions, powers, square roots and absolute values. This fact has then been used to prove the existence of a singular value decomposition (SVD) and a max-algebraic QR decomposition (QRD) of a matrix in the symmetrized max-plus algebra. These decompositions are max-algebraic analogues of basic matrix decompositions from linear algebra that are used in many contemporary algorithms for the identification of linear systems. It is obvious that the proof technique that has been used to prove the existence of the max-algebraic SVD and the max-algebraic QRD can also be used to prove the existence of max-algebraic analogues of many other real matrix decompositions from linear algebra such as e.g. the eigenvalue decomposition for symmetric matrices, the LU decomposition, the Schur decomposition, the Hessenberg decomposition and so on.

We have defined a rank based on the max-algebraic SVD and indicated how the max-algebraic SVD might be used in the identification of max-linear time-invariant discrete event systems. Finally we have used the fact that a system of multivariate max-algebraic polynomial equalities and inequalities can be transformed into an ELCP to derive a method to compute all max-algebraic singular value decompositions of a matrix. The ELCP technique can also be used to compute max-algebraic QRDs and other max-algebraic matrix decompositions.

Topics for future research are: further investigation of the properties of the max-algebraic SVD and the max-algebraic QRD, development of efficient algorithms to compute a (minimal) max-algebraic SVD of a matrix, and application of the max-algebraic SVD and other matrix decompositions in the system theory for max-linear time-invariant discrete event systems. Furthermore, it is obvious that many other decompositions and properties of matrices in linear algebra also have a max-algebraic analogue, especially if we make use of the correspondence between $(\mathcal{C}_e, +, \cdot)$ and $\mathbb{T}_{\max}$. This will also be a topic for further research.

Chapter 8

General Conclusions

8.1 Overview of the Contributions

In this thesis we have discussed some topics that were all related in one way or another to the minimal state space realization problem for max-linear time-invariant discrete event systems (DESs). One of the foundations of the research that has been described in this thesis is the analogy between state space descriptions for linear time-invariant systems on the one hand and state space descriptions for max-linear time-invariant DESs on the other hand. However, if we compare linear system theory and max-algebraic system theory for DESs, then we immediately notice that there are not yet max-algebraic analogues or alternatives for many algorithms and procedures that are frequently used in linear system theory.

In this thesis we have developed some tools that can be used in the max-algebraic system theory for DESs. We have shown that many fundamental max-algebraic problems that arise when one wants to further develop the max-algebraic system theory for DESs can be cast as a mathematical programming problem, viz. the Extended Linear Complementarity Problem. This has allowed us to derive a method to solve the minimal state space realization problem for max-linear time-invariant DESs, which can be considered as one of the fundamental problems in max-algebraic system theory for DESs.

Let us now briefly review the main contributions of this thesis.

In Chapter 3 we have presented the Extended Linear Complementarity Problem (ELCP) and established a link between the ELCP and other linear complementarity problems. We have shown that the ELCP can be considered as a unifying framework for the Linear Complementarity Problem and its generalizations. Furthermore, we have made a thorough study of the solution set of the general ELCP and developed an algorithm to find all the solutions of an ELCP. We have presented the results of some experiments that show how factors such as the number of variables, the number of inequalities, the number of extreme generators and the order in which the inequalities are processed influence the execution time of our ELCP algorithm. We have also shown that the ELCP is an NP-hard problem.

In this thesis we have shown that the ELCP can be considered as a powerful framework for describing a large class of max-algebraic problems: In Chapter 4 we have proved that the problem of finding all finite solutions of a system of multivariate max-algebraic polynomial equalities and inequalities is equivalent to an ELCP. In Chapters 4, 5, 6 and 7 we have used this result to show that many problems in the max-plus algebra, the symmetrized max-plus algebra and the max-min-plus algebra such as calculating max-algebraic matrix factorizations, solving systems of (homogeneous) max-linear balances, performing max-algebraic state space transformations, determining partial or minimal state space realizations of the impulse response of a max-linear time-invariant DES, constructing matrices with a given max-algebraic characteristic polynomial, determining max-algebraic singular value decompositions or max-algebraic QR decompositions of a given matrix, mixed max-min problems, max-max and max-min problems can be reformulated as an ELCP or solved using the ELCP approach.

In Chapter 5 we have derived necessary conditions for the coefficients of the max-algebraic characteristic polynomial of a matrix with entries in $\mathbb{R}_\varepsilon$. For square matrices with entries in $\mathbb{R}_\varepsilon$ and with a dimension that is less than or equal to 4 we have also derived necessary and sufficient conditions for the coefficients of the max-algebraic characteristic polynomial. If we have a max-algebraic polynomial with a degree that is less than or equal to 4, these results allow us to check whether the given max-algebraic polynomial can be the max-algebraic characteristic polynomial of a matrix with entries in $\mathbb{R}_\varepsilon$, and to construct such a matrix, if it exists. We have also shown that in theory we can use the ELCP approach to solve the problem of constructing a matrix with a given max-algebraic characteristic polynomial.

In Chapter 6 we have presented transformations that enable us to find equivalent max-algebraic state space realizations of the input-output behavior or the impulse response of a max-linear time-invariant DES. Next we have used our results on the coefficients of the max-algebraic characteristic polynomial of a matrix with entries in $\mathbb{R}_\varepsilon$ to derive a procedure to determine the minimal system order of a max-linear time-invariant DES starting from its impulse response. We have shown that we can use the ELCP approach to compute all fixed order partial state space realizations of a given impulse response. Furthermore, we can also use the ELCP approach to determine all minimal state space realizations of a given impulse response.

In Chapter 7 we have established a link between a ring of real functions with conventional addition and multiplication as basic operations and the symmetrized max-plus algebra. We have used this link to introduce the max-complex structure $\mathbb{T}_{\max}$ and to prove the existence of the max-algebraic singular value decomposition (SVD) and the max-algebraic QR decomposition (QRD), which can be considered as the max-algebraic analogues of two basic matrix decompositions from linear algebra that arise in many algorithms for solving fundamental problems in linear system theory. We have studied some properties of the max-algebraic SVD and we have indicated how the max-algebraic SVD might be used in the identification of max-linear time-invariant DESs. We have also shown that the problem of determining a max-algebraic SVD or a max-algebraic QRD of a matrix can be solved using the ELCP approach.

Finally we want to remark that for max-linear time-invariant DESs we can also write down a time-invariant state space model that is "linear" in the min-plus-algebra (See e.g. [3, 26]). The basic operations of this algebra are minimization and addition. Since minimization and maximization can be considered as dual operations, it is obvious that all the results for max-linear time-invariant models that have been obtained in this thesis can easily be adapted for min-linear time-invariant models.

8.2 Open Problems and Suggestions for Further Research

There are still many open problems left and many topics require further investigation:

We have not yet found general necessary and sufficient conditions for the coefficients of the max-algebraic characteristic polynomial of a square matrix of arbitrary size with entries in $\mathbb{R}_\varepsilon$.

In connection with max-algebraic state space transformations it is still an open question whether there exist transformations that provide a link between two arbitrary state space realizations of the input-output behavior or the impulse response of a given max-linear time-invariant DES.

We do not know how to determine a minimal set of Markov parameters such that any minimal state space realization of this minimal set is also a minimal realization of the entire impulse response (without using an enumerative approach). Furthermore, if the sequence $\{\mathcal{R}_{\min}(G,N)\}_{N=1}^{\infty}$ of the sets of the minimal state space realizations of the first N Markov parameters of a given impulse response G becomes stationary from a certain index N_0 on, it is still an open question whether and how we can determine this index without explicitly computing the terms of the sequence $\{\mathcal{R}_{\min}(G,N)\}_{N=1}^{\infty}$.

The technique that has been used to prove the existence of the max-algebraic SVD and the max-algebraic QRD can also be used to prove the existence of a max-algebraic analogue of many other matrix decompositions from linear algebra such as e.g. the eigenvalue decomposition for symmetric matrices, the LU decomposition, the Schur decomposition and the Hessenberg decomposition. The properties of these max-algebraic matrix decompositions certainly deserve further investigation. We should also determine whether and how these decompositions can be used in max-algebraic system theory, especially in connection with the minimal state space realization problem and the identification problem for max-linear time-invariant DESs. Furthermore, we have to develop efficient algorithms to compute max-algebraic SVDs and other max-algebraic matrix decompositions of a given matrix. The properties of the structure $\mathbb{T}_{\max}$ and max-algebraic matrix decompositions in this extended structure could also be a topic for further research.

In practice we can only solve small-sized ELCPs with our ELCP algorithm since the execution time of this algorithm increases very rapidly as the number of variables and (in)equalities grows. Therefore, there certainly is a need for more efficient algorithms to find one solution of an ELCP. Since the general ELCP is an NP-hard problem, it is very unlikely that we can develop a polynomial time algorithm to solve the ELCP. Nevertheless, we could try to develop efficient algorithms to solve special cases of the ELCP.

In this thesis we have shown that many max-algebraic problems can be reformulated as an ELCP or solved using the ELCP approach. By showing that a problem can be reformulated as an ELCP, we show that in general the set of all the (finite) solutions of the given problem consists of the union of faces of a polyhedron. This insight in the geometrical structure of the solution set of the problem might lead to the development of more efficient algorithms to solve that particular problem. Although we have shown that in general the problem of solving a system of multivariate max-algebraic polynomial equalities and inequalities is NP-hard, it may also be interesting to determine which subclasses of this general problem can be solved by polynomial time algorithms. Furthermore, it is still an open question whether problems such as determining a partial or a minimal state space realization of the impulse response of a max-linear time-invariant DES, constructing a matrix with a given max-algebraic characteristic polynomial, determining a max-algebraic singular value decomposition or a max-algebraic QR decomposition of a given matrix, and so on are also NP-hard.

Since only certain subclasses of DESs can be described by a max-linear time-invariant state space model, we should also investigate how and whether the results that have been presented in this thesis can be generalized to DESs that cannot be described by a max-linear or a time-invariant model such as DESs with variable or stochastic processing and transportation times, or with variable routing. It is obvious that the complexity of the basic problems of the system theory for this kind of DESs will even be higher than the complexity of the basic problems of the system theory for max-linear time-invariant DESs.

This complexity appears to be one of the inherent characteristics of DESs and it also explains why there is such a wide gap between theory and practice in the field of DESs: in almost all frameworks and methodologies to model, to analyze or to control DESs many basic problems are characterized by the fact that the execution time of even the most efficient current algorithms to solve the problem increases exponentially as the size of the problem increases. This brings us back to the trade-off that was already mentioned in the introduction of this thesis: the more accurate the model is, the less we can analytically say about its properties and the more complex the problems related to this model are. This trade-off is one of the reasons why at present there is such a wide gap between mathematical theories for DESs on the one hand, and the use and the control of DESs in practice on the other hand: the problems for which the execution time of the most efficient current algorithms to solve the problem increases exponentially as the size of the problem increases are precisely those problems for which industry needs solutions and vice versa. Developing methods to deal with the inherent complexity of the problems that appear in connection with the modeling, analyzing and controlling of DESs is certainly one of the major challenges for DES research.

Our work clearly reveals the need for efficient algorithms to solve some basic max-algebraic problems. We hope that our research will show directions and possible approaches to develop such algorithms. Moreover, it should be clear that much research is still needed in order to get a complete system theory for (max-linear time-invariant) DESs that can be compared with the system theory for (linear) time-driven systems. We hope that with this work we have made a contribution to the enhancement of the max-algebraic system theory for DESs.

Appendix A

The Signed Version of the Max-Algebraic Characteristic Equation

In this appendix we discuss an alternative version of the max-algebraic characteristic equation of a matrix that has been introduced by Olsder and Roos [115, 116, 125, 126]. We show that the derivation of Olsder and Roos is not entirely correct and we give the correct formulas for the coefficients of this alternative version of the max-algebraic characteristic equation. We also give a counterexample for a conjecture of Olsder [115] in which he states necessary and sufficient conditions for the coefficients of the alternative version of the max-algebraic characteristic equation of a matrix with entries in $\mathbb{R}_\varepsilon$.

A.1 The Signed Version of the Max-Algebraic Characteristic Equation of a Matrix with Entries in $\mathbb{R}_\varepsilon$

In this section we recapitulate the reasoning Olsder and Roos have used in [126] to derive another version of the max-algebraic characteristic equation of a matrix with entries in $\mathbb{R}_\varepsilon$ (See also [3]). We show where their reasoning goes wrong and how it can be corrected.

If $A \in \mathbb{R}_\varepsilon^{n \times n}$ then z^A is a real n by n matrix-valued function with domain of definition $\mathbb{R}_0^+$ that is defined by $(z^A)_{ij} = z^{a_{ij}}$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n$. By definition we have $z^\varepsilon = 0$ for all $z \in \mathbb{R}_0^+$. Note that this mapping is closely related to the mapping $\mathcal{F}$ that has been introduced in Section 7.2.

The *dominant* of A is defined as follows:

$$\mathrm{dom}_{\oplus} A = \begin{cases} \text{the highest exponent in } \det z^A & \text{if } \det z^A \neq 0\,, \\ \varepsilon & \text{otherwise.} \end{cases}$$

The characteristic polynomial of the matrix-valued function z^A (in conventional linear algebra) is given by

$$\det\left(\lambda(z) I_n - z^A\right) = \lambda^n(z) + \gamma_1(z)\lambda^{n-1}(z) + \ldots + \gamma_{n-1}(z)\lambda(z) + \gamma_n(z)$$

with

$$\gamma_k(z) = (-1)^k \sum_{\varphi \in \mathcal{C}_n^k} \det z^{A_{\varphi\varphi}} \quad . \tag{A.1}$$

The Cayley-Hamilton theorem applied to the matrix-valued function z^A yields

$$(z^A)^n + \gamma_1(z)(z^A)^{n-1} + \ldots + \gamma_{n-1}(z)(z^A) + \gamma_n(z) I_n = 0 \tag{A.2}$$

for all $z \in \mathbb{R}_0^+$. In [126] Olsder and Roos claim that the highest degree in (A.1) is equal to $\max\{\mathrm{dom}_{\oplus} A_{\varphi\varphi} \,|\, \varphi \in \mathcal{C}_n^k\}$ and that

$$\gamma_k(z) \;\sim\; (-1)^k \,\bar{\gamma}_k\, z^{\max\left\{\mathrm{dom}_{\oplus} A_{\varphi\varphi} \,\middle|\, \varphi \in \mathcal{C}_n^k\right\}}\ ,\ z \rightarrow \infty$$

where $\bar{\gamma}_k$ is equal to the number of even permutations that contribute to the highest degree in (A.1) minus the number of odd permutations that contribute to the highest degree. However, the highest degree in (A.1) is not necessarily equal to $\max\left\{\mathrm{dom}_{\oplus} A_{\varphi\varphi} \,\middle|\, \varphi \in \mathcal{C}_n^k\right\}$, since if the number of even permutations that contribute to $z^{\max\left\{\mathrm{dom}_{\oplus} A_{\varphi\varphi} \,\middle|\, \varphi \in \mathcal{C}_n^k\right\}}$ is equal to the number of odd permutations that contribute to $z^{\max\left\{\mathrm{dom}_{\oplus} A_{\varphi\varphi} \,\middle|\, \varphi \in \mathcal{C}_n^k\right\}}$, the term $z^{\max\left\{\mathrm{dom}_{\oplus} A_{\varphi\varphi} \,\middle|\, \varphi \in \mathcal{C}_n^k\right\}}$ disappears. We shall illustrate this with an example:

Example A.1.1 Consider again the matrix A of Example 2.2.6:

$$A = \begin{bmatrix} -2 & 1 & \varepsilon \\ 1 & 0 & 1 \\ \varepsilon & 0 & 2 \end{bmatrix} .$$

The matrix-valued function z^A is given by

$$z^A = \begin{bmatrix} z^{-2} & z & 0 \\ z & 1 & z \\ 0 & 1 & z^2 \end{bmatrix} .$$

We have

$$\det z^{A_{\{1,2\},\{1,2\}}} = \begin{bmatrix} z^{-2} & z \\ z & 1 \end{bmatrix} = z^{-2} - z^2$$

$$\det z^{A_{\{1,3\},\{1,3\}}} = \begin{bmatrix} z^{-2} & 0 \\ 0 & z^2 \end{bmatrix} = 1$$

$$\det z^{A_{\{2,3\},\{2,3\}}} = \begin{bmatrix} 1 & z \\ 1 & z^2 \end{bmatrix} = z^2 - z \enspace .$$

Hence, $\mathrm{dom}_\oplus\, A_{\{1,2\},\{1,2\}} = 2$, $\mathrm{dom}_\oplus\, A_{\{1,3\},\{1,3\}} = 0$ and $\mathrm{dom}_\oplus\, A_{\{2,3\},\{2,3\}} = 2$. However,

$$\begin{aligned}\gamma_2(z) &= (-1)^2 \left(\det z^{A_{\{1,2\},\{1,2\}}} + \det z^{A_{\{1,3\},\{1,3\}}} + \det z^{A_{\{2,3\},\{2,3\}}}\right) \\ &= z^{-2} - z^2 + 1 + z^2 - z \;=\; -z + 1 + z^{-2} \enspace .\end{aligned}$$

So the highest degree in $\gamma_2(z)$ is equal to 1 whereas $\max\{ \mathrm{dom}_\oplus\, A_{\varphi\varphi} \,|\, \varphi \in \mathcal{C}_3^2 \} = 2 \neq 1$. □

The highest degree term in (A.1) can be determined as follows. Define

$$\Gamma_k = \left\{ \xi \;\middle|\; \exists \{i_1, i_2, \ldots, i_k\} \in \mathcal{C}_n^k,\, \exists \sigma \in \mathcal{P}_k \text{ such that } \xi = \sum_{r=1}^{k} a_{i_r i_{\sigma(r)}} \right\}$$

for $k = 1, 2, \ldots, n$. For every $k \in \{1, 2, \ldots, n\}$ and for every $\xi \in \Gamma_k$ we define

$$I_k^{\mathrm{e}}(\xi) = \#\left\{ \sigma \in \mathcal{P}_{k,\mathrm{even}} \;\middle|\; \exists \{i_1, i_2, \ldots, i_k\} \in \mathcal{C}_n^k \text{ such that } \sum_{r=1}^{k} a_{i_r i_{\sigma(r)}} = \xi \right\}$$

$$I_k^{\mathrm{o}}(\xi) = \#\left\{ \sigma \in \mathcal{P}_{k,\mathrm{odd}} \;\middle|\; \exists \{i_1, i_2, \ldots, i_k\} \in \mathcal{C}_n^k \text{ such that } \sum_{r=1}^{k} a_{i_r i_{\sigma(r)}} = \xi \right\}$$

$$I_k(\xi) = I_k^{\mathrm{e}}(\xi) - I_k^{\mathrm{o}}(\xi) \enspace .$$

Since (A.1) can be rewritten as

$$\begin{aligned}\gamma_k(z) &= (-1)^k \sum_{\{i_1,\ldots,i_k\} \in \mathcal{C}_n^k} \sum_{\sigma \in \mathcal{P}_k} \mathrm{sgn}\,(\sigma) \prod_{r=1}^{k} (z^A)_{i_r i_{\sigma(r)}} \\ &= (-1)^k \sum_{\{i_1,\ldots,i_k\} \in \mathcal{C}_n^k} \sum_{\sigma \in \mathcal{P}_k} \mathrm{sgn}\,(\sigma)\; z^{\left(\sum_{r=1}^{k} a_{i_r i_{\sigma(r)}}\right)} \enspace ,\end{aligned}$$

the highest degree that appears in $\gamma_k(z)$ is given by

$$c_k \stackrel{\mathrm{def}}{=} \max\{ \xi \in \Gamma_k \,|\, I_k(\xi) \neq 0 \}$$

and the coefficients of the characteristic equation of z^A satisfy:

$$\gamma_k(z) \;\sim\; (-1)^k\, I_k(c_k)\, z^{c_k} \;,\; z \to \infty \enspace .$$

Define $\hat{\gamma}_k = (-1)^k I_k(c_k)$ for $k = 1, 2, \ldots, n$. Let $\mathcal{I} = \{\, k \mid \hat{\gamma}_k > 0 \,\}$ and $\mathcal{J} = \{\, k \mid \hat{\gamma}_k < 0 \,\}$. It is easy to verify that we always have $1 \in \mathcal{I}$.
If A is a square matrix with entries in $\mathbb{R}_\varepsilon$ then we have

$$\left(z^A\right)^k \sim z^{\left(A^{\otimes^k}\right)} \;, \; z \to \infty \; . \tag{A.3}$$

As a consequence, (A.2) results in

$$z^{\left(A^{\otimes^n}\right)} + \sum_{k \in \mathcal{I}} \hat{\gamma}_k \, z^{c_k} z^{\left(A^{\otimes^{n-k}}\right)} \sim \sum_{k \in \mathcal{J}} \hat{\gamma}_k \, z^{c_k} z^{\left(A^{\otimes^{n-k}}\right)} \;, \; z \to \infty \; .$$

Since all the terms of this expression have positive coefficients, comparison of the highest degree terms of corresponding entries on the left-hand and the right-hand side of this expression leads to the following identity in $\mathbb{R}_{\max}$:

$$A^{\otimes^n} \oplus \bigoplus_{k \in \mathcal{I}} c_k A^{\otimes^{n-k}} = \bigoplus_{k \in \mathcal{J}} c_k A^{\otimes^{n-k}} \; .$$

This equation can be considered as a max-algebraic version of the Cayley-Hamilton theorem if we would define the max-algebraic characteristic equation of A as

$$\lambda^{\otimes^n} \oplus \bigoplus_{k \in \mathcal{I}} c_k \lambda^{\otimes^{n-k}} = \bigoplus_{k \in \mathcal{J}} c_k \lambda^{\otimes^{n-k}} \; . \tag{A.4}$$

Note that $\mathcal{I} \cap \mathcal{J} = \emptyset$. If we define

$$b_k = \begin{cases} c_k & \text{if } k \in \mathcal{I}\,, \\ \ominus c_k & \text{if } k \in \mathcal{J}, \\ \varepsilon & \text{otherwise,} \end{cases}$$

for $k = 1, 2, \ldots, n$, if we change the equality in (A.4) into a balance and if we move all the terms to the left-hand side, then (A.4) can be rewritten as $\lambda^{\otimes^n} \oplus \bigoplus_{k=1}^{n} b_k \otimes \lambda^{\otimes^{n-k}} \nabla \varepsilon$. Since this balance resembles the max-algebraic characteristic equation of A and since all the coefficients of this balance are signed, we call (A.4) the *signed version of the max-algebraic characteristic equation* of A.

Let us now calculate the signed version of the max-algebraic characteristic equation of the matrix A of Example A.1.1.

Example A.1.2 For the matrix

$$A = \begin{bmatrix} -2 & 1 & \varepsilon \\ 1 & 0 & 1 \\ \varepsilon & 0 & 2 \end{bmatrix}$$

we have $\Gamma_1 = \{2, 0, -2\}$, $\Gamma_2 = \{2, 1, 0, -2, \varepsilon\}$, $\Gamma_3 = \{4, 0, -1, \varepsilon\}$ and

$$\begin{array}{lllll} I_1(2) = 1, & I_1(0) = 1, & I_1(-2) = 1, & & \\ I_2(2) = 0, & I_2(1) = -1, & I_2(0) = 1, & I_2(-2) = 1, & I_2(\varepsilon) = -1, \\ I_3(4) = -1, & I_3(0) = 1, & I_3(-1) = -1, & I_3(\varepsilon) = 1 \ . & \end{array}$$

Hence, $c_1 = 2$, $c_2 = 1$ and $c_3 = 4$. Since $\hat{\gamma}_1 = -1$, $\hat{\gamma}_2 = -1$ and $\hat{\gamma}_3 = 1$, we have $\mathcal{I} = \{3\}$ and $\mathcal{J} = \{1, 2\}$. So the signed version of the max-algebraic characteristic equation of A is:

$$\lambda^{\otimes^3} \oplus 4 = 2 \otimes \lambda^{\otimes^2} \oplus 1 \otimes \lambda \ .$$

Furthermore,

$$A^{\otimes^3} \oplus 4 \otimes E_3 = \begin{bmatrix} 4 & 3 & 4 \\ 3 & 4 & 5 \\ 3 & 4 & 6 \end{bmatrix} = 2 \otimes A^{\otimes^2} \oplus 1 \otimes A \ .$$

So A satisfies the signed version of its max-algebraic characteristic equation. Note that the max-algebraic eigenvalue $\lambda = 2$ that was already calculated in Example 2.2.9 also satisfies the signed version of the max-algebraic characteristic equation of A:

$$2^{\otimes^3} \oplus 4 = 6 \oplus 4 = 6 = 6 \oplus 3 = 2 \otimes 2^{\otimes^2} \oplus 1 \otimes 2 \ . \qquad \square$$

Remarks

1. The signed version of the max-algebraic characteristic equation has only been derived for matrices with entries in $\mathbb{R}_\varepsilon$ since in general (A.3) does not hold for a matrix with entries in $\mathbb{S}$ or in $\mathbb{S}^\vee$ (unless we would redefine the $\oplus$ operator such that terms with equal max-absolute values but with opposite signs would be cancelled).

2. The definition of the coefficients a_k of the max-algebraic characteristic equation of A as given by (5.1) and the coefficients c_k of the signed version of the max-algebraic characteristic equation are similar. Apart from the signs, we could say that the only difference between the a_k's and the c_k's is that terms with equal max-absolute value but with opposite signs are cancelled when we calculate the c_k's. As a consequence, we have

$$\begin{array}{lll} c_k = |a_k|_\oplus \text{ and } k \in \mathcal{I} & & \text{if } a_k \text{ is max-positive,} \\ c_k = |a_k|_\oplus \text{ and } k \in \mathcal{J} & & \text{if } a_k \text{ is max-negative,} \\ c_k \leqslant |a_k|_\oplus & & \text{if } a_k \text{ is balanced.} \end{array}$$

A similar reasoning also leads to $\mathrm{dom}_\oplus A \leqslant |\det_\oplus A|_\oplus$.

3. For the matrix A of Example A.1.1 we have $c_2 = 1$ whereas the dominants of the principal 2 by 2 submatrices of A: $A_{\{1,2\},\{1,2\}}$, $A_{\{1,3\},\{1,3\}}$ and $A_{\{2,3\},\{2,3\}}$ are 2, 0 and 2 respectively. This shows that in general the dominant cannot be used to define the coefficients of the signed version of the max-algebraic characteristic equation.

4. Define $d_k = \max\{\,\xi \,|\, \xi \in \Gamma_k\,\}$ and $\delta_k = I_k(d_k)$ for $k = 1, 2, \ldots, n$. Note that d_k is equal to $\max\{\mathrm{dom}_\oplus A_{\varphi\varphi} \,|\, \varphi \in \mathcal{C}_n^k\,\}$ and that δ_k is equal to $\bar{\gamma}_k$. If we follow the formulas of [3, 126] literally, we should set c_k equal to d_k if $\delta_k = 0$ and put k in $\mathcal{J}$.
   In [115, 116, 125] a derivation that is similar to that of [126] has been presented, but there c_k was set equal to ε if $\delta_k = 0$.
   However, the following example shows that in general neither of these definitions leads to a valid max-algebraic analogue of the Cayley-Hamilton theorem: when we take these definitions for the coefficients of the max-algebraic characteristic equation then there exist matrices that do not satisfy their max-algebraic characteristic equation.

   Example A.1.3 For the matrix

$$A = \begin{bmatrix} \varepsilon & \varepsilon & 0 & \varepsilon \\ 0 & 0 & \varepsilon & \varepsilon \\ 1 & 1 & 1 & \varepsilon \\ 1 & 1 & 1 & -1 \end{bmatrix}$$

   we have $d_1 = 1$, $\delta_1 = 1$, $d_2 = 1$, $\delta_2 = 0$, $d_3 = 1$, $\delta_3 = 0$, $d_4 = 0$ and $\delta_4 = 0$. The definition of [126] would result in

$$\lambda^{\otimes^4} = 1 \otimes \lambda^{\otimes^3} \,\oplus\, 1 \otimes \lambda^{\otimes^2} \,\oplus\, 1 \otimes \lambda \,\oplus\, 0 \ . \tag{A.5}$$

   However, since

$$A^{\otimes^4} = \begin{bmatrix} 3 & 3 & 3 & \varepsilon \\ 2 & 2 & 2 & \varepsilon \\ 4 & 4 & 4 & \varepsilon \\ 4 & 4 & 4 & -4 \end{bmatrix}$$

   and

$$1 \otimes A^{\otimes^3} \,\oplus\, 1 \otimes A^{\otimes^2} \,\oplus\, 1 \otimes A \,\oplus\, 0 \otimes E_4 = \begin{bmatrix} 3 & 3 & 3 & \varepsilon \\ 2 & 2 & 2 & \varepsilon \\ 4 & 4 & 4 & \varepsilon \\ 4 & 4 & 4 & 0 \end{bmatrix} \ ,$$

   the matrix A does not satisfy (A.5).
   The definition of [115, 116, 125] would result in the following max-algebraic characteristic equation for A:

$$\lambda^{\otimes^4} = 1 \otimes \lambda^{\otimes^3} \ . \tag{A.6}$$

However, since

$$1 \otimes A^{\otimes^3} = \begin{bmatrix} 3 & 3 & 3 & \varepsilon \\ 2 & 2 & 2 & \varepsilon \\ 4 & 4 & 4 & \varepsilon \\ 4 & 4 & 4 & -2 \end{bmatrix} ,$$

the matrix A does not satisfy (A.6) either.
The signed version of the max-algebraic characteristic equation of A is given by

$$\lambda^{\otimes^4} \ \oplus \ 0 \otimes \lambda^{\otimes^2} = 1 \otimes \lambda^{\otimes^3} \ .$$

Note that A satisfies this equation since

$$A^{\otimes^4} \ \oplus \ 0 \otimes A^{\otimes^2} = \begin{bmatrix} 3 & 3 & 3 & \varepsilon \\ 2 & 2 & 2 & \varepsilon \\ 4 & 4 & 4 & \varepsilon \\ 4 & 4 & 4 & -2 \end{bmatrix} = 1 \otimes A^{\otimes^3} \ . \qquad \square$$

A.2 A Counterexample for a Conjecture of Olsder

In [115] Olsder states that if an equation of the form

$$\lambda^{\otimes^3} \ \oplus \ c_1 \otimes \lambda = c_2 \otimes \lambda^{\otimes^2} \ \oplus \ c_0 \tag{A.7}$$

has less than three (possibly coinciding) solutions [in $\mathbb{R}_\varepsilon$][1], then it cannot be [the signed version of] the max-algebraic characteristic equation of a matrix [with entries in $\mathbb{R}_\varepsilon$]. Next he proposes the following conjecture:

Conjecture A.2.1 *A monic equation of degree n with the highest and one but highest order terms at different sides of the equality sign is [the signed version of] a [max-algebraic] characteristic equation of an $n \times n$ matrix [with entries in $\mathbb{R}_\varepsilon$] if and only if this equation has the maximum number of possibly coinciding real solutions. With "possibly coinciding" is meant that an arbitrary small perturbation of the coefficients exists, in the usual $||\cdot||_2$ norm, such that the perturbed equation has the maximum number of solutions which are all different.*

Now we show by a counterexample that the statement made in [115] about (A.7) does not hold. Hence, Conjecture A.2.1 does not hold either. Note that for this example the different (correct and wrong) definitions for the coefficients of the max-algebraic characteristic equation all yield the same result.

[1] The words between square brackets have been added to make the formulation consistent with the one that has been used in the previous section and in the previous chapters.

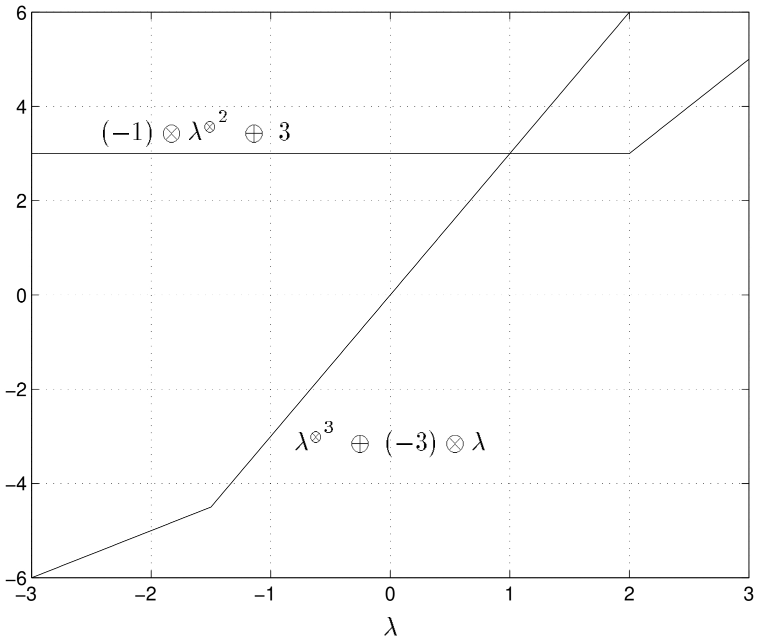


Figure A.1: The graphs of the max-algebraic polynomials on the left-hand and the right-hand side of (A.8).

Example A.2.2 Consider

$$A = \begin{bmatrix} -1 & \varepsilon & 3 \\ 0 & -2 & \varepsilon \\ \varepsilon & 0 & \varepsilon \end{bmatrix} .$$

The different definitions for the coefficients of the max-algebraic characteristic equation discussed in Section A.1 all result in the following max-algebraic characteristic equation for A:

$$\lambda^{\otimes^3} \oplus (-3) \otimes \lambda = (-1) \otimes \lambda^{\otimes^2} \oplus 3 \ . \tag{A.8}$$

In Figure A.1 we have plotted the graphs of the max-algebraic polynomials on the left-hand and the right-hand side of (A.8). Clearly, (A.8) has only one simple root, viz. $\lambda^* = 1$. Hence, Olsder's statement that (A.7) cannot be the signed version of the max-algebraic characteristic equation of a matrix with entries in $\mathbb{R}_\varepsilon$ if it has less than three (possibly coinciding) solutions in $\mathbb{R}_\varepsilon$, does not hold. As a consequence, Conjecture A.2.1 does not hold either.
The "regular" max-algebraic characteristic equation of the matrix A (cf. Definition 5.1.1) is given by

$$\lambda^{\otimes^3} \ominus (-1) \otimes \lambda^{\otimes^2} \oplus (-3) \otimes \lambda \ominus 3 \,\nabla\, \varepsilon \ .$$

Note that the coefficients of this equation correspond to those of (A.8) and that they satisfy the necessary and sufficient conditions of Proposition 5.3.3 for the coefficients of the max-algebraic characteristic equation of a 3 by 3 matrix with entries in $\mathbb{R}_\varepsilon$ since

$$\begin{aligned} a_1^{\oplus} &= \varepsilon \\ a_2^{\oplus} &= -3 \leqslant -2 = (-1) \otimes (-1) = a_1^{\ominus} \otimes a_1^{\ominus} \\ a_3^{\oplus} &= \varepsilon \leqslant \varepsilon = (-1) \otimes \varepsilon = a_1^{\ominus} \otimes a_2^{\ominus} \ . \end{aligned}$$

□

Appendix B

Proofs of Some Propositions of Chapter 5

In this appendix we prove some of the propositions of Chapter 5. We derive an efficient algorithm to compute the coefficients of the max-algebraic characteristic polynomial of a max-algebraic upper Hessenberg matrix with zeros on the first subdiagonal. Next we propose a conjecture on the max-algebraic characteristic polynomial of a matrix with entries in $\mathbb{R}_\varepsilon$ and we develop a heuristic algorithm to construct matrices with a given max-algebraic characteristic polynomial that is based on this conjecture.

B.1 Proof of Proposition 5.2.6

Let $A \in \mathbb{R}_\varepsilon^{n \times n}$ and let $\lambda^{\otimes^n} \oplus \bigoplus_{k=1}^{n} a_k \otimes \lambda^{\otimes^{n-k}}$ be the MACP of A. In this section and in the next section we shall use the following expressions for the max-positive and the max-negative contributions to the coefficients a_1, a_2, a_3 and a_4:

$$a_1^{\text{neg}} = \bigoplus_{\{i_1\} \in \mathcal{C}_n^1} a_{i_1 i_1} \tag{B.1}$$

$$a_2^{\text{pos}} = \bigoplus_{\{j_1, j_2\} \in \mathcal{C}_n^2} a_{j_1 j_1} \otimes a_{j_2 j_2} \tag{B.2}$$

$$a_2^{\text{neg}} = \bigoplus_{\{j_1, j_2\} \in \mathcal{C}_n^2} a_{j_1 j_2} \otimes a_{j_2 j_1} \tag{B.3}$$

$$a_3^{\text{pos}} = \bigoplus_{\{k_1, k_2, k_3\} \in \mathcal{C}_n^3} a_{k_1 k_1} \otimes a_{k_2 k_3} \otimes a_{k_3 k_2} \tag{B.4}$$

$$a_3^{\text{neg}} = \bigoplus_{\{k_1,k_2,k_3\}\in\mathcal{C}_n^3} a_{k_1k_1} \otimes a_{k_2k_2} \otimes a_{k_3k_3} \ \oplus$$

$$\bigoplus_{\{k_1,k_2,k_3\}\in\mathcal{C}_n^3} a_{k_1k_2} \otimes a_{k_2k_3} \otimes a_{k_3k_1} \tag{B.5}$$

$$a_4^{\text{pos}} = \bigoplus_{\{l_1,l_2,l_3,l_4\}\in\mathcal{C}_n^4} a_{l_1l_1} \otimes a_{l_2l_2} \otimes a_{l_3l_3} \otimes a_{l_4l_4} \ \oplus$$

$$\bigoplus_{\{l_1,l_2,l_3,l_4\}\in\mathcal{C}_n^4} a_{l_1l_1} \otimes a_{l_2l_3} \otimes a_{l_3l_4} \otimes a_{l_4l_2} \ \oplus$$

$$\bigoplus_{\{l_1,l_2,l_3,l_4\}\in\mathcal{C}_n^4} a_{l_1l_2} \otimes a_{l_2l_1} \otimes a_{l_3l_4} \otimes a_{l_4l_3} \tag{B.6}$$

$$a_4^{\text{neg}} = \bigoplus_{\{l_1,l_2,l_3,l_4\}\in\mathcal{C}_n^4} a_{l_1l_1} \otimes a_{l_2l_2} \otimes a_{l_3l_4} \otimes a_{l_4l_3} \ \oplus$$

$$\bigoplus_{\{l_1,l_2,l_3,l_4\}\in\mathcal{C}_n^4} a_{l_1l_2} \otimes a_{l_2l_3} \otimes a_{l_3l_4} \otimes a_{l_4l_1} \ . \tag{B.7}$$

These expressions can be derived from (5.2) – (5.7).
The following lemma gives some additional necessary conditions for the coefficients of the MACP of a matrix with entries in $\mathbb{R}_\varepsilon$.

Lemma B.1.1 *Let $A \in \mathbb{R}_\varepsilon^{n\times n}$ and let $\lambda^{\otimes^n} \oplus \bigoplus_{k=1}^{n} a_k \otimes \lambda^{\otimes^{n-k}}$ be the MACP of A. If $a_4^\oplus > a_1^\ominus \otimes a_3^\ominus$ and $a_4^\oplus \geqslant a_1^\ominus \otimes a_3^\oplus$ then we have*

(1) $a_4^\oplus \neq \varepsilon$

(2) $a_4^\oplus = \bigoplus_{\{l_1,l_2,l_3,l_4\}\in\mathcal{C}_n^4} a_{l_1l_2} \otimes a_{l_2l_1} \otimes a_{l_3l_4} \otimes a_{l_4l_3}$

(3) $a_1^\ominus \otimes a_1^\ominus \leqslant a_2^\ominus$

(4) $a_2^\oplus \leqslant a_2^\ominus$

(5) $a_4^\oplus \leqslant a_2^\ominus \otimes a_2^\ominus$.

Proof:

(1) We can only have $a_4^\oplus > a_1^\ominus \otimes a_3^\ominus$ if $a_4^\oplus \neq \varepsilon$.

(2) If $a_4^\oplus \neq \varepsilon$ then $a_4^\oplus = a_4^{\text{pos}}$. So it follows from (B.6) that $a_4^\oplus = t_1 \oplus t_2 \oplus t_3$ with

$$t_1 = \bigoplus_{\{l_1,l_2,l_3,l_4\}\in\mathcal{C}_n^4} a_{l_1l_1} \otimes a_{l_2l_2} \otimes a_{l_3l_3} \otimes a_{l_4l_4}$$

$$t_2 = \bigoplus_{\{l_1,l_2,l_3,l_4\}\in\mathcal{C}_n^4} a_{l_1l_1} \otimes a_{l_2l_3} \otimes a_{l_3l_4} \otimes a_{l_4l_2}$$

$$t_3 = \bigoplus_{\{l_1,l_2,l_3,l_4\}\in\mathcal{C}_n^4} a_{l_1l_2} \otimes a_{l_2l_1} \otimes a_{l_3l_4} \otimes a_{l_4l_3} \quad .$$

Now we prove that $a_4^\oplus > a_1^\ominus \otimes a_3^\ominus$ and $a_4^\oplus \geqslant a_1^\ominus \otimes a_3^\oplus$ imply that $a_4^\oplus > a_1^{\text{neg}} \otimes a_3^{\text{neg}}$. By Proposition 5.2.1 we have $a_1^\ominus = a_1^{\text{neg}}$. If $a_3^{\text{pos}} > a_3^{\text{neg}}$ then we have $a_3^\oplus > a_3^{\text{neg}}$ and then $a_4^\oplus \geqslant a_1^\ominus \otimes a_3^\oplus$ implies that $a_4^\oplus > a_1^{\text{neg}} \otimes a_3^{\text{neg}}$. On the other hand, if $a_3^{\text{pos}} \leqslant a_3^{\text{neg}}$ then $a_3^\ominus = a_3^{\text{neg}}$ and then $a_4^\oplus > a_1^\ominus \otimes a_3^\ominus$ results in $a_4^\oplus > a_1^{\text{neg}} \otimes a_3^{\text{neg}}$.
From (B.1) and (B.5) it follows that $a_1^{\text{neg}} \otimes a_3^{\text{neg}} = t_4 \oplus t_5$ with

$$t_4 = \bigoplus_{\substack{\{i_1\}\in\mathcal{C}_n^1\\ \{k_1,k_2,k_3\}\in\mathcal{C}_n^3}} a_{i_1i_1} \otimes a_{k_1k_1} \otimes a_{k_2k_2} \otimes a_{k_3k_3}$$

$$t_5 = \bigoplus_{\substack{\{i_1\}\in\mathcal{C}_n^1\\ \{k_1,k_2,k_3\}\in\mathcal{C}_n^3}} a_{i_1i_1} \otimes a_{k_1k_2} \otimes a_{k_2k_3} \otimes a_{k_3k_1} \quad .$$

If we compare t_1 and t_4, we see that every term of t_1 also appears in t_4. Hence, $t_1 \leqslant t_4$. Analogously, we find that $t_2 \leqslant t_5$. If we combine these inequalities, we get $t_1 \oplus t_2 \leqslant t_4 \oplus t_5$.
We have already shown that the conditions $a_4^\oplus > a_1^\ominus \otimes a_3^\ominus$ and $a_4^\oplus \geqslant a_1^\ominus \otimes a_3^\oplus$ imply that $a_4^\oplus > a_1^{\text{neg}} \otimes a_3^{\text{neg}}$ or equivalently $t_1 \oplus t_2 \oplus t_3 > t_4 \oplus t_5$. If we compare this strict inequality with the inequality $t_1 \oplus t_2 \leqslant t_4 \oplus t_5$, we conclude that $t_1 < t_3$ and $t_2 < t_3$. Hence,

$$a_4^\oplus = t_3 = \bigoplus_{\{l_1,l_2,l_3,l_4\}\in\mathcal{C}_n^4} a_{l_1l_2} \otimes a_{l_2l_1} \otimes a_{l_3l_4} \otimes a_{l_4l_3} \quad . \tag{B.8}$$

(3) First we prove that the conditions $a_4^\oplus > a_1^\ominus \otimes a_3^\ominus$ and $a_4^\oplus \geqslant a_1^\ominus \otimes a_3^\oplus$ imply that $a_4^\oplus \geqslant a_1^{\text{neg}} \otimes a_3^{\text{pos}}$: By Proposition 5.2.1 we have $a_1^\ominus = a_1^{\text{neg}}$. If $a_3^{\text{neg}} \geqslant a_3^{\text{pos}}$ then $a_3^\ominus \geqslant a_3^{\text{pos}}$ and then $a_4^\oplus > a_1^\ominus \otimes a_3^\ominus$ implies that $a_4^\oplus > a_1^{\text{neg}} \otimes a_3^{\text{pos}}$. On the other hand, if $a_3^{\text{neg}} < a_3^{\text{pos}}$ then we have $a_3^\oplus = a_3^{\text{pos}}$ and then $a_4^\oplus \geqslant a_1^\ominus \otimes a_3^\oplus$ results in $a_4^\oplus \geqslant a_1^{\text{neg}} \otimes a_3^{\text{pos}}$.
Suppose that the maximum in (B.8) is reached for $l_1 = \delta_1$, $l_2 = \delta_2$, $l_3 = \delta_3$ and $l_4 = \delta_4$ and that $a_{\delta_1\delta_2} \otimes a_{\delta_2\delta_1} \geqslant a_{\delta_3\delta_4} \otimes a_{\delta_4\delta_3}$. Note that $a_{\delta_1\delta_2}$, $a_{\delta_2\delta_1}$, $a_{\delta_3\delta_4}$ and $a_{\delta_4\delta_3}$ are different from ε since $a_4^\oplus \neq \varepsilon$.
We have

$$a_1^{\text{neg}} \otimes a_3^{\text{pos}} = \bigoplus_{\substack{\{i_1\}\in\mathcal{C}_n^1\\ \{k_1,k_2,k_3\}\in\mathcal{C}_n^3}} a_{i_1i_1} \otimes a_{k_1k_1} \otimes a_{k_2k_3} \otimes a_{k_3k_2} \quad . \tag{B.9}$$

Now we consider the terms of (B.9) for which $k_2 = \delta_3$ and $k_3 = \delta_4$. Then we know that $k_1 \neq \delta_3$ and $k_1 \neq \delta_4$. Since $a_1^{\text{neg}} \otimes a_3^{\text{pos}} \leqslant a_4^{\text{pos}}$, we have

$$\bigoplus_{\substack{\{i_1\},\{k_1\} \in \mathcal{C}_n^1 \\ k_1 \neq \delta_3,\, k_1 \neq \delta_4}} a_{i_1 i_1} \otimes a_{k_1 k_1} \otimes a_{\delta_3 \delta_4} \otimes a_{\delta_4 \delta_3} \leqslant a_{\delta_1 \delta_2} \otimes a_{\delta_2 \delta_1} \otimes a_{\delta_3 \delta_4} \otimes a_{\delta_4 \delta_3} \ .$$

Since $a_{\delta_3 \delta_4} \otimes a_{\delta_4 \delta_3} \neq \varepsilon$, we can max-divide both terms of this inequality by $a_{\delta_3 \delta_4} \otimes a_{\delta_4 \delta_3}$. This results in

$$\bigoplus_{\substack{\{i_1\},\{k_1\} \in \mathcal{C}_n^1 \\ k_1 \neq \delta_3,\, k_1 \neq \delta_4}} a_{i_1 i_1} \otimes a_{k_1 k_1} \leqslant a_{\delta_1 \delta_2} \otimes a_{\delta_2 \delta_1} \leqslant a_2^{\text{neg}} \ . \tag{B.10}$$

Analogously we find

$$\bigoplus_{\substack{\{i_1\},\{k_1\} \in \mathcal{C}_n^1 \\ k_1 \neq \delta_1,\, k_1 \neq \delta_2}} a_{i_1 i_1} \otimes a_{k_1 k_1} \leqslant a_2^{\text{neg}} \ . \tag{B.11}$$

Since $\delta_i \neq \delta_j$ for all i, j with $i \neq j$, (B.10) and (B.11) result in

$$\bigoplus_{\{i_1\},\{k_1\} \in \mathcal{C}_n^1} a_{i_1 i_1} \otimes a_{k_1 k_1} \leqslant a_2^{\text{neg}} \ .$$

Hence,

$$a_1^{\ominus} \otimes a_1^{\ominus} = a_1^{\text{neg}} \otimes a_1^{\text{neg}} = \bigoplus_{\{i_1\},\{k_1\} \in \mathcal{C}_n^1} a_{i_1 i_1} \otimes a_{k_1 k_1} \leqslant a_2^{\text{neg}} \ . \tag{B.12}$$

By Proposition 5.2.3 we have $a_2^{\text{pos}} \leqslant a_1^{\text{neg}} \otimes a_1^{\text{neg}}$. If we combine this with (B.12), we get $a_2^{\text{pos}} \leqslant a_2^{\text{neg}}$ or $a_2^{\text{neg}} = a_2^{\ominus}$. Hence, $a_1^{\ominus} \otimes a_1^{\ominus} \leqslant a_2^{\ominus}$.

(4) Since $a_2^{\text{pos}} \leqslant a_2^{\text{neg}}$, we have $a_2^{\oplus} \leqslant a_2^{\ominus}$.

(5) By Proposition 5.2.3 we have $a_4^{\text{pos}} \leqslant a_1^{\text{neg}} \otimes a_3^{\text{neg}} \oplus a_2^{\text{neg}} \otimes a_2^{\text{neg}}$. However, we already know that $a_4^{\text{pos}} = a_4^{\oplus} > a_1^{\text{neg}} \otimes a_3^{\text{neg}}$. Therefore, we have $a_4^{\oplus} = a_4^{\text{pos}} \leqslant a_2^{\text{neg}} \otimes a_2^{\text{neg}}$ and thus also $a_4^{\oplus} \leqslant a_2^{\ominus} \otimes a_2^{\ominus}$ since $a_2^{\ominus} = a_2^{\text{neg}}$ if the conditions of this lemma are satisfied. □

Now we prove Proposition 5.2.6.

Proof of Proposition 5.2.6 : We have to prove that if $A \in \mathbb{R}_\varepsilon^{n \times n}$ with $n \geqslant 4$ then the coefficients of the MACP of A always fall into exactly one of the following three cases:

Case A: $a_4^{\oplus} \leqslant a_1^{\ominus} \otimes a_3^{\ominus}$ or $a_4^{\oplus} < a_1^{\ominus} \otimes a_3^{\oplus}$,

Case B: $a_4^{\oplus} > a_1^{\ominus} \otimes a_3^{\ominus}$ and $a_4^{\oplus} \geqslant a_1^{\ominus} \otimes a_3^{\oplus}$ and $a_4^{\oplus} \leqslant a_2^{\ominus} \otimes a_2^{\ominus}$ and ($a_1^{\ominus} = \varepsilon$ or $a_2^{\oplus} = \varepsilon$ or $a_4^{\ominus} = a_4^{\oplus}$) ,

Case C: $a_4^{\oplus} > a_1^{\ominus} \otimes a_3^{\ominus}$ and $a_4^{\oplus} \geqslant a_1^{\ominus} \otimes a_3^{\oplus}$ and $a_4^{\oplus} \leqslant a_2^{\ominus} \otimes a_2^{\ominus}$ and $a_2^{\oplus} \neq \varepsilon$ and $a_4^{\ominus} = \varepsilon$.

From Proposition 5.2.5 it follows that

(a) $a_4^{\oplus} \leqslant a_1^{\ominus} \otimes a_3^{\ominus}$ or

(b) $a_4^{\oplus} < a_1^{\ominus} \otimes a_3^{\oplus}$ or

(c) $a_4^{\oplus} \leqslant a_2^{\ominus} \otimes a_2^{\ominus}$ or

(d) $a_4^{\oplus} < a_2^{\oplus} \otimes a_2^{\oplus}$ or

(e) $a_4^{\oplus} < a_2^{\oplus} \otimes a_2^{\ominus}$.

This means that we can distinguish between the following mutually exclusive cases:

Case 1 : (a) holds or (b) holds.
Case 2 : (a) and (b) do not hold, but (c) holds.
Case 3 : (a), (b) and (c) do not hold, but (d) holds.
Case 4 : (a), (b), (c) and (d) do not hold, but (e) holds.

From Lemma B.1.1(5) it follows that if (a) and (b) do not hold, (c) holds. Hence, Cases 3 and 4 cannot occur.
Case 1 corresponds to Case A and Case 2 corresponds to Cases B and C.
Assume that we are in Case 2. Now we have to prove that there are two mutually exclusive subcases: Case B and Case C. By Lemma B.1.1(1) we have $a_4^{\oplus} \neq \varepsilon$ in Case 2. Hence, either $a_4^{\ominus} = a_4^{\oplus}$ or $a_4^{\ominus} = \varepsilon$.
If $a_1^{\ominus} = \varepsilon$ or $a_2^{\oplus} = \varepsilon$ or $a_4^{\ominus} = a_4^{\oplus}$, then we are in Case B.
Otherwise, we have $a_1^{\ominus} \neq \varepsilon$, $a_2^{\oplus} \neq \varepsilon$ and $a_4^{\ominus} = \varepsilon$. The condition $a_2^{\oplus} \neq \varepsilon$ implies that $a_1^{\ominus} \neq \varepsilon$ since $a_2^{\oplus} \leqslant a_1^{\ominus} \otimes a_1^{\ominus}$ by Proposition 5.2.4. Hence, we may drop the condition $a_1^{\ominus} \neq \varepsilon$.
So we have shown that the coefficients of the MACP of A always fall into one of the three mutually exclusive cases A, B or C. □

B.2 Proof of Proposition 5.3.4

B.2.1 Necessary Conditions

In this subsection we prove that the conditions of Proposition 5.3.4 for Case B and Case C are necessary.
First we prove that the conditions for Case B are necessary:

Proposition B.2.1 *Let $A \in \mathbb{R}_\varepsilon^{n \times n}$ with $n \geqslant 4$ and let $\lambda^{\otimes^n} \oplus \bigoplus_{k=1}^{n} a_k \otimes \lambda^{\otimes^{n-k}}$ be the MACP of A. If $a_4^\oplus > a_1^\ominus \otimes a_3^\ominus$ and $a_4^\oplus \geqslant a_1^\ominus \otimes a_3^\oplus$ then at least one of the following statements holds:*

(1) $a_1^\ominus \otimes a_4^\oplus \leqslant a_2^\ominus \otimes a_3^\oplus$ *or*

(2) $a_1^\ominus \otimes a_4^\oplus < a_2^\ominus \otimes a_3^\ominus$.

Proof: If $a_4^\oplus > a_1^\ominus \otimes a_3^\ominus$ then we have $a_4^\oplus \neq \varepsilon$ and thus $a_4^\oplus = a_4^{\text{pos}}$. From (B.1) and Lemma B.1.1(2) it follows that

$$a_1^{\text{neg}} \otimes a_4^{\text{pos}} = a_1^{\text{neg}} \otimes a_4^\oplus = \bigoplus_{\substack{\{i_1\} \in \mathcal{C}_n^1 \\ \{l_1,l_2,l_3,l_4\} \in \mathcal{C}_n^4}} a_{i_1 i_1} \otimes a_{l_1 l_2} \otimes a_{l_2 l_1} \otimes a_{l_3 l_4} \otimes a_{l_4 l_3} \ .$$

If we combine (B.3) and (B.4), we obtain

$$a_2^{\text{neg}} \otimes a_3^{\text{pos}} = \bigoplus_{\substack{\{j_1,j_2\} \in \mathcal{C}_n^2 \\ \{k_1,k_2,k_3\} \in \mathcal{C}_n^3}} a_{j_1 j_2} \otimes a_{j_2 j_1} \otimes a_{k_1 k_1} \otimes a_{k_2 k_3} \otimes a_{k_3 k_2} \ .$$

Consider an arbitrary term $a_{i_1 i_1} \otimes a_{l_1 l_2} \otimes a_{l_2 l_1} \otimes a_{l_3 l_4} \otimes a_{l_4 l_3}$ of $a_1^{\text{neg}} \otimes a_4^{\text{pos}}$.
If $i_1 = l_1$ or if $i_1 = l_2$ then we have $i_1 \neq l_3$ and $i_1 \neq l_4$. So in this case the term $a_{i_1 i_1} \otimes a_{l_1 l_2} \otimes a_{l_2 l_1} \otimes a_{l_3 l_4} \otimes a_{l_4 l_3}$ corresponds to the term of $a_2^{\text{neg}} \otimes a_3^{\text{pos}}$ with e.g. $j_1 = l_1$, $j_2 = l_2$, $k_1 = i_1$, $k_2 = l_3$ and $k_3 = l_4$.
Otherwise, we have $i_1 \neq l_1$ and $i_1 \neq l_2$ and then the term $a_{i_1 i_1} \otimes a_{l_1 l_2} \otimes a_{l_2 l_1} \otimes a_{l_3 l_4} \otimes a_{l_4 l_3}$ corresponds to the term of $a_2^{\text{neg}} \otimes a_3^{\text{pos}}$ with e.g. $j_1 = l_3$, $j_2 = l_4$, $k_1 = i_1$, $k_2 = l_1$ and $k_3 = l_2$.
So every term of $a_1^{\text{neg}} \otimes a_4^{\text{pos}}$ also appears in $a_2^{\text{neg}} \otimes a_3^{\text{pos}}$. Hence, $a_1^{\text{neg}} \otimes a_4^{\text{pos}} \leqslant a_2^{\text{neg}} \otimes a_3^{\text{pos}}$.
We have $a_1^\ominus = a_1^{\text{neg}}$ by Proposition 5.2.1. Since the condition $a_4^\oplus > a_1^\ominus \otimes a_3^\ominus$ implies that $a_4^\oplus \neq \varepsilon$, we have $a_4^{\text{pos}} = a_4^\oplus$. Since $a_2^\oplus \leqslant a_2^\ominus$ by Lemma B.1.1(4), we have $a_2^{\text{neg}} = a_2^\ominus$.
If $a_3^{\text{pos}} \geqslant a_3^{\text{neg}}$ then we have $a_3^\oplus = a_3^{\text{pos}}$ and then $a_1^{\text{neg}} \otimes a_4^{\text{pos}} \leqslant a_2^{\text{neg}} \otimes a_3^{\text{pos}}$ results in $a_1^\ominus \otimes a_4^\oplus \leqslant a_2^\ominus \otimes a_3^\oplus$.
On the other hand, if $a_3^{\text{pos}} < a_3^{\text{neg}}$ then $a_3^\ominus > a_3^{\text{pos}}$ and this leads to $a_1^\ominus \otimes a_4^\oplus < a_2^\ominus \otimes a_3^\ominus$. □

The conditions for Case C are also necessary:

Proposition B.2.2 *Let $A \in \mathbb{R}_\varepsilon^{n \times n}$ with $n \geqslant 4$ and let $\lambda^{\otimes^n} \oplus \bigoplus_{k=1}^{n} a_k \otimes \lambda^{\otimes^{n-k}}$ be the MACP of A. If $a_4^\oplus > a_1^\ominus \otimes a_3^\ominus$ and $a_4^\oplus \geqslant a_1^\ominus \otimes a_3^\oplus$ and $a_2^\oplus \neq \varepsilon$ and $a_4^\ominus = \varepsilon$ then we have*

(1) $a_1^\ominus \otimes a_4^\oplus = a_2^\ominus \otimes a_3^\oplus$ *and*

(2) $a_1^\ominus \otimes a_3^\oplus = a_2^\ominus \otimes a_2^\oplus$.

Proof:

(1) By Lemma B.1.1(4) we have $a_2^{\oplus} \leqslant a_2^{\ominus}$. Since $a_2^{\oplus} \neq \varepsilon$, this implies that $a_2^{\oplus} = a_2^{\ominus} = a_2^{\mathrm{pos}} = a_2^{\mathrm{neg}}$ in Case C. Furthermore, the condition $a_4^{\oplus} > a_1^{\ominus} \otimes a_3^{\ominus}$ implies that

$$a_4^{\oplus} = a_4^{\mathrm{pos}} \neq \varepsilon \ . \tag{B.13}$$

Now we prove that the conditions $a_4^{\oplus} > a_1^{\ominus} \otimes a_3^{\ominus}$, $a_4^{\oplus} \geqslant a_1^{\ominus} \otimes a_3^{\oplus}$ and $a_2^{\oplus} = a_2^{\ominus}$ imply that $a_4^{\mathrm{pos}} \leqslant a_3^{\mathrm{pos}} \otimes a_1^{\mathrm{neg}}$.

We have $a_2^{\mathrm{pos}} = \bigoplus_{\{j_1,j_2\} \in \mathcal{C}_n^2} a_{j_1 j_1} \otimes a_{j_2 j_2}$. Suppose that the maximum is reached for $j_1 = \gamma_1$ and $j_2 = \gamma_2$. So $a_2^{\mathrm{pos}} = a_{\gamma_1\gamma_1} \otimes a_{\gamma_2\gamma_2}$. Since

$$a_2^{\mathrm{pos}} = a_2^{\mathrm{neg}} = \bigoplus_{\{j_1,j_2\} \in \mathcal{C}_n^2} a_{j_1 j_2} \otimes a_{j_2 j_1} \ ,$$

this implies that $a_{j_1 j_2} \otimes a_{j_2 j_1} \leqslant a_{\gamma_1\gamma_1} \otimes a_{\gamma_2\gamma_2}$ for all $j_1, j_2 \in \{1, 2, \ldots, n\}$. By Lemma B.1.1(2) we have

$$a_4^{\mathrm{pos}} = a_4^{\oplus} = \bigoplus_{\{l_1,l_2,l_3,l_4\} \in \mathcal{C}_n^4} a_{l_1 l_2} \otimes a_{l_2 l_1} \otimes a_{l_3 l_4} \otimes a_{l_4 l_3} \ . \tag{B.14}$$

From (B.1) and (B.4) it follows that

$$a_1^{\mathrm{neg}} \otimes a_3^{\mathrm{pos}} = \bigoplus_{\substack{\{i_1\} \in \mathcal{C}_n^1 \\ \{k_1,k_2,k_3\} \in \mathcal{C}_n^3}} a_{i_1 i_1} \otimes a_{k_1 k_1} \otimes a_{k_2 k_3} \otimes a_{k_3 k_2} \ .$$

Consider an arbitrary term $a_{l_1 l_2} \otimes a_{l_2 l_1} \otimes a_{l_3 l_4} \otimes a_{l_4 l_3}$ of a_4^{pos}.
If $\gamma_2 = l_1$ or $\gamma_2 = l_2$, then we have $\gamma_2 \neq l_3$ and $\gamma_2 \neq l_4$ and then $a_{l_1 l_2} \otimes a_{l_2 l_1} \otimes a_{l_3 l_4} \otimes a_{l_4 l_3}$ is less than or equal to $a_{\gamma_1\gamma_1} \otimes a_{\gamma_2\gamma_2} \otimes a_{l_3 l_4} \otimes a_{l_4 l_3}$, which corresponds to the term of $a_1^{\mathrm{neg}} \otimes a_3^{\mathrm{pos}}$ with $i_1 = \gamma_1$, $k_1 = \gamma_2$, $k_2 = l_3$ and $k_3 = l_4$.
On the other hand, if $\gamma_2 \neq l_1$ and $\gamma_2 \neq l_2$, then $a_{l_1 l_2} \otimes a_{l_2 l_1} \otimes a_{l_3 l_4} \otimes a_{l_4 l_3}$ is less than or equal to the term of $a_1^{\mathrm{neg}} \otimes a_3^{\mathrm{pos}}$ with $i_1 = \gamma_1$, $k_1 = \gamma_2$, $k_2 = l_1$ and $k_3 = l_2$.
Hence,

$$a_4^{\mathrm{pos}} \leqslant a_1^{\mathrm{neg}} \otimes a_3^{\mathrm{pos}} \ . \tag{B.15}$$

If $a_3^{\mathrm{pos}} < a_3^{\mathrm{neg}}$, we have $a_3^{\ominus} > a_3^{\mathrm{pos}}$ and then the condition $a_4^{\oplus} > a_1^{\ominus} \otimes a_3^{\ominus}$ implies that $a_4^{\mathrm{pos}} = a_4^{\oplus} > a_1^{\mathrm{neg}} \otimes a_3^{\mathrm{pos}}$ but this is in contradiction with (B.15). This means that we always have $a_3^{\mathrm{pos}} \geqslant a_3^{\mathrm{neg}}$ in Case C. Hence,

$$a_3^{\oplus} = a_3^{\mathrm{pos}} \tag{B.16}$$

and thus $a_4^{\oplus} \leqslant a_1^{\ominus} \otimes a_3^{\oplus}$. If we combine this with the condition $a_4^{\oplus} \geqslant a_1^{\ominus} \otimes a_3^{\oplus}$, we get

$$a_4^{\oplus} = a_1^{\ominus} \otimes a_3^{\oplus} \quad . \tag{B.17}$$

In general we have $a_2^{\oplus} \leqslant a_1^{\ominus} \otimes a_1^{\ominus}$ by Proposition 5.2.4. However, we also have $a_1^{\ominus} \otimes a_1^{\ominus} \leqslant a_2^{\ominus}$ by Lemma B.1.1(3) and since $a_2^{\oplus} = a_2^{\ominus}$ in Case C, this implies that

$$a_1^{\ominus} \otimes a_1^{\ominus} = a_2^{\oplus} = a_2^{\ominus} \tag{B.18}$$

in Case C. Hence, $a_1^{\ominus} \otimes a_4^{\oplus} = a_1^{\ominus} \otimes a_1^{\ominus} \otimes a_3^{\oplus} = a_2^{\ominus} \otimes a_3^{\oplus}$.

(2) If $a_4^{\ominus} = \varepsilon$ then $a_4^{\text{neg}} < a_4^{\text{pos}}$ since $a_4^{\oplus} \neq \varepsilon$ (cf. (B.13)). Assume that the maximum in (B.14) is reached for $l_1 = \delta_1$, $l_2 = \delta_2$, $l_3 = \delta_3$ and $l_4 = \delta_4$. Since $a_4^{\oplus} \neq \varepsilon$, we have $a_{\delta_1\delta_2} \otimes a_{\delta_2\delta_1} \neq \varepsilon$ and $a_{\delta_3\delta_4} \otimes a_{\delta_4\delta_3} \neq \varepsilon$.
Since $a_4^{\text{neg}} < a_4^{\text{pos}}$ the first max-algebraic summation on the right-hand side of (B.7) is also less than $a_4^{\text{pos}} = a_{\delta_1\delta_2} \otimes a_{\delta_2\delta_1} \otimes a_{\delta_3\delta_4} \otimes a_{\delta_4\delta_3}$. Hence,

$$\bigoplus_{\{l_1,l_2,l_3,l_4\}\in\mathcal{C}_n^4} a_{l_1l_1} \otimes a_{l_2l_2} \otimes a_{l_3l_4} \otimes a_{l_4l_3} < a_{\delta_1\delta_2} \otimes a_{\delta_2\delta_1} \otimes a_{\delta_3\delta_4} \otimes a_{\delta_4\delta_3} \quad .$$

Now we consider the terms on the left-hand side of this inequality for which $l_3 = \delta_3$ and $l_4 = \delta_4$. We have

$$\bigoplus_{\substack{\{l_1,l_2\}\in\mathcal{C}_n^2 \\ l_1,l_2\notin\{\delta_1,\delta_2\}}} a_{l_1l_1} \otimes a_{l_2l_2} \otimes a_{\delta_3\delta_4} \otimes a_{\delta_4\delta_3} \; < \; a_{\delta_1\delta_2} \otimes a_{\delta_2\delta_1} \otimes a_{\delta_3\delta_4} \otimes a_{\delta_4\delta_3} \quad .$$

Since $a_{\delta_3\delta_4} \otimes a_{\delta_4\delta_3} \neq \varepsilon$, this results in

$$\bigoplus_{\substack{\{l_1,l_2\}\in\mathcal{C}_n^2 \\ l_1,l_2\notin\{\delta_1,\delta_2\}}} a_{l_1l_1} \otimes a_{l_2l_2} \; < \; a_{\delta_1\delta_2} \otimes a_{\delta_2\delta_1} \; \leqslant \; a_2^{\text{neg}} \; = \; a_2^{\text{pos}} \quad . \tag{B.19}$$

Using an analogous reasoning but with $l_3 = \delta_1$ and $l_4 = \delta_2$ we find

$$\bigoplus_{\substack{\{l_1,l_2\}\in\mathcal{C}_n^2 \\ l_1,l_2\notin\{\delta_3,\delta_4\}}} a_{l_1l_1} \otimes a_{l_2l_2} \; < \; a_2^{\text{pos}} \quad . \tag{B.20}$$

We have $\delta_i \neq \delta_j$ for all i, j with $i \neq j$. So if we combine (B.19), (B.20) and (B.2), we get

$$a_2^{\text{pos}} = a_{\delta_1\delta_1} \otimes a_{\delta_3\delta_3} \;\oplus\; a_{\delta_1\delta_1} \otimes a_{\delta_4\delta_4} \;\oplus\; a_{\delta_2\delta_2} \otimes a_{\delta_3\delta_3} \;\oplus\; a_{\delta_2\delta_2} \otimes a_{\delta_4\delta_4}$$

since all the other terms of the form $a_{j_1j_1} \otimes a_{j_2j_2}$ with $\{j_1, j_2\} \in \mathcal{C}_n^2$ are less than a_2^{pos} by (B.19) and (B.20).

We may assume without loss of generality that $a_2^{\oplus} = a_2^{\text{pos}} = a_{\delta_1\delta_1} \otimes a_{\delta_3\delta_3}$.
Now we show by contradiction that $a_1^{\ominus} = a_{\delta_1\delta_1} = a_{\delta_3\delta_3}$.
Assume that $a_{\delta_1\delta_1} \neq a_{\delta_3\delta_3}$. We may assume without loss of generality that $a_{\delta_1\delta_1} > a_{\delta_3\delta_3}$. This implies that $a_1^{\ominus} \geqslant a_{\delta_1\delta_1}$ and $a_1^{\ominus} > a_{\delta_3\delta_3}$ and thus $a_1^{\ominus} \otimes a_1^{\ominus} > a_{\delta_1\delta_1} \otimes a_{\delta_3\delta_3} = a_2^{\oplus}$ but this is in contradiction with the fact that $a_2^{\oplus} = a_1^{\ominus} \otimes a_1^{\ominus}$ (cf. (B.18)). Hence, our assumption was wrong and therefore we have $a_{\delta_1\delta_1} = a_{\delta_3\delta_3}$.
Since $a_2^{\oplus} = a_1^{\ominus} \otimes a_1^{\ominus}$ in Case C, this implies that

$$a_1^{\ominus} = a_{\delta_1\delta_1} = a_{\delta_3\delta_3} \quad .$$

Since $a_{\delta_1\delta_2} \otimes a_{\delta_2\delta_1} \leqslant a_2^{\text{neg}} = a_2^{\text{pos}} = a_{\delta_1\delta_1} \otimes a_{\delta_3\delta_3}$, we have

$$\begin{aligned} a_4^{\oplus} &= a_{\delta_1\delta_2} \otimes a_{\delta_2\delta_1} \otimes a_{\delta_3\delta_4} \otimes a_{\delta_4\delta_3} \\ &\leqslant a_{\delta_3\delta_3} \otimes a_{\delta_1\delta_1} \otimes a_{\delta_3\delta_4} \otimes a_{\delta_4\delta_3} \\ &\leqslant a_1^{\text{neg}} \otimes a_3^{\text{pos}} = a_1^{\ominus} \otimes a_3^{\oplus} \qquad \text{(by (B.16))} \quad . \end{aligned}$$

Since $a_4^{\oplus} = a_1^{\ominus} \otimes a_3^{\oplus}$ by (B.17), all the inequalities in this expression should be equalities. Since $a_{\delta_3\delta_4} \otimes a_{\delta_4\delta_3} \neq \varepsilon$, this leads to

$$a_{\delta_1\delta_2} \otimes a_{\delta_2\delta_1} = a_{\delta_3\delta_3} \otimes a_{\delta_1\delta_1} = a_1^{\ominus} \otimes a_1^{\ominus} = a_2^{\ominus} \quad .$$

Analogously we find $a_2^{\ominus} = a_{\delta_3\delta_4} \otimes a_{\delta_4\delta_3}$.
Hence, $a_4^{\oplus} = a_2^{\ominus} \otimes a_2^{\ominus}$, and since $a_2^{\oplus} = a_2^{\ominus}$ in Case C and since $a_1^{\ominus} \otimes a_3^{\oplus} = a_4^{\oplus}$ by (B.17), this results in

$$a_1^{\ominus} \otimes a_3^{\oplus} = a_4^{\oplus} = a_2^{\ominus} \otimes a_2^{\ominus} = a_2^{\oplus} \otimes a_2^{\ominus} \quad . \qquad \square$$

B.2.2 Sufficient Conditions

First we derive some extra conditions that automatically follow from the necessary and sufficient conditions stated in Proposition 5.3.4.

Lemma B.2.3 *If the numbers $a_1, a_2, a_3, a_4 \in \mathbb{S}$ satisfy the conditions for Case B or Case C of Proposition 5.2.6 and if they also satisfy all the corresponding necessary and sufficient conditions of Proposition 5.3.4 then we have*

(1) $a_4^{\oplus} \neq \varepsilon$

(2) $a_2^{\ominus} \neq \varepsilon$

(3) $a_2^{\oplus} \leqslant a_2^{\ominus}$

(4) $a_1^{\ominus} \otimes a_1^{\ominus} \leqslant a_2^{\ominus}$.

Proof:

(1) The condition $a_4^{\oplus} > a_1^{\ominus} \otimes a_3^{\ominus}$ can only be satisfied if $a_4^{\oplus} \neq \varepsilon$.

(2) Since $a_4^{\oplus} \leqslant a_2^{\ominus} \otimes a_2^{\ominus}$ and $a_4^{\oplus} \neq \varepsilon$, we have $a_2^{\ominus} \neq \varepsilon$.

(3) If $a_2^\ominus \neq \varepsilon$ then $a_2^\oplus \leqslant a_2^\ominus$.

(4) We prove by contradiction that $a_1^\ominus \otimes a_1^\ominus \leqslant a_2^\ominus$. Assume that $a_2^\ominus < a_1^\ominus \otimes a_1^\ominus$. Note that this implies that $a_1^\ominus \neq \varepsilon$.
If $a_2^\ominus < a_1^\ominus \otimes a_1^\ominus$ then the first necessary and sufficient condition for Case B: $a_1^\ominus \otimes a_4^\oplus \leqslant a_2^\ominus \otimes a_3^\oplus$ leads to $a_1^\ominus \otimes a_4^\oplus \leqslant a_2^\ominus \otimes a_3^\oplus < a_1^\ominus \otimes a_1^\ominus \otimes a_3^\oplus$ and thus $a_4^\oplus < a_1^\ominus \otimes a_3^\oplus$ since $a_1^\ominus \neq \varepsilon$. But this is in contradiction with the fact that $a_4^\oplus \geqslant a_1^\ominus \otimes a_3^\oplus$ in Case B. The second necessary and sufficient condition of Case B: $a_1^\ominus \otimes a_4^\oplus < a_2^\ominus \otimes a_3^\ominus$ would result in $a_1^\ominus \otimes a_4^\oplus < a_1^\ominus \otimes a_1^\ominus \otimes a_3^\ominus$ and thus $a_4^\oplus < a_1^\ominus \otimes a_3^\ominus$, whereas $a_4^\oplus > a_1^\ominus \otimes a_3^\ominus$ in Case B.
The second necessary and sufficient condition for Case C: $a_1^\ominus \otimes a_4^\oplus = a_2^\ominus \otimes a_3^\oplus$ would lead to $a_4^\oplus < a_1^\ominus \otimes a_3^\oplus$, which is impossible since $a_4^\oplus > a_1^\ominus \otimes a_3^\oplus$ in Case C.
So our initial assumption was wrong. Hence, $a_1^\ominus \otimes a_1^\ominus \leqslant a_2^\ominus$. □

Lemma B.2.4 *If the numbers $a_1, a_2, a_3, a_4 \in \mathbb{S}$ satisfy the conditions for Case C of Proposition 5.2.6 and if they also satisfy both the general and the case-specific necessary and sufficient conditions for Case C of Proposition 5.3.4, then we have*

(1) $a_1^\ominus \neq \varepsilon$

(2) $a_2^\oplus = a_2^\ominus = a_1^\ominus \otimes a_1^\ominus$

(3) $a_3^\oplus = a_1^\ominus \otimes a_2^\ominus = \left(a_1^\ominus\right)^{\otimes^3}$

(4) $a_4^\oplus = a_2^\ominus \otimes a_2^\ominus = a_1^\ominus \otimes a_3^\oplus = \left(a_1^\ominus\right)^{\otimes^4}$

(5) $a_3^\ominus = \varepsilon$.

Proof:

(1) The second general necessary and sufficient condition of Proposition 5.3.4 states that $a_2^\oplus \leqslant a_1^\ominus \otimes a_1^\ominus$. Since $a_2^\oplus \neq \varepsilon$ in Case C, this implies that $a_1^\ominus \neq \varepsilon$.

(2) Since $a_2^\oplus \neq \varepsilon$ in Case C and since $a_2^\ominus \neq \varepsilon$ by Lemma B.2.3(2), we have $a_2^\oplus = a_2^\ominus$.
Furthermore, we have $a_1^\ominus \otimes a_1^\ominus \leqslant a_2^\ominus$ by Lemma B.2.3(4). If we combine this with the condition $a_2^\oplus \leqslant a_1^\ominus \otimes a_1^\ominus$, we get $a_2^\oplus \leqslant a_1^\ominus \otimes a_1^\ominus \leqslant a_2^\ominus$. Since $a_2^\oplus = a_2^\ominus$, this results in $a_2^\oplus = a_1^\ominus \otimes a_1^\ominus = a_2^\ominus$.

(3) If we max-multiply the first necessary and sufficient condition for Case C: $a_1^\ominus \otimes a_3^\oplus = a_2^\ominus \otimes a_2^\oplus$ by $a_1^\ominus$, we get $a_1^\ominus \otimes a_1^\ominus \otimes a_3^\oplus = a_1^\ominus \otimes a_2^\ominus \otimes a_2^\oplus$. Since $a_1^\ominus \otimes a_1^\ominus = a_2^\oplus = a_2^\ominus \neq \varepsilon$, this leads to $a_3^\oplus = a_1^\ominus \otimes a_2^\ominus = a_1^\ominus \otimes a_1^\ominus \otimes a_1^\ominus = \left(a_1^\ominus\right)^{\otimes^3}$.

(4) Using a similar reasoning, the condition $a_1^\ominus \otimes a_4^\oplus = a_2^\ominus \otimes a_3^\oplus$ leads to $a_4^\oplus = a_1^\ominus \otimes a_3^\oplus = a_1^\ominus \otimes \left(a_1^\ominus\right)^{\otimes^3} = \left(a_1^\ominus\right)^{\otimes^4} = a_2^\ominus \otimes a_2^\ominus$.

(5) We always have $a_4^\oplus > a_1^\ominus \otimes a_3^\ominus$ in Case C. If we combine this with $a_4^\oplus = a_1^\ominus \otimes a_3^\oplus$ and $a_1^\ominus \neq \varepsilon$ (cf. steps (4) and (1) of this proof), then we get $a_3^\oplus > a_3^\ominus$. Hence, $a_3^\ominus = \varepsilon$. □

Now we demonstrate that the conditions of Proposition 5.3.4 are sufficient by showing that if the coefficients of a given max-algebraic polynomial satisfy all the conditions for respectively Case A, B or C of Proposition 5.3.4, the corresponding matrix B_A, B_B or B_C has the given max-algebraic polynomial as its MACP.

Proposition B.2.5 *Consider the max-algebraic polynomial*

$$\lambda^{\otimes^4} \oplus a_1 \otimes \lambda^{\otimes^3} \oplus a_2 \otimes \lambda^{\otimes^2} \oplus a_3 \otimes \lambda \oplus a_4 \ . \qquad \text{(B.21)}$$

Suppose that the coefficients of this max-algebraic polynomial satisfy the conditions of Proposition 5.2.6. If the following conditions are also satisfied:

$$\begin{aligned}
&a_1^\oplus = \varepsilon\\
&a_2^\oplus \leqslant a_1^\ominus \otimes a_1^\ominus\\
&a_3^\oplus \leqslant a_1^\ominus \otimes a_2^\ominus \ \textit{or}\ a_3^\oplus < a_1^\ominus \otimes a_2^\oplus\\
&\textit{for Case A: no extra conditions}\\
&\textit{for Case B: } a_1^\ominus \otimes a_4^\oplus \leqslant a_2^\ominus \otimes a_3^\oplus \ \textit{or}\ a_1^\ominus \otimes a_4^\oplus < a_2^\ominus \otimes a_3^\ominus\\
&\textit{for Case C: } a_1^\ominus \otimes a_3^\oplus = a_2^\ominus \otimes a_2^\oplus \ \textit{and}\ a_1^\ominus \otimes a_4^\oplus = a_2^\ominus \otimes a_3^\oplus \ ,
\end{aligned}$$

then the following matrices have (B.21) as their MACP:

$$B_\mathrm{A} = \begin{bmatrix} a_1^\ominus & a_2^\ominus & a_3^\ominus & a_4^\ominus \\ 0 & \kappa_{1,2} & \kappa_{1,3} & \kappa_{1,4} \\ \varepsilon & 0 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 0 & \varepsilon \end{bmatrix} \qquad \textit{for Case A},$$

$$B_\mathrm{B} = \begin{bmatrix} a_1^\ominus & a_2^\ominus & a_3^\ominus & a_4^\ominus \\ 0 & \kappa_{1,2} & \kappa_{1,3} & \varepsilon \\ \varepsilon & 0 & \varepsilon & \kappa_{2,4} \\ \varepsilon & \varepsilon & 0 & \varepsilon \end{bmatrix} \qquad \textit{for Case B},$$

$$B_\mathrm{C} = \begin{bmatrix} a_1^\ominus & a_2^\ominus & \varepsilon & \varepsilon \\ 0 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 0 & \kappa_{2,3} & \kappa_{2,4} \\ \varepsilon & \varepsilon & 0 & \varepsilon \end{bmatrix} \qquad \textit{for Case C}.$$

Proof:
We prove that the MACP

$$\lambda^{\otimes^4} \oplus b_1 \otimes \lambda^{\otimes^3} \oplus b_2 \otimes \lambda^{\otimes^2} \oplus b_3 \otimes \lambda \oplus b_4$$

of B_{A}, B_{B} or B_{C} coincides with (B.21) if all the conditions for respectively Case A, B or C are satisfied.
From the proofs of the propositions for the 2 by 2 and the 3 by 3 case we already know that $\kappa_{1,2} \leqslant a_1^{\ominus}$, $a_2^{\oplus} \ominus a_2^{\ominus} \ominus \kappa_{1,3} = a_2^{\oplus} \ominus a_2^{\ominus}$, $a_1^{\ominus} \otimes \kappa_{1,2} = a_2^{\oplus}$ and $a_1^{\ominus} \otimes \kappa_{1,3} = a_3^{\oplus}$.

Case A:

First we prove that in this case we have $a_1^{\ominus} \otimes \kappa_{1,4} = a_4^{\oplus}$ and $a_3^{\oplus} \ominus a_3^{\ominus} \ominus \kappa_{1,4} = a_3^{\oplus} \ominus a_3^{\ominus}$.
If $a_1^{\ominus} = \varepsilon$ then $\kappa_{1,4} = \varepsilon$. Furthermore, if $a_1^{\ominus} = \varepsilon$ then $a_4^{\oplus} = \varepsilon$ since $a_4^{\oplus} \leqslant a_1^{\ominus} \otimes a_3^{\ominus}$ or $a_4^{\oplus} < a_1^{\ominus} \otimes a_3^{\oplus}$ in Case A. So if $a_1^{\ominus} = \varepsilon$ then we have $a_1^{\ominus} \otimes \kappa_{1,4} = \varepsilon = a_4^{\oplus}$ and $\kappa_{1,4} = \varepsilon$ and thus also $a_3^{\oplus} \ominus a_3^{\ominus} \ominus \kappa_{1,4} = a_3^{\oplus} \ominus a_3^{\ominus}$.

If $a_1^{\ominus} \neq \varepsilon$, then $\kappa_{1,4} = \left\lfloor \dfrac{a_4^{\oplus}}{a_1^{\ominus}} \right\rfloor$ and thus $a_1^{\ominus} \otimes \kappa_{1,4} = a_4^{\oplus}$. If $a_1^{\ominus} \neq \varepsilon$, then the condition $a_4^{\oplus} \leqslant a_1^{\ominus} \otimes a_3^{\ominus}$ implies that $\kappa_{1,4} \leqslant a_3^{\ominus}$, and the condition $a_4^{\oplus} < a_1^{\ominus} \otimes a_3^{\oplus}$ implies that $\kappa_{1,4} < a_3^{\oplus}$. Hence, $a_3^{\oplus} \ominus a_3^{\ominus} \ominus \kappa_{1,4} = a_3^{\oplus} \ominus a_3^{\ominus}$ also holds in this case.

As a consequence, we have

$$
\begin{aligned}
b_1 &= \ominus a_1^{\ominus} \ominus \kappa_{1,2} = \ominus a_1^{\ominus} = a_1 \\
b_2 &= a_1^{\ominus} \otimes \kappa_{1,2} \ominus a_2^{\ominus} \ominus \kappa_{1,3} = a_2^{\oplus} \ominus a_2^{\ominus} \ominus \kappa_{1,3} = a_2^{\oplus} \ominus a_2^{\ominus} = a_2 \\
b_3 &= a_1^{\ominus} \otimes \kappa_{1,3} \ominus a_3^{\ominus} \ominus \kappa_{1,4} = a_3^{\oplus} \ominus a_3^{\ominus} \ominus \kappa_{1,4} = a_3^{\oplus} \ominus a_3^{\ominus} = a_3 \\
b_4 &= a_1^{\ominus} \otimes \kappa_{1,4} \ominus a_4^{\ominus} = a_4^{\oplus} \ominus a_4^{\ominus} = a_4 \quad .
\end{aligned}
$$

So (B.21) is the MACP of B_{A}.

Case B:

Since $a_4^{\oplus} \leqslant a_2^{\ominus} \otimes a_2^{\ominus}$ in Case B and since $a_2^{\ominus} \neq \varepsilon$ by Lemma B.2.3(2), we have $\kappa_{2,4} \leqslant a_2^{\ominus}$.

Now we use the necessary and sufficient conditions for Case B to prove that $a_1^{\ominus} \otimes \kappa_{2,4} \leqslant a_3^{\oplus}$ if $a_3^{\oplus} \geqslant a_3^{\ominus}$ and that $a_1^{\ominus} \otimes \kappa_{2,4} < a_3^{\ominus}$ if $a_3^{\oplus} < a_3^{\ominus}$.
If $a_3^{\oplus} \geqslant a_3^{\ominus}$ and if one of the necessary and sufficient conditions for Case B: $a_1^{\ominus} \otimes a_4^{\oplus} \leqslant a_2^{\ominus} \otimes a_3^{\oplus}$ or $a_1^{\ominus} \otimes a_4^{\oplus} < a_2^{\ominus} \otimes a_3^{\ominus}$ is fulfilled, we have $a_1^{\ominus} \otimes a_4^{\oplus} \leqslant a_2^{\ominus} \otimes a_3^{\oplus}$.
Since $a_2^{\ominus} \neq \varepsilon$, this results in $a_1^{\ominus} \otimes \kappa_{2,4} = a_1^{\ominus} \otimes \left\lfloor \dfrac{a_4^{\oplus}}{a_2^{\ominus}} \right\rfloor \leqslant a_3^{\oplus}$.

On the other hand, if $a_3^{\oplus} < a_3^{\ominus}$ then the necessary and sufficient conditions for Case B lead to $a_1^{\ominus} \otimes a_4^{\oplus} < a_2^{\ominus} \otimes a_3^{\ominus}$. As a consequence, we have $a_1^{\ominus} \otimes \kappa_{2,4} < a_3^{\ominus}$.
Hence, we always have $a_3^{\oplus} \oplus a_1^{\ominus} \otimes \kappa_{2,4} \ominus a_3^{\ominus} = a_3^{\oplus} \ominus a_3^{\ominus}$.

Now we prove that under the conditions of Case B we have $a_1^{\ominus} \otimes \kappa_{1,2} \otimes \kappa_{2,4} \leqslant a_4^{\ominus}$.
If $a_1^{\ominus} = \varepsilon$ or if $a_2^{\oplus} = \varepsilon$ then $a_1^{\ominus} \otimes \kappa_{1,2} \otimes \kappa_{2,4} = \varepsilon \leqslant a_4^{\ominus}$.
Otherwise, we have $a_1^{\ominus} \neq \varepsilon$, $a_2^{\oplus} \neq \varepsilon$ and $a_4^{\oplus} = a_4^{\ominus}$. Since $a_2^{\ominus} \neq \varepsilon$ by Lemma B.2.3(2) this implies that $a_2^{\oplus} = a_2^{\ominus}$. Hence, $a_1^{\ominus} \otimes \kappa_{1,2} \otimes \kappa_{2,4} =$

$a_1^\ominus \otimes \frac{a_2^\oplus}{a_1^\ominus} \otimes \frac{a_4^\oplus}{a_2^\ominus} = a_4^\oplus = a_4^\ominus$.
We also have $a_2^\ominus \otimes \kappa_{2,4} = a_4^\oplus$ since $a_2^\ominus \neq \varepsilon$.
Since the coefficients of the MACP of B_B are given by

$$\begin{aligned}
b_1 &= \ominus a_1^\ominus \ominus \kappa_{1,2} = \ominus a_1^\ominus = a_1 \\
b_2 &= a_1^\ominus \otimes \kappa_{1,2} \ominus a_2^\ominus \ominus \kappa_{1,3} \ominus \kappa_{2,4} = a_2^\oplus \ominus a_2^\ominus \ominus \kappa_{1,3} \ominus \kappa_{2,4} \\
&= a_2^\oplus \ominus a_2^\ominus = a_2 \\
b_3 &= a_1^\ominus \otimes \kappa_{1,3} \oplus a_1^\ominus \otimes \kappa_{2,4} \oplus \kappa_{1,2} \otimes \kappa_{2,4} \ominus a_3^\ominus \\
&= a_3^\oplus \oplus (a_1^\ominus \oplus \kappa_{1,2}) \otimes \kappa_{2,4} \ominus a_3^\ominus \\
&= a_3^\oplus \oplus a_1^\ominus \otimes \kappa_{2,4} \ominus a_3^\ominus = a_3^\oplus \ominus a_3^\ominus = a_3 \\
b_4 &= a_2^\ominus \otimes \kappa_{2,4} \ominus a_4^\ominus \ominus a_1^\ominus \otimes \kappa_{1,2} \otimes \kappa_{2,4} \\
&= a_4^\oplus \ominus a_4^\ominus \ominus a_4^\ominus = a_4 \ ,
\end{aligned}$$

the max-algebraic polynomial (B.21) is the MACP of B_B.

Case C:

We always have $a_2^\ominus \neq \varepsilon$ in Case C. Furthermore, $a_1^\ominus \neq \varepsilon$ by Lemma B.2.4(1) and $a_2^\oplus = a_1^\ominus \otimes a_1^\ominus$ by Lemma B.2.4(2). As a consequence, the condition $a_1^\ominus \otimes a_3^\oplus = a_2^\ominus \otimes a_2^\oplus$ implies that $\kappa_{2,3} = \frac{a_3^\oplus}{a_2^\ominus} = \frac{a_2^\oplus}{a_1^\ominus} = a_1^\ominus$ and $a_1^\ominus \otimes \kappa_{2,3} = a_1^\ominus \otimes \frac{a_3^\oplus}{a_2^\ominus} = a_2^\oplus$.
Since $a_4^\oplus \leqslant a_2^\ominus \otimes a_2^\ominus$ and $a_2^\ominus \neq \varepsilon$ in Case C, we have $\kappa_{2,4} \leqslant a_2^\ominus$.
The condition $a_1^\ominus \otimes a_4^\oplus = a_2^\ominus \otimes a_3^\oplus$ leads to $a_1^\ominus \otimes \kappa_{2,4} = a_1^\ominus \otimes \frac{a_4^\oplus}{a_2^\ominus} = a_3^\oplus$.
We also have $a_2^\ominus \otimes \kappa_{2,3} = a_3^\oplus$ and $a_2^\ominus \otimes \kappa_{2,4} = a_4^\oplus$. Furthermore, $a_3^\ominus = \varepsilon$ by Lemma B.2.4(5). Moreover, $a_4^\ominus = \varepsilon$ in Case C.
Now we have

$$\begin{aligned}
b_1 &= \ominus a_1^\ominus \ominus \kappa_{2,3} = \ominus a_1^\ominus = a_1 \\
b_2 &= a_1^\ominus \otimes \kappa_{2,3} \ominus a_2^\ominus \ominus \kappa_{2,4} = a_2^\oplus \ominus a_2^\ominus = a_2 \\
b_3 &= a_1^\ominus \otimes \kappa_{2,4} \oplus a_2^\ominus \otimes \kappa_{2,3} = a_3^\oplus = a_3 \\
b_4 &= a_2^\ominus \otimes \kappa_{2,4} = a_4^\oplus = a_4 \ .
\end{aligned}$$

This means that (B.21) is the MACP of B_C. □

B.3 The Max-Algebraic Characteristic Polynomial of a Max-Algebraic Upper Hessenberg Matrix with Zeros on the First Subdiagonal

In Chapter 5 and in the previous sections of this appendix we have encountered matrices with the following structure:

$$A = \begin{bmatrix} \kappa_{0,1} & \kappa_{0,2} & \kappa_{0,3} & \cdots & \kappa_{0,n-1} & \kappa_{0,n} \\ 0 & \kappa_{1,2} & \kappa_{1,3} & \cdots & \kappa_{1,n-1} & \kappa_{1,n} \\ \varepsilon & 0 & \kappa_{2,3} & \cdots & \kappa_{2,n-1} & \kappa_{2,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \varepsilon & \varepsilon & \varepsilon & \cdots & \kappa_{n-2,n-1} & \kappa_{n-2,n} \\ \varepsilon & \varepsilon & \varepsilon & \cdots & 0 & \kappa_{n-1,n} \end{bmatrix} . \tag{B.22}$$

A matrix of this form will be called a *max-algebraic upper Hessenberg matrix with zeros on the first subdiagonal.*

Let us now derive some formulas to compute the coefficients of the MACP of a max-algebraic upper Hessenberg matrix with zeros on the first subdiagonal. Let $A \in \mathbb{R}_\varepsilon^{n\times n}$ be a max-algebraic upper Hessenberg matrix of the form (B.22). Let $m \in \{1, 2, \ldots, n\}$ and $k \in \{0, 1, \ldots, m\}$. We represent the coefficient of $\lambda^{\otimes^{m-k}}$ in the MACP of the submatrix $A_{\{1,2,\ldots,m\},\{1,2,\ldots,m\}}$ of A by $a_k(m)$. So

$$a_k(m) \;=\; (\ominus 0)^{\otimes^k} \otimes \bigoplus_{\{i_1,\ldots,i_k\}\in\mathcal{C}_m^k} \det\nolimits_\oplus A_{\{i_1,\ldots,i_k\},\{i_1,\ldots,i_k\}} \;.$$

Note that

$$a_0(m) = 0 \tag{B.23}$$

$$a_1(m) = \ominus \bigoplus_{i=1}^{m} \kappa_{i-1,i} \tag{B.24}$$

for all m.

We shall use the following lemma:

Lemma B.3.1 *Let $A \in \mathbb{R}_\varepsilon^{n\times n}$ be a max-algebraic upper Hessenberg matrix of the form (B.22) and let $k \in \{1, 2, \ldots, n\}$. Then we have*

$$(\ominus 0)^{\otimes^s} \otimes \bigoplus_{\{i_1,i_2,\ldots,i_s\}\in\mathcal{C}_{n-k+s}^s} \det\nolimits_\oplus A_{\{i_1,i_2,\ldots,i_s,n-k+s+1\},\{i_1,i_2,\ldots,i_s,n\}}$$
$$= \bigoplus_{r=n-k}^{n-k+s} a_{k-n+r}(r) \otimes \kappa_{r,n} \tag{B.25}$$

for $s = 1, 2, \ldots, k-1$.

Proof: We shall prove this lemma by induction. Let us denote the left-hand side of (B.25) by $\rho(s)$.
Note that the value of s should always be less than or equal to $k-1$ since the row index $n-k+s+1$ should be less than or equal to n.
Define

$$\gamma_{i,j} = \begin{cases} 0 & \text{if } i = j+1\,, \\ \varepsilon & \text{otherwise,} \end{cases}$$

for $i = 2, 3, \ldots, n$ and $j = 1, 2, \ldots, n-1$.
First we prove that (B.25) holds for $s = 1$. We have

$$\begin{aligned} \rho(1) &= (\ominus 0) \otimes \bigoplus_{\{i_1\} \in \mathcal{C}^1_{n-k+1}} \det\nolimits_{\oplus} A_{\{i_1, n-k+2\}, \{i_1, n\}} \\ &= \ominus \bigoplus_{r=1}^{n-k+1} \det\nolimits_{\oplus} \begin{bmatrix} \kappa_{r-1,r} & \kappa_{r-1,n} \\ \gamma_{n-k+2,r} & \kappa_{n-k+1,n} \end{bmatrix} \\ &= \ominus \bigoplus_{r=1}^{n-k+1} \kappa_{r-1,r} \otimes \kappa_{n-k+1,n} \;\oplus\; \bigoplus_{r=1}^{n-k+1} \gamma_{n-k+2,r} \otimes \kappa_{r-1,n} \quad . \end{aligned}$$

For any $r \in \{1, 2, \ldots, n-k+1\}$ we have $\gamma_{n-k+2,r} = 0$ if $n-k+2 = r+1$ and $\gamma_{n-k+2,r} = \varepsilon$ otherwise. Hence,

$$\rho(1) = \left(\ominus \bigoplus_{r=1}^{n-k+1} \kappa_{r-1,r} \right) \otimes \kappa_{n-k+1,n} \;\oplus\; \kappa_{n-k,n} \quad .$$

By (B.23) and (B.24) this results in

$$\begin{aligned} \rho(1) &= a_1(n-k+1) \otimes \kappa_{n-k+1,n} \;\oplus\; a_0(n-k) \otimes \kappa_{n-k,n} \\ &= \bigoplus_{r=n-k}^{n-k+1} a_{k-n+r}(r) \otimes \kappa_{r,n} \quad . \end{aligned}$$

So (B.25) holds for $s = 1$.
Let $S \in \{2, 3, \ldots, k-1\}$. Now we assume that formula (B.25) holds for $s = S-1$ and we prove that it also holds for $s = S$:

$$\begin{aligned} \rho(S) &= (\ominus 0)^{\otimes^S} \otimes \bigoplus_{\{i_1, \ldots, i_S\} \in \mathcal{C}^S_{n-k+S}} \det\nolimits_{\oplus} A_{\{i_1, \ldots, i_S, n-k+S+1\}, \{i_1, \ldots, i_S, n\}} \\ &= (\ominus 0)^{\otimes^S} \otimes \\ &\qquad \bigoplus_{\{i_1, \ldots, i_S\} \in \mathcal{C}^S_{n-k+S}} \det\nolimits_{\oplus} \begin{bmatrix} \kappa_{i_1-1,i_1} & \ldots & \kappa_{i_1-1,i_S} & \kappa_{i_1-1,n} \\ \gamma_{i_2,i_1} & \ldots & \kappa_{i_2-1,i_S} & \kappa_{i_2-1,n} \\ \vdots & \ddots & \vdots & \vdots \\ \varepsilon & \ldots & \gamma_{n-k+S+1,i_S} & \kappa_{n-k+S,n} \end{bmatrix} \end{aligned}$$

$$\begin{aligned}
&= (\ominus 0)^{\otimes^{S}} \otimes \bigoplus_{\{i_1,\ldots,i_S\}\in\mathcal{C}^S_{n-k+S}} \det\nolimits_{\oplus} A_{\{i_1,\ldots,i_S\},\{i_1,\ldots,i_S\}} \otimes \kappa_{n-k+S,n} \;\oplus \\
&\quad (\ominus 0)^{\otimes^{S+1}} \otimes \bigoplus_{\{i_1,\ldots,i_S\}\in\mathcal{C}^S_{n-k+S}} \gamma_{n-k+S+1,i_S} \otimes \det\nolimits_{\oplus} A_{\{i_1,\ldots,i_S\},\{i_1,\ldots,i_{S-1},n\}} \\
&= a_S(n-k+S) \otimes \kappa_{n-k+S,n} \;\oplus \\
&\quad (\ominus 0)^{\otimes^{S-1}} \otimes \bigoplus_{\{i_1,\ldots,i_{S-1}\}\in\mathcal{C}^{S-1}_{n-k+S-1}} \det\nolimits_{\oplus} A_{\{i_1,\ldots,i_{S-1},n-k+S\},\{i_1,\ldots,i_{S-1},n\}}
\end{aligned}$$

since $\gamma_{n-k+S+1,i_S} = 0$ if $n-k+S = i_S$ and $\gamma_{n-k+S+1,i_S} = \varepsilon$ otherwise. By the induction hypothesis (B.25) holds for $s = S-1$. Hence,

$$\begin{aligned}
\rho(S) &= a_S(n-k+S) \otimes \kappa_{n-k+S,n} \;\oplus \bigoplus_{r=k-n}^{n-k+S-1} a_{k-n+r}(r) \otimes \kappa_{r,n} \\
&= \bigoplus_{r=n-k}^{n-k+S} a_{k-n+r}(r) \otimes \kappa_{r,n} \quad .
\end{aligned}$$

So now we have proved that (B.25) holds for $s = 1,2,\ldots,k-1$. □

Proposition B.3.2 *If $A \in \mathbb{R}_\varepsilon^{n\times n}$ is a max-algebraic upper Hessenberg matrix of the form (B.22) then we have*

$$a_0^{\text{pos}}(m) = 0 \tag{B.26}$$

$$a_0^{\text{neg}}(m) = \varepsilon \tag{B.27}$$

$$a_k^{\text{pos}}(m) = a_k^{\text{pos}}(m-1) \;\oplus \bigoplus_{r=m-k}^{m-1} a_{k-n+r}^{\text{neg}}(r) \otimes \kappa_{r,m} \tag{B.28}$$

$$a_k^{\text{neg}}(m) = a_k^{\text{neg}}(m-1) \;\oplus \bigoplus_{r=m-k}^{m-1} a_{k-n+r}^{\text{pos}}(r) \otimes \kappa_{r,m} \tag{B.29}$$

for $m = 1,2,\ldots,n$ and $k = 1,2,\ldots,m$, where $a_m^{\text{pos}}(m-1) = a_m^{\text{neg}}(m-1) = \varepsilon$ for $m = 1,2,\ldots,n$ by definition.

Proof: Formulas (B.26) and (B.27) are a direct consequence of (B.23). Consider an arbitrary $m \in \{1,2,\ldots,n\}$ and an arbitrary $k \in \{1,2,\ldots,m\}$. We have

$$\begin{aligned}
a_k(m) &= (\ominus 0)^{\otimes^{k}} \otimes \bigoplus_{\{i_1,\ldots,i_k\}\in\mathcal{C}^k_m} \det\nolimits_{\oplus} A_{\{i_1,\ldots,i_k\},\{i_1,\ldots,i_k\}} \\
&= (\ominus 0)^{\otimes^{k}} \otimes \bigoplus_{\{i_1,\ldots,i_k\}\in\mathcal{C}^k_{m-1}} \det\nolimits_{\oplus} A_{\{i_1,\ldots,i_k\},\{i_1,\ldots,i_k\}} \;\oplus
\end{aligned}$$

$$(\ominus 0)^{\otimes^k} \otimes \bigoplus_{\{i_1,\ldots,i_{k-1}\}\in\mathcal{C}_{m-1}^{k-1}} \det\nolimits_\oplus A_{\{i_1,\ldots,i_{k-1},m\},\{i_1,\ldots,i_{k-1},m\}} \ .$$

If we take into account that $a_m(m-1) = \varepsilon$ by definition and if we use Lemma B.3.1 with $n = m$ and $s = k-1$, we obtain

$$a_k(m) \ = \ a_k(m-1) \ \oplus \ (\ominus 0) \otimes \bigoplus_{r=m-k}^{m-1} a_{k-m+r}(r) \otimes \kappa_{r,m} \ . \tag{B.30}$$

If we extract the max-positive and the max-negative contributions to this expression, we get formulas (B.28) and (B.29). □

If $A \in \mathbb{R}_\varepsilon^{n\times n}$ is a max-algebraic upper Hessenberg matrix with zeros on the first subdiagonal then we can use the recursive formulas of Proposition B.3.2 to develop an efficient algorithm to compute the coefficients of the MACP of A.

Consider the following scheme:

$$\begin{array}{cccccc} & m=1 & m=2 & m=3 & \ldots & m=n \\ k=1 & a_1(1) & a_1(2) & a_1(3) & \ldots & a_1(n) \\ k=2 & & a_2(2) & a_2(3) & \ldots & a_2(n) \\ k=3 & & & a_3(3) & \ldots & a_3(n) \\ \vdots & & & & \ddots & \vdots \\ k=n & & & & & a_n(n) \ . \end{array} \tag{B.31}$$

Note that the coefficient $a_k(m)$ is on the $(m-k+1)$st diagonal of this scheme. The max-positive and the max-negative contributions to the entries of this scheme can be computed diagonal by diagonal with formulas (B.28) and (B.29). The last column of the scheme contains the coefficients of the MACP of A. It is easy to verify that the complexity of this algorithm to compute the coefficients of the MACP of a max-algebraic upper Hessenberg matrix $A \in \mathbb{R}_\varepsilon^{n\times n}$ with zeros on the first subdiagonal is $O(n^3)$.

In Section B.4 we shall use the following lemma:

Lemma B.3.3 *Let $A \in \mathbb{R}_\varepsilon$ be a max-algebraic upper Hessenberg matrix of the form (B.22) and let $\lambda^{\otimes^n} \oplus \bigoplus_{k=1}^{n} a_k \otimes \lambda^{\otimes^{n-k}}$ be the MACP of A. Let $k \in \{1,2,\ldots,n\}$. Then the max-positive and the max-negative contributions to a_k are given by*

$$a_k^{\mathrm{pos}} = \bigoplus_{l=1}^{\lfloor \frac{k}{2} \rfloor} \bigoplus_{(i_1,\ldots,i_{2l},\delta_1,\ldots,\delta_{2l})\in\Phi(n,k,2l)} \bigotimes_{s=1}^{2l} \kappa_{i_s-1,i_s+\delta_s-1} \tag{B.32}$$

and

$$a_k^{\mathrm{neg}} = \bigoplus_{l=0}^{\left\lfloor \frac{k-1}{2} \right\rfloor} \bigoplus_{(i_1,\ldots,i_{2l+1},\delta_1,\ldots,\delta_{2l+1}) \in \Phi(n,k,2l+1)} \bigotimes_{s=1}^{2l+1} \kappa_{i_s-1,i_s+\delta_s-1} \qquad \text{(B.33)}$$

respectively, where the empty max-algebraic sum $\bigoplus_{l=1}^{0} \ldots$ *is equal to* ε *by definition and where*

$$\begin{aligned}\Phi(n,k,l) = \Big\{ (i_1,i_2,\ldots,i_l,\delta_1,\delta_2,\ldots,\delta_l) \,\Big|\, & \{i_1,i_2,\ldots,i_l\} \in \mathcal{C}_n^l, \\ & i_1 < i_2 < \ldots < i_l,\ \ \delta_1,\delta_2,\ldots,\delta_l \in \mathbb{N}_0,\ \ \delta_s \leqslant i_{s+1} - i_s \ \text{ for} \\ & s = 1,2,\ldots,l,\ \ \delta_l \leqslant n - i_l + 1 \ \text{ and } \sum_{s=1}^{l} \delta_s = k \Big\} \ .\end{aligned}$$

Proof: We shall prove by induction that

$$a_k(m) = \bigoplus_{l=1}^{k} \left((\ominus 0)^{\otimes^l} \otimes \bigoplus_{(i_1,\ldots,i_l,\delta_1,\ldots,\delta_l) \in \Phi(m,k,l)} \bigotimes_{s=1}^{l} \kappa_{i_s-1,i_s+\delta_s-1} \right) \qquad \text{(B.34)}$$

for $m = 1,2,\ldots,n$ and $k = 1,2,\ldots,m$. If we extract the max-positive and max-negative contributions to (B.34) with $m = n$ and $k \in \{1,2,\ldots,n\}$, we get formulas (B.32) and (B.33).

First we prove that (B.34) holds for $k = 1$ and for $m = 1,2,\ldots,n$. We have

$$\begin{aligned}\Phi(m,1,1) &= \big\{ (i_1,\delta_1) \,\big|\, \{i_1\} \in \mathcal{C}_m^1,\ \ \delta_1 \in \mathbb{N}_0,\ \ \delta_1 \leqslant m - i_1 + 1 \ \text{ and } \ \delta_1 = 1 \big\} \\ &= \big\{ (s,1) \,\big|\, s = 1,2,\ldots,m \big\} \ .\end{aligned}$$

So for $k = 1$ the right-hand side of (B.34) is equal to $(\ominus 0) \otimes \bigoplus_{s=1}^{m} \kappa_{s-1,s}$, which is equal to $a_1(m)$ by (B.24). So formula (B.34) holds for $k = 1$ and for $m = 1,2,\ldots,n$.

Consider again scheme (B.31). In the first step of this proof we have proved that formula (B.34) holds for all the coefficients on the first row of this scheme. Recall that the coefficients of the scheme can be computed diagonal by diagonal with formula (B.30). Now we prove by induction that (B.34) holds for $m = 2,3,\ldots,n$ and $k = 2,3,\ldots,m$.

Let $M \in \{2,3,\ldots,n\}$ and $K \in \{2,3,\ldots,M\}$. Assume that (B.34) holds for all the coefficients $a_k(m)$ that are on the $(M-K)$th diagonal (if $K \neq M$) and for all the coefficients $a_k(m)$ that are on $(M-K+1)$st diagonal but that have smaller k and m indices than $a_K(M)$. We now prove that (B.34) also holds for

$a_K(M)$. We start from formula (B.30) with $k = K$ and $m = M$:

$$a_K(M) = a_K(M-1) \ominus \bigoplus_{r=M-K}^{M-1} a_{K-M+r}(r) \otimes \kappa_{r,M} \ . \tag{B.35}$$

Note that we do not allow $k = 0$ in (B.34). Therefore, we extract the term of (B.35) that contains $a_0(M-K)$ and put it apart. Since $a_0(M-K)$ is equal to 0, this leads to

$$a_K(M) = a_K(M-1) \ominus \kappa_{M-K,K} \ominus \bigoplus_{r=M-K+1}^{M-1} a_{K-M+r}(r) \otimes \kappa_{r,M} \ . \tag{B.36}$$

Assume that $K \neq M$. By the induction hypothesis the right-hand side of (B.36) is equal to

$$\bigoplus_{l=1}^{K} \left((\ominus 0)^{\otimes^l} \otimes \bigoplus_{(i_1,\ldots,i_l,\delta_1,\ldots,\delta_l) \in \Phi(M-1,K,l)} \bigotimes_{s=1}^{l} \kappa_{i_s-1,i_s+\delta_s-1} \right) \ominus \kappa_{M-K,M} \ominus$$
$$\bigoplus_{r=M-K+1}^{M-1} \bigoplus_{l=1}^{K-M+r} \left((\ominus 0)^{\otimes^l} \otimes \bigoplus_{(i_1,\ldots,i_l,\delta_1,\ldots,\delta_l) \in \Phi(r,K-M+r,l)} \left(\bigotimes_{s=1}^{l} \kappa_{i_s-1,i_s+\delta_s-1} \right) \otimes \kappa_{r,M} \right) . \tag{B.37}$$

Let us now see what happens if $a_K(M)$ is on the first diagonal of scheme (B.31) or equivalently if $K = M$. If $K = M$, the first term of the right-hand side of (B.36) is equal to ε by definition (cf. Proposition B.3.2). Now we prove by contradiction that $\Phi(M-1, M, l) = \emptyset$ for $l = 1, 2, \ldots, M$.
Consider an arbitrary $l \in \{1, 2, \ldots, M\}$. If $(i_1, \ldots, i_l, \delta_1, \ldots, \delta_l) \in \Phi(M-1, M, l)$ then we have $\delta_1 \leqslant i_2 - i_1$, $\delta_2 \leqslant i_2 - i_1$, $\ldots$, $\delta_{l-1} \leqslant i_l - i_{l-1}$, $\delta_l \leqslant M - i_l$ and thus $\delta_1 + \delta_2 + \ldots + \delta_l \leqslant M - i_1$. Since $i_1 \geqslant 1$, this implies that $\delta_1 + \delta_2 + \ldots + \delta_l \leqslant M - 1$, which is in contradiction with the condition $\delta_1 + \delta_2 + \ldots + \delta_l = M$. Hence, $\Phi(M-1, M, l) = \emptyset$.

Since the empty max-algebraic sum $\bigoplus_{\phi \in \emptyset} \ldots$ is equal to ε by definition, the first max-algebraic summation of (B.37) is also equal to ε. So we could say that (B.37) also holds if $k = m = K = M$.
Now we show that expression (B.37) is equal to

$$\bigoplus_{l=1}^{K} \left((\ominus 0)^{\otimes^l} \otimes \bigoplus_{(i_1,\ldots,i_l,\delta_1,\ldots,\delta_l) \in \Phi(M,K,l)} \bigotimes_{s=1}^{l} \kappa_{i_s-1,i_s+\delta_s-1} \right) \tag{B.38}$$

by proving that every term of (B.37) also appears in (B.38) and vice versa.

- Since $\Phi(M-1,K,l) \subseteq \Phi(M,K,l)$ for all l, every term of the first max-algebraic summation of (B.37) also appears in (B.38).

- The term $\ominus\, \kappa_{M-K,M}$ corresponds to the pair $(M-K+1,K) \in \Phi(M,K,1)$. So this term also appears in (B.37).

- Let $r \in \{M-K+1, M-K+2, \ldots, M-1\}$, $l \in \{1,2,\ldots,K-M+r\}$ and $(i_1,\ldots,i_l,\delta_1,\ldots,\delta_l) \in \Phi(r,K-M+r,l)$. If we define $l_1 = l+1$, $i_{l_1} = r+1$ and $\delta_{l_1} = M-r$, then we have

$$\ominus\left((\ominus 0)^{\otimes^{l}} \otimes \left(\bigotimes_{s=1}^{l} \kappa_{i_s-1,i_s+\delta_s-1} \right) \otimes \kappa_{r,M} \right) = (\ominus 0)^{\otimes^{l_1}} \otimes \bigotimes_{s=1}^{l_1} \kappa_{i_s-1,i_s+\delta_s-1} \quad . \tag{B.39}$$

  Since $l \leqslant K-1$, we have $l_1 \leqslant K$. Let us prove that $(i_1,\ldots,i_{l_1},\delta_1,\ldots,\delta_{l_1}) \in \Phi(M,K,l_1)$. Since $i_{l_1} = r+1 \leqslant M$, $\{i_1,\ldots,i_l\} \in \mathcal{C}_r^l$ and $i_1 < i_2 < \ldots < i_l \leqslant r$, we have $\{i_1,\ldots,i_{l_1}\} \in \mathcal{C}_M^l$ and $i_1 < i_2 < \ldots < i_{l_1}$. The condition $\delta_l \leqslant r - i_l + 1$ results in $\delta_l \leqslant i_{l_1} - i_{l_1-1}$. Furthermore, $\delta_{l_1} = M - r \in \mathbb{N}_0$ and $\delta_{l_1} \leqslant M - i_{l_1} + 1 = M - r$. We have $\delta_1 + \delta_2 + \ldots + \delta_l = K - M + r$, which leads to $\delta_1 + \delta_2 + \ldots + \delta_{l_1} = (K-M+r) + (M-r) = K$. Hence, $(i_1,\ldots,i_{l_1},\delta_1,\ldots,\delta_{l_1}) \in \Phi(M,K,l_1)$. As a consequence, term (B.39) also appears in (B.38).

So now we have proved that every term of (B.37) appears in (B.38).
Now we consider an arbitrary term $t_{l\phi}$ of (B.38) that corresponds to a certain value of l and to the $2l$-tuple $\phi = (i_1,\ldots,i_l,\delta_1,\ldots,\delta_l) \in \Phi(M,K,l)$.

- If $\delta_l \leqslant M - i_l$ then $i_l \neq M$ since otherwise we would have $\delta_l = 0$, which is not allowed. Hence, $\{i_1,\ldots,i_l\} \in \mathcal{C}_{M-1}^l$. So $\phi \in \Phi(M-1,K,l)$, which means that $t_{l\phi}$ appears in the first max-algebraic summation of (B.37).

- Now assume that $\delta_l > M - i_l$. This is only possible if $i_l + \delta_l - 1 = M$.
  If $l = 1$, we have $\delta_l = \delta_1 = K$ and thus $i_l = M - K + 1$. Hence, $t_{l\phi} = (\ominus 0) \otimes \kappa_{M-K,M}$. This term also appears in (B.37).
  From now on we assume that $l > 1$. We define r such that $i_l = r+1$ and thus also $\delta_l = M - r$. Since $\delta_1 + \delta_2 + \ldots + \delta_l = K$, $l > 1$ and $\delta_1, \delta_2, \ldots, \delta_l \in \mathbb{N}_0$, we have $1 \leqslant \delta_l \leqslant K-1$. Hence, $r \in \{M-K+1, M-K+2, \ldots, M-1\}$.
  If we define $l_1 = l - 1$, then we have $l_1 \geqslant 1$ and

$$\begin{aligned} t_{l\phi} &= (\ominus 0)^{\otimes^{l}} \otimes \bigotimes_{s=1}^{l} \kappa_{i_s-1,i_s+\delta_s-1} \\ &= \ominus\left((\ominus 0)^{\otimes^{l_1}} \otimes \left(\bigotimes_{s=1}^{l_1} \kappa_{i_s-1,i_s+\delta_s-1} \right) \otimes \kappa_{r,M} \right) \quad . \end{aligned} \tag{B.40}$$

Since $\{i_1, \ldots, i_{l_1+1}\} \in \mathcal{C}_M^{l_1+1}$ and $i_1 < i_2 < \ldots < i_{l_1+1} = r + 1$, we have $\{i_1, \ldots, i_{l_1}\} \in \mathcal{C}_r^{l_1}$. The condition $\delta_{l-1} \leqslant i_l - i_{l-1}$ results in $\delta_{l_1} \leqslant r - i_{l_1} + 1$. Furthermore, $\delta_1 + \delta_2 + \ldots + \delta_{l_1} = K - \delta_l = K - M + r$. Hence, $(i_1, \ldots, i_{l_1}, \delta_1, \ldots, \delta_{l_1}) \in \Phi(r, K - M + r, l_1)$. Since $\delta_1 + \delta_2 + \ldots + \delta_{l_1} = K - M + r$ and $\delta_1, \delta_2, \ldots, \delta_{l_1} \geqslant 1$, we have $l_1 \leqslant K - M + r$. Hence, term (B.40) also appears in (B.37).

So every term of (B.38) appears in (B.37).
As a consequence, expressions (B.37) and (B.38) are equal, which means that $a_K(M)$ is also equal to (B.38).
Hence, (B.34) holds for $m = 1, 2, \ldots, n$ and $k = 1, 2, \ldots, m$. □

Let us illustrate formulas (B.32) and (B.33) by writing down the formulas for the max-positive and the max-negative contributions to the coefficients of the MACP of the matrix

$$A = \begin{bmatrix} \kappa_{0,1} & \kappa_{0,2} & \kappa_{0,3} \\ 0 & \kappa_{1,2} & \kappa_{1,3} \\ \varepsilon & 0 & \kappa_{2,3} \end{bmatrix} .$$

We have

$$\begin{aligned}
a_1^{\text{pos}} &= \varepsilon \\
a_1^{\text{neg}} &= \kappa_{0,1} \ \oplus \ \kappa_{1,2} \ \oplus \ \kappa_{2,3} \\
a_2^{\text{pos}} &= \kappa_{0,1} \otimes \kappa_{1,2} \ \oplus \ \kappa_{0,1} \otimes \kappa_{2,3} \ \oplus \ \kappa_{1,2} \otimes \kappa_{2,3} \\
a_2^{\text{neg}} &= \kappa_{0,2} \ \oplus \ \kappa_{1,3} \\
a_3^{\text{pos}} &= \kappa_{0,1} \otimes \kappa_{1,3} \ \oplus \ \kappa_{0,2} \otimes \kappa_{2,3} \\
a_3^{\text{neg}} &= \kappa_{0,3} \ \oplus \ \kappa_{0,1} \otimes \kappa_{1,2} \otimes \kappa_{2,3} \ .
\end{aligned}$$

If we consider a_k^{pos} with k equal to 2 or 3, then we see that each term consists of an even number of factors; the sum of the differences between the second and the first index of the factors of each term is equal to k and the sequence of the indices of the factors of each term is nondecreasing.
Each term of a_k^{neg} with k equal to 1, 2 or 3 consists of an odd number of factors; the sum of the differences between the second and the first index of each factor is equal to k and the sequence of the indices of the factors of each term is nondecreasing.

B.4 A Heuristic Algorithm for the Construction of Matrices with a Given Max-Algebraic Characteristic Polynomial

In this section we present a heuristic algorithm that will in most cases result in a matrix with entries in $\mathbb{R}_\varepsilon$ and with a MACP that is equal to a given max-algebraic polynomial.

Extrapolating the results of Section 5.3 and supported by many examples we state the following conjecture:

Conjecture B.4.1 *If* $\lambda^{\otimes^n} \oplus \bigoplus_{k=1}^{n} a_k \otimes \lambda^{\otimes^{n-k}}$ *is the MACP of an n by n matrix with entries in $\mathbb{R}_\varepsilon$ then it is also the MACP of an n by n max-algebraic upper Hessenberg matrix with zeros on the first subdiagonal.*

So this conjecture states that if a given max-algebraic polynomial of degree n is the MACP of a matrix with entries in $\mathbb{R}_\varepsilon$, we can always construct a matrix $A \in \mathbb{R}_\varepsilon^{n\times n}$ of the form (B.22) such that the MACP of A is equal to the given polynomial.

If Conjecture B.4.1 would hold, we could use the formulas of Lemma B.3.3 or Proposition B.3.2 to obtain a system of multivariate max-algebraic polynomial equalities and inequalities to construct a matrix $A \in \mathbb{R}_\varepsilon^{n\times n}$ of the form (B.22) with a MACP that coincides with a given monic nth degree max-algebraic polynomial.

If we take the formulas of Lemma B.3.3, we get an ELCP with $\dfrac{n^2}{2} + \dfrac{n}{2} + 1$ variables and $O\left(\left(\dfrac{3+\sqrt{5}}{2}\right)^n\right)$ inequalities. So this ELCP has less variables than the full ELCP derived in Section 5.4, which has $n^2 + 1$ variables. The number of inequalities of the ELCP based on the formulas of Lemma B.3.3 is much smaller than the number of inequalities of the full ELCP, which lies between $e(n! - n)$ and $en! + 1$.

The number of inequalities of the ELCP that results from the formulas of Proposition B.3.2 is $O(n^3)$, which is much smaller than the number of inequalities of the full ELCP. It is also considerably smaller than the number of inequalities of the ELCP based on the formulas of Lemma B.3.3. If we use the ELCP algorithm of Section 3.4 to solve the ELCP then this will have a positive effect on the execution time of the algorithm. However, this effect is counteracted by the fact that there are now $\dfrac{3}{2}n^2 - \dfrac{3}{2}n + 2$ variables: the entries of the upper triangular part of A, the dummy variables $a_1^{\text{neg}}(1), \ldots, a_1^{\text{neg}}(n-1)$, $a_k^{\text{pos}}(m)$, $a_k^{\text{neg}}(m)$ for $m = 2, \ldots, n-1$ and $k = 2, \ldots, m$ and an extra variable α to make the ELCP homogeneous. So the number of variables of the ELCP that results from the formulas of Proposition B.3.2 is greater than the number of variables of the full ELCP derived in Section 5.4 and it is about 2 to 3 times the number of variables of the ELCP based on the formulas of Lemma B.3.3. Since in general the average execution time of our ELCP algorithm depends polynomially on the number of inequalities and more or less exponentially on the number of variables, the fact that we have more variables in the ELCP based on the formulas of Lemma B.3.3 completely annihilates the positive effect of having a much smaller number of inequalities.

Furthermore, experiments show that in general the solution set of an ELCP that is based on the formulas of Proposition B.3.2 is so complex that it takes even more time to solve this ELCP with our ELCP algorithm than to solve the corresponding full ELCP. In practice it seems that our ELCP algorithm performs best on ELCPs based on the formulas of Lemma B.3.3.

The ELCP approach based on Conjecture B.4.1 and the formulas of Lemma B.3.3 allows us to tackle larger problems than the full ELCP approach. However, the major disadvantage of this approach stays its computational complexity, which means that it can only be used for max-algebraic polynomials with a small degree.

Therefore, we now present a heuristic algorithm based on Conjecture B.4.1 to construct a matrix $B \in \mathbb{R}_\varepsilon^{n \times n}$ of the form (B.22) such that the MACP of B will be equal to a given monic nth degree max-algebraic polynomial $\lambda^{\otimes^n} \oplus \bigoplus_{k=1}^{n} a_k \otimes \lambda^{\otimes^{n-k}}$. This algorithm will on the average be much faster than the ELCP-based algorithms and it will allow us to deal with max-algebraic polynomials with a larger degree. If a result is returned, it is right. But it is possible that sometimes no result is returned although there is a solution. Note that it is advisable to check whether the coefficients of the given max-algebraic polynomial satisfy the necessary conditions of Propositions 5.2.5 and 5.2.6 before executing the algorithm.

In the initialization step of the heuristic algorithm, we reconstruct the a_k^{neg}'s by setting $a_1^{\mathrm{neg}} = a_1^{\ominus}$ and $a_k^{\mathrm{neg}} = (-\delta) \otimes a_k^{\oplus} \oplus a_k^{\ominus} = \max(a_k^{\oplus} - \delta, a_k^{\ominus})$ for $k = 2, 3, \ldots, n$ where δ is a small positive real number.

Consider

$$T_1 = \begin{bmatrix} a_1^{\mathrm{neg}} & a_2^{\mathrm{neg}} & a_3^{\mathrm{neg}} & \ldots & a_{n-1}^{\mathrm{neg}} & a_n^{\mathrm{neg}} \\ 0 & a_1^{\mathrm{neg}} & a_2^{\mathrm{neg}} & \ldots & a_{n-2}^{\mathrm{neg}} & a_{n-1}^{\mathrm{neg}} \\ \varepsilon & 0 & a_1^{\mathrm{neg}} & \ldots & a_{n-3}^{\mathrm{neg}} & a_{n-2}^{\mathrm{neg}} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ \varepsilon & \varepsilon & \varepsilon & \ldots & 0 & a_1^{\mathrm{neg}} \end{bmatrix}$$

and

$$T_2 = \begin{bmatrix} \varepsilon & \varepsilon & \varepsilon & \ldots & \varepsilon \\ \varepsilon & \mu_{1,2} & \mu_{1,3} & \ldots & \mu_{1,n} \\ \varepsilon & \varepsilon & \mu_{2,3} & \ldots & \mu_{2,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \varepsilon & \varepsilon & \varepsilon & \ldots & \mu_{n-1,n} \end{bmatrix}$$

where

$$\mu_{i,j} = \begin{cases} \dfrac{a_j^{\oplus}}{a_i^{\mathrm{neg}}} & \text{if } a_i^{\mathrm{neg}} \neq \varepsilon\,, \\ \varepsilon & \text{if } a_i^{\mathrm{neg}} = \varepsilon\,. \end{cases}$$

In each major step of the heuristic algorithm we shall select entries of T_1 and T_2 to compose a new matrix until the MACP of this matrix coincides with the given max-algebraic polynomial or until all possible choices have been considered. In the latter case the algorithm will return an empty matrix to indicate that it did not find a solution.

We shall give the heuristic algorithm in the same pseudo programming language as the one that has been used to describe the ELCP algorithm of Section 3.4. We use a function called maxcharpoly to compute some or all of the coefficients of the MACP of a matrix: if $\lambda^{\otimes^n} \oplus \bigoplus_{k=1}^{n} c_k \otimes \lambda^{\otimes^{n-k}}$ is the MACP of a matrix $C \in \mathbb{R}_\varepsilon^{n\times n}$ and if $1 \leqslant k \leqslant n$ then maxcharpoly(C,k) returns the column vector $\begin{bmatrix} c_1 & c_2 & \ldots & c_k \end{bmatrix}^T$. Since all the intermediate matrices that are constructed in the heuristic algorithm are max-algebraic upper Hessenberg matrices with zeros on the first subdiagonal, we can use the efficient algorithm based on Proposition B.3.2 to calculate the coefficients of the MACP of these matrices.

A heuristic algorithm to construct a matrix with a given MACP

Input: n, $a = \begin{bmatrix} a_1 & a_2 & \ldots & a_n \end{bmatrix}^T$, δ

Initialization:

$a^{\mathrm{neg}} \leftarrow (-\delta) \otimes a^{\oplus} \;\oplus\; a^{\ominus}$ { *Reconstruct the* a_k^{neg}*'s.* }

{ *Construct the* μ_{ij}*'s.* }

for $i = 1, 2, \ldots, n-1$ do

  if $a_i^{\mathrm{neg}} \neq \varepsilon$ **then**

    $\forall j \in \{i+1, i+2, \ldots, n\} : \mu_{ij} \leftarrow \left\lfloor \dfrac{a_j^{\oplus}}{a_i^{\mathrm{neg}}} \right\rfloor$

  else

    $\forall j \in \{i+1, i+2, \ldots, n\} : \mu_{ij} \leftarrow \varepsilon$

  end if

end for

$\tau \leftarrow O_{n\times 1}$ { *Initialize the offset vector.* }

Main loop:

continue $\leftarrow 1$

while *continue* $= 1$ **do**

  { *Construct a template matrix* T *that has zeros on the first* }

  { *subdiagonal and that contains the* a_k^{neg}*'s on positions that* }

  { *are indicated by the offset vector* τ*.* }

  $T \leftarrow \mathcal{E}_{n\times n}$

  $\forall k \in \{1, 2, \ldots, n-1\} : t_{k+1,k} \leftarrow 0$

  $\forall k \in \{1, 2, \ldots, n\} : t_{1+\tau_k, k+\tau_k} \leftarrow a_k^{\mathrm{neg}}$

```
B ← T
ok ← 1
col ← 2
while ok = 1 and col ⩽ n do
    α ← {1, 2, ..., col}
    c ← maxcharpoly(B, col)
    if c ≠ a_α then
        { We continue putting one of the μ_{i,col}'s in column   }
        { col of B until the first col coefficients of the MACP }
        { of B match the first col coefficients of the given     }
        { max-algebraic polynomial.                              }
        found ← 0
        row ← 2
        while found = 0 and row ⩽ col do
            if b_{row,col} = ε and μ_{row−1,col} ≠ ε then
                b_{row,col} ← μ_{row−1,col}
                c ← maxcharpoly(B, col)
                if c = a_α then
                    found ← 1
                else
                    b_{row,col} ← ε
                end if
            end if
            row ← row + 1
        end while
        if found = 0 then
            ok ← 0
        end if
    end if
    col ← col + 1
end while
if ok = 1 then
    continue ← 0                    { A valid result has been found. }
else
    { Adapt the offset vector. }
    pos ← 1
    maxel ← n − 1
    while continue = 1 and τ_pos = maxel do
```

$pos \leftarrow pos + 1$
$maxel \leftarrow maxel - 1$
if $pos = n$ **then**
{ *All possible offset vectors have been considered* }
{ *but no solution has been found.* }
$continue \leftarrow 0$
$B \leftarrow [\,]$
end if
end while
if $continue = 1$ **then**
$\tau_{pos} \leftarrow \tau_{pos} + 1$
$\forall k \in \{1, 2, \ldots, pos - 1\} : \tau_k \leftarrow 0$
end if
end if
end while
Output: B

Note that the worst case complexity of this heuristic algorithm is $O(n!\, n^5)$ where n is the degree of the given max-algebraic polynomial. This means that in its present form this algorithm should not be used for max-algebraic polynomials with a large degree. However, experiments have shown that the average execution time of the heuristic algorithm is considerably smaller than that of an algorithm that uses the ELCP approach, and that the heuristic algorithm enables us to deal with max-algebraic polynomials with a larger degree.

We have tried out the heuristic algorithm as follows. For a given maximal dimension N we have generated a sequence of random matrices of size n by n with $5 \leqslant n \leqslant N$ and with integer entries distributed uniformly in the interval

Trial	Number of matrices	N	Number of successful runs
1	20000	8	19989
2	20000	9	19991
3	20000	10	19987
4	20000	11	19987
5	5000	12	5000

Table B.1: The results of some experiments with the heuristic algorithm to construct a matrix with a given MACP for random matrices of size n by n with $5 \leqslant n \leqslant N$.

$[-2n, 2n]$. Next we have computed the coefficients of the MACP of each of these matrices and used them as input for the algorithm. The results are summed up in Table B.1. A run is called successful if the algorithm succeeded in computing a matrix that had the same MACP as the random matrix. For the 5th trial we have only considered 5000 matrices in order to limit the time needed to perform this experiment. If we consider all the runs of the experiment then we see that the algorithm was successful in about 99.95% of the cases.
In 45 out of the 46 cases where the heuristic algorithm did not return a result, we managed to transform the original matrix into an upper Hessenberg matrix with zeros on the first subdiagonal that had same MACP as the original random matrix by applying transformations that preserve the MACP. In the remaining case (with $n = 6$) we had to use the ELCP algorithm based on the formulas of Lemma B.3.3 to construct an upper Hessenberg matrix with zeros on the first subdiagonal that had same MACP as the original random matrix.

Appendix C

Proofs of Some Lemmas of Chapter 6

In this appendix we prove some of the lemmas of Chapter 6. We also state a generalization by Gaubert of Theorem 6.3.5 and we illustrate this generalized theorem with an example.

C.1 Proof of Lemma 6.3.7

Before proving Lemma 6.3.7 we first give some additional definitions and lemmas.
The least common multiple of two positive integers c_1 and c_2 is denoted by $\operatorname{lcm}(c_1, c_2)$. If γ is a set of positive integers then the least common multiple of the elements of γ is denoted by $\operatorname{lcm}\gamma$.
Recall that if a matrix A is irreducible then there exist integers $k_0 \in \mathbb{N}$ and $c \in \mathbb{N}_0$ such that

$$A^{\otimes^{k+c}} = \lambda^{\otimes^{c}} \otimes A^{\otimes^{k}} \qquad \text{for all } k \geqslant k_0 \tag{C.1}$$

(cf. Theorem 2.2.8). The smallest possible positive integer c for which (C.1) holds is called the *cyclicity* of the matrix A. Note that the only irreducible matrix that has a max-algebraic eigenvalue that is equal to ε is the 1 by 1 max-algebraic zero matrix $[\,\varepsilon\,]$.

Lemma C.1.1 *Let $A \in \mathbb{R}_\varepsilon^{n\times n}$ be an irreducible matrix with $A \neq \mathcal{E}_{n\times n}$. If λ is the max-algebraic eigenvalue of A and if c is the cyclicity of A then we have*

$$\forall i, j \in \{1, 2, \ldots, n\}\,, \exists q_0 \in \mathbb{N} \text{ such that}$$

$$\left(A^{\otimes^{q_0+rc}}\right)_{ij} = \lambda^{\otimes^{rc}} \otimes \left(A^{\otimes^{q_0}}\right)_{ij} \neq \varepsilon \qquad \text{for all } r \in \mathbb{N}\,. \tag{C.2}$$

Proof: If n is equal to 1, the proof of the lemma is trivial. So from now on we assume that $n > 1$.
Since A is irreducible, its precedence graph $\mathcal{G}(A)$ is strongly connected. So if we select two arbitrary different vertices u and v of $\mathcal{G}(A)$ then there exists a path P_1 from u to v. There also exists a path P_2 from v to u. The concatenation of P_1 and P_2 yields a path from u to u. So if we consider two arbitrary (not necessarily different) vertices v_1 and v_2 of $\mathcal{G}(A)$, then there always exists a path from v_1 to v_2.
Consider arbitrary indices $i, j \in \{1, 2, \ldots, n\}$. Let P be a path from vertex j of $\mathcal{G}(A)$ to vertex i of $\mathcal{G}(A)$ and let C be a path from vertex i to vertex i. Let l be the length of P and let m be the length of C. Recall that if $k \in \mathbb{N}_0$ then $(A^{\otimes^k})_{ij}$ is the maximal weight of all paths of $\mathcal{G}(A)$ from vertex j to vertex i with length k; if there does not exist a path from vertex j to vertex i with length k then $(A^{\otimes^k})_{ij}$ is equal to ε.
Consider an arbitrary $p \in \mathbb{N}$. The concatenation of P and p times C yields a path from vertex j to vertex i with length $l + pm$. Hence, $\left(A^{\otimes^{l+pm}}\right)_{ij} \neq \varepsilon$ for any $p \in \mathbb{N}$.
Since A is irreducible, there exist integers $k_0 \in \mathbb{N}$ and $c \in \mathbb{N}_0$ such that $A^{\otimes^{k+c}} = \lambda^{\otimes^c} \otimes A^{\otimes^k}$ for all $k \geqslant k_0$. So if we select p such that $q_0 \stackrel{\text{def}}{=} l + mp \geqslant k_0$ then we have $A^{\otimes^{q_0+rc}} = \lambda^{\otimes^c} \otimes A^{\otimes^{q_0+(r-1)c}} = \ldots = \lambda^{\otimes^{rc}} \otimes A^{\otimes^{q_0}}$ for any $r \in \mathbb{N}$. Furthermore, $\lambda \neq \varepsilon$ since $A \neq \mathcal{E}_{n \times n}$.
As a consequence, we have $\left(A^{\otimes^{q_0+rc}}\right)_{ij} = \lambda^{\otimes^{rc}} \otimes \left(A^{\otimes^{q_0}}\right)_{ij} \neq \varepsilon$ for all $r \in \mathbb{N}$. □

Lemma C.1.2 *If $A \in \mathbb{R}_\varepsilon^{n\times n}$ is not irreducible then there exists a max-algebraic permutation matrix $P \in \mathbb{R}_\varepsilon^{n\times n}$ such that the matrix $\hat{A} = P \otimes A \otimes P^T$ is a max-algebraic block upper triangular matrix of the form*

$$\hat{A} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} & \ldots & \hat{A}_{1l} \\ \mathcal{E} & \hat{A}_{22} & \ldots & \hat{A}_{2l} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{E} & \mathcal{E} & \ldots & \hat{A}_{ll} \end{bmatrix} \tag{C.3}$$

with $l > 1$ and where the matrices $\hat{A}_{11}, \hat{A}_{22}, \ldots, \hat{A}_{ll}$ are square and irreducible.

Proof: See e.g. [3]. This lemma is also the max-algebraic equivalent of a result of [75]. □

Lemma C.1.3 *Consider $A \in \mathbb{R}_\varepsilon^{n\times n}$ and let $T \in \mathbb{R}_\varepsilon^{n\times n}$ be a max-invertible matrix. Then A and $T \otimes A \otimes T^{\otimes^{-1}}$ have the same max-algebraic eigenvalues.*

Proof: First we prove that every max-algebraic eigenvalue of A is also a max-algebraic eigenvalue of $T \otimes A \otimes T^{\otimes^{-1}}$. Let λ be a max-algebraic eigenvalue of A and let x be a corresponding max-algebraic eigenvector. We have $(T \otimes$

$A \otimes T^{\otimes^{-1}}) \otimes (T \otimes x) = T \otimes A \otimes x = T \otimes \lambda \otimes x = \lambda \otimes (T \otimes x)$. So λ is a max-algebraic eigenvalue of $T \otimes A \otimes T^{\otimes^{-1}}$ (and $T \otimes x$ is a corresponding max-algebraic eigenvector).

In order to prove that every max-algebraic eigenvalue of $T \otimes A \otimes T^{\otimes^{-1}}$ is also a max-algebraic eigenvalue of A, we define $\tilde{A} = T \otimes A \otimes T^{\otimes^{-1}}$ and $\tilde{T} = T^{\otimes^{-1}}$. Now it follows from the first part of this proof that every max-algebraic eigenvalue of $\tilde{A}$ is also a max-algebraic eigenvalue of $\tilde{T} \otimes \tilde{A} \otimes \tilde{T}^{\otimes^{-1}} = A$. □

Consider $A \in \mathbb{R}_\varepsilon^{n \times n}$. Assume that the max-algebraic permutation matrix P transforms A into a max-algebraic block upper triangular matrix $\hat{A} = P \otimes A \otimes P^T$ of the form (C.3) with $l \geqslant 1$ and where the matrices $\hat{A}_{11}$, $\hat{A}_{22}$, ..., $\hat{A}_{ll}$ are irreducible. Note that if A is irreducible then we can take $P = E_n$. Furthermore, l is equal to 1 if and only if A is irreducible.

Let λ_i be the max-algebraic eigenvalue of $\hat{A}_{ii}$ for $i = 1, 2, \ldots, l$. Although in general the matrix $\hat{A}$ is not uniquely defined, the set $\{ \lambda_i \mid i = 1, 2, \ldots, l \}$ will be the same for any max-algebraic block upper triangular matrix $P \otimes A \otimes P^T$ with irreducible diagonal blocks where P is a max-algebraic permutation matrix. So the set of the max-algebraic eigenvalues of $\hat{A}_{11}$, $\hat{A}_{22}$, ..., $\hat{A}_{ll}$ does not depend on P. This set will be denoted by $L(A)$. Note that $L(A) = L(\hat{A})$.

It can be shown (see e.g. [54]) that every max-algebraic eigenvalue of A belongs to $L(A)$ and that $\max L(A)$ and λ_l are always max-algebraic eigenvalues of $\hat{A}$ (and also of A by Lemma C.1.3). So the largest max-algebraic eigenvalue of A is equal to $\max L(A)$.

The following lemma is an extension of Lemma C.1.1 and a corrected version of a lemma that can be found in [149]:

Lemma C.1.4 *Let $\hat{A} \in \mathbb{R}_\varepsilon^{n \times n}$ be a matrix of the form (C.3) where the matrices $\hat{A}_{11}$, $\hat{A}_{22}$, ..., $\hat{A}_{ll}$ are irreducible. Let λ_i and c_i be respectively the max-algebraic eigenvalue and the cyclicity of $\hat{A}_{ii}$ for $i = 1, 2, \ldots, l$. Define sets α_1, α_2, ..., α_l such that $\hat{A}_{\alpha_i \alpha_j} = \hat{A}_{ij}$ for all i, j with $i \leqslant j$.*
Define

$$S_{ij}(s) = \Big\{ \{i_0, i_1, \ldots, i_s\} \subset \mathbb{N} \,\Big|\, i = i_0 < i_1 < \ldots < i_s = j \text{ and } \hat{A}_{i_t i_{t+1}} \neq \mathcal{E} \text{ for } t = 0, 1, \ldots, s-1 \Big\}$$

for all i, j with $i < j$ and for $s = 1, 2, \ldots, j - i$.
Let $\lambda_{ii} = \lambda_i$ and $c_{ii} = c_i$ for $i = 1, 2, \ldots, n$. Define

$$S_{ij} = \big\{ t \,\big|\, \exists s \in \{1, 2, \ldots, j-i\} \text{ and } \exists \gamma \in S_{ij}(s) \text{ such that } t \in \gamma \big\}$$

$$\lambda_{ij} = \begin{cases} \max\{ \lambda_t \mid t \in S_{ij} \} & \textit{if } S_{ij} \neq \emptyset\,, \\ \varepsilon & \textit{if } S_{ij} = \emptyset\,, \end{cases}$$

$$c_{ij} = \begin{cases} \operatorname{lcm}\{ c_t \mid t \in S_{ij} \} & \textit{if } S_{ij} \neq \emptyset\,, \\ 1 & \textit{if } S_{ij} = \emptyset\,, \end{cases}$$

for all i, j with $i < j$.
Let $i, j \in \{1, 2, \ldots, l\}$ with $i \leqslant j$. If $\lambda_{ij} \neq \varepsilon$ then we have

$$\begin{gathered}\forall u \in \alpha_i, \forall v \in \alpha_j, \exists q_0 \in \mathbb{N} \textit{ such that} \\ \left(\left(\hat{A}^{\otimes^{q_0 + rc_{ij}}}\right)_{\alpha_i\alpha_j}\right)_{uv} = \lambda_{ij}{}^{\otimes^{rc_{ij}}} \otimes \left(\left(\hat{A}^{\otimes^{q_0}}\right)_{\alpha_i\alpha_j}\right)_{uv} \neq \varepsilon \end{gathered} \tag{C.4}$$

for all $r \in \mathbb{N}$.
Furthermore, there exists an integer $K \in \mathbb{N}$ such that

$$\left(\hat{A}^{\otimes^{k+c_{ij}}}\right)_{\alpha_i\alpha_j} = \lambda_{ij}{}^{\otimes^{c_{ij}}} \otimes \left(\hat{A}^{\otimes^{k}}\right)_{\alpha_i\alpha_j} \tag{C.5}$$

for all $k \geqslant K$ and for all i, j with $i \leqslant j$.

Proof: If $l = 1$ then (C.4) corresponds to Lemma C.1.1. Furthermore, if $l = 1$ then (C.5) is a direct consequence of (C.1).

From now on we assume that $l > 1$. Define $F_{ij}(k) = (A^{\otimes^k})_{\alpha_i\alpha_j}$ for all $k \in \mathbb{N}$ and for all i, j with $i \leqslant j$.
Consider arbitrary indices $i, j \in \{1, 2, \ldots, l\}$ with $i \leqslant j$.
We have $F_{ii}(k) = \hat{A}_{ii}^{\otimes^k}$ for all $k \in \mathbb{N}$. If $\lambda_{ii} \neq \varepsilon$ then $\hat{A}_{ii} \neq \mathcal{E}$ and then it follows from Lemma C.1.1 that

$$\begin{aligned} &\forall u, v \in \alpha_i, \exists q_0 \in \mathbb{N} \text{ such that} \\ &\qquad \big(F_{ii}(q_0 + rc_{ii})\big)_{uv} = \big(\hat{A}_{ii}^{\otimes^{q_0 + rc_{ii}}}\big)_{uv} = \lambda_{ii}{}^{\otimes^{rc_{ii}}} \otimes \big(\hat{A}_{ii}^{\otimes^{q_0}}\big)_{uv} \\ &\qquad\qquad = \lambda_{ii}{}^{\otimes^{rc_{ii}}} \otimes \big(F_{ii}(q_0)\big)_{uv} \neq \varepsilon \qquad \text{for all } r \in \mathbb{N}\,. \end{aligned}$$

So (C.4) holds for all i, j with $i = j$.
Furthermore, by Theorem 2.2.8 there exists an integer $k_{0i} \in \mathbb{N}$ such that $F_{ii}(k + c_{ii}) = \lambda_{ii}{}^{\otimes^{c_{ii}}} \otimes F_{ii}(k)$ for all $k \geqslant k_{0i}$. So if we select $K \in \mathbb{N}$ such that $K \geqslant k_{0i}$ for all i then (C.5) holds for all i, j with $i = j$.
From now on we assume that $i \neq j$. Then we have

$$\begin{aligned} F_{ij}(k) &= \bigoplus_{s=1}^{j-i} \bigoplus_{i = i_0 < i_1 < \ldots < i_s = j} \bigoplus_{\substack{p_0, p_1, \ldots, p_s \in \mathbb{N} \\ p_0 + \ldots + p_s = k - s}} \hat{A}_{i_0 i_0}^{\otimes^{p_0}} \otimes \hat{A}_{i_0 i_1} \otimes \hat{A}_{i_1 i_1}^{\otimes^{p_1}} \otimes \ldots \\ &\qquad\qquad \otimes \hat{A}_{i_{s-1} i_s} \otimes \hat{A}_{i_s i_s}^{\otimes^{p_s}} \\ &= \bigoplus_{s=1}^{j-i} \bigoplus_{\{i_0, i_1, \ldots, i_s\} \in S_{ij}(s)} \bigoplus_{\substack{p_0, p_1, \ldots, p_s \in \mathbb{N} \\ p_0 + \ldots + p_s = k - s}} F_{i_0 i_0}(p_0) \otimes \hat{A}_{i_0 i_1} \otimes \\ &\qquad\qquad F_{i_1 i_1}(p_1) \otimes \ldots \otimes \hat{A}_{i_{s-1} i_s} \otimes F_{i_s i_s}(p_s) \end{aligned} \tag{C.6}$$

for all $k \in \mathbb{N}$ where the empty max-algebraic sum $\bigoplus_{\{i_0,i_1,\ldots,i_s\}\in\emptyset} \ldots$ is equal to $\mathcal{E}$ by definition.
If $S_{ij} = \emptyset$ then we have $F_{ij}(k) = \mathcal{E}$ for all $k \in \mathbb{N}$. So (C.5) holds if $S_{ij} = \emptyset$.
From now on we assume that $S_{ij} \neq \emptyset$.
If $\lambda_{ij} = \varepsilon$ then we have $\hat{A}_{i_t i_t} = [\varepsilon]$ for all $t \in S_{ij}$ and then it follows from (C.6) that $F_{ij}(k) = \mathcal{E}$ for all $k \in \mathbb{N}$ with $k > j - i$. So if we select K such that $K > l - 1$ then (C.5) also holds if $\lambda_{ij} = \varepsilon$ and $S_{ij} \neq \emptyset$.
From now on we assume that $\lambda_{ij} \neq \varepsilon$. Since $S_{ij}(s) \neq \emptyset$, there exists an index $i_y \in S_{ij}$ such that $\lambda_{ij} = \lambda_{i_y}$ and $c_{ij} = w c_{i_y}$ for some $w \in \mathbb{N}_0$.

Consider an arbitrary integer $s \in \{1, 2, \ldots, j-i\}$ and an arbitrary set $\{i_0, i_1, \ldots, i_s\} \in S_{ij}(s)$ such that $i_y \in \{i_0, i_1, \ldots, i_s\}$. Since $S_{ij}(s) \neq \emptyset$ and $\lambda_{ij} \neq \varepsilon$, there exist indices u_t, v_t such that $(\hat{A}_{i_{t-1} i_t})_{u_t v_t} \neq \varepsilon$ for $t = 1, 2, \ldots, s$. Let $v_0 = u$ and $u_{s+1} = v$. Define $\varphi = \{\, t \mid \hat{A}_{i_t i_t} \neq \mathcal{E} \,\}$ and $\varphi^{\mathrm{c}} = \{0, 1, \ldots, s\} \backslash \varphi$. Note that $y \in \varphi$. If $t \in \varphi^{\mathrm{c}}$ then $\hat{A}_{i_t i_t} = [\varepsilon]$, $\lambda_{i_t} = \varepsilon$ and $u_{t+1} = v_t$. If $t \in \varphi$ then $\lambda_{i_t} \neq \varepsilon$.
From Lemma C.1.1 it follows that for each $t \in \varphi$ there exists an integer $p_t \in \mathbb{N}$ such that

$$\left(F_{i_t i_t}(p_t + r c_{i_t})\right)_{v_t u_{t+1}} \neq \varepsilon \tag{C.7}$$

for all $r \in \mathbb{N}$. Recall that $\varepsilon^{\otimes^0} = 0$ by definition. So if we set $p_t = 0$ for all $t \in \varphi^{\mathrm{c}}$, then it follows from (C.6) and (C.7) that

$$\begin{aligned}
\left(F_{ij}(p_0 + p_1 + \ldots + p_s + s + r w c_{i_y})\right)_{uv} \geqslant \left(F_{i_0 i_0}(p_0)\right)_{u u_1} \otimes \\
\left(\hat{A}_{i_0 i_1}\right)_{u_1 v_1} \otimes \left(F_{i_1 i_1}(p_1)\right)_{v_1 u_2} \otimes \ldots \otimes \left(F_{i_y i_y}(p_y + r w c_{i_y})\right)_{v_y u_{y+1}} \otimes \ldots \\
\otimes \left(\hat{A}_{i_{s-1} i_s}\right)_{u_s v_s} \otimes \left(F_{i_s i_s}(p_s)\right)_{v_s v} \neq \varepsilon \qquad \text{for all } r \in \mathbb{N}\,.
\end{aligned} \tag{C.8}$$

Since $\lambda_{ij} = \lambda_{i_y} \geqslant \lambda_t$ for all $t \in \{0, 1, \ldots, s\}$, there exist a set $\{i_0, i_1, \ldots, i_s\}$, an index i_y and integers p_0, p_1, $\ldots$, p_s, $r_0 \in \mathbb{N}$ and $w \in \mathbb{N}_0$ such that (C.8) holds with equality for all $r \geqslant r_0$ and such that $c_{ij} = w c_{i_y}$. Furthermore,

$$\left(F_{i_y i_y}(p_y + r c_{i_y})\right)_{v_y u_{y+1}} = \lambda_{i_y}{}^{\otimes^{r c_{i_y}}} \otimes \left(F_{i_y i_y}(p_y)\right)_{v_y u_{y+1}}$$

for all $r \in \mathbb{N}$ by Lemma C.1.1. So if we define $q_0 = p_0 + p_1 + \ldots + p_t + s + r_0 c_{i_y}$ then we have

$$\left(F_{ij}(q_0 + r c_{i_y})\right)_{uv} = \lambda_{i_y}{}^{\otimes^{r c_{i_y}}} \otimes \left(F_{ij}(q_0)\right)_{uv} \neq \varepsilon \qquad \text{for all } r \in \mathbb{N}\,. \tag{C.9}$$

So now we have proved that (C.4) holds for all i, j with $i \leqslant j$.

Consider again an arbitrary integer $s \in \{1, 2, \ldots, j-i\}$ and an arbitrary set

$\{i_0, i_1, \ldots, i_s\} \in S_{ij}(s)$ such that $i_y \in \{i_0, i_1, \ldots, i_s\}$. From Theorem 2.2.8 it follows that for each $t \in \{1, 2, \ldots, s\}$ there exists an integer $k_t \in \mathbb{N}$ such that

$$F_{i_t i_t}(k + c_{i_t}) = \lambda_{i_t}{}^{\otimes^{c_{i_t}}} \otimes F_{i_t i_t}(k) \qquad \text{for all } k \geqslant k_t \,.$$

If we combine this with (C.6) and if we use the same reasoning as the one that has been used to prove (C.9) then we find that there exists an integer $k_{0ij} \in \mathbb{N}$ such that

$$F_{ij}(k + c_{ij}) = \lambda_{ij}{}^{\otimes^{c_{ij}}} \otimes F_{ij}(k) \qquad \text{for all } k \geqslant k_{0ij} \,.$$

So if we select $K \in \mathbb{N}$ such that $K \geqslant k_{0ij}$ for all i, j with $i < j$ then (C.5) also holds if $\lambda_{ij} \neq \varepsilon$ and $S_{ij} \neq \emptyset$. □

Note that the combination of Lemma C.1.2 and Lemma C.1.4 yields a generalization of the cyclicity property (C.1) of irreducible matrices.
Consider a matrix $A \in \mathbb{R}_\varepsilon^{n \times n}$. By Lemma C.1.2 there exists a max-algebraic permutation matrix P that transforms A into a max-algebraic block upper triangular matrix $\hat{A}$ with irreducible diagonal blocks. Since $\hat{A} = P \otimes A \otimes P^T$, we have $A = P^T \otimes \hat{A} \otimes P$ and thus $A^{\otimes^k} = P^T \otimes \hat{A}^{\otimes^k} \otimes P$ for all $k \in \mathbb{N}$. Left max-multiplication of a matrix by a max-algebraic permutation matrix corresponds to a permutation of the rows of the matrix, and right max-multiplication of a matrix by a max-algebraic permutation matrix corresponds to a permutation of the columns of the matrix. Therefore, it follows from Lemma C.1.4 that for any $i, j \in \{1, 2, \ldots, n\}$ the sequence $\{(A^{\otimes^k})_{ij}\}_{k=0}^{\infty}$ is ultimately geometric.

Let (A, B, C) be a realization of the impulse response $G = \{G_k\}_{k=0}^{\infty}$ of a max-linear time-invariant DES with m inputs and p outputs. Consider arbitrary indices i, j with $i \in \{1, 2, \ldots, p\}$ and $j \in \{1, 2, \ldots, m\}$. It is easy to verify that if all the terms of the sequence $\{(G_k)_{ij}\}_{k=0}^{\infty}$ are finite then the corresponding λ_s's in (6.4) are uniquely defined. Only if the sequence $\{(G_k)_{ij}\}_{k=0}^{\infty}$ contains terms that are equal to ε it is possible that some of the corresponding λ_s's in (6.4) are not uniquely defined. In that case we select the smallest possible value for these λ_s's, i.e. ε. For a given impulse response G the set of the smallest possible values for the λ_s's in (6.4) will be denoted by $L^*(G)$.
From Lemma C.1.4 it follows that each finite $\lambda_s^* \in L^*(G)$ corresponds to some $\lambda_i \in L(A)$. Now we show that if (A,B,C) is a minimal realization of G and if $\lambda^* = \max L^*(G) \neq \varepsilon$ then λ^* is the largest max-algebraic eigenvalue of A, i.e. $\max L^*(G) = \max L(A)$.

Proof of Lemma 6.3.7 : Assume that G is the impulse response of a system with m inputs and p outputs.
First we assume that A is irreducible. Then there exist integers $k_0 \in \mathbb{N}$ and $c \in \mathbb{N}_0$ such that $A^{\otimes^{k+c}} = \lambda^{\otimes^c} \otimes A^{\otimes^k}$ for all $k \geqslant k_0$ where λ is the unique max-algebraic eigenvalue of A. Hence, $G_{k+c} = \lambda^{\otimes^c} \otimes G_k$ for all $k \geqslant k_0$. Since $\max L^*(G) \neq \varepsilon$, we have $B \neq \mathcal{E}$ and $C \neq \mathcal{E}$. As a consequence, we have

$L^*(G) = \{\lambda\}$ and $\max L^*(G) = \lambda$.

Now we consider the case where A is reducible. In general we can transform A into a max-algebraic block upper diagonal matrix of the form (C.3) with l diagonal blocks. Since the proof for the general case is rather long, we shall only consider the case $l = 2$. Note however that the proof that will be presented here contains all the necessary ingredients to prove the general case.

If $l = 2$ then there exists a max-algebraic diagonal matrix P such that

$$\hat{A} = P \otimes A \otimes P^T = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \varepsilon & \hat{A}_{21} \end{bmatrix}$$

where $\hat{A}_{11} \in \mathbb{R}_\varepsilon^{n_1 \times n_1}$ and $\hat{A}_{22} \in \mathbb{R}_\varepsilon^{n_2 \times n_2}$ are irreducible. Let λ_i, c_i, α_i, λ_{ij} and c_{ij} for $i = 1,2$ and $j = 1,2$ with $i \leqslant j$ be defined as in Lemma C.1.4.

If we define $\hat{B} = P \otimes B$ and $\hat{C} = C \otimes P^T$ then the triple $(\hat{A}, \hat{B}, \hat{C})$ is also a minimal realization of G by Proposition 6.2.1. Assume that $\hat{B}$ and $\hat{C}$ are partitioned as follows:

$$\hat{B} = \begin{bmatrix} \hat{B}_{11} \\ \hat{B}_{21} \end{bmatrix} \text{ and } \hat{C} = \begin{bmatrix} \hat{C}_{11} & \hat{C}_{12} \end{bmatrix}$$

with $\hat{B}_{i1} \in \mathbb{R}_\varepsilon^{n_i \times m}$ and $\hat{C}_{1i} \in \mathbb{R}_\varepsilon^{p \times n_i}$ for $i = 1,2$.

Define $F(k) = \hat{A}^{\otimes^k}$ for all $k \in \mathbb{N}$. Assume that $F(k)$ is partitioned as follows:

$$F(k) = \begin{bmatrix} F_{11}(k) & F_{12}(k) \\ \varepsilon & F_{22}(k) \end{bmatrix}$$

where $F_{ij}(k)$ has the same size as $\hat{A}_{ij}$ for all i,j with $i < j$.

Let $c = \operatorname{lcm}(c_1, c_2)$. As a consequence of Lemma C.1.4 there exists an integer $k_0 \in \mathbb{N}$ such that

$$F_{11}(k+c) = {\lambda_1}^{\otimes^c} \otimes F_{11}(k) \tag{C.10}$$

$$F_{22}(k+c) = {\lambda_2}^{\otimes^c} \otimes F_{22}(k) \tag{C.11}$$

$$F_{12}(k+c) = {\lambda_{12}}^{\otimes^c} \otimes F_{12}(k) \tag{C.12}$$

for all $k \geqslant k_0$.

Since

$$G_k = \hat{C}_{11} \otimes F_{11}(k) \otimes \hat{B}_{11} \oplus \hat{C}_{11} \otimes F_{12}(k) \otimes \hat{B}_{21} \oplus \hat{C}_{12} \otimes F_{22}(k) \otimes \hat{B}_{21}$$

for all $k \in \mathbb{N}$, it follows from (C.10) – (C.12) that

$$\begin{aligned} G_{k+c} = {} & {\lambda_1}^{\otimes^c} \otimes \hat{C}_{11} \otimes F_{11}(k) \otimes \hat{B}_{11} \oplus {\lambda_{12}}^{\otimes^c} \otimes \hat{C}_{11} \otimes F_{12}(k) \otimes \hat{B}_{21} \oplus \\ & {\lambda_2}^{\otimes^c} \otimes \hat{C}_{12} \otimes F_{22}(k) \otimes \hat{B}_{21} \end{aligned} \tag{C.13}$$

for all $k \geqslant k_0$. Note that $\lambda_{12} = \lambda_1 \oplus \lambda_2$ if $\hat{A}_{12} \neq \mathcal{E}$ and $\lambda_{12} = \varepsilon$ if $\hat{A}_{12} = \mathcal{E}$. Therefore, (C.13) implies that $L^*(G) \subseteq \{\lambda_1, \lambda_2, \varepsilon\}$. Define $\lambda^* = \max L^*(G)$. Since $\max L^*(G) \neq \varepsilon$, we have $\lambda^* = \lambda_1$ or $\lambda^* = \lambda_2$. So if $\lambda_1 = \lambda_2$ then the lemma is proved.
From now on we assume that $\lambda_1 \neq \lambda_2$.
We distinguish between two cases: $\hat{A}_{12} = \mathcal{E}$ and $\hat{A}_{12} \neq \mathcal{E}$.

Case 1: $\hat{A}_{12} = \mathcal{E}$.

We may assume without loss of generality that $\lambda_1 \geqslant \lambda_2$ since we can always permute the diagonal blocks of $\hat{A}$ by performing an additional similarity transformation with a max-algebraic permutation matrix on the triple $(\hat{A}, \hat{B}, \hat{C})$ if necessary.
If $\hat{A}_{12} = \mathcal{E}$ then $F_{12}(k) = \mathcal{E}$ for all $k \in \mathbb{N}$. Hence, $G_k = \hat{C}_{11} \otimes F_{11}(k) \otimes \hat{B}_{11} \oplus \hat{C}_{12} \otimes F_{22}(k) \otimes \hat{B}_{21}$ for all $k \in \mathbb{N}$. Let us now prove by contradiction that $\lambda^* = \lambda_1$.
Assume that $\lambda^* \neq \lambda_1$. This implies that $\lambda^* = \lambda_2 < \lambda_1$. Hence, $\lambda_1 \neq \varepsilon$. Now we prove by contradiction that we can only have $\lambda^* \neq \lambda_1$ if $\hat{C}_{11} = \mathcal{E}$ or $\hat{B}_{11} = \mathcal{E}$.
Suppose that $\hat{C}_{11} \neq \mathcal{E}$ and $\hat{B}_{11} \neq \mathcal{E}$. Then there exist indices i_1, j_1, i_2, j_2 such that $(\hat{C}_{11})_{i_1 j_1} \neq \varepsilon$ and $(\hat{B}_{11})_{i_2 j_2} \neq \varepsilon$. Since $\lambda_1 \neq \varepsilon$, it follows from Lemma C.1.1 that there exists an integer $q_0 \in \mathbb{N}$ such that

$$\big(F_{11}(q_0 + rc)\big)_{j_1 i_2} = {\lambda_1}^{\otimes^{rc}} \otimes \big(F_{11}(q_0)\big)_{j_1 i_2} \neq \varepsilon \qquad \text{for all } r \in \mathbb{N}\,.$$

Since $\lambda_1 > \lambda_2$, this implies that there exists an integer $r_0 \in \mathbb{N}$ such that

$$(G_{q_0+rc})_{i_1 j_2} \geqslant {\lambda_1}^{\otimes^{rc}} \otimes (G_{q_0})_{i_1 j_2} \neq \varepsilon \qquad \text{for all } r \geqslant r_0\,.$$

But then we have $\lambda^* = \max L^*(G) \geqslant \lambda_1$, which is in contradiction with our assumption that $\lambda^* < \lambda_1$.
So we can only have $\lambda^* < \lambda_1$ if $\hat{C}_{11} = \mathcal{E}$ or $\hat{B}_{11} = \mathcal{E}$. However, in that case we have $G_k = \hat{C}_{12} \otimes \hat{A}_{22}^{\otimes^k} \otimes \hat{B}_{21}$ for all $k \in \mathbb{N}$. This implies that $(\hat{A}_{22}, \hat{B}_{21}, \hat{C}_{12})$ is also a realization of G. However, this would contradict the fact that $(\hat{A}, \hat{B}, \hat{C})$ is a minimal realization. Therefore, our assumption that $\lambda^* \neq \lambda_1$ was wrong.
So if $\hat{A}_{12} = \mathcal{E}$ then we have $\lambda^* = \lambda_1 = \max L(A)$.

Case 2: $\hat{A}_{12} \neq \mathcal{E}$.

First we consider the case $\lambda_1 > \lambda_2$. Hence, $\lambda_1 \neq \varepsilon$ and $\lambda_{12} = \lambda_1$. Now we prove by contradiction that $\lambda^* = \lambda_1$.
Assume that $\lambda^* \neq \lambda_1$. Hence, $\lambda^* = \lambda_2 < \lambda_1$. Now we prove by contradiction that this is only possible if $\hat{C}_{11} = \mathcal{E}$.
If $\hat{C}_{11} \neq \mathcal{E}$ then there exist indices i_1 and j_1 such that $(\hat{C}_{11})_{i_1 j_1} \neq \varepsilon$. Clearly, $\hat{B} \neq \mathcal{E}$ since otherwise we would have $G_k = \mathcal{E}$ for all $k \in \mathbb{N}$ and thus also $\max L^*(G) = \varepsilon$. So there exist indices i_2, j_2, i_3, j_3 such that $(\hat{B}_{11})_{i_2 j_2} \neq \varepsilon$ or

$(\hat{B}_{21})_{i_3 j_3} \neq \varepsilon$. If $(\hat{B}_{11})_{i_2 j_2} \neq \varepsilon$ then we can use the same reasoning as in Case 1 to prove that $\lambda^* \geqslant \lambda_1$, which is in contradiction with our assumption that $\lambda^* < \lambda_1$. So from now on we assume that $\hat{B}_{11} = \mathcal{E}$ and that $(\hat{B}_{21})_{i_3 j_3} \neq \varepsilon$. Since $\lambda_{12} = \lambda_1 \neq \varepsilon$, $c_{12} = c_1$ and $c = \mathrm{lcm}(c_1, c_2) = w c_{12}$ for some $w \in \mathbb{N}_0$, it follows from Lemma C.1.4 that there exists an integer $q_0 \in \mathbb{N}$ such that

$$\left(F_{12}(q_0 + rc)\right)_{j_1 i_3} = {\lambda_1}^{\otimes^{rc}} \otimes \left(F_{12}(q_0)\right)_{j_1 i_3} \neq \varepsilon \qquad \text{for all } r \in \mathbb{N} \,.$$

Since $\lambda_1 > \lambda_2$, this implies that there exists an integer $r_0 \in \mathbb{N}$ such that

$$(G_{q_0 + rc})_{i_1 j_3} \geqslant {\lambda_1}^{\otimes^{rc}} \otimes (G_{q_0})_{i_1 j_3} \neq \varepsilon \qquad \text{for all } r \geqslant r_0 \,.$$

Hence, $\lambda^* = \max L^*(G) \geqslant \lambda_1$, but this is also in contradiction with our assumption that $\lambda^* < \lambda_1$.
So we can only have $\lambda^* < \lambda_1$ if $\hat{C}_{11} = \mathcal{E}$. However, then we have $G_k = \hat{C}_{12} \otimes \hat{A}_{22}^{\otimes^k} \otimes \hat{B}_{21}$ for all $k \in \mathbb{N}$. So $(\hat{A}_{22}, \hat{B}_{21}, \hat{C}_{12})$ is also a realization of G. Since this contradicts the fact that $(\hat{A}, \hat{B}, \hat{C})$ is a minimal realization, our assumption that $\lambda^* \neq \lambda_1$ was wrong.
So in this case we also have $\lambda^* = \lambda_1 = \max L(A)$.

If $\lambda_1 < \lambda_2$ then we can use a similar reasoning as for the case $\lambda_1 > \lambda_2$ to prove that $\lambda^* = \lambda_2 = \max L(A)$.

So λ^* is equal to the largest max-algebraic eigenvalue of A. □

C.2 Proof of Lemma 6.3.8

Now we prove that if $\lambda \neq \varepsilon$ is the largest max-algebraic eigenvalue of a matrix $A \in \mathbb{R}_\varepsilon^{n \times n}$ then there exists a max-invertible matrix $T \in \mathbb{R}_\varepsilon^{n \times n}$ such that $\left\| T \otimes A \otimes T^{\otimes^{-1}} \right\|_\oplus = \lambda$.

Proof of Lemma 6.3.8: First we assume that A is irreducible. Now we apply a reasoning that is similar to the one used to prove Lemma 3.2.8 of [57] and Lemma 3.3 of [99]. Let u be a max-algebraic eigenvector of A that corresponds to the max-algebraic eigenvalue λ. Since the matrix A is irreducible, the components of u are finite [3, 20, 54], which implies that their max-algebraic inverses are defined. If we define a max-algebraic diagonal matrix $T \in \mathbb{R}_\varepsilon^{n \times n}$ such that $t_{ii} = {u_i}^{\otimes^{-1}}$ for $i = 1, 2, \ldots, n$, then T is max-invertible and

$$\begin{aligned}
\left\| T \otimes A \otimes T^{\otimes^{-1}} \right\|_\oplus &= \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{n} {u_i}^{\otimes^{-1}} \otimes a_{ij} \otimes u_j \\
&= \bigoplus_{i=1}^{n} \left({u_i}^{\otimes^{-1}} \otimes \left(\bigoplus_{j=1}^{n} a_{ij} \otimes u_j \right) \right)
\end{aligned}$$

$$
\begin{aligned}
&= \bigoplus_{i=1}^{n} {u_i}^{\otimes^{-1}} \otimes (A \otimes u)_i \\
&= \bigoplus_{i=1}^{n} {u_i}^{\otimes^{-1}} \otimes \lambda \otimes u_i \\
&= \lambda \ .
\end{aligned}
$$

So now we have proved that the lemma holds if A is irreducible.

From now on we assume that A is not irreducible. By Lemma C.1.2 there exists a max-algebraic permutation matrix $P \in \mathbb{R}_\varepsilon^{n \times n}$ such that $\hat{A} = P \otimes A \otimes P^T$ is a max-algebraic block upper triangular matrix of the form (C.3) with irreducible diagonal blocks $\hat{A}_{11}$, $\hat{A}_{22}$, ..., $\hat{A}_{ll}$. Let n_i be the number of rows of $\hat{A}_{ii}$ for $i = 1, 2, \ldots, l$. Note that $\hat{A}_{ij} \in \mathbb{R}_\varepsilon^{n_i \times n_j}$ for all i, j with $i \leqslant j$.

Let λ_i be the max-algebraic eigenvalue of $\hat{A}_{ii}$ for $i = 1, 2, \ldots, l$. Note that $\lambda = \max\{\lambda_1, \lambda_2, \ldots, \lambda_l\}$. Let u_i be a max-algebraic eigenvector of $\hat{A}_{ii}$ that corresponds to the max-algebraic eigenvalue λ_i for $i = 1, 2, \ldots, l$. Since the matrices $\hat{A}_{11}$, $\hat{A}_{22}$, ..., $\hat{A}_{ll}$ are irreducible, the components of u_1, u_2, ..., u_l are finite. Now we define l diagonal matrices D_1, D_2, ..., D_l with $D_i \in \mathbb{R}_\varepsilon^{n_i \times n_i}$ for $i = 1, 2, \ldots, l$ such that $(D_i)_{jj} = (u_i)_j{}^{\otimes^{-1}}$ for $i = 1, 2, \ldots, l$ and $j = 1, 2, \ldots, n_i$. Let

$$
D = \begin{bmatrix} D_1 & \varepsilon & \ldots & \varepsilon \\ \varepsilon & D_2 & \ldots & \varepsilon \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon & \varepsilon & \ldots & D_l \end{bmatrix}
$$

and

$$
\bar{A} = D \otimes \hat{A} \otimes D^{\otimes^{-1}} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} & \ldots & \bar{A}_{1l} \\ \varepsilon & \bar{A}_{22} & \ldots & \bar{A}_{2l} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon & \varepsilon & \ldots & \bar{A}_{ll} \end{bmatrix}
$$

with $\bar{A}_{ij} \in \mathbb{R}_\varepsilon^{n_i \times n_j}$ for all i, j with $i \leqslant j$. From the first part of this proof it follows that $\|\bar{A}_{ii}\|_\oplus = \lambda_i$ for all i. Hence, $\bigoplus_{i=1}^{l} \|\bar{A}_{ii}\|_\oplus = \lambda$. Now we define l real numbers $\alpha_1, \alpha_2, \ldots, \alpha_l$ such that

$$
\begin{aligned}
\alpha_1 &= 0 \\
\alpha_j &= \lambda^{\otimes^{-1}} \otimes \left(0 \oplus \left(\bigoplus_{i=1}^{j-1} \|\bar{A}_{ij}\|_\oplus \otimes \alpha_i \right) \right) \qquad \text{for } j = 2, 3, \ldots, l \, .
\end{aligned}
$$

Note that $\lambda^{\otimes^{-1}}$ is defined and finite since λ is finite. As a consequence, the α_j's are also defined and finite.
The diagonal matrix

$$\bar{D} = \begin{bmatrix} \alpha_1 \otimes E_{n_1} & \mathcal{E} & \dots & \mathcal{E} \\ \mathcal{E} & \alpha_2 \otimes E_{n_2} & \dots & \mathcal{E} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{E} & \mathcal{E} & \dots & \alpha_l \otimes E_{n_l} \end{bmatrix}$$

is max-invertible since all the α_j's are finite. If we define

$$\tilde{A} = \bar{D} \otimes \bar{A} \otimes \bar{D}^{\otimes^{-1}} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} & \dots & \tilde{A}_{1l} \\ \mathcal{E} & \tilde{A}_{22} & \dots & \tilde{A}_{2l} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{E} & \mathcal{E} & \dots & \tilde{A}_{ll} \end{bmatrix}$$

with $\tilde{A}_{ij} \in \mathbb{R}_\varepsilon^{n_i \times n_j}$ for all i, j with $i \leqslant j$ then we have $\tilde{A}_{ii} = \bar{A}_{ii}$ for all i. Hence, $\displaystyle\bigoplus_{i=1}^{l} \|\tilde{A}_{ii}\|_\oplus = \lambda$.
Consider arbitrary indices $i, j \in \{1, 2, \dots, l\}$ with $i < j$.
If $\|\bar{A}_{ij}\|_\oplus = \varepsilon$ then we have $\tilde{A}_{ij} = \bar{A}_{ij} = \mathcal{E}$ and thus $\|\tilde{A}_{ij}\|_\oplus = \varepsilon$.
Now assume that $\|\bar{A}_{ij}\|_\oplus \neq \varepsilon$. Since $\alpha_j \geqslant \lambda^{\otimes^{-1}} \otimes \left\|\bar{A}_{ij}\right\|_\oplus \otimes \alpha_i$, we have

$$\alpha_j{}^{\otimes^{-1}} \leqslant \lambda \otimes \left(\|\bar{A}_{ij}\|_\oplus\right)^{\otimes^{-1}} \otimes \alpha_i{}^{\otimes^{-1}} \quad . \tag{C.14}$$

We have

$$\begin{aligned} \|\tilde{A}_{ij}\|_\oplus &= \|\alpha_i \otimes \bar{A}_{ij} \otimes \alpha_j{}^{\otimes^{-1}}\|_\oplus \\ &= \alpha_i \otimes \|\bar{A}_{ij}\|_\oplus \otimes \alpha_j{}^{\otimes^{-1}} \\ &\leqslant \alpha_i \otimes \|\bar{A}_{ij}\|_\oplus \otimes \lambda \otimes \left(\|\bar{A}_{ij}\|_\oplus\right)^{\otimes^{-1}} \otimes \alpha_i{}^{\otimes^{-1}} \qquad \text{(by (C.14))} \\ &\leqslant \lambda \ . \end{aligned}$$

So if we define $T = \bar{D} \otimes D \otimes P$, then T is max-invertible, $\tilde{A} = T \otimes A \otimes T^{\otimes^{-1}}$ and $\|T \otimes A \otimes T^{\otimes^{-1}}\|_\oplus = \|\tilde{A}\|_\oplus = \displaystyle\bigoplus_{i=1}^{l} \|\tilde{A}_{ii}\|_\oplus \oplus \bigoplus_{i=1}^{l-1} \bigoplus_{j=i+1}^{l} \|\tilde{A}_{ij}\|_\oplus = \lambda$. □

C.3 An Upper Bound for the Minimal System Order

The max-algebraic sum of sequences is defined as follows. If $G = \{G_k\}_{k=0}^\infty$ and $H = \{H_k\}_{k=0}^\infty$ with $G_k, H_k \in \mathbb{R}_\varepsilon^{l \times m}$ for all $k \in \mathbb{N}$, then $G \oplus H$ is also a sequence

with $(G \oplus H)_k = G_k \oplus H_k$ for all $k \in \mathbb{N}$.
From Theorem 6.1.3 it follows that the impulse response of a max-linear time-invariant DES can always be considered as the max-algebraic sum of a finite number of ultimately geometric impulse responses (See also [3, 54, 56]).
Now we state the generalization of Theorem 6.3.5 by Gaubert:

Theorem C.3.1 *Let g be the impulse response of a max-linear time-invariant SISO DES with $g \neq \{\varepsilon\}_{k=0}^{\infty}$. Let $g_1, g_2, \ldots, g_s$ be ultimately geometric sequences such that $g = g_1 \oplus g_2 \oplus \ldots \oplus g_s$. Then there exists a state space realization of g of order $\sum_{i=1}^{s} \mathrm{rank}_{\oplus,\mathrm{wc}}\left(H(g_i)\right)$.*

Proof: See [54, 56]. □

Example C.3.2 Let us now apply Theorem C.3.1 to the impulse response $g = \{g_k\}_{k=0}^{\infty}$ of Example 6.4.3. We have

$$g_k = \begin{cases} 0 & \text{if } k \text{ is even,} \\ k & \text{if } k \text{ is odd,} \end{cases}$$

for $k = 0, 1, 2, \ldots$. Clearly, this sequence is not ultimately geometric. However, it can be considered as the max-algebraic sum of two ultimately geometric sequences $g_1 = \{(g_1)_k\}_{k=0}^{\infty}$ and $g_2 = \{(g_2)_k\}_{k=0}^{\infty}$ that are defined as follows:

$$(g_1)_k = 0 \quad \text{and} \quad (g_2)_k = \begin{cases} \varepsilon & \text{if } k \text{ is even,} \\ k & \text{if } k \text{ is odd,} \end{cases}$$

for $k = 0, 1, 2, \ldots$. Note that $(g_1)_{k+1} = 0 \otimes (g_1)_k$ and $(g_2)_{k+2} = 2 \otimes (g_2)_k = 1^{\otimes^2} \otimes (g_2)_k$ for all $k \in \mathbb{N}$.
If we define $H_1 = H(g_1)$ and $H_2 = H(g_2)$, then we have

$$H_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad \text{and } H_2 = \begin{bmatrix} \varepsilon & 1 & \varepsilon & 3 & \varepsilon & 5 & \ldots \\ 1 & \varepsilon & 3 & \varepsilon & 5 & \varepsilon & \ldots \\ \varepsilon & 3 & \varepsilon & 5 & \varepsilon & 7 & \ldots \\ 3 & \varepsilon & 5 & \varepsilon & 7 & \varepsilon & \ldots \\ \varepsilon & 5 & \varepsilon & 7 & \varepsilon & 9 & \ldots \\ 5 & \varepsilon & 7 & \varepsilon & 9 & \varepsilon & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} .$$

We have $\mathrm{rank}_{\oplus,\mathrm{wc}}\left(H(g_1)\right) = 1$ since all the columns of H_1 are equal to the first column.
We have $(H_2)_{.,2k+1} = (2k) \otimes (H_2)_{.,1}$ and $(H_2)_{.,2k+2} = (2k) \otimes (H_2)_{.,2}$ for all $k \in \mathbb{N}$. Furthermore, it is impossible to find a number $\alpha \in \mathbb{R}_\varepsilon$ such that $(H_2)_{.,1} = \alpha \otimes (H_2)_{.,2}$ or $(H_2)_{.,2} = \alpha \otimes (H_2)_{.,1}$. Hence, $\mathrm{rank}_{\oplus,\mathrm{wc}}\left(H_2\right) = 2$. So it

follows from Theorem C.3.1 that g can be described by a 3rd order max-linear time-invariant state space model.

In order to construct a 3rd order state space realization of g, we first construct state space realizations of g_1 and g_2. It is easy to verify that g_1 can be realized by the triple $(A_1, B_1, C_1) = ([\,0\,], [\,0\,], [\,0\,])$ and that g_2 can be realized by the triple (A_2, B_2, C_2) with

$$A_2 = \begin{bmatrix} \varepsilon & 2 \\ 0 & \varepsilon \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ \varepsilon \end{bmatrix} \text{ and } C_2 = \begin{bmatrix} \varepsilon & 1 \end{bmatrix} .$$

These realizations have been obtained by applying the methods discussed in [33, 54, 56].
We can merge these two realizations as follows [54, 56]: if we define

$$A = \begin{bmatrix} A_1 & \mathcal{E} \\ \mathcal{E} & A_2 \end{bmatrix} = \begin{bmatrix} 0 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 2 \\ \varepsilon & 0 & \varepsilon \end{bmatrix}$$

$$B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \varepsilon \end{bmatrix}$$

$$C = \begin{bmatrix} C_1 & C_2 \end{bmatrix} = \begin{bmatrix} 0 & \varepsilon & 1 \end{bmatrix} ,$$

then (A, B, C) is a realization of g. □

Appendix D

The Max-Algebraic SVD and the Max-Algebraic QRD: Proofs, Examples and Extensions

First we prove some lemmas and propositions of Chapter 7. Next we give an example of the calculation of the max-algebraic QRD of a matrix with the mapping $\mathcal{F}$ and with the ELCP approach. Finally we propose some extensions of the max-algebraic SVD and the max-algebraic QRD and we show that in theory these extended decompositions can also be computed using the ELCP approach.

D.1 Proof of Lemma 7.3.2

In this section we show that functions that belong to the class $\mathcal{S}_{\mathrm{e}}$ are asymptotically equivalent to an exponential in the neighborhood of ∞. We shall use the following lemma:

Lemma D.1.1 *If $f \in \mathcal{S}_{\mathrm{e}}$ is a series, i.e. if there exists a positive real number K such that $f(x) = \sum_{i=0}^{\infty} \alpha_i e^{a_i x}$ for all $x \geqslant K$ with $\alpha_i \in \mathbb{R}_0$, $a_i \in \mathbb{R}$, $a_i > a_{i+1}$ for all i and where the series converges absolutely for every $x \geqslant K$, then the series $\sum_{i=0}^{\infty} \alpha_i e^{a_i x}$ converges uniformly in $[K, \infty)$.*

Proof: Since $f(x)$ can be written as a series, we have $a_0 \neq \varepsilon$. Hence,

$$\sum_{i=0}^{\infty} \alpha_i e^{a_i x} = \alpha_0 e^{a_0 x} \left(1 + \sum_{i=1}^{\infty} \frac{\alpha_i}{\alpha_0} e^{(a_i - a_0)x} \right) = \alpha_0 e^{a_0 x} \left(1 + \sum_{i=1}^{\infty} \gamma_i e^{c_i x} \right)$$

with $\gamma_i = \dfrac{\alpha_i}{\alpha_0} \in \mathbb{R}_0$ and $c_i = a_i - a_0 < 0$ for all $i \in \mathbb{N}_0$. Since $\displaystyle\sum_{i=0}^{\infty} \alpha_i e^{a_i K}$ converges absolutely, $\displaystyle\sum_{i=1}^{\infty} \gamma_i e^{c_i K}$ also converges absolutely.

If $x \geqslant K$ then we have $e^{c_i x} \leqslant e^{c_i K}$ for all $i \in \mathbb{N}_0$ since $c_i < 0$ for all i. Hence, $|\gamma_i e^{c_i x}| < |\gamma_i e^{c_i K}|$ for all $x \geqslant K$ and for all $i \in \mathbb{N}_0$. We already know that $\displaystyle\sum_{i=1}^{\infty} |\gamma_i e^{c_i K}|$ converges. Now we can apply the Weierstrass M-test (See e.g. [83, 104]). As a consequence, the series $\displaystyle\sum_{i=1}^{\infty} \gamma_i e^{c_i x}$ converges uniformly in $[K, \infty)$, which implies that the series $\displaystyle\sum_{i=0}^{\infty} \alpha_i e^{a_i x}$ also converges uniformly in $[K, \infty)$. □

Proof of Lemma 7.3.2: If $f \in \mathcal{S}_{\mathrm{e}}$ then there exists a positive real number K such that $f(x) = \displaystyle\sum_{i=0}^{n} \alpha_i e^{a_i x}$ for all $x \geqslant K$ with $n \in \mathbb{N} \cup \{\infty\}$, $\alpha_i \in \mathbb{R}_0$ and $a_i \in \mathbb{R}_\varepsilon$ for all i. If $n = \infty$ then f is a series that converges absolutely in $[K, \infty)$.

If $a_0 = \varepsilon$ then there exists a real number K such that $f(x) = 0$ for all $x \geqslant K$ and then we have $f(x) \sim 0 = 1 \cdot e^{\varepsilon x}$, $x \to \infty$ by Definition 7.1.4.

If $n = 1$ then $f(x) = \alpha_0 e^{a_0 x}$ and thus $f(x) \sim \alpha_0 e^{a_0 x}$, $x \to \infty$ with $\alpha_0 \in \mathbb{R}_0$ and $a_0 \in \mathbb{R}_\varepsilon$.

From now on we assume that $n > 1$ and $a_0 \neq \varepsilon$. Then we can rewrite $f(x)$ as

$$f(x) = \alpha_0 e^{a_0 x} \left(1 + \sum_{i=1}^{n} \frac{\alpha_i}{\alpha_0} e^{(a_i - a_0)x} \right) = \alpha_0 e^{a_0 x} (\, 1 + p(x)\,)$$

with $p(x) = \displaystyle\sum_{i=1}^{n} \gamma_i e^{c_i x}$ where $\gamma_i = \dfrac{\alpha_i}{\alpha_0} \in \mathbb{R}_0$ and $c_i = a_i - a_0 < 0$ for all i. Note that $p \in \mathcal{S}_{\mathrm{e}}$. Since $c_i < 0$ for all i, we have

$$\lim_{x \to \infty} p(x) = \lim_{x \to \infty} \left(\sum_{i=1}^{n} \gamma_i e^{c_i x} \right) = \sum_{i=1}^{n} \left(\lim_{x \to \infty} \gamma_i e^{c_i x} \right) = 0 \; . \qquad \text{(D.1)}$$

If $n = \infty$ then the series $\displaystyle\sum_{i=1}^{\infty} \gamma_i e^{c_i x}$ converges uniformly in $[K, \infty)$ by Lemma D.1.1. As a consequence, we may also interchange the summation and the limit

in (D.1) if $n = \infty$ (cf. [83]).
Now we have

$$\lim_{x\to\infty} \frac{f(x)}{\alpha_0 e^{a_0 x}} = \lim_{x\to\infty} \frac{\alpha_0 e^{a_0 x}(1+p(x))}{\alpha_0 e^{a_0 x}} = \lim_{x\to\infty} (1+p(x)) = 1$$

and thus $f(x) \sim \alpha_0 e^{a_0 x}$, $x \to \infty$ where $\alpha_0 \in \mathbb{R}_0$ and $a_0 \in \mathbb{R}$. □

D.2 Proof of Proposition 7.3.3

In this section we show that $\mathcal{S}_\mathrm{e}$ is closed under elementary operations such as additions, multiplications, subtractions, divisions, square roots and absolute values.

Proof of Proposition 7.3.3: If f and g belong to $\mathcal{S}_\mathrm{e}$ then we may assume without loss of generality that the domains of definition of f and g coincide, since we can always restrict the functions f and g to $\operatorname{dom} f \cap \operatorname{dom} g$ and since the restricted functions also belong to $\mathcal{S}_\mathrm{e}$.
Since f and g belong to $\mathcal{S}_\mathrm{e}$, there exists a positive real number K such that

$$f(x) = \sum_{i=0}^{n} \alpha_i e^{a_i x} \quad \text{and} \quad g(x) = \sum_{j=0}^{m} \beta_j e^{b_j x} \qquad \text{for all } x \geqslant K$$

with $m, n \in \mathbb{N} \cup \{\infty\}$, $\alpha_i, \beta_j \in \mathbb{R}_0$ and $a_i, b_j \in \mathbb{R}_\varepsilon$ for all i, j. If m or n is equal to ∞ then the corresponding series converges absolutely in $[K, \infty)$.
We may assume without loss of generality that both m and n are equal to ∞. If m or n are finite then we can always add dummy terms of the form $0 \cdot e^{\varepsilon x}$ to $f(x)$ or $g(x)$. Afterwards we can remove terms of the form $re^{\varepsilon x}$ with $r \in \mathbb{R}$ to obtain an expression with non-zero coefficients and decreasing exponents. So from now on we assume that both f and g are absolute convergent series of exponentials.

If $a_0 = \varepsilon$ then we have $f(x) = 0$ for all $x \geqslant K$, which means that $|f(x)| = 0$ for all $x \geqslant K$. So if $a_0 = \varepsilon$ then $|f|$ belongs to $\mathcal{S}_\mathrm{e}$.
If $a_0 \neq \varepsilon$ then there exists a real number $L \geqslant K$ such that either $f(x) > 0$ or $f(x) < 0$ for all $x \geqslant L$ since $f(x) \sim \alpha_0 e^{a_0 x}$, $x \to \infty$ with $\alpha_0 \neq 0$ by Lemma 7.3.2. Hence, either $|f(x)| = f(x)$ or $|f(x)| = -f(x)$ for all $x \geqslant L$. So in this case $|f|$ also belongs to $\mathcal{S}_\mathrm{e}$.

Since f and g are analytic in $[K, \infty)$, the functions ρf, $f+g$, $f-g$, $f \cdot g$ and f^l are also analytic in $[K, \infty)$ for any $\rho \in \mathbb{R}$ and any $l \in \mathbb{N}$.

Now we prove that these functions can be written as a sum of exponentials or as an absolutely convergent series of exponentials.
Consider an arbitrary $\rho \in \mathbb{R}$. If $\rho = 0$ then $\rho f(x) = 0$ for all $x \geqslant K$ and thus $\rho f \in \mathcal{S}_\mathrm{e}$.

If $\rho \neq 0$ then we have $\rho f(x) = \sum_{i=0}^{\infty} (\rho\alpha_i) e^{a_i x}$. The series $\sum_{i=0}^{\infty} (\rho\alpha_i) e^{a_i x}$ also converges absolutely in $[K, \infty)$ and has the same exponents as $f(x)$. Hence, $\rho f \in \mathcal{S}_{\mathrm{e}}$.

The sum function $f + g$ is a series of exponentials since

$$f(x) + g(x) = \sum_{i=0}^{\infty} \alpha_i e^{a_i x} + \sum_{j=0}^{\infty} \beta_j e^{b_j x} \ .$$

Furthermore, this series converges absolutely for every $x \geqslant K$. Therefore, the sum of the series does not change if we rearrange the terms [83]. So $f(x)+g(x)$ can be written in the form of Definition 7.3.1 by reordering the terms, adding up terms with equal exponents and removing terms of the form $re^{\varepsilon x}$ with $r \in \mathbb{R}$, if there are any. If the result is a series then the sequence of exponents is decreasing and unbounded from below. So $f + g \in \mathcal{S}_{\mathrm{e}}$.

Since $f - g = f + (-1)g$, the function $f - g$ also belongs to $\mathcal{S}_{\mathrm{e}}$.

The series corresponding to f and g converge absolutely for every $x \geqslant K$. Therefore, their Cauchy product will also converge absolutely for every $x \geqslant K$ and it will be equal to fg [83]:

$$f(x)g(x) = \sum_{i=0}^{\infty} \sum_{j=0}^{i} \alpha_j \beta_{i-j} e^{(a_j + b_{i-j})x} \qquad \text{for all } x \geqslant K \, .$$

Using the same procedure as for $f + g$, we can also write this product in the form (7.11) or (7.12). So $fg \in \mathcal{S}_{\mathrm{e}}$.

Let $l \in \mathbb{N}$. If $l = 0$ then $f^l = 0 \in \mathcal{S}_{\mathrm{e}}$ and if $l = 1$ then $f^l = f \in \mathcal{S}_{\mathrm{e}}$. If $l > 1$, we can make repeated use of the fact that $fg \in \mathcal{S}_{\mathrm{e}}$ if $f, g \in \mathcal{S}_{\mathrm{e}}$ to prove that f^l also belongs to $\mathcal{S}_{\mathrm{e}}$.

If there exists a real number P such that $f(x) \neq 0$ for all $x \geqslant P$ then $\dfrac{1}{f}$ and $\dfrac{g}{f}$ are defined and analytic in $[P, \infty)$. Note that we may assume without loss of generality that $P \geqslant K$. Furthermore, since the function f restricted to the interval $[P, \infty)$ also belongs to $\mathcal{S}_{\mathrm{e}}$, we may assume without loss of generality that the domain of definition of f is $[P, \infty)$.

If $f(x) \neq 0$ for all $x \geqslant P$ then we have $a_0 \neq \varepsilon$. As a consequence, we can rewrite $f(x)$ as

$$f(x) = \sum_{i=0}^{\infty} \alpha_i e^{a_i x} = \alpha_0 e^{a_0 x} \left(1 + \sum_{i=1}^{\infty} \frac{\alpha_i}{\alpha_0} e^{(a_i - a_0)x} \right) = \alpha_0 e^{a_0 x} (\, 1 + p(x)\,)$$

with $p(x) = \displaystyle\sum_{i=1}^{\infty} \gamma_i e^{c_i x}$ where $\gamma_i = \dfrac{\alpha_i}{\alpha_0} \in \mathbb{R}_0$ and $c_i = a_i - a_0 < 0$ for all i. Note that p is defined in $[P, \infty)$ and that $p \in \mathcal{S}_{\mathrm{e}}$.

If $c_1 = \varepsilon$ then $p(x) = 0$ and $\dfrac{1}{f(x)} = \dfrac{1}{\alpha_0} e^{-a_0 x}$ for all $x \geqslant P$. Hence, $\dfrac{1}{f} \in \mathcal{S}_{\mathrm{e}}$.

Now assume that $c_1 \neq \varepsilon$. Since $\{c_i\}_{i=1}^{\infty}$ is a non-increasing sequence of negative numbers with $\lim_{i \to \infty} c_i = \varepsilon = -\infty$ and since p converges uniformly in $[P, \infty)$ by Lemma D.1.1, we have $\lim_{x \to \infty} p(x) = 0$ (cf. (D.1)). So $|\, p(x) \,|$ will be less than 1 if x is large enough, say if $x \geqslant M$. If we use the Taylor series expansion of $\dfrac{1}{1+x}$, we obtain

$$\frac{1}{1+p(x)} = \sum_{k=0}^{\infty} (-1)^k p^k(x) \qquad \text{if } |p(x)| < 1\,. \tag{D.2}$$

We already know that $p \in \mathcal{S}_{\mathrm{e}}$. Hence, $p^k \in \mathcal{S}_{\mathrm{e}}$ for all $k \in \mathbb{N}$. We have $|p(x)| < 1$ for all $x \geqslant M$. Moreover, for any $k \in \mathbb{N}$ the highest exponent in p^k is equal to kc_1, which implies that the dominant exponent of p^k tends to $-\infty$ as k tends to ∞. As a consequence, the coefficients and the exponents of more and more successive terms of the partial sum function s_n that is defined by $s_n(x) = \sum_{k=0}^{n} (-1)^k p^k(x)$ for $x \in [M, \infty)$ will not change any more as n becomes larger and larger. Therefore, the series on the right-hand side of (D.2) also is a sum of exponentials:

$$\frac{1}{1+p(x)} = \sum_{k=0}^{\infty} (-1)^k \left(\sum_{i=1}^{\infty} \gamma_i e^{c_i x} \right)^k = \sum_{k=0}^{\infty} d_i e^{\delta_i x} \qquad \text{for all } x \geqslant M\,.$$

Note that the set of exponents of this series will have no finite accumulation point since the highest exponent in p^k is equal to kc_1. First we prove that this series also converges absolutely. Define $p^*(x) = \sum_{i=1}^{\infty} |\gamma_i| e^{c_i x}$ for all $x \geqslant P$. Since the terms of the series p^* are the absolute values of the terms of the series p and since p converges absolutely in $[P, \infty)$, p^* also converges absolutely in $[P, \infty)$. By Lemma D.1.1 p^* also converges uniformly in $[P, \infty)$. Furthermore, $\{c_i\}_{i=1}^{\infty}$ is a non-increasing and unbounded sequence of negative numbers. As a consequence, we have $\lim_{x \to \infty} p^*(x) = 0$ (cf. (D.1)). So $|\, p^*(x) \,|$ will be less than 1 if x is large enough, say if $x \geqslant N$. Therefore, we have

$$\frac{1}{1+p^*(x)} = \sum_{k=0}^{\infty} (-1)^k \left(p^*(x)\right)^k \qquad \text{for all } x \geqslant N\,.$$

This series converges absolutely in $[N, \infty)$. Since

$$\sum_{k=0}^{\infty} |d_i| e^{\delta_i x} \leqslant \sum_{k=0}^{\infty} \left(\sum_{i=1}^{\infty} |\gamma_i| e^{c_i x} \right)^k = \sum_{k=0}^{\infty} \left| \left(p^*(x)\right)^k \right| \ ,$$

the series $\sum_{k=0}^{\infty} d_i e^{\delta_i x}$ also converges absolutely for any $x \in [N, \infty)$. Since this series converges absolutely, we can reorder the terms. After reordering the

terms, adding up terms with the same exponents and removing terms of the form $re^{\varepsilon x}$ with $r \in \mathbb{R}$ if necessary, the sequence of exponents will be decreasing and unbounded from below.
This implies that $\dfrac{1}{1+p} \in \mathcal{S}_{\mathrm{e}}$ and thus also $\dfrac{1}{f} \in \mathcal{S}_{\mathrm{e}}$.
As a consequence, $\dfrac{g}{f} = g\dfrac{1}{f}$ also belongs to $\mathcal{S}_{\mathrm{e}}$.
If there exists a real number Q such that $f(x) > 0$ for all $x \geqslant Q$ then the function $\sqrt{f}$ is defined and analytic in $[Q, \infty)$. We may assume without loss of generality that $Q \geqslant K$ and that the domain of definition of f is $[Q, \infty)$.
If $a_0 = \varepsilon$ then we have $\sqrt{f(x)} = 0$ for all $x \geqslant Q$ and thus $\sqrt{f} \in \mathcal{S}_{\mathrm{e}}$.
If $a_0 \neq \varepsilon$ then $\alpha_0 > 0$ and then we can rewrite $\sqrt{f(x)}$ as

$$\sqrt{f(x)} = \sqrt{\alpha_0}\ e^{\frac{1}{2}a_0 x} \sqrt{1+p(x)}\ .$$

Now we can use the Taylor series expansion of $\sqrt{1+x}$. This leads to

$$\sqrt{1+p(x)} = \sum_{k=0}^{\infty} \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}-k\right)\ k!}\, p^k(x) \qquad \text{if } |p(x)| < 1\,,$$

where Γ is the gamma function. If we apply the same reasoning as for $\dfrac{1}{1+p}$, we find that $\sqrt{1+p} \in \mathcal{S}_{\mathrm{e}}$ and thus also $\sqrt{f} \in \mathcal{S}_{\mathrm{e}}$. □

D.3 Proof of Proposition 7.5.1

In this section we prove that a matrix $A \in \mathbb{S}^{m \times n}$ with finite entries always has a max-algebraic SVD with finite singular values and finite singular vectors.

Proof of Proposition 7.5.1: Define $\tilde{A} = \mathcal{F}(A, \cdot)$. In the proof of Theorem 7.3.5 we have shown that there exists a path of approximate SVDs $\tilde{U}\tilde{\Sigma}\tilde{V}^T$ of $\tilde{A}$ on some interval $[L, \infty)$ where the entries of $\tilde{U}$, $\tilde{\Sigma}$ and $\tilde{V}^T$ are asymptotically equivalent to an exponential in the neighborhood of ∞. If we apply the reverse mapping $\mathcal{R}$, we obtain a max-algebraic SVD of A: $A \mathrel{\nabla} U \otimes \Sigma \otimes V^T$.
If all the singular values and all the components of the singular vectors of this max-algebraic SVD are finite then the proposition is proved.
Now we show how a max-algebraic SVD that contains singular values that are equal to ε or singular vectors with components that are equal to ε can be transformed into a max-algebraic SVD $\hat{U} \otimes \hat{\Sigma} \otimes \hat{V}^T$ of A with finite singular values and vectors. This will be done in three steps. First we make all the singular values finite, next we make the components of the left singular vectors finite, and finally we make the components of the right singular vectors finite.

Step 1: We make all the singular values finite.

Let $r = \min(m, n)$. Since $U \otimes \Sigma \otimes V^T$ is a max-algebraic SVD of A, we have

$$a_{ij}^{\oplus} \oplus \bigoplus_{k=1}^{r} u_{ik}^{\oplus} \otimes \sigma_k \otimes v_{jk}^{\ominus} \oplus \bigoplus_{k=1}^{r} u_{ik}^{\ominus} \otimes \sigma_k \otimes v_{jk}^{\oplus}$$
$$= a_{ij}^{\ominus} \oplus \bigoplus_{k=1}^{r} u_{ik}^{\oplus} \otimes \sigma_k \otimes v_{jk}^{\oplus} \oplus \bigoplus_{k=1}^{r} u_{ik}^{\ominus} \otimes \sigma_k \otimes v_{jk}^{\ominus} \qquad \text{(D.3)}$$

for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$ (cf. (7.26)). Since $|a_{ij}|_{\oplus}$ is finite for all i, j, (D.3) will still hold if we augment some of the terms that appear in one of the four max-algebraic summations of (D.3) provided that these terms stay less than or equal to $|a_{ij}|_{\oplus}$. Since $|u_{jk}|_{\oplus} \leqslant 0$ and $|v_{jk}|_{\oplus} \leqslant 0$ for all j, k by Corollary 7.4.4, this condition will be satisfied as long as any new singular value $\hat{\sigma}_k$ that corresponds to a term that has been augmented stays less than or equal to $|a_{ij}|_{\oplus}$. If we define $f = \min\limits_{i,j} |a_{ij}|_{\oplus}$ then f is finite. Assume that $\sigma_l = \sigma_{l+1} = \ldots = \sigma_r = \varepsilon$. If we set

$$\hat{\sigma}_i = \begin{cases} \sigma_j & \text{if } i \in \{1, 2, \ldots, l-1\} , \\ f & \text{if } i \in \{l, l+1, \ldots, r\} , \end{cases}$$

for $i = 1, 2, \ldots, r$ and if $\hat{\Sigma}$ is a m by n max-algebraic diagonal matrix with $\hat{\Sigma}_{ii} = \hat{\sigma}_i$ for $i = 1, 2, \ldots, r$, then we have $A \nabla U \otimes \hat{\Sigma} \otimes V$. Since the $\hat{\sigma}_i$'s are ordered and since we did not change the other equations, we now have a max-algebraic SVD of A for which all the singular values are finite.

Step 2: We make the components of the left singular vectors finite.

Define $\mathcal{I} = \{ (i, k) \,|\, u_{ik} = \varepsilon \}$. Let $\hat{U}$ be the matrix obtained by replacing each entry u_{ik} of U that is equal to ε by M or by $\ominus M$ where M is a negative real number the exact value of which will be determined later on and where the exact max-algebraic sign of $\hat{u}_{ik}$ will also be determined later on. So

$$\hat{u}_{ik} = \begin{cases} u_{ik} & \text{if } (i, k) \notin \mathcal{I} , \\ M \text{ or } \ominus M & \text{if } (i, k) \in \mathcal{I} , \end{cases}$$

for all i, k. Let $\hat{u}_i$ be the ith column of $\hat{U}$ for $i = 1, 2, \ldots, m$.
By Corollary 7.4.4 we have $|v_{jk}|_{\oplus} \leqslant 0$ for all j, k. So if we consider (D.3) with the σ_k's replaced by the $\hat{\sigma}_k$'s then it is obvious that this equation will still hold if the entries of the vectors $\hat{u}_i$ satisfy

$$|\hat{u}_{ik}|_{\oplus} = \hat{u}_{ik}^{\oplus} \oplus \hat{u}_{ik}^{\ominus} \leqslant \min_j |a_{ij}|_{\oplus} - \hat{\sigma}_k \qquad \text{for all } (i, k) \in \mathcal{I} . \qquad \text{(D.4)}$$

If we define $g = \min\limits_{i,j} |a_{ij}|_{\oplus} - \hat{\sigma}_1$, then g is finite. Since $\hat{\sigma}_k \leqslant \hat{\sigma}_1$ for $k = 1, 2, \ldots, r$, condition (D.4) will be fulfilled if

$$|\hat{u}_{ik}|_{\oplus} \leqslant g \qquad \text{for all } (i, k) \in \mathcal{I} . \qquad \text{(D.5)}$$

Since $\tilde{U}(s)$ is an orthogonal matrix for $s \geqslant L$ and since the entries of $\tilde{U}$ are analytic and thus also continuous, we have either $\det \tilde{U}(s) = 1$ or $\det \tilde{U}(s) = -1$ for all $s \geqslant L$. The entries of an orthogonal matrix always lie in the interval $[-1, 1]$. Therefore, all the dominant exponents of the entries of $\tilde{U}$ are less than or equal to 0. So $|\det \tilde{U}(s)|$ can only be equal to 1 for $s \geqslant L$ if there exists a permutation φ of the set $\{1, 2, \ldots, m\}$ such that

$$\prod_{i=1}^{m} \tilde{u}_{i\varphi(i)}(s) \sim c \, , \; s \to \infty$$

for some $c \in \mathbb{R}_0$ or equivalently

$$\tilde{u}_{i\varphi(i)}(s) \sim c_i \, , \; s \to \infty \qquad \text{for } i = 1, 2, \ldots, m \tag{D.6}$$

with $c_1, c_2, \ldots, c_m \in \mathbb{R}_0$. If we apply the reverse mapping $\mathcal{R}$ to (D.6), we get

$$u_{i\varphi(i)} = 0 \quad \text{or} \quad u_{i\varphi(i)} = \ominus 0 \qquad \text{for } i = 1, 2, \ldots, m \tag{D.7}$$

since $\mathcal{R}(c_i) = 0$ if $c_i > 0$ and $\mathcal{R}(c_i) = \ominus 0$ if $c_i < 0$.
Now we permute the columns of U such that the entries that are equal to 0 in max-absolute value will be on the main diagonal. This can be done as follows. We define a max-algebraic permutation matrix $P \in \mathbb{R}_\varepsilon^{m \times m}$ such that

$$p_{ij} = \begin{cases} 0 & \text{if } i = \varphi(j)\,, \\ \varepsilon & \text{otherwise,} \end{cases}$$

for all i, j. If we define $W = U \otimes P$ then we have

$$w_{ii} = 0 \quad \text{ or } \quad w_{ii} = \ominus 0 \qquad \text{for } i = 1, 2, \ldots, m\,.$$

Let w_i be the ith column of W for $i = 1, 2, \ldots, m$. Since W contains the same columns as U but in a (possibly) different order, we have $w_i^T \otimes w_i \,\nabla\, 0$ for all i and $w_i^T \otimes w_j \,\nabla\, \varepsilon$ for all i, j with $i \neq j$. Hence, $W^T \otimes W \,\nabla\, E_m$. From Proposition 7.4.3 it follows that $w_i^T \otimes w_i = \|w_i\|_\oplus = 0$ for all i.

Now we copy all the entries of W to $\hat{W}$. We shall update the columns of $\hat{W}$ in two steps. First we shall make all max-algebraic inner products of two different columns of $\hat{W}$ finite. Next we shall make the entries of $\hat{W}$ that are still equal to ε finite.

Step 2a: We make all max-algebraic inner products of two different columns of $\hat{W}$ finite.

Define

$$h = \min\{\, |w_i^T \otimes w_j|_\oplus \,|\, w_i^T \otimes w_j \neq \varepsilon \,\}$$

and $M = \min(g, h-1)$. Note that h and M are finite. Since $|w_{ij}|_{\oplus} \leqslant 0$ for all i, j by Corollary 7.4.4, we have $|w_i^T \otimes w_j|_{\oplus} \leqslant 0$ for all i, j. Hence, $h \leqslant 0$ and $M < 0$. Furthermore, if $w_i^T \otimes w_j \neq \varepsilon$ then $|w_i^T \otimes w_j|_{\oplus} > M$.
Consider the following algorithm in which some of the entries of $\hat{W}$ that are equal to ε are replaced by M or by $\ominus M$:

Input: m, W, $\hat{W}$, M
    for $i = 1, 2, \ldots, m-1$ **do**
        for $j = i+1, 2, \ldots, m$ **do**
            if $w_i^T \otimes w_j = \varepsilon$ **then**
                $\hat{w}_{ij} \leftarrow M$
                $\hat{w}_{ji} \leftarrow (\ominus M) \otimes \hat{w}_{ii} \otimes \hat{w}_{jj}$
            end if
        end for
    end for
Output: $\hat{W}$

Now we prove that this algorithm results in

$$\hat{w}_i^T \otimes \hat{w}_j = w_i^T \otimes w_j \qquad \text{if } |w_i^T \otimes w_j|_{\oplus} \neq \varepsilon\,, \tag{D.8}$$

$$\hat{w}_i^T \otimes \hat{w}_j = M^{\bullet} \qquad \text{if } |w_i^T \otimes w_j|_{\oplus} = \varepsilon\,. \tag{D.9}$$

First we prove (D.8).
Assume that $w_i^T \otimes w_j$ is finite. Later on we shall prove that only infinite entries of $\hat{W}$ are replaced by M or by $\ominus M$ and that the finite entries of $\hat{W}$ do not change if we execute the algorithm given above. As a consequence, we have

$$\begin{aligned}
\hat{w}_i^T \otimes \hat{w}_j &= \bigoplus_{w_{ki} \neq \varepsilon \text{ and } w_{kj} \neq \varepsilon} \hat{w}_{ki} \otimes \hat{w}_{kj} \;\oplus \bigoplus_{w_{ki} = \varepsilon \text{ or } w_{kj} = \varepsilon} \hat{w}_{ki} \otimes \hat{w}_{kj} \\
&= \bigoplus_{w_{ki} \neq \varepsilon \text{ and } w_{kj} \neq \varepsilon} w_{ki} \otimes w_{kj} \;\oplus \bigoplus_{w_{ki} = \varepsilon \text{ or } w_{kj} = \varepsilon} \hat{w}_{ki} \otimes \hat{w}_{kj} \\
&= w_i^T \otimes w_j \;\oplus\; q_{ij}
\end{aligned} \tag{D.10}$$

where $q_{ij} = \displaystyle\bigoplus_{w_{ki} = \varepsilon \text{ or } w_{kj} = \varepsilon} \hat{w}_{ki} \otimes \hat{w}_{kj}$. Since

$$|\hat{w}_{ki}|_{\oplus} \leqslant M \text{ and } |\hat{w}_{kj}|_{\oplus} \leqslant 0 \qquad \text{if } w_{ki} = \varepsilon\,,$$

$$|\hat{w}_{ki}|_{\oplus} \leqslant 0 \text{ and } |\hat{w}_{kj}|_{\oplus} \leqslant M \qquad \text{if } w_{kj} = \varepsilon\,,$$

we have

$$
\begin{aligned}
|q_{ij}|_{\oplus} &= \bigoplus_{w_{ki}=\varepsilon \text{ or } w_{kj}=\varepsilon} |\hat{w}_{ki} \otimes \hat{w}_{kj}|_{\oplus} && \text{(by Proposition 2.3.5)} \\
&= \bigoplus_{w_{ki}=\varepsilon \text{ or } w_{kj}=\varepsilon} |\hat{w}_{ki}|_{\oplus} \otimes |\hat{w}_{kj}|_{\oplus} && \text{(by Proposition 2.3.5)} \\
&\leqslant M \\
&< |w_i^T \otimes w_j|_{\oplus} \; .
\end{aligned}
$$

If we combine this with (D.10), we obtain $\hat{w}_i^T \otimes \hat{w}_j = w_i^T \otimes w_j$. So the values of the finite max-algebraic inner products do not change.

Now we prove (D.9).
Assume that $w_i^T \otimes w_j = \varepsilon$. Then we have $w_{si} \otimes w_{sj} = \varepsilon$ for $s = 1, 2, \ldots, m$ or equivalently

$$
w_{si} = \varepsilon \;\text{ or }\; w_{sj} = \varepsilon \qquad \text{for } s = 1, 2, \ldots, m\,. \tag{D.11}
$$

Since $|w_{ii}|_{\oplus}$ and $|w_{jj}|_{\oplus}$ are equal to 0, this implies that both w_{ij} and w_{ji} are equal to ε.
It is possible that some of the infinite components of $\hat{w}_i$ and $\hat{w}_j$ have already been replaced by M or $\ominus M$. However, $\hat{w}_{ij}$ and $\hat{w}_{ji}$ are still equal to ε since each ordered pair of indices (i, j) is encountered only once in the algorithm. So we only replace infinite entries of $\hat{W}$ by M or $\ominus M$ if we execute the algorithm.
In the algorithm $\hat{w}_{ij}$ is replaced by M and $\hat{w}_{ji}$ is replaced by $(\ominus M) \otimes \hat{w}_{ii} \otimes \hat{w}_{jj}$. Hence,

$$
|\hat{w}_{ji}|_{\oplus} = |(\ominus M) \otimes \hat{w}_{ii} \otimes \hat{w}_{jj}|_{\oplus} = M \otimes |\hat{w}_{ii}|_{\oplus} \otimes |\hat{w}_{jj}|_{\oplus} = M \otimes 0 \otimes 0 = M
$$

since $|a \otimes b|_{\oplus} = |a|_{\oplus} \otimes |b|_{\oplus}$ for all $a, b \in \mathbb{S}$ by Proposition 2.3.5.
Since M, $\hat{w}_{ii}$ and $\hat{w}_{jj}$ are signed, $\hat{w}_{ji}$ is also signed. So either $\hat{w}_{ji} = M$ or $\hat{w}_{ji} = \ominus M$.
Now we have

$$
\begin{aligned}
\hat{w}_i^T \otimes \hat{w}_j &= \hat{w}_{ii} \otimes \hat{w}_{ij} \;\oplus\; \hat{w}_{ji} \otimes \hat{w}_{jj} \;\oplus\; \bigoplus_{\substack{s=1 \\ s \neq i, s \neq j}}^{m} \hat{w}_{si} \otimes \hat{w}_{sj} \\
&= \hat{w}_{ii} \otimes M \;\oplus\; (\ominus M) \otimes \hat{w}_{ii} \otimes \hat{w}_{jj} \otimes \hat{w}_{jj} \;\oplus\; t_{ij}
\end{aligned} \tag{D.12}
$$

where $t_{ij} = \bigoplus_{\substack{s=1 \\ s \neq i, s \neq j}}^{m} \hat{w}_{si} \otimes \hat{w}_{sj}$. By (D.11) we have

$$
w_{si} = \varepsilon \text{ and thus } |\hat{w}_{si}|_{\oplus} \leqslant M \quad \text{ or } \quad w_{sj} = \varepsilon \text{ and thus } |\hat{w}_{sj}|_{\oplus} \leqslant M
$$

for $s = 1, 2, \ldots, m$. Furthermore, since $|w_{pq}|_{\oplus} \leqslant 0$ for all p, q by Corollary 7.4.4 and since $M < 0$, we have $|\hat{w}_{pq}|_{\oplus} \leqslant 0$ for all p, q. Hence, $|\hat{w}_{si} \otimes \hat{w}_{sj}|_{\oplus} \leqslant M$ for all s. As a consequence, we have $|t_{ij}|_{\oplus} \leqslant M$.
Since $\hat{w}_{jj}$ is signed and since $|\hat{w}_{jj}|_{\oplus} = 0$, we have either $\hat{w}_{jj} = 0$ or $\hat{w}_{jj} = \ominus 0$. Hence, $\hat{w}_{jj} \otimes \hat{w}_{jj} = 0$. Therefore, (D.12) results in

$$\hat{w}_i^T \otimes \hat{w}_j = M \otimes \hat{w}_{ii} \ \oplus \ (\ominus M) \otimes \hat{w}_{ii} \ \oplus \ t_{ij} \ .$$

Since $\hat{w}_{ii}$ is either 0 or $\ominus 0$ and since $|t_{ij}|_{\oplus} \leqslant M$, this leads to

$$\hat{w}_i^T \otimes \hat{w}_j = M^{\bullet} \ \oplus \ t_{ij} = M^{\bullet} \ .$$

So now all max-algebraic inner products of two columns of $\tilde{W}$ are finite.

Step 2b: We make the remaining entries of $\hat{W}$ that are equal to ε finite by replacing them by M.

As already explained above this does not change the value of the finite max-algebraic inner products $\hat{w}_i^T \otimes \hat{w}_j = w_i^T \otimes w_j$. Furthermore, since the other max-algebraic inner products $\hat{w}_i^T \otimes \hat{w}_j$ are already equal to $M^{\bullet}$, their value does not change either. So (D.8) and (D.9) still hold.

If we define $\hat{U} = \hat{W} \otimes P^T$, then $\hat{U}$ contains the same columns as $\hat{W}$ but in a (possibly) different order. As a consequence, we have

$$\begin{aligned}
&\hat{u}_i^T \otimes \hat{u}_i = u_i^T \otimes u_i = 0 && \text{for all } i\,,\\
&\hat{u}_i^T \otimes \hat{u}_j = u_i^T \otimes u_j \ \nabla\ \varepsilon && \text{for all } i,j \text{ with } i \neq j \text{ and } |u_i^T \otimes u_j|_{\oplus} \neq \varepsilon\,,\\
&\hat{u}_i^T \otimes \hat{u}_j = M^{\bullet} \ \nabla\ \varepsilon && \text{for all } i,j \text{ with } i \neq j \text{ and } |u_i^T \otimes u_j|_{\oplus} = \varepsilon\,.
\end{aligned}$$

So now we have a finite matrix $\hat{U}$ that satisfies $\hat{U}^T \otimes \hat{U} \ \nabla\ E_m$. Furthermore, condition (D.5) is satisfied since $M \leqslant g$. This implies that (D.3) still holds. Therefore, we now have a max-algebraic SVD of A with finite singular values and finite left singular vectors.

Step 3: Finally we make the components of the right singular vectors finite.

Using a reasoning that is analogous to the one of Step 2 of this proof we can transform the right singular vectors v_i into right singular vectors $\hat{v}_i$ with finite entries.

This yields a max-algebraic SVD $\hat{U} \otimes \hat{\Sigma} \otimes \hat{V}$ of A with finite singular values and finite singular vectors. □

Note that we cannot use Proposition 4.1.4 to prove Proposition 7.5.1 since even if all the entries of A are finite, the system of multivariate max-algebraic polynomial equalities and inequalities consisting of (7.34) – (7.35) and (7.45) – (7.55) still contains equations with right-hand sides that are equal to ε, viz. (7.34) and (7.35).

D.4 A Worked Example of the Max-Algebraic QRD

In this section we compute a max-algebraic QRD of a matrix, first with the mapping $\mathcal{F}$ and the Givens algorithm, and then with the ELCP approach.

Example D.4.1 Let us compute a max-algebraic QRD of

$$B = \begin{bmatrix} 2 & \ominus 0 & -1 \\ \ominus 3 & 1^{\bullet} & \varepsilon \end{bmatrix}$$

using the mapping $\mathcal{F}$. We define $\tilde{B} = \mathcal{F}(B, \cdot)$ where we take all the coefficients of the exponentials equal to 1:

$$\tilde{B}(s) = \begin{bmatrix} e^{2s} & -1 & e^{-s} \\ -e^{3s} & e^{s} & 0 \end{bmatrix} \qquad \text{for all } s \in \mathbb{R}_0^+ .$$

If we use the Givens QR algorithm, we get a path of QR decompositions $\tilde{Q}\tilde{R}$ of $\tilde{B}$ with

$$\tilde{Q}(s) = \begin{bmatrix} \dfrac{e^{-s}}{\sqrt{1+e^{-2s}}} & \dfrac{1}{\sqrt{1+e^{-2s}}} \\ \dfrac{-1}{\sqrt{1+e^{-2s}}} & \dfrac{e^{-s}}{\sqrt{1+e^{-2s}}} \end{bmatrix}$$

$$\tilde{R}(s) = \begin{bmatrix} e^{3s}\sqrt{1+e^{-2s}} & -e^{s}\sqrt{1+e^{-2s}} & \dfrac{e^{-2s}}{\sqrt{1+e^{-2s}}} \\ 0 & 0 & \dfrac{e^{-s}}{\sqrt{1+e^{-2s}}} \end{bmatrix}$$

for all $s \in \mathbb{R}_0^+$. Hence,

$$\tilde{Q}(s) \sim \begin{bmatrix} e^{-s} & 1 \\ -1 & e^{-s} \end{bmatrix}, \quad s \to \infty$$

$$\tilde{R}(s) \sim \begin{bmatrix} e^{3s} & -e^{s} & e^{-2s} \\ 0 & 0 & e^{-s} \end{bmatrix}, \quad s \to \infty \; .$$

If we define $Q = \mathcal{R}(\tilde{Q})$ and $R = \mathcal{R}(\tilde{R})$, we obtain

$$Q = \begin{bmatrix} -1 & 0 \\ \ominus 0 & -1 \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} 3 & \ominus 1 & -2 \\ \varepsilon & \varepsilon & -1 \end{bmatrix} .$$

We have

$$Q \otimes R = \begin{bmatrix} 2 & \ominus 0 & -1 \\ \ominus 3 & 1 & (-2)^{\bullet} \end{bmatrix} \nabla B$$

$$Q^T \otimes Q = \begin{bmatrix} 0 & (-1)^\bullet \\ (-1)^\bullet & 0 \end{bmatrix} \nabla E_2$$

and $\|R\|_\oplus = 3 = \|B\|_\oplus$.
Note that $Q \otimes Q^T = Q^T \otimes Q \nabla E_2$ and $\det_\oplus Q = 0 \not\nabla \varepsilon$ (cf. Section D.5). □

Example D.4.2 Now we use the ELCP technique to compute max-algebraic QRDs of the matrix B of the previous example.
To reduce the number of variables and equations we shall only compute normalized max-algebraic QRDs of B: we require that the diagonal entries of Q belong to $\mathbb{S}^\oplus$. Afterwards we shall reconstruct the set of all the max-algebraic QRDs of B from the set of the normalized max-algebraic QRDs of B.
We introduce a matrix $T \in \mathbb{R}_\varepsilon^{2\times 3}$ such that $T = B^\oplus \oplus Q^\oplus \otimes R^\ominus \oplus Q^\ominus \otimes R^\oplus$ and a symmetric matrix $P \in \mathbb{R}_\varepsilon^{2\times 2}$ such that $P = (Q^\oplus)^T \otimes Q^\oplus \oplus (Q^\ominus)^T \otimes Q^\ominus$. Note that $p_{11} = p_{22} = 0$ since we also have $P = E_2 \oplus (Q^\oplus)^T \otimes Q^\ominus \oplus (Q^\ominus)^T \otimes Q^\oplus$.
Just like in Example 7.5.5 we apply the technique of Remark 7.5.3 and we introduce new variables $q_{12}^\boxplus$, $q_{12}^\boxminus$, $q_{21}^\boxplus$, $q_{21}^\boxminus$ and $r_{ik}^\boxplus, r_{ik}^\boxminus$ for $i = 1,2$ and $k = 1,2,3$ such that

$$q_{ij} = q_{ij}^\boxplus \ominus q_{ij}^\boxminus \quad \text{and} \quad q_{ij}^\boxplus \otimes q_{ij}^\boxminus \leqslant -\xi \qquad \text{for all } i,j \text{ with } i \neq j\,,$$

$$r_{ik} = r_{ik}^\boxplus \ominus r_{ik}^\boxminus \quad \text{and} \quad r_{ik}^\boxplus \otimes r_{ik}^\boxminus \leqslant -\xi \qquad \text{for all } i,k\,,$$

where ξ is a large positive number. All the variables are put in one large column vector x:

$$x = [\; q_{11} \;\; q_{12}^\boxplus \;\; q_{21}^\boxplus \;\; q_{22} \;\; q_{12}^\boxminus \;\; q_{21}^\boxminus \;\; r_{11}^\boxplus \;\; r_{12}^\boxplus \;\; r_{13}^\boxplus \;\; r_{22}^\boxplus \;\; r_{23}^\boxplus$$
$$r_{11}^\boxminus \;\; r_{12}^\boxminus \;\; r_{13}^\boxminus \;\; r_{22}^\boxminus \;\; r_{23}^\boxminus \;\; t_{11} \;\; t_{12} \;\; t_{13} \;\; t_{21} \;\; t_{22} \;\; t_{23} \;\; p_{12} \;]^T \;.$$

Since B_{32} is equal to ε, we have to use the procedure given in Remark 7.5.4 and see how the set of the normalized max-algebraic QRDs of the matrix

$$B(\xi) = \begin{bmatrix} 2 & \ominus 0 & -1 \\ \ominus 3 & 1^\bullet & (-\xi)^\bullet \end{bmatrix}$$

evolves as ξ goes to ∞.
If we use the ELCP algorithm of Section 3.4 to compute the solution set of the corresponding ELCP for some values of ξ that are greater than e.g. 1000, then we observe that the components of the finite points depend affinely on ξ, that there are no central generators, and that the extreme generators and the pairs of maximal cross-complementary subsets are the same for all the values of ξ. For any $\xi \geqslant 1000$, the generators and the finite points are given by Table D.1 and the pairs of maximal cross-complementary subsets are given by Table D.2. Since the $q_{21}^\boxplus$ component of all the finite points tends to ε as ξ tends to ∞, x_1^e becomes redundant if ξ goes to ∞. This also holds for x_2^e, x_5^e, x_7^e and x_9^e.

	$\mathcal{X}^{\mathrm{ext}}(\xi)$									$\mathcal{X}^{\mathrm{fin}}(\xi)$		
	x_1^{e}	x_2^{e}	x_3^{e}	x_4^{e}	x_5^{e}	x_6^{e}	x_7^{e}	x_8^{e}	x_9^{e}	$x_1^{\mathrm{f}}(\xi)$	$x_2^{\mathrm{f}}(\xi)$	$x_3^{\mathrm{f}}(\xi)$
q_{11}	0	0	0	0	0	0	0	0	0	-1	-1	-1
$q_{12}^{\boxplus}$	0	0	0	0	0	0	0	0	0	0	0	0
$q_{21}^{\boxplus}$	-1	0	0	0	0	0	0	0	0	$-\xi$	$-\xi$	$-\xi$
q_{22}	0	0	0	0	0	0	0	0	0	-1	-1	-1
$q_{12}^{\boxminus}$	0	-1	0	0	0	0	0	0	0	$-\xi$	$-\xi$	$-\xi$
$q_{21}^{\boxminus}$	0	0	0	0	0	0	0	0	0	0	0	0
$r_{11}^{\boxplus}$	0	0	0	0	0	0	0	0	0	3	3	3
$r_{12}^{\boxplus}$	0	0	-1	0	0	0	0	0	0	1	$-\xi-1$	$-\xi-1$
$r_{13}^{\boxplus}$	0	0	0	0	0	0	0	0	0	-2	-2	-2
$r_{22}^{\boxplus}$	0	0	0	-1	0	0	0	0	0	$-\xi$	0	$-\xi$
$r_{23}^{\boxplus}$	0	0	0	0	0	0	0	0	0	-1	-1	-1
$r_{11}^{\boxminus}$	0	0	0	0	-1	0	0	0	0	$-\xi-3$	$-\xi-3$	$-\xi-3$
$r_{12}^{\boxminus}$	0	0	0	0	0	-1	0	0	0	$-\xi-1$	1	1
$r_{13}^{\boxminus}$	0	0	0	0	0	0	-1	0	0	$-\xi+2$	$-\xi+2$	$-\xi+2$
$r_{22}^{\boxminus}$	0	0	0	0	0	0	0	-1	0	0	$-\xi$	0
$r_{23}^{\boxminus}$	0	0	0	0	0	0	0	0	-1	$-\xi+1$	$-\xi+1$	$-\xi+1$
t_{11}	0	0	0	0	0	0	0	0	0	2	2	2
t_{12}	0	0	0	0	0	0	0	0	0	0	0	0
t_{13}	0	0	0	0	0	0	0	0	0	-1	-1	-1
t_{21}	0	0	0	0	0	0	0	0	0	3	3	3
t_{22}	0	0	0	0	0	0	0	0	0	1	1	1
t_{23}	0	0	0	0	0	0	0	0	0	-2	-2	-2
p_{12}	0	0	0	0	0	0	0	0	0	-1	-1	-1

Table D.1: The generators and the finite points of the ELCP of Example D.4.2 for $\xi \geqslant 1000$.

s	$\mathcal{X}_s^{\mathrm{ext}}(\xi)$	$\mathcal{X}_s^{\mathrm{fin}}(\xi)$
1	$\{x_1^{\mathrm{e}}, x_2^{\mathrm{e}}, x_3^{\mathrm{e}}, x_4^{\mathrm{e}}, x_5^{\mathrm{e}}, x_6^{\mathrm{e}}, x_7^{\mathrm{e}}, x_9^{\mathrm{e}}\}$	$\{x_1^{\mathrm{f}}(\xi), x_3^{\mathrm{f}}(\xi)\}$
2	$\{x_1^{\mathrm{e}}, x_2^{\mathrm{e}}, x_3^{\mathrm{e}}, x_4^{\mathrm{e}}, x_5^{\mathrm{e}}, x_7^{\mathrm{e}}, x_8^{\mathrm{e}}, x_9^{\mathrm{e}}\}$	$\{x_2^{\mathrm{f}}(\xi), x_3^{\mathrm{f}}(\xi)\}$

Table D.2: The pairs of maximal cross-complementary subsets of the sets $\mathcal{X}^{\mathrm{ext}}(\xi)$ and $\mathcal{X}^{\mathrm{fin}}(\xi)$ of Example D.4.2 for $\xi \geqslant 1000$.

Furthermore, we may remove $x_4^{\rm e}$ from the set $\mathcal{X}_1^{\rm ext}$ and $x_3^{\rm e}$ from the set $\mathcal{X}_2^{\rm ext}$ if $\xi = \infty$. So if we define $\tilde{x}_1^{\rm e} = x_3^{\rm e}$, $\tilde{x}_2^{\rm e} = x_4^{\rm e}$, $\tilde{x}_3^{\rm e} = x_6^{\rm e}$, $\tilde{x}_4^{\rm e} = x_8^{\rm e}$, and $\tilde{x}_i^{\rm f} = \lim\limits_{\xi \to \infty} x_i^{\rm f}(\xi)$ for $i = 1, 2, 3$, then the set of all the normalized QRDs of B is described by the generators and the "finite" points of Table D.3 and the set

$$\tilde{\Lambda} = \left\{ \Big(\{\tilde{x}_1^{\rm e}, \tilde{x}_3^{\rm e}\}, \{\tilde{x}_1^{\rm f}, \tilde{x}_3^{\rm f}\} \Big), \Big(\{\tilde{x}_2^{\rm e}, \tilde{x}_4^{\rm e}\}, \{\tilde{x}_2^{\rm f}, \tilde{x}_3^{\rm f}\} \Big) \right\}$$

of ordered pairs of maximal cross-complementary subsets of $\tilde{\mathcal{X}}^{\rm ext}$ and $\tilde{\mathcal{X}}^{\rm fin}$.
Note that we have $q_{12}^{\boxplus} \otimes q_{12}^{\boxminus} = \varepsilon$, $q_{21}^{\boxplus} \otimes q_{21}^{\boxminus} = \varepsilon$ and $r_{ij}^{\boxplus} \otimes r_{ij}^{\boxminus} = \varepsilon$ for all i, j if $\xi = \infty$.
The "finite" point $\tilde{x}_1^{\rm f}$ corresponds to

$$Q_1 = \begin{bmatrix} -1 & 0 \\ \ominus 0 & -1 \end{bmatrix} \quad \text{and} \quad R_1 = \begin{bmatrix} 3 & 1 & -2 \\ \varepsilon & \ominus 0 & -1 \end{bmatrix} . \tag{D.13}$$

We have

$$Q_1 \otimes R_1 = \begin{bmatrix} 2 & 0^\bullet & -1 \\ \ominus 3 & \ominus 1 & (-2)^\bullet \end{bmatrix} \nabla\, B$$

$$Q_1^T \otimes Q_1 = \begin{bmatrix} 0 & (-1)^\bullet \\ (-1)^\bullet & 0 \end{bmatrix} \nabla\, E_2$$

and $\|R_1\|_\oplus = 3 = \|B\|_\oplus$.
Since $\tilde{x}_1^{\rm e}$ and $\tilde{x}_1^{\rm f}$ are cross-complementary, we may replace $(R_1)_{12}$ by any $\rho \in \mathbb{R}$ with $\rho \leqslant 1$ or by ε.
All the other max-algebraic QRDs of B can be obtained from the normalized max-algebraic QRDs of B by replacing the first column $Q_{.,1}$ of the Q matrix by $\ominus Q_{.,1}$ and the first row $R_{1,.}$ of the R matrix by $\ominus R_{1,.}$, or by replacing $Q_{.,2}$ by $\ominus Q_{.,2}$ and $R_{2,.}$ by $\ominus R_{2,.}$, or by a combination of these replacements.
If $Q_2 \otimes R_2$ and $Q_3 \otimes R_3$ are the max-algebraic QRDs of B that correspond to respectively $\tilde{x}_2^{\rm f}$ and $\tilde{x}_3^{\rm f}$, then the result of Example D.4.1 corresponds to the max-algebraic QRD of B obtained by setting $(R_2)_{22} = \varepsilon$ or $(R_3)_{22} = \varepsilon$. □

D.5 Extensions of the Max-Algebraic SVD

In this section we propose possible extensions of the definition of the max-algebraic SVD.
If $U \in \mathbb{R}^{m \times n}$ then we have $U^T U = I_m$ if and only if $U U^T = I_m$. However, in the symmetrized max-plus algebra $U^T \otimes U \nabla E_m$ does not always imply that $U \otimes U^T \nabla E_m$ as is shown by the following example:

Example D.5.1 Consider

$$U = \begin{bmatrix} 0 & 0 & -1 & -1 \\ -1 & -1 & \ominus 0 & \ominus 0 \\ \ominus 0 & 0 & \varepsilon & -1 \\ \varepsilon & \varepsilon & \ominus 0 & 0 \end{bmatrix} .$$

	$\tilde{\mathcal{X}}^{\mathrm{ext}}$				$\tilde{\mathcal{X}}^{\mathrm{fin}}$		
	$\tilde{x}_1^{\mathrm{e}}$	$\tilde{x}_2^{\mathrm{e}}$	$\tilde{x}_3^{\mathrm{e}}$	$\tilde{x}_4^{\mathrm{e}}$	$\tilde{x}_1^{\mathrm{f}}$	$\tilde{x}_2^{\mathrm{f}}$	$\tilde{x}_3^{\mathrm{f}}$
q_{11}	0	0	0	0	-1	-1	-1
$q_{12}^{\boxplus}$	0	0	0	0	0	0	0
$q_{21}^{\boxplus}$	0	0	0	0	ε	ε	ε
q_{22}	0	0	0	0	-1	-1	-1
$q_{12}^{\boxminus}$	0	0	0	0	ε	ε	ε
$q_{21}^{\boxminus}$	0	0	0	0	0	0	0
$r_{11}^{\boxplus}$	0	0	0	0	3	3	3
$r_{12}^{\boxplus}$	-1	0	0	0	1	ε	ε
$r_{13}^{\boxplus}$	0	0	0	0	-2	-2	-2
$r_{22}^{\boxplus}$	0	-1	0	0	ε	0	ε
$r_{23}^{\boxplus}$	0	0	0	0	-1	-1	-1
$r_{11}^{\boxminus}$	0	0	0	0	ε	ε	ε
$r_{12}^{\boxminus}$	0	0	-1	0	ε	1	1
$r_{13}^{\boxminus}$	0	0	0	0	ε	ε	ε
$r_{22}^{\boxminus}$	0	0	0	-1	0	ε	0
$r_{23}^{\boxminus}$	0	0	0	0	ε	ε	ε
t_{11}	0	0	0	0	2	2	2
t_{12}	0	0	0	0	0	0	0
t_{13}	0	0	0	0	-1	-1	-1
t_{21}	0	0	0	0	3	3	3
t_{22}	0	0	0	0	1	1	1
t_{23}	0	0	0	0	-2	-2	-2
p_{12}	0	0	0	0	-1	-1	-1

Table D.3: The generators and the "finite" points of the ELCP of Example D.4.2 for $\xi = \infty$.

We have

$$U^T \otimes U = \begin{bmatrix} 0 & 0^\bullet & (-1)^\bullet & (-1)^\bullet \\ 0^\bullet & 0 & (-1)^\bullet & (-1)^\bullet \\ (-1)^\bullet & (-1)^\bullet & 0 & 0^\bullet \\ (-1)^\bullet & (-1)^\bullet & 0^\bullet & 0 \end{bmatrix} \ \nabla \ E_4 \ ,$$

but

$$U \otimes U^T = \begin{bmatrix} 0 & (-1)^\bullet & 0^\bullet & (-1)^\bullet \\ (-1)^\bullet & 0 & (-1)^\bullet & 0^\bullet \\ 0^\bullet & (-1)^\bullet & 0 & -1 \\ (-1)^\bullet & 0^\bullet & -1 & 0 \end{bmatrix} \not\nabla E_4$$

since $(U \otimes U^T)_{34} = (U \otimes U^T)_{43} = -1 \not\nabla \varepsilon$. □

In the proof of the existence theorem of the max-algebraic SVD we have seen that for every matrix $A \in \mathbb{S}^{m \times n}$ (with finite entries) there is at least one max-algebraic SVD that corresponds to a path of approximate SVDs $\tilde{U}\tilde{\Sigma}\tilde{V}^T$ of $\tilde{A} = \mathcal{F}(A, \cdot)$ on some interval $[L, \infty)$. So if $s \geqslant L$ then $\tilde{U}(s)$ satisfies both $\tilde{U}^T(s)\,\tilde{U}(s) = I_m$ and $\tilde{U}(s)\,\tilde{U}^T(s) = I_m$, and $\tilde{V}(s)$ satisfies both $\tilde{V}^T(s)\,\tilde{V}(s) = I_n$ and $\tilde{V}(s)\,\tilde{V}^T(s) = I_n$. Therefore, we could add two extra conditions to the definition of the max-algebraic SVD: $U \otimes U^T \nabla E_m$ and $V \otimes V^T \nabla E_n$.

Furthermore, the left singular vectors of the path of approximate SVDs $\tilde{U}\tilde{\Sigma}\tilde{V}^T$ will be linearly independent in every point of $[L, \infty)$ since $\tilde{U}^T(s)\,\tilde{U}(s) = I_m$ for every $s \geqslant L$. The right singular vectors will also be linearly independent. However, in the symmetrized max-plus algebra the condition $U^T \otimes U \nabla E_m$ does not always guarantee that the columns of U are max-linearly independent — even if the entries U are signed — as is shown by the following example:

Example D.5.2 Consider

$$U = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \ominus 0 & 0 \\ 0 & 0 & \ominus 0 \end{bmatrix} .$$

We have

$$U^T \otimes U = U \otimes U^T = \begin{bmatrix} 0 & 0^\bullet & 0^\bullet \\ 0^\bullet & 0 & 0^\bullet \\ 0^\bullet & 0^\bullet & 0 \end{bmatrix} \nabla E_3 .$$

Furthermore, $\det_\oplus U = 0^\bullet$. So by Theorem 2.3.15 there exists a signed solution of $\alpha_1 \otimes u_1 \oplus \alpha_2 \otimes u_2 \oplus \alpha_3 \otimes u_3 \nabla \mathcal{E}_{3\times 1}$ (If we use the algorithm of [54] to solve this system of homogeneous max-linear balances, we obtain $\alpha_1 = 0$, $\alpha_2 = \ominus 0$ and $\alpha_3 = \ominus 0$). This means that the vectors u_1, u_2 and u_3 are max-linearly dependent (cf. Definition 2.3.16). □

If we want the left singular vectors to be max-linearly independent and if we also want the right singular vectors to be max-linearly independent, we should have $\det_\oplus U \not\nabla \varepsilon$ and $\det_\oplus V \not\nabla \varepsilon$ by Theorem 2.3.15. So we could also add these conditions to the definition of the max-algebraic SVD. Note that these conditions also imply that the rows of U and V are max-linearly independent since $\det_\oplus U = \det_\oplus U^T$. This leads to:

Proposition D.5.3 (The extended max-algebraic SVD) *Let $A \in \mathbb{S}^{m\times n}$ and let $r = \min(m,n)$. Then there exist a max-algebraic diagonal matrix $\Sigma \in \mathbb{R}_\varepsilon^{m\times n}$ and matrices $U \in (\mathbb{S}^\vee)^{m\times m}$ and $V \in (\mathbb{S}^\vee)^{n\times n}$ such that $A \mathrel{\nabla} U \otimes \Sigma \otimes V^T$ with $U^T \otimes U \mathrel{\nabla} E_m$, $U \otimes U^T \mathrel{\nabla} E_m$, $V^T \otimes V \mathrel{\nabla} E_n$, $V \otimes V^T \mathrel{\nabla} E_n$; where the rows and the columns of U and V are max-linearly independent or equivalently $\det_\oplus U \not\nabla \varepsilon$ and $\det_\oplus V \not\nabla \varepsilon$; and with $\|A\|_\oplus \geqslant \sigma_1 \geqslant \sigma_2 \geqslant \ldots \geqslant \sigma_r$, where $\sigma_i = (\Sigma)_{ii}$ for all i.*
Every decomposition $A \mathrel{\nabla} U \otimes \Sigma \otimes V^T$ that satisfies all the conditions given above is called an extended max-algebraic SVD of A.

It is obvious that we can also add similar conditions to the definition of the max-algebraic QRD. This leads to:

Proposition D.5.4 (The extended max-algebraic QRD) *If $A \in \mathbb{S}^{m\times n}$ then there exist a matrix $Q \in (\mathbb{S}^\vee)^{m\times m}$ and a max-algebraic upper triangular matrix $R \in (\mathbb{S}^\vee)^{m\times n}$ such that $A \mathrel{\nabla} Q \otimes R$ with $Q^T \otimes Q \mathrel{\nabla} E_m$, $Q \otimes Q^T \mathrel{\nabla} E_m$, $\det_\oplus Q \not\nabla \varepsilon$ and $\|R\|_\oplus \leqslant \|A\|_\oplus$.*
Every decomposition $A \mathrel{\nabla} Q \otimes R$ that satisfies all the conditions given above is called an extended max-algebraic QRD of A.

Note that the decompositions that have been computed in Examples 7.3.9 and D.4.1 satisfy all the conditions of respectively Proposition D.5.3 and Proposition D.5.4. Hence, the max-algebraic SVD of Example 7.3.9 is also an extended max-algebraic SVD and the max-algebraic QRD of Example D.4.1 is also an extended max-algebraic QRD.

Let us now show that the extended max-algebraic SVD and the extended max-algebraic QRD also result in a system of multivariate max-algebraic polynomial equalities and inequalities.
If we use a reasoning similar to the one made for $U^T \otimes U \mathrel{\nabla} E_m$, then the conditions

$$U \otimes U^T \mathrel{\nabla} E_m \tag{D.14}$$

$$V \otimes V^T \mathrel{\nabla} E_n \tag{D.15}$$

also yield multivariate max-algebraic polynomial equalities that could be added to the system (7.34) – (7.35), (7.45) – (7.55).
The conditions $\det_\oplus U \not\nabla \varepsilon$ and $\det_\oplus V \not\nabla \varepsilon$ can be rewritten as

$$(\det\nolimits_\oplus U)^\oplus \otimes (\det\nolimits_\oplus U)^\ominus = \varepsilon \tag{D.16}$$

$$(\det\nolimits_\oplus V)^\oplus \otimes (\det\nolimits_\oplus V)^\ominus = \varepsilon \tag{D.17}$$

where

$$(\det\nolimits_\oplus U)^\oplus = \bigoplus_{\substack{\varphi \cup \psi \,\in\, \mathcal{P}_{n,\text{even}} \\ \text{dom}\,\varphi \cap \text{dom}\,\psi = \emptyset,\ \#\text{dom}\,\psi \text{ is even}}} \bigotimes_{i \in \text{dom}\,\varphi} u_{i\varphi(i)}^\oplus \otimes \bigotimes_{j \in \text{dom}\,\psi} u_{j\psi(j)}^\ominus \quad \oplus$$

$$\bigoplus_{\substack{\varphi\cup\psi\,\in\,\mathcal{P}_{n,\text{odd}}\\ \text{dom}\,\varphi\cap\text{dom}\,\psi=\emptyset,\ \#\,\text{dom}\,\psi\ \text{is odd}}} \bigotimes_{i\,\in\,\text{dom}\,\varphi} u^{\oplus}_{i\varphi(i)} \otimes \bigotimes_{j\,\in\,\text{dom}\,\psi} u^{\ominus}_{j\psi(j)}$$

and

$$(\det\nolimits_{\oplus} U)^{\ominus} \;=\; \bigoplus_{\substack{\varphi\cup\psi\,\in\,\mathcal{P}_{n,\text{even}}\\ \text{dom}\,\varphi\cap\text{dom}\,\psi=\emptyset,\ \#\,\text{dom}\,\psi\ \text{is odd}}} \bigotimes_{i\,\in\,\text{dom}\,\varphi} u^{\oplus}_{i\varphi(i)} \otimes \bigotimes_{j\,\in\,\text{dom}\,\psi} u^{\ominus}_{j\psi(j)} \;\oplus$$
$$\bigoplus_{\substack{\varphi\cup\psi\,\in\,\mathcal{P}_{n,\text{odd}}\\ \text{dom}\,\varphi\cap\text{dom}\,\psi=\emptyset,\ \#\,\text{dom}\,\psi\ \text{is even}}} \bigotimes_{i\,\in\,\text{dom}\,\varphi} u^{\oplus}_{i\varphi(i)} \otimes \bigotimes_{j\,\in\,\text{dom}\,\psi} u^{\ominus}_{j\psi(j)} \;.$$

Analogous expressions exist for $(\det_{\oplus} V)^{\oplus}$ and $(\det_{\oplus} V)^{\ominus}$. So (D.16) and (D.17) can be considered as multivariate max-algebraic polynomial equalities. If we also add these constraints to the system (7.34) – (7.35), (7.45) – (7.55), we still have a system of multivariate max-algebraic polynomial equalities and inequalities.

If the matrix A has finite entries, we can use a reasoning that is analogous to the one of the proof of Proposition 7.5.1 to show that there exists at least one extended max-algebraic SVD $U \otimes \Sigma \otimes V^T$ of A with finite singular values and finite singular vectors that also satisfies (D.14) and (D.15).
As a direct consequence of (D.7) the max-algebraic determinant of the matrix U of the proof of Proposition 7.5.1 satisfies $|\det_{\oplus} U|_{\oplus} = 0$. Since $M < 0$ and since the entries of U are less than or equal to 0 in max-absolute value, the value of $\det_{\oplus} U$ will not change if we replace the infinite entries of U by M or by $\ominus M$. This also holds for V. So we can still use the procedure of the proof of Proposition 7.5.1 to obtain an extended max-algebraic SVD with finite singular values and finite singular vectors for a matrix with finite entries.
This means that in theory we can use the ELCP approach to compute all the extended max-algebraic SVDs of a given matrix. However, we have to point out that the conditions (D.16) and (D.17) would yield such a large number of extra inequalities that in practice it will be impossible to solve the resulting ELCP in a reasonable amount of CPU time with the ELCP algorithm of Section 3.4.

Using a similar reasoning as for the extended max-algebraic SVD it can be shown that in theory we can still use the ELCP approach to solve the system of multivariate max-algebraic polynomial equalities and inequalities that corresponds to the extended max-algebraic QRDs of a given matrix.

Appendix E

An Informal Introduction to the Symmetrized Max-Plus Algebra

In this appendix we give an informal and intuitive introduction to the symmetrized max-plus algebra. We also give the solution set of some elementary balances and we present some extra worked examples that illustrate the properties of the max-algebraic minus operator, the balance operator and the balance relation.

E.1 The Symmetrized Max-Plus Algebra

In contrast to linear algebra, there exist no inverse elements with respect to $\oplus$ in $\mathbb{R}_{\max}$. Consider e.g. the following equation:

$$x \oplus 3 = 2 \tag{E.1}$$

or equivalently

$$\max(x, 3) = 2 \ .$$

Clearly, this equation has no solutions in $\mathbb{R}_\varepsilon$.
Therefore, we now introduce the symmetrized max-plus algebra (Note that a formal introduction to the symmetrized max-plus algebra can be found in Section 2.3).

First we define two new elements for every $x \in \mathbb{R}_\varepsilon$: $\ominus x$ and $x^\bullet$. This gives rise to three different sets of elements:

- $\mathbb{S}^\oplus = \mathbb{R}_\varepsilon$: the set of the max-positive or max-zero numbers,

- $\mathbb{S}^{\ominus} = \{\ominus x \mid x \in \mathbb{R}_\varepsilon\}$: the set of the max-negative or max-zero numbers,
- $\mathbb{S}^{\bullet} = \{x^{\bullet} \mid x \in \mathbb{R}_\varepsilon\}$: the set of the balanced numbers.

This yields the following extension of $\mathbb{R}_\varepsilon$:

$$\mathbb{S} = \mathbb{S}^{\oplus} \cup \mathbb{S}^{\ominus} \cup \mathbb{S}^{\bullet} \ .$$

We have $\varepsilon = \ominus\varepsilon = \varepsilon^{\bullet}$. Hence, $\mathbb{S}^{\oplus} \cap \mathbb{S}^{\ominus} \cap \mathbb{S}^{\bullet} = \{\varepsilon\}$. The elements of $\mathbb{S}^{\oplus} \cup \mathbb{S}^{\ominus}$ are called signed.
The $\oplus$ and the $\otimes$ operation can be extended to $\mathbb{S}$. The resulting structure $\mathbb{S}_{\max} = (\mathbb{S}, \oplus, \otimes)$ is called the symmetrized max-plus algebra. The symmetrized max-plus algebra is a dioid.
If $x, y \in \mathbb{R}_\varepsilon$ then we have:

$$x \oplus (\ominus y) = x \qquad \text{if } x > y\,, \tag{E.2}$$

$$x \oplus (\ominus y) = \ominus y \qquad \text{if } x < y\,, \tag{E.3}$$

$$x \oplus (\ominus x) = x^{\bullet}\,. \tag{E.4}$$

Furthermore, for any $a, b \in \mathbb{S}$ we have

$$a^{\bullet} = (\ominus a)^{\bullet} = (a^{\bullet})^{\bullet} \tag{E.5}$$

$$(\ominus a) \otimes (\ominus b) = a \otimes b \tag{E.6}$$

$$a \otimes b^{\bullet} = (a \otimes b)^{\bullet} \tag{E.7}$$

$$\ominus(\ominus a) = a \tag{E.8}$$

$$\ominus(a \oplus b) = (\ominus a) \oplus (\ominus b) \tag{E.9}$$

$$\ominus(a \otimes b) = (\ominus a) \otimes b \ . \tag{E.10}$$

The last three properties allow us to write $a \ominus b$ instead of $a \oplus (\ominus b)$. Note that the $\ominus$ operator has many properties that are similar to properties of the $-$ operator from conventional algebra.
If we have to evaluate max-algebraic sums that contain balanced numbers, we rewrite the balanced numbers as follows:

$$a^{\bullet} = a \ominus a \tag{E.11}$$

and then we apply the properties of $\oplus$ and the rules (E.2) – (E.4).

Let us illustrate the rules given above with some examples.

Example E.1.1 We have

$$
\begin{aligned}
\ominus 0 \oplus 8 \ominus 3^{\bullet} &= \ominus 0 \oplus 8 \ominus (3 \ominus 3) && \text{(by (E.11))}\\
&= \ominus 0 \oplus 8 \ominus 3 \oplus (\ominus(\ominus 3)) && \text{(by (E.9))}\\
&= \ominus 0 \oplus 8 \ominus 3 \oplus 3 && \text{(by (E.8))}\\
&= 8 \oplus 3 \ominus 0 \ominus 3 && \text{(since } \oplus \text{ is commutative in } \mathbb{S})\\
&= (8 \oplus 3) \ominus (0 \oplus 3) && \text{(by (E.9))}\\
&= 8 \ominus 3 && \text{(by the definition of } \oplus)\\
&= 8 && \text{(by (E.2))} \ . \qquad \square
\end{aligned}
$$

Example E.1.2 We have

$$
\begin{aligned}
(\ominus 1) \otimes (4 \oplus 7 \ominus 9) &= (\ominus 1) \otimes (7 \ominus 9) && \text{(by the definition of } \oplus)\\
&= (\ominus 1) \otimes (\ominus 9) && \text{(by (E.3))}\\
&= 1 \otimes 9 && \text{(by (E.6))}\\
&= 10 && \text{(by the definition of } \otimes) \ . \qquad \square
\end{aligned}
$$

Let $a \in \mathbb{S}$. The max-positive part $a^{\oplus}$ and the max-negative part $a^{\ominus}$ of a are defined as follows:

- if $a \in \mathbb{S}^{\oplus}$ then $a^{\oplus} = a$ and $a^{\ominus} = \varepsilon$,
- if $a \in \mathbb{S}^{\ominus}$ then $a^{\oplus} = \varepsilon$ and $a^{\ominus} = \ominus a$,
- if $a \in \mathbb{S}^{\bullet}$ then there exists a number $b \in \mathbb{R}_{\varepsilon}$ such that $a = b^{\bullet}$ and then $a^{\oplus} = a^{\ominus} = b$.

So we have $a = a^{\oplus} \ominus a^{\ominus}$ and $a^{\oplus}, a^{\ominus} \in \mathbb{R}_{\varepsilon}$. The max-absolute value of $a \in \mathbb{S}$ is defined by $|a|_{\oplus} = a^{\oplus} \oplus a^{\ominus}$.

Example E.1.3 Let $a = \ominus 4$. Since $a \in \mathbb{S}^{\ominus}$, we have $a^{\oplus} = \varepsilon$, $a^{\ominus} = 4$ and $|a|_{\oplus} = 4$.
If $b = 2^{\bullet}$ then $b \in \mathbb{S}^{\bullet}$. Furthermore, $b^{\oplus} = 2$, $b^{\ominus} = 2$ and $|b|_{\oplus} = 2$. □

Since $\ominus$ is not cancellative — i.e. in general $a \ominus a \neq \varepsilon$, the zero element for $\oplus$ — we use balances (∇) instead of equalities in the symmetrized max-plus algebra. The balance relation is defined as follows:

If $a, b \in \mathbb{S}$ then we have $a \,\nabla\, b$ if and only if $a^{\oplus} \oplus b^{\ominus} = a^{\ominus} \oplus b^{\oplus}$.

Note that $a \ominus a \,\nabla\, \varepsilon$ for all $a \in \mathbb{S}$.

Example E.1.4 We have $1^{\oplus} = 1$, $1^{\ominus} = \varepsilon$, $(1^{\bullet})^{\oplus} = (1^{\bullet})^{\ominus} = 1$. This implies that $1^{\oplus} \oplus (1^{\bullet})^{\ominus} = 1 \oplus 1 = 1 = \varepsilon \oplus 1 = 1^{\ominus} \oplus (1^{\bullet})^{\oplus}$. Hence, $1 \,\nabla\, 1^{\bullet}$.
We have $3 \not\nabla \ominus 0$ since $3^{\oplus} \oplus (\ominus 0)^{\ominus} = 3 \oplus 0 = 3 \neq \varepsilon = \varepsilon \oplus \varepsilon = 3^{\ominus} \oplus (\ominus 0)^{\oplus}$. □

For balances we have the following basic rules:

Rule 1: An element with an $\ominus$ sign can be transferred to the other side of a balance as follows:

$$\forall a, b, c \in \mathbb{S} : \; a \ominus c \, \nabla \, b \quad \Leftrightarrow \quad a \, \nabla \, b \oplus c \ .$$

Rule 2: If both sides of a balance are signed then we may replace the balance by an equality:

$$\forall a, b \in \mathbb{S}^{\oplus} \cup \mathbb{S}^{\ominus} : \; a \, \nabla \, b \;\Leftrightarrow\; a = b \ .$$

In order to illustrate the concepts and the rules that have been introduced above we shall now check whether the numbers $\ominus 3$, $2^{\bullet}$ and $4^{\bullet}$ are solutions of the balance

$$x \oplus 3 \,\nabla\, 2 \tag{E.12}$$

that corresponds to equation (E.1) (with the equality replaced by a balance).

- We have

$$\begin{aligned} \ominus 3 \oplus 3 \,\nabla\, 2 \quad &\Leftrightarrow \quad 3 \,\nabla\, 2 \oplus 3 && \text{(by Rule 1)} \\ &\Leftrightarrow \quad 3 \,\nabla\, 3 && \text{(by the definition of } \oplus\text{)} \\ &\Leftrightarrow \quad 3 = 3 && \text{(by Rule 2)} \ , \end{aligned}$$

  and since the last expression holds, $\ominus 3$ is a solution of the balance (E.12).

- Since

$$\begin{aligned} 2^{\bullet} \oplus 3 \,\nabla\, 2 \quad &\Leftrightarrow \quad 2 \ominus 2 \oplus 3 \,\nabla\, 2 && \text{(by (E.11))} \\ &\Leftrightarrow \quad 2 \oplus 3 \,\nabla\, 2 \oplus 2 && \text{(by Rule 1)} \\ &\Leftrightarrow \quad 3 \,\nabla\, 2 && \text{(by the definition of } \oplus\text{)} \\ &\Leftrightarrow \quad 3 = 2 && \text{(by Rule 2)} \end{aligned}$$

  and since $3 \neq 2$, $2^{\bullet}$ is not a solution of (E.12).

- Since

$$\begin{aligned} 4^{\bullet} \oplus 3 \,\nabla\, 2 \quad &\Leftrightarrow \quad 4 \ominus 4 \oplus 3 \,\nabla\, 2 && \text{(by (E.11))} \\ &\Leftrightarrow \quad 4 \oplus 3 \,\nabla\, 2 \oplus 4 && \text{(by Rule 1)} \\ &\Leftrightarrow \quad 4 \,\nabla\, 4 && \text{(by the definition of } \oplus\text{)} \\ &\Leftrightarrow \quad 4 = 4 && \text{(by Rule 2)} \ , \end{aligned}$$

  $4^{\bullet}$ is also a solution of (E.12).

E.2 Some Elementary Balances

Now we give the solution set of some basic balances (with $a \in \mathbb{R}$ and $x \in \mathbb{S}$):

- Consider the balance $x \nabla a$. The solution set of this balance is given by

$$\{a\} \cup \{b^\bullet \,|\, b \in \mathbb{R}_\varepsilon \text{ and } b \geqslant a\} \ .$$

- Now consider the balance $x \nabla \ominus a$. The solution set of this balance is given by

$$\{\ominus a\} \cup \{b^\bullet \,|\, b \in \mathbb{R}_\varepsilon \text{ and } b \geqslant a\} \ .$$

- The solution set of the balance $x \nabla a^\bullet$ is given by

$$\{b \,|\, b \in \mathbb{R}_\varepsilon \text{ and } b \leqslant a\} \cup \{\ominus c \,|\, c \in \mathbb{R}_\varepsilon \text{ and } c \leqslant a\} \cup \{d^\bullet \,|\, d \in \mathbb{R}_\varepsilon\} \ .$$

- The solution set of $x \nabla \varepsilon$ is given by

$$\{\varepsilon\} \cup \{b^\bullet \,|\, b \in \mathbb{R}_\varepsilon\} \ .$$

Now we can solve the balance (E.12). We have

$$\begin{aligned} x \oplus 3 \nabla 2 \quad &\Leftrightarrow \quad x \nabla 2 \ominus 3 && \text{(by Rule 1)} \\ &\Leftrightarrow \quad x \nabla \ominus 3 && \text{(by (E.3))} \ . \end{aligned}$$

Hence, the solution set of (E.12) is given by

$$\mathcal{S} = \{\ominus 3\} \cup \{b^\bullet \,|\, b \in \mathbb{R}_\varepsilon \text{ and } b \geqslant 3\} \ .$$

Note that $\ominus 3$, $4^\bullet \in \mathcal{S}$ and $2^\bullet \notin \mathcal{S}$, which corresponds to what we have found above.

E.3 Worked Examples

Now we present some extra worked examples in which we use the results of the previous section to solve balances.

- The solution set of the balance $x \nabla 2$ is given by

$$\{2\} \cup \{b^\bullet \,|\, b \in \mathbb{R}_\varepsilon \text{ and } b \geqslant 2\} \ .$$

  Therefore, we should have $3^\bullet \nabla 2$. Let us verify this. Since

$$\begin{aligned} 3^\bullet \nabla 2 \quad &\Leftrightarrow \quad 3 \ominus 3 \nabla 2 \\ &\Leftrightarrow \quad 3 \nabla 2 \oplus 3 \end{aligned}$$

$$
\begin{aligned}
&\Leftrightarrow \quad 3 \nabla 3 \\
&\Leftrightarrow \quad 3 = 3 \ ,
\end{aligned}
$$

we really have $3^\bullet \nabla 2$.
Note that we could also have verified this using the definition of the balance relation.

- Consider the balance $x \oplus 4 \nabla \ominus 3$. The number x will be a solution of this balance if and only if

$$
\begin{aligned}
x \nabla \ominus 3 \ominus 4 \quad &\Leftrightarrow \quad x \nabla \ominus(3 \oplus 4) \\
&\Leftrightarrow \quad x \nabla \ominus 4 \ .
\end{aligned}
$$

  So x is a solution of the balance if $x = \ominus 4$ or if $x = b^\bullet$ with $b \in \mathbb{R}_\varepsilon$ and $b \geqslant 4$.

- We have

$$
\begin{aligned}
\ominus x \oplus 4^\bullet \nabla 5 \quad &\Leftrightarrow \quad 4 \ominus 4 \ominus 5 \nabla x \\
&\Leftrightarrow \quad x \nabla 4 \ominus (4 \oplus 5) \\
&\Leftrightarrow \quad x \nabla 4 \ominus 5 \\
&\Leftrightarrow \quad x \nabla \ominus 5 \ .
\end{aligned}
$$

  So the solution set of the balance $\ominus x \oplus 4^\bullet \nabla 5$ is given by

$$
\{\ominus 5\} \cup \{ b^\bullet \mid b \in \mathbb{R}_\varepsilon \text{ and } b \geqslant 5 \} \ .
$$

- Consider the balance $x \oplus 4 \ominus 3 \nabla 6^\bullet$. The number x is a solution of this balance if and only if

$$
\begin{aligned}
x \nabla 6 \ominus 6 \ominus 4 \oplus 3 \quad &\Leftrightarrow \quad x \nabla (6 \oplus 3) \ominus (6 \oplus 4) \\
&\Leftrightarrow \quad x \nabla 6 \ominus 6 \\
&\Leftrightarrow \quad x \nabla 6^\bullet \ .
\end{aligned}
$$

  So the solution set of the balance $x \oplus 4 \ominus 3 \nabla 6^\bullet$ is given by

$$
\{ b \mid b \in \mathbb{R}_\varepsilon \text{ and } b \leqslant 6 \} \cup \{ \ominus c \mid c \in \mathbb{R}_\varepsilon \text{ and } c \leqslant 6 \} \cup \{ d^\bullet \mid d \in \mathbb{R}_\varepsilon \} \ .
$$

As an exercise, the reader could now try to verify that the following expressions hold:

- $4 \nabla 6^\bullet$,
- $1^\bullet \nabla \varepsilon$,
- $3 \not\nabla 2^\bullet \ominus 0$,
- $9 \ominus 9 \oplus 1 \nabla 5 \ominus 8$.

Conventional algebra		Max-plus algebra
$+$	$\leftrightarrow$	$\oplus$
$\times$	$\leftrightarrow$	$\otimes$
0	$\leftrightarrow$	ε
1	$\leftrightarrow$	0

Conventional algebra		Symmetrized max-plus algebra
$-$	$\leftrightarrow$	$\ominus$
$=$	$\leftrightarrow$	∇
0	$\leftrightarrow$	$a^{\bullet}$ $(a \in \mathbb{R}_{\varepsilon})$

Table E.1: Some analogies between conventional algebra and the (symmetrized) max-plus algebra.

E.4 Some Analogies between Conventional Algebra and the (Symmetrized) Max-Plus Algebra

In Table E.1 we have listed some of the analogies between conventional algebra and the (symmetrized) max-plus algebra. Let us show by some examples how this table can be used.

The identity element for $+$ in conventional algebra is 0. From the information given in the table we can deduce that this corresponds to the fact that the identity element for $\oplus$ in the max-plus algebra is ε.

Let us now consider a more complicated example. In conventional linear algebra the determinant of a matrix $A \in \mathbb{R}^{n \times n}$ is defined by

$$\det A = \sum_{\sigma \in \mathcal{P}_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)}$$

with

$$\operatorname{sgn}(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ is even,} \\ -1 & \text{if } \sigma \text{ is odd.} \end{cases}$$

If we translate all the operations that appear in these formulas according to the key given in Table E.1, we obtain the following definition for the max-algebraic determinant of a matrix $A \in \mathbb{S}^{n \times n}$:

$$\det\nolimits_{\oplus} A = \bigoplus_{\sigma \in \mathcal{P}_n} \operatorname{sgn}_{\oplus}(\sigma) \otimes \bigotimes_{i=1}^{n} a_{i\sigma(i)}$$

with

$$\operatorname{sgn}_{\oplus}(\sigma) = \begin{cases} 0 & \text{if } \sigma \text{ is even,} \\ \ominus 0 & \text{if } \sigma \text{ is odd,} \end{cases}$$

(cf. Definitions 2.3.11 and 2.3.12).
Note however that in the symmetrized max-plus algebra we (often) get balances instead of equalities. In conventional algebra we have e.g. $a - a = 0$ for all $a \in \mathbb{R}$ whereas in the symmetrized max-plus algebra we have $a \ominus a \ \nabla\ \varepsilon$ for all $a \in \mathbb{S}$.

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Bart De Schutter was born on October 18, 1968 in Ekeren, Belgium. He received the degree of electrotechnical-mechanical engineering in control theory from the Katholieke Universiteit Leuven, Belgium in July 1991. Since then he is working as a researcher at the Electrical Engineering Department of the Katholieke Universiteit Leuven, first on an I.W.O.N.L. (Belgian Institute for the Advancement of Scientific Research in Industry and Agriculture) grant, and after his military service, as a research assistant of the N.F.W.O. (Belgian National Fund for Scientific Research).