The Generalized Linear Complementarity Problem Applied to the Complete Analysis of Resistive Piecewise-Linear Circuits

LIEVEN VANDENBERGHE, MEMBER, IEEE, BART L. DE MOOR, MEMBER, IEEE, AND JOOS VANDEWALLE, SENIOR MEMBER, IEEE

Abstract —An important application of complementarity theory consists in solving sets of piecewise-linear equations and hence in the analysis of piecewise-linear resistive circuits. In this paper we show how a generalized version of the linear complementarity problem (LCP) can be used to analyze a broad class of piecewise-linear circuits. One can allow nonlinear resistors that are neither voltage nor current controlled, and no restrictions to the linear part of the circuit have to be made. As a second contribution, we describe an algorithm for the solution of the generalized complementarity problem and show how it can be applied to yield a complete description of the dc solution set as well as of driving-point and transfer characteristics.

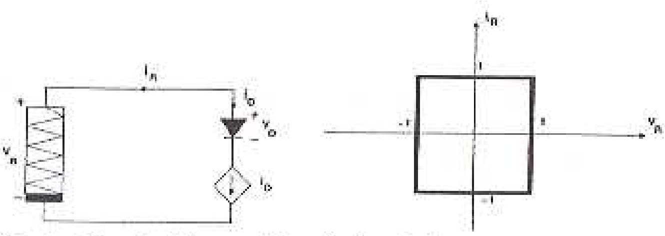

Fig. 1. Piecewise-linear resistive circuit and the current–voltage characteristic of the nonlinear resistor.

I. Introduction

It is well known that the geometry of the solution set of a piecewise-linear resistive circuit can be very complicated. The circuit can have multiple solutions, the solution set can be continuous or unbounded. The simple circuit in Fig. 1, for instance, possesses an infinite number of solutions. All points on the lower half of the square resistor characteristic are acceptable operating points (Fig. 2). The solution set is even unbounded, for when the diode is blocking, the voltage across the diode can take any nonpositive value.

During the past two decades, the description and analysis of piecewise-linear resistive circuits has been a prolific topic in nonlinear circuits literature. Several researchers have demonstrated how a piecewise-linear approximation to nonlinear device characteristics can be exploited in very efficient solution schemes. None of these methods, however, succeeds in determining the complete solution set for problems as in Fig. 1.

Chua's canonical piecewise-linear analysis is based on the formulation of the circuit equations in a canonical form, consisting of linear terms and absolute values and on Katzenelson's algorithm for the solution of sets of piecewise linear equations [1], [2].

Another approach makes use of the possibility of formulating the network equations of a piecewise-linear circuit as a linear complementarity problem (LCP) [6]–[8]. This can be most easily understood by first synthesizing the piecewise-linear resistors with linear elements, constant sources and ideal diodes. The diode equations constitute the "complementarity" conditions in the LCP. Another way of deriving the LCP associated with the circuit, is to refer to the general equivalence between LCP and sets of piecewise-linear equations, as exposed by Eaves [9], [10]. Following an analogous reasoning as Eaves in [10], Chua's canonical equation, for instance, can easily be written as an LCP.

The most popular algorithms for the solution of an LCP (e.g., Lemke's pivoting algorithm) as well as Katzenelson's algorithm, are homotopy methods that generate a path in the solution space, leading from an initial point to a solution of the problem. In consequence, they can only find one solution at the same time, which is a severe disadvantage. Determining the complete solution set would require trying all possible initial points.

This drawback is particularly striking in the determination of driving-point and transfer characteristics, where a so-called *breakpoint hopping* is performed, to trace the curve from one initial breakpoint onwards. Unconnected parts of the characteristic cannot be determined in this way, unless a point is given on each part. A famous example of a driving-point plot with unconnected parts is

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L. Vandenberghe is with ESAT Laboratory, Department of Electrical Engineering, Katholieke Universiteit Leuven, B-3030 Heverlee, Belgium.

B. L. De Moor is with the Computer Science Department and the Information Systems Laboratory, Stanford University, Stanford, CA, on leave from the ESAT Laboratory, Department of Electrical Engineering, Katholieke Universiteit Leuven, B-3030 Heverlee, Belgium.

J. Vandewalle is with the ESAT Laboratory, Department of Electrical Engineering, Katholieke Universiteit Leuven, B-3030 Heverlee, Belgium.

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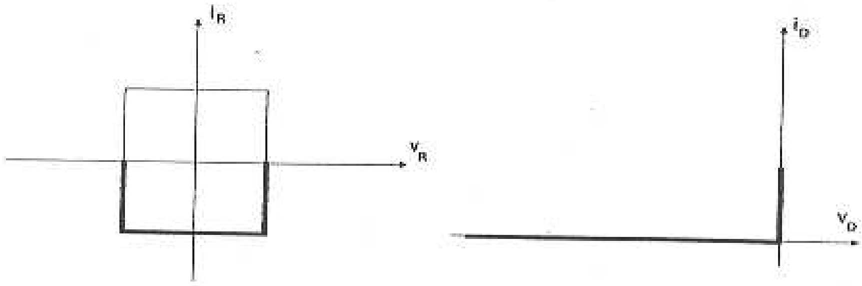

Fig. 2. The allowed parts of the $i-v$ characteristics of the resistors in Fig. 1. The solutions form a continuous and unbounded set.

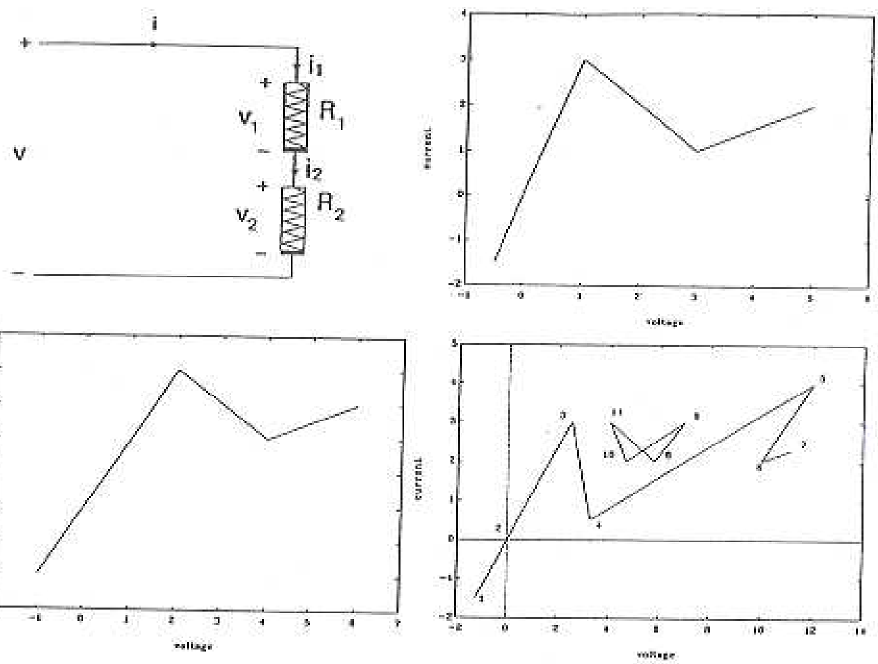

Fig. 3. The series connection of two PWL resistors, their $i-v$-characteristics and the composite $i-v$ plot, which is formed by two unconnected parts.

the $I-V$ characteristic of the series connection of two tunnel diodes. Fig. 3 shows an example where the diode characteristics are approximated by a piecewise linear curve. A more exotic example is borrowed from [5] and depicted in Fig. 4. The driving-point plot consists of the full square plus two unbounded branches.

An alternative to the above-mentioned methods would therefore consist in an exhaustive trial of all possibilities [12]. A refined version [13] of this method is based on a circuit representation in a special structured form and allows to reduce the number of linear systems to be solved.

In this paper, we present *a new method for finding all solutions to the LCP, and hence of a PWL resistive circuit, without exhaustive searching.* Moreover, we admit a generalized form of the LCP, which allows us to include a broader class of circuits and to analyze circuits as those presented in the examples of this introduction. In Section II, the circuit elements allowed in the analysis are described, and a global closed-form expression for one-dimensional piecewise-linear curves is given. This description has the advantage of being applicable to resistors that are neither current nor voltage controlled, a case generally excluded in piecewise-linear analysis. In the subsequent two sections, it is shown how the circuit equations of a piecewise linear circuit, can be formulated in a common form called the generalized linear complementarity problem (GLCP), the definition of which is given in Section V. The versatility and generality of this approach is demonstrated in Section 6 by a number of examples. An algorithm for the solution of the GLCP has recently been proposed and is summarized in the Appendix.

II. Description of Piecewise-Linear Circuit Elements

The circuits under study may contain the following elements.

1) all possible linear resistive elements,

2) piecewise-linear two terminal resistors (the resistors are not required to be either voltage or current controlled),

3) piecewise-linear controlled sources (all four types) with one controlling variable (the characteristics may be multi-valued).

An extension to multiterminal nonlinear resistors and controlled sources with more than one controlling variable is possible [17], [18], but will be omitted here for the sake of

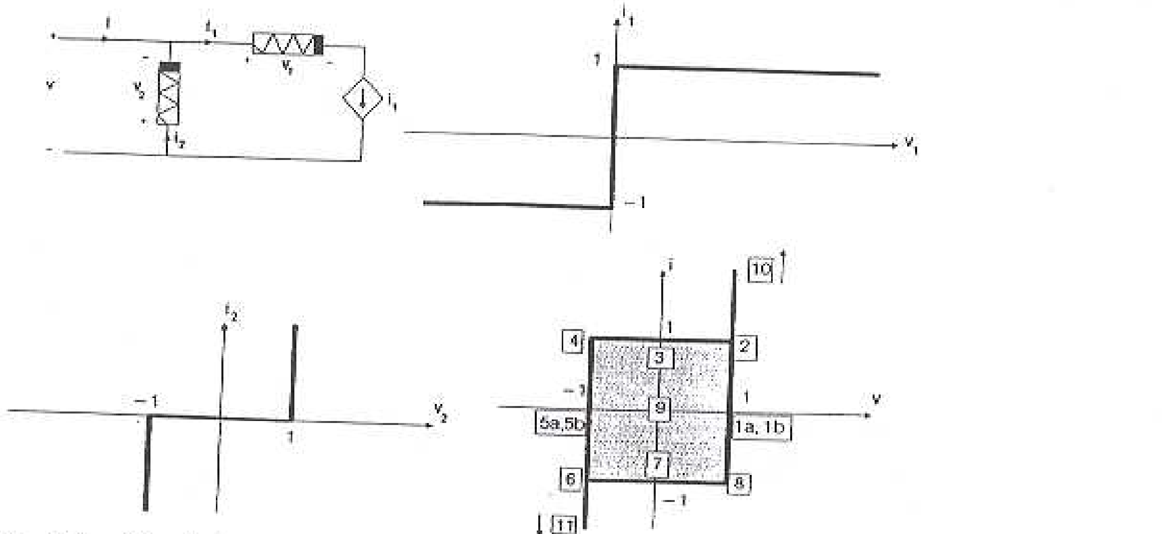

Fig. 4. PWL resistive circuit, the individual resistor characteristics, and the composite driving-point plot.

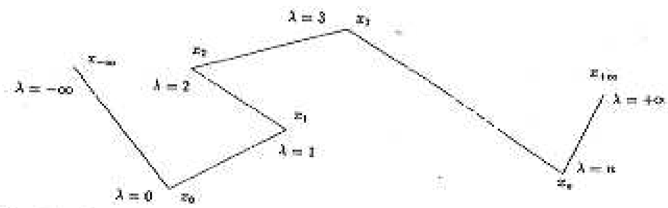

Fig. 5. One-dimensional PWL curve in an ambient space of dimension m, and parametrization with a parameter λ.

clarity. The basic technique for the formulation of the GLCP associated to a circuit, is the parametrization given below.

We first introduce the following notation for a real number x:

$$x^+ = \max(x,0);\; x^- = \max(-x,0).$$

The same notation can be used for an n-vector x when all operations are assumed to be performed componentwise. An equivalent definition is

$$x = x^+ - x^-$$
$$x^+ \geqslant 0;\; x^- \geqslant 0$$
$$(x^+)^t \cdot x^- = 0.$$

Given a one-dimensisonal PWL curve in R^m (Fig. 5), characterized by a set of $n+1$ breakpoints $x_0, x_1, \cdots, x_n$ and two directions $x_{-\infty}$ and $x_{+\infty}$, the parametrization can proceed as follows.

- Assign a parameter λ running from $-\infty$ to $+\infty$ [11].
- The part between $\lambda = -\infty$ and $\lambda = 1$ can be expressed as

$$x = x_0 + x_{-\infty} \cdot \lambda^- + (x_1 - x_0)\lambda^+.$$

- The direction of the curve between x_0 and x_1, is $x_1 - x_0$. From $\lambda = 1$ onwards the direction of the curve has to be corrected. This can be done by adding a term $(x_2 - 2x_1 + x_0)\cdot(\lambda - 1)^+$:

$$x = x_0 + x_{-\infty} \cdot \lambda^- + (x_1 - x_0)\cdot \lambda^+ + (x_2 - 2x_1 + x_0)\cdot(\lambda - 1)^+.$$

This formula is now valid between $-\infty$ and $\lambda = 2$.

- This can be continued to describe the complete curve:

$$x = x_0 + x_{-\infty} \cdot \lambda^- + (x_1 - x_0)\cdot \lambda^+ + \sum_{k=2}^{n} (x_k - 2x_{k-1} + x_{k-2})\cdot(\lambda - k + 1)^+ + (x_{+\infty} - x_n + x_{n-1})\cdot(\lambda - n)^+ \tag{1}$$

- We define auxiliary variables

$$\lambda_i = \lambda - i$$
$$\lambda_i^+ - \lambda_i^- = \lambda^+ - \lambda^- - i \tag{2}$$

for $i = 1, \cdots, n$. Substituting the variables $\lambda_1, \cdots, \lambda_n$ in (1) and adding the compatibility conditions (2) delivers the complete PWL description:

$$x = x_0 + x_{-\infty} \cdot \lambda^- + (x_1 - x_0)\cdot \lambda^+ + \sum_{k=2}^{n} (x_k - 2x_{k-1} + x_{k-2})\cdot \lambda_{k-1}^+ + (x_{+\infty} - x_n + x_{n-1})\cdot \lambda_n^+ \tag{3}$$

$$\lambda_j^+ - \lambda_j^- = \lambda^+ - \lambda^- - j, \qquad j = 1 \cdots n \tag{3.a}$$

$$\lambda^+, \lambda^-, \lambda_j^+, \lambda_j^- \geqslant 0, \qquad j = 1 \cdots n \tag{3.b}$$

$$\lambda^+ \cdot \lambda^- = 0,\; \lambda_j^+ \cdot \lambda_j^- = 0, \qquad j = 1 \cdots n. \tag{3.c}$$

In the sequel a more compact matrix equation will be used:

$$x = x_0 + X^- \cdot \Lambda^- + X^+ \cdot \Lambda^+$$
$$E \cdot (\Lambda^+ - \Lambda^-) = e$$
$$\Lambda^- \geqslant 0;\; \Lambda^+ \geqslant 0$$
$$(\Lambda^-)^t \cdot \Lambda^+ = 0 \tag{4}$$

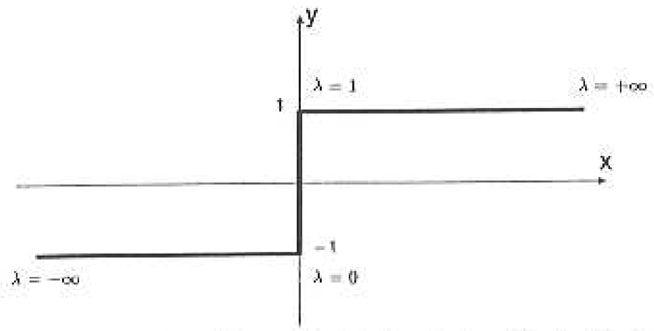

Fig. 6. Step-function of Example 1. The function is multi-valued in the origin.

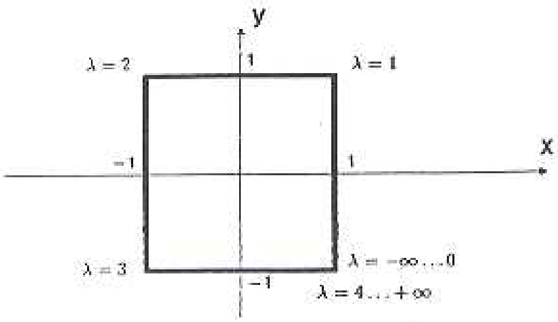

Fig. 7. PWL-curve in Example 2.

where

$$\Lambda^- = [\lambda^- \quad \lambda_1^- \quad \cdots \quad \lambda_n^-]'$$
$$\Lambda^+ = [\lambda^+ \quad \lambda_1^+ \quad \cdots \quad \lambda_n^+]'$$

and

$$X^- = [x_{-\infty} \quad 0 \quad \cdots \quad 0] \in R^{m\times(n+1)}$$
$$X^+ = [x_1 - x_0 \quad \cdots \quad x_{+\infty} - x_n + x_{n-1}] \in R^{m\times(n+1)}$$
$$E = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 \\ 1 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 0 & 0 & \cdots & -1 \end{bmatrix} \in R^{n\times(n+1)}$$
$$e = [1 \quad 2 \quad \cdots \quad n]'.$$

Example 1: The Step-Function (Fig. 6)

The application of the formula above (3) yields

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \cdot \lambda^- + \begin{bmatrix} 0-0 \\ 1-(-1) \end{bmatrix} \cdot \lambda^+ + \begin{bmatrix} 1-0+0 \\ 0-1+(-1) \end{bmatrix} \cdot \lambda_1^+$$

or

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \cdot \lambda^- + \begin{bmatrix} 0 \\ 2 \end{bmatrix} \cdot \lambda^+ + \begin{bmatrix} 1 \\ -2 \end{bmatrix} \cdot \lambda_1^+$$
$$\lambda^+, \lambda^-, \lambda_1^+, \lambda_1^- \geqslant 0$$
$$\lambda_1^+ - \lambda_1^- = \lambda^+ - \lambda^- - 1$$
$$\lambda^+ \cdot \lambda^- + \lambda_1^+ \cdot \lambda_1^- = 0.$$

Example 2: The Edges of the Square (Fig. 7)

This is an example of a curve which is not controlled by one of the variables. It can be parametrized by assigning a coordinate λ as in Fig. 3:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} \cdot \lambda^+ + \begin{bmatrix} -2 \\ -2 \end{bmatrix} \cdot \lambda_1^+ + \begin{bmatrix} 2 \\ -2 \end{bmatrix} \cdot \lambda_2^+ + \begin{bmatrix} 2 \\ 2 \end{bmatrix} \cdot \lambda_3^+ - \begin{bmatrix} 2 \\ 0 \end{bmatrix} \cdot \lambda_4^+$$
$$\lambda_1^+ - \lambda_1^- = \lambda^+ - \lambda^- - 1$$
$$\lambda_2^+ - \lambda_2^- = \lambda^+ - \lambda^- - 2$$
$$\lambda_3^+ - \lambda_3^- = \lambda^+ - \lambda^- - 3$$
$$\lambda_4^+ - \lambda_4^- = \lambda^+ - \lambda^- - 4$$
$$\lambda^+ \cdot \lambda^- = \lambda_1^+ \cdot \lambda_1^- = \lambda_2^+ \cdot \lambda_2^- = \lambda_3^+ \cdot \lambda_3^- = \lambda_4^+ \cdot \lambda_4^- = 0$$
$$\lambda^+, \lambda^- \geqslant 0, \ \lambda_i^+, \lambda_i^- \geqslant 0, \qquad i = 1, \cdots, 4.$$

III. Determination of the Operating Points of PWL Circuits

The parametrization just derived can be used to cast the circuit equations of a PWL resistive circuit in the standard form of a generalized linear complementarity problem (GLCP), as will now be shown. In the next section we will demonstrate that the extension to driving-point analysis and transfer characteristics only requires minor modifications.

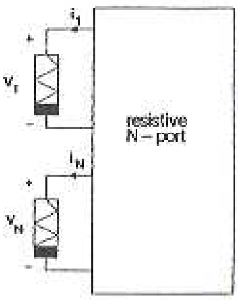

Fig. 8. All nonlinear resistors are extracted from the circuit. The remaining N-port is linear.

We follow the classical procedure of extracting all nonlinear elements out of the circuit as shown in Fig. 8. The nonlinear elements will be assumed to be uncoupled resistors, the changes to include coupling being obvious. Let N be the number of nonlinear resistors. The resulting N-port contains only linear resistive elements and independent sources, and can always be described by its constraint matrix description:

$$C \cdot [i_1 \quad v_1 \quad i_2 \quad v_2 \quad \cdots \quad i_N \quad v_N]' = b \qquad (5)$$

where C is a matrix with $2N$ columns and an arbitrary number of rows. Contrary to most existing methods we are not going to split the ports of the circuit in so-called voltage- and current-ports and to require that a particular hybrid representation of the N-port exists. Nor do we assume the resistors to be either voltage- or current-controlled. The GLCP to be solved can easily be derived by taking the PWL description of each resistor ($k=1,\cdots,N$):

$$\begin{bmatrix} i_k \\ v_k \end{bmatrix} = \begin{bmatrix} i_{k0} \\ v_{k0} \end{bmatrix} + X_k^- \cdot \Lambda_k^- + X_k^+ \cdot \Lambda_k^+$$
$$E_k \cdot \Lambda_k^+ - E_k \cdot \Lambda_k^- = e_k$$
$$\Lambda_k^-, \Lambda_k^+ \geqslant 0$$
$$(\Lambda_k^+)' \cdot \Lambda_k^- = 0 \tag{6}$$

and substituting in (5):

$$C\begin{bmatrix} X_1^- & 0 & \cdots & 0 \\ 0 & X_2^- & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & X_N^- \end{bmatrix}\begin{bmatrix} \Lambda_1^- \\ \vdots \\ \Lambda_N^- \end{bmatrix} + C\begin{bmatrix} X_1^+ & 0 & \cdots & 0 \\ 0 & X_2^+ & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & X_N^+ \end{bmatrix}\begin{bmatrix} \Lambda_1^+ \\ \vdots \\ \Lambda_N^+ \end{bmatrix} = b - C\begin{bmatrix} i_{00} \\ v_{00} \\ i_{10} \\ v_{10} \\ \vdots \\ i_{N0} \\ v_{N0} \end{bmatrix}$$
$$\begin{bmatrix} E_1 & 0 & \cdots & 0 \\ 0 & E_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & E_N \end{bmatrix}\begin{bmatrix} \Lambda_1^- \\ \vdots \\ \Lambda_N^- \end{bmatrix} - \begin{bmatrix} E_1 & 0 & \cdots & 0 \\ 0 & E_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & E_N \end{bmatrix}\begin{bmatrix} \Lambda_1^- \\ \vdots \\ \Lambda_N^- \end{bmatrix} = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_N \end{bmatrix}$$
$$\Lambda_i^- \geqslant 0,\ \Lambda_i^+ \geqslant 0, \qquad \text{for } i=1,\cdots,N$$
$$(\Lambda_i^+)' \cdot \Lambda_i^- = 0, \qquad \text{for } i=1,\cdots,N. \tag{7}$$

Solving this GLCP will yield the complete solution set.

IV. Driving-Point and Transfer Characteristics

The analysis given in the previous section can easily be extended to the determination of the driving-point (DP) plot (i_{IN} versus v_{IN}-plot) of a one port. Together with all nonlinear resistors, we extract the input branch. The constraint matrix description of the remaining ($N+1$)-port is

$$C \cdot [i_{\mathrm{IN}} \ \ v_{\mathrm{IN}} \ \ i_1 \ \ v_1 \ \ i_2 \ \ v_2 \ \ \cdots \ \ i_N \ \ v_N]' = b \tag{8}$$

In order to formulate this as a GLCP we write i_{IN} and v_{IN} as the sum of two complementary variables:

$$i_{\mathrm{IN}} = i_{\mathrm{IN}}^+ - i_{\mathrm{IN}}^-$$
$$v_{\mathrm{IN}} = v_{\mathrm{IN}}^+ - v_{\mathrm{IN}}^-$$
$$i_{\mathrm{IN}}^-, v_{\mathrm{IN}}^-, i_{\mathrm{IN}}^+, v_{\mathrm{IN}}^+ \geqslant 0$$
$$i_{\mathrm{IN}}^- \cdot i_{\mathrm{IN}}^+ + v_{\mathrm{IN}}^- \cdot v_{\mathrm{IN}}^+ = 0.$$

Together with the PWL description of the nonlinear resistors (6) this yields

$$C\begin{bmatrix} -1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & & X_1^- & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & X_N^- \end{bmatrix}\begin{bmatrix} i_{\mathrm{IN}}^- \\ v_{\mathrm{IN}}^- \\ \Lambda_1^- \\ \vdots \\ \Lambda_N^- \end{bmatrix} + C\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & X_1^+ & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & X_N^+ \end{bmatrix}\begin{bmatrix} i_{\mathrm{IN}}^+ \\ v_{\mathrm{IN}}^+ \\ \Lambda_1^+ \\ \vdots \\ \Lambda_N^+ \end{bmatrix} = b - C\begin{bmatrix} 0 \\ 0 \\ i_{00} \\ v_{00} \\ i_{10} \\ v_{10} \\ \vdots \\ i_{N0} \\ v_{N0} \end{bmatrix}$$
$$\begin{bmatrix} E_1 & 0 & \cdots & 0 \\ 0 & E_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & E_N \end{bmatrix}\begin{bmatrix} \Lambda_1^- \\ \vdots \\ \Lambda_N^- \end{bmatrix} - \begin{bmatrix} E_1 & 0 & \cdots & 0 \\ 0 & E_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & E_N \end{bmatrix}\begin{bmatrix} \Lambda_1^- \\ \vdots \\ \Lambda_N^- \end{bmatrix} = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_N \end{bmatrix}$$
$$\Lambda_i^- \geqslant 0,\ \Lambda_i^+ \geqslant 0, \qquad \text{for } i=1,\cdots,N$$
$$v_{\mathrm{IN}}^-, i_{\mathrm{IN}}^-, v_{\mathrm{IN}}^+, i_{\mathrm{IN}}^+ \geqslant 0$$
$$(\Lambda_i^+)' \cdot \Lambda_i^- = 0, \qquad \text{for } i=1,\cdots,N$$
$$v_{\mathrm{IN}}^-, v_{\mathrm{IN}}^+ + i_{\mathrm{IN}}^- \cdot i_{\mathrm{IN}}^+ = 0. \tag{9}$$

It is important to note that this GLCP will in general be rectangular (cf. the remark in Section V).

The extension to the computation of transfer characteristics is straightforward. For instance, in order to trace the $v_{OUT}-v_{IN}$-characteristic of a circuit the input and the output branch are extracted together with all nonlinear resistors and apply the same reasoning to the resulting $(N+2)$ port.

V. The Generalized Linear Complementarity Problem

The term GLCP for the problems encountered above is justified by a number of differences with the classical formulation.

The Linear Complementarity Problem: Given $N \in R^{n \times n}$ and $q \in R^n$, find all solutions $w \in R^n$ and $z \in R^n$ to

$$\begin{aligned} w + Nz &= q \\ w \geqslant 0, z &\geqslant 0 \\ w^t \cdot z &= 0 \end{aligned} \tag{10}$$

where the shorthand notation $x \geqslant 0$ for vector inequalities holds componentwise. This problem is well known in mathematical programming as a unifying description of a large class of problems, including linear and quadratic programming [14], fixed-point problems and sets of piecewise-linear equations [9], [10], bimatrix equilibrium points [15] and variational inequalities [16]. A generalization to this formulation has been given in [17]. For our present purposes, the following form will be sufficient.

The Generalized Linear Complementarity Problem (GLCP)

Given $M \in R^{m \times n}$, $N \in R^{m \times n}$, $q \in R^m$, find all $w \in R^n$, $z \in R^n$ and $\alpha \in R$ such that

$$\begin{aligned} Mw + Nz &= q \cdot \alpha \\ w, z \geqslant 0, &\qquad \alpha \geqslant 0 \\ w^t \cdot z &= 0. \end{aligned} \tag{11}$$

There are three distinct differences between this formulation and (10):

- GLCP (11) allows rectangular LCP's ($m < n$). In the previous section it is shown how these occur in the determination of driving-point and transfer characteristics.
- Even if $m = n$, we do not exclude cases where no reordering of the variables w and z provides an invertible matrix M.
- The nonnegative scalar α has been added to allow for solutions at infinity, i.e., directions where the solution set is unbounded. The normalized solution set is the intersection of the solution set of (11) and the hyperplane $\alpha = 1$.

VI. The Solution Set of Linear Systems of Constrained Equalities and of the GLCP

The algorithm we propose for the solution of (11) is discussed in Appendix A. In this section the geometric and algebraic properties of the solution set are described. These properties are important for the interpretation of the obtained solution set.

Consider first the set of linear equations with nonnegativity constraints:

$$\begin{aligned} &A \cdot x = 0, \text{ with given } A \in R^{m \times n} \\ &x \geqslant 0. \end{aligned} \tag{12}$$

Definition 1: The solution set $\mathscr{L}_A = \{x \geqslant 0 | Ax = 0\}$ is a *polyhedral cone* (the intersection of the first orthant in R^n and the subspace $\ker A$).

$\mathscr{L}_A$ can be defined completely by all positive linear combinations of the extreme rays $\{v^1, v^2, \cdots, v^q\}$ of the polyhedral cone.

Definition 2: A vector $v^i \in R^n$ is an *extreme ray* of $\mathscr{L}_A$ if there exists a hyperplane $V = \{x \in R^n | h^t \cdot x = 0\}$ such that $V \cap \mathscr{L}_A = \{x | x = \lambda v^i, \lambda \geqslant 0\}$.

Theorem 1: A *necessary and sufficient* condition for a solution $v \in \mathscr{L}_A$ to be an extreme ray is that no other solutions possess zeros at the same positions as v. In other words: call $\mathscr{I}_v = \{k | v_k = 0\}$, the set of indexes k where $v_k = 0$, then v is an extreme ray iff there does not exist a solution w with $\mathscr{I}_v \subseteq \mathscr{I}_w$.

Corollary 1: If the rows of A are independent, then a *necessary* condition for extremity is that the number of zeros in v is greater than or equal to $n - m - 1$: $\#\mathscr{I}_v \geqslant n - m - 1$.

Definition 3: Two extreme rays v and w are *adjacent* if there exists a supporting hyperplane $V = \{x | h^t \cdot x = 0, h^t \cdot z \geqslant 0, \forall z \in \mathscr{L}_A\}$ such that $V \cap \mathscr{L}_A = \{x = \lambda_1 v + \lambda_2 w, \lambda_1, \lambda_2 \geqslant 0\}$. The set of all convex combinations of two adjacent rays is called a *two-dimensional face* of the cone.

Theorem 2: A *necessary and sufficient* condition for two extreme solutions v and w to be adjacent is that there exist no other extreme solutions with zeros at the same positions as the common zeros of v and w. Call $\mathscr{I}_{v,w} = \{k | v_k = 0$ and $w_k = 0\}$, the set of indexes of the common zeros of w and v, then v and w are adjacent iff there exist no other extreme solutions z with $\mathscr{I}_{v,w} \subseteq \mathscr{I}_z$.

Corollary 2: If the rows of A are linearly independent, then a *necessary* condition for extreme solutions to be adjacent is that the number of common zeros in v and w is greater than or equal to $n - m - 2$: $\#\mathscr{I}_{v,w} \geqslant n - m - 2$.

We are now ready to return to the GLCP (11), which will be treated as a set of linear equations

$$[M \quad N \quad -q] \begin{bmatrix} w \\ z \\ \alpha \end{bmatrix} = 0, \qquad w \geqslant 0;\ z \geqslant 0;\ \alpha \geqslant 0 \tag{13}$$

with extra constraints ($w^t \cdot z = 0$). The solution set of (13) is then normalized by taking the intersection with the hyperplane $\alpha = 1$. This intersection is a *generalized polytope* determined by the set of vertices $v^i = [(w^i)^t \quad (z^i)^t \quad \alpha^i]^t$, with $\alpha^i = 1$ for a finite vertex or with $\alpha^i = 0$ for a vertex at infinity, where the polytope becomes unbounded.

Theorem 3: The solution set of the GLCP (11) consists of all nonnegative combinations of vertices determined by (13) with the following restrictions:

- all vertices that are not complementary ($(w^i)^t \cdot z^i \neq 0$) should be discarded,
- only convex combinations of cross-complementary vertices are allowed (vertices v^i and v^k with $(w^i)^t \cdot z^k = (w^k)^t \cdot z^i = 0$).

VII. Examples

The first two examples deal with the determination of operating points, and in Examples 5 and 6 driving-point plots are traced.

Example 3: We compute the solutions to the circuit in Fig. 9. The constraint matrix description is simply:

$$[1 \quad 1]\begin{bmatrix} i \\ v \end{bmatrix} = 0 \tag{14}$$

and the resistor description:

$$\begin{bmatrix} i \\ v \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix}\lambda^- + \begin{bmatrix} -2 \\ 0 \end{bmatrix}\lambda^+ + \begin{bmatrix} 2 \\ 1 \end{bmatrix}\lambda_1^+. \tag{15}$$

$$\lambda_1^+ - \lambda_1^- = \lambda^+ - \lambda^- - 1$$

Substituting this in (14) yields the GLCP:

$$\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} \lambda^- \\ \lambda_1^- \end{bmatrix} + \begin{bmatrix} 2 & -3 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \lambda^+ \\ \lambda_1^+ \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

or

$$\begin{bmatrix} 1 & 0 & 2 & -3 & -1 \\ -1 & 1 & 1 & -1 & -1 \end{bmatrix}\begin{bmatrix} \lambda^- \\ \lambda_1^- \\ \lambda^+ \\ \lambda_1^+ \\ \alpha \end{bmatrix} = 0 \tag{16}$$

$$\lambda^-, \lambda_1^-, \lambda^+, \lambda_1^-, \alpha \geq 0$$

$$\lambda^+\lambda^- + \lambda_1^+\lambda_1^- = 0.$$

This GLCP is solved in Appendix A as an example. The normalized solution set and the corresponding points on the $i-v$ characteristic of $\mathscr{R}_1$ are given in Table I.

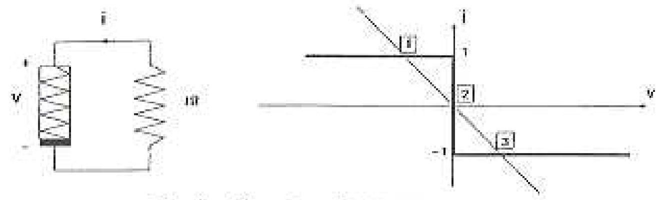

Fig. 9. Circuit studied in Example 3.

TABLE I
Solution Set of Example 3 (See Fig. 9)

	1	2	3
λ^-	1	0	0
λ_1^-	2	0.5	0
λ^+	0	0.5	2
λ_1^+	0	0	1
α	1	1	1
i	1	0	-1
v	-1	0	1

Example 4: The $i-v$ characteristic for the resistor $\mathscr{R}_1$ in Fig. 1 has already been given in Section IV:

$$\begin{bmatrix} i_R \\ v_R \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix}\cdot\lambda^+ + \begin{bmatrix} -2 \\ -2 \end{bmatrix}\cdot\lambda_1^+$$

$$+ \begin{bmatrix} -2 \\ 2 \end{bmatrix}\cdot\lambda_2^+ + \begin{bmatrix} 2 \\ 2 \end{bmatrix}\cdot\lambda_3^+ - \begin{bmatrix} 0 \\ 2 \end{bmatrix}\cdot\lambda_4^+$$

$$\lambda_1^+ - \lambda_1^- = \lambda^+ - \lambda^- - 1$$

$$\lambda_2^+ - \lambda_2^- = \lambda^+ - \lambda^- - 2$$

$$\lambda_3^+ - \lambda_3^- = \lambda^+ - \lambda^- - 3$$

$$\lambda_4^+ - \lambda_4^- = \lambda^+ - \lambda^- - 4$$

$$\lambda^+\cdot\lambda^- = \lambda_1^+\cdot\lambda_1^- = \lambda_2^+\cdot\lambda_2^- = \lambda_3^+\cdot\lambda_3^- = \lambda_4^+\cdot\lambda_4^- = 0.$$

The description of the diode is trivial:

$$\begin{bmatrix} i_D \\ v_D \end{bmatrix} = \begin{bmatrix} \lambda_D^+ \\ -\lambda_D^- \end{bmatrix}$$

$$\lambda_D^+, \lambda_D^- \geq 0$$

$$\lambda_D^+\cdot\lambda_D^- = 0.$$

Substituting these descriptions in the network equation:

$$[1 \quad 0 \quad 1 \quad 0]\begin{bmatrix} i_R \\ v_R \\ i_D \\ v_D \end{bmatrix} = 0$$

leads to the GLCP:

$$\begin{bmatrix} 2 & -2 & -2 & 2 & 0 & 1 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} \lambda^+ \\ \lambda_1^+ \\ \lambda_2^+ \\ \lambda_3^+ \\ \lambda_4^+ \\ \lambda_D^+ \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} \lambda^- \\ \lambda_1^- \\ \lambda_2^- \\ \lambda_3^- \\ \lambda_4^- \\ \lambda_D^- \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

$$\lambda_i^+, \lambda_i^- \geq 0, \qquad i = 1, \cdots, 4$$

$$\lambda^+, \lambda^-, \lambda_D^+, \lambda_D^- \geq 0$$

$$\lambda^+\cdot\lambda^- = \lambda_1^+\cdot\lambda_1^- = \lambda_2^+\cdot\lambda_2^- = \lambda_3^+\cdot\lambda_3^- = \lambda_4^+\cdot\lambda_4^-$$

$$= \lambda_D^-\cdot\lambda_D^+ = 0.$$

The solution of this GLCP is given in Table II. The interpretation of this solution set is more complex than in the previous example because of the appearance of cross-complementary solutions (cf. Section VI). The significance of the finite vertices 1–5 is most easily understood. These

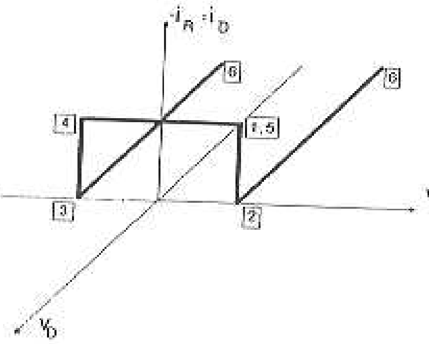

Fig. 10. The complete solution set for the circuit in Fig. 1.

solutions are represented in the solution space in Fig. 10. Solution **1** corresponds to the point $\lambda = 0$ of $\mathscr{R}_1$, solution **2** to the point $\lambda = 0.5$. Moreover these two solutions are cross-complementary and all convex combinations of these are valid solutions (the piece from **1** to **2** on the $i-v$ characteristics). The next solution is **3** ($\lambda = 2.5$), followed by **4** ($\lambda = 3$) and **5** ($\lambda = 4$). Checking cross-complementarity reveals that all combinations of **3** and **4**, and of **4** and **5** belong to the solution set. Vertex **6** lies at infinity (pointing in the direction $[i_D \quad v_D] = [0 \quad -1]$, cf. Fig. 10) and is cross-complementary with solutions **2** and **3**. The physical meaning is that, when the diode is blocking, the voltage over it is an arbitrary negative number (a point of the form $[i_D \quad v_D] = [0 \quad 0] + \beta[0 \quad -1]$, with $\beta \geqslant 0$). The meaning of vertices **7** and **8** is less essential: **7**, combined with the cross-complementary solution **1** yields any point with $\lambda \in (-\infty, 0]$ and **8**, together with **5**, any point with $\lambda \in [0, +\infty)$. Physically, these solutions coincide with **1**, resp. **5**.

Example 5: Series Connection of Two Tunnel Diodes: We trace the $i-v$ characteristic of the one-port made up by a series connection of the two (PWL) tunneldiodes in Fig. 3. The PWL description of the resistors reads

$$\mathscr{R}_1: \quad \begin{aligned} \begin{bmatrix} i_1 \\ v_1 \end{bmatrix} &= \begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} -3 \\ -1 \end{bmatrix} \cdot \lambda^- + \begin{bmatrix} -2.5 \\ 2 \end{bmatrix} \cdot \lambda^+ + \begin{bmatrix} 3.5 \\ 0 \end{bmatrix} \cdot \lambda_1^+ \\ \lambda_1^+ - \lambda_1^- &= \lambda^+ - \lambda^- - 1 \\ \lambda^+, \lambda^-, \lambda_1^+, \lambda_1^- &\geqslant 0 \\ \lambda^+ \cdot \lambda^- + \lambda_1^+ \cdot \lambda_1^- &= 0 \end{aligned}$$

$$\mathscr{R}_2: \quad \begin{aligned} \begin{bmatrix} i_2 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 4 \\ 2 \end{bmatrix} + \begin{bmatrix} -4 \\ -2 \end{bmatrix} \cdot \mu^- + \begin{bmatrix} -2 \\ 2 \end{bmatrix} \cdot \mu^+ + \begin{bmatrix} 3 \\ 0 \end{bmatrix} \cdot \mu_1^+ \\ \mu_1^+ - \mu_1^- &= \mu^+ - \mu^- - 1 \\ \mu^+, \mu^-, \mu_1^+, \mu_1^- &\geqslant 0 \\ \mu^+ \cdot \mu^- + \mu_1^+ \cdot \mu_1^- &= 0. \end{aligned}$$

Substituting in the constraint matrix description

$$\begin{bmatrix} 0 & 1 & 0 & -1 & 0 & -1 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} i \\ v \\ i_1 \\ v_1 \\ i_2 \\ v_2 \end{bmatrix} = 0$$

TABLE II
SOLUTION SET OF EXAMPLE 4 (SEE FIG. 1)

	1	2	3	4	5	6	7	8
λ^+	0	0.5	2.5	3	4	0	0	1
λ_1^+	0	0	1.5	2	3	0	0	1
λ_2^+	0	0	0.5	1	2	0	0	1
λ_3^+	0	0	0	0	1	0	0	1
λ_4^+	0	0	0	0	0	0	0	1
λ_D^+	1	0	0	1	1	0	0	0
λ^-	0	0	0	0	0	0	1	0
λ_1^-	1	0.5	0	0	0	0	1	0
λ_2^-	2	1.5	0	0	0	0	1	0
λ_3^-	3	2.5	0.5	0	0	0	1	0
λ_4^-	4	3.5	1.5	1	0	0	1	0
λ_D^-	0	0	0	0	0	1	0	0
σ	1	1	1	1	1	0	0	0

TABLE III
SOLUTION SET OF EXAMPLE 5 (SEE FIG. 3)

	1	2	3	4	5	6	7	8	9	10	11
i^-	0.77	0	0	0	0	0	0	0	0	0	0
v^-	0.64	0	0	0	0	0	0	0	0	0	0
λ^-	0.26	1	0	0	0	0	0	0	0	0.33	0
λ_1^-	0.26	2	1	0	0	0	0	0.6	1	1.33	1
μ^-	0.19	1	0.25	0.88	0	0	0	0	0	0	0
μ_1^-	0.19	2	1.25	1.88	1	0	0	0	0	0	0.5
i^+	0	0	3	0.5	4	2	0.29	2	3	2	3
v^+	0	0	2.5	3.25	12	10	1.48	5.8	7	4.67	4
λ^+	0	0	0	1	4.5	2.5	0.29	0.4	0	0	0
λ_1^+	0	0	0	0	3.5	1.5	0.29	0	0	0	0
μ^+	0	0	0	0	0	1	0.29	1	2	1	0.5
μ_1^+	0	0	0	0	0	0	0.29	0	1	0	0
σ	0	1	1	1	1	1	0	1	1	1	1
i	-0.77	0	3	0.50	4	2	0.29	2	3	2	3
v	-0.64	0	2.5	3.25	12	10	1.48	5.8	7	4.67	4

yields the GLCP

$$\begin{bmatrix} -1 & 0 & 3 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 2 & 0 \\ -1 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} i^- \\ v^- \\ \lambda^- \\ \lambda_1^- \\ \mu^- \\ \mu_1^- \end{bmatrix} + \begin{bmatrix} 1 & 0 & 2.5 & -3.5 & 0 & 0 \\ 0 & 1 & -2 & 0 & -2 & 0 \\ 1 & 0 & 0 & 0 & 2 & -3 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} i^+ \\ v^+ \\ \lambda^+ \\ \lambda_1^+ \\ \mu^+ \\ \mu_1^+ \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 4 \\ 1 \\ 1 \end{bmatrix}$$

$$v^-, i^-, \lambda^-, \lambda_1^-, \mu^-, \mu_1^- \geqslant 0$$

$$v^+, i^+, \lambda^+, \lambda_1^+, \mu^+, \mu_1^+ \geqslant 0$$

$$v^- \cdot v^+ + i^- \cdot i^+ + \lambda^- \cdot \lambda^+ + \lambda_1^- \cdot \lambda_1^+ + \mu^- \cdot \mu^+ + \mu_1^- \cdot \mu_1^+ = 0.$$

The GLCP algorithm produces 11 solutions (see Table III). Verifying cross-complementarity conditions allows to trace the complete DP-plot which consists of two pieces: **1–2–3–4–5–6–7** and **8–9–10–11–8** (Fig. 3).

Example 6: The resistors in Fig. 4 are described as

$$\mathscr{R}_1: \begin{bmatrix} i_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} \lambda^- + \begin{bmatrix} 2 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda^+ \\ \lambda_1^+ \end{bmatrix}$$

TABLE IV
SOLUTION SET OF EXAMPLE 6 (SEE FIG. 4)

	1a	1b	2	3	4	5a	5b	6	7	8	9	10	11	12	13
i^+	0	0	1	1	1	0	0	0	0	0	0	1	0	0	0
v^+	1	1	1	0	0	0	0	0	0	1	0	0	0	0	0
λ^+	0	0.5	1	1	1	0.5	1	0	0	0	0.5	0	0	1	0
λ_1^+	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
μ^+	0	0	0	0.5	1	1	2	1	0.5	0	0.5	0	1	0	0
μ_1^+	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0
i^-	0	0	0	0	0	0	0	1	1	1	0	0	1	0	0
v^-	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0
λ^-	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
λ_1^-	1	0.5	0	0	0	0.5	0	1	1	1	0.5	0	0	0	1
μ^-	1	0	0	0	0	0	0	0	0	0	0	1	0	0	0
μ_1^-	2	1	1	0.5	0	0	0	0	0.5	1	0.5	1	0	0	0
α	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0
i	0	0	1	1	1	0	0	-1	-1	-1	0	1	-1	0	0
v	1	1	1	0	-1	-1	-1	-1	0	0	0	0	0	0	0

with $\lambda^+ - \lambda^- - \lambda_1^+ + \lambda_1^- = 1$, and

$$\mathscr{R}_2: \begin{bmatrix} i_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \mu^- + \begin{bmatrix} 0 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} \mu^+ \\ \mu_1^+ \end{bmatrix}$$

with $\mu^+ - \mu^- - \mu_1^+ + \mu_1^- = 1$. The network equations are

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} i \\ v \\ i_1 \\ v_1 \\ i_2 \\ v_2 \end{bmatrix} = 0.$$

We combine these equations into the GLCP:

$$\begin{bmatrix} 1 & 0 & -2 & 2 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 & -2 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} i^+ \\ v^+ \\ \lambda^+ \\ \lambda_1^+ \\ \mu^+ \\ \mu_1^+ \end{bmatrix}$$

$$+ \begin{bmatrix} -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} i^- \\ v^- \\ \lambda^- \\ \lambda_1^- \\ \mu^- \\ \mu_1^- \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

together with the appropriate nonnegativity and complementarity conditions (see Table IV). These solutions are represented in a $i-v$ plane in Fig. 4. Solutions **1a** and **1b** are coinciding, as are solutions **5a** and **5b**. The cross-complementarity rule indicates that the full square belongs to the solution set. Solutions **10** and **11** are directions, solutions **12** and **13** are of mere mathematical significance.

VIII. CONCLUSIONS

The method presented in this paper gives a complete description of a PWL resistive circuit: the result is a list of extreme points of the solution set along with a rule to decide whether two or more extreme points should be

TABLE V
THE ALGORITHM APPLIED TO EXAMPLE 3 (SEE APPENDIX A)

S^k	S^0					S^1					S^2		
	1	2	3	4	5	1	2	3	4	5	1	2	3
λ^-	1	0	0	0	0	0	3	1	0	0	1	0	0
λ_1^-	0	1	0	0	0	1	0	0	0	0	2	1	0
λ^+	0	0	1	0	0	0	0	0	3	1	0	1	4
λ_1^+	0	0	0	1	0	0	1	0	2	0	0	0	2
α	0	0	0	0	1	0	0	1	0	2	1	2	2
$a_{k+1}^t S^k$	1	0	2	-3	-1	1	-4	-2	1	-1			

connected. The most prominent advantages of this method are as follows:

- A complete description of the solution set is given, even when a continuum of solutions occurs, or when the solution set is unbounded in some direction.
- No restrictions are imposed on the linear part of the circuit, or on the existence of any hybrid representation.
- We allow nonlinear resistors which are neither voltage nor current controlled.

As far as is known to the authors, no algorithm has been presented with the same versatility and generality. A prototype program for this algorithm has been implemented. A more practical software implementation is currently being investigated along with its use in circuit simulation for finding all dc operating points. The results should also be useful in studying geometric properties of nonlinear resistive networks [5], dynamics, chaos, and neural networks.

APPENDIX A
THE GLCP ALGORITHM

In 1953 Motzkin *et al.* [19] proposed an algorithm for the solution of sets of linear inequalities that can easily be adapted for the solution of linear equalities and hence for the GLCP [20], [17]. We first describe the inductive algorithm for the solution of (12).

Call $a_i^t =$ the ith row of A, then we denote with $S^k \subset R^{n \times q_k}$ the matrix formed by the q_k extreme rays of the solution set of

$$a_i^t x = 0, \qquad i = 1, \cdots, k$$
$$x \geqslant 0 \tag{17}$$

- $S^0 = I_n$, the initial set of extreme rays generates the first orthant in R^n.
- The iteration describes how S^k is updated into S^{k+1}, when a new equality

$$a_{k+1}^t x = 0 \tag{18}$$

is added. Put $(s^{k+1})^t = a_{k+1}^t \cdot S^k$, a $1 \times q_k$ matrix. For each element in $(s^{k+1})^t$ three possibilities exist:

Case 1: $s_j^{k+1} = 0$, indicating that S_j^k (jth column in S^k) lies in the hyperplane $a_{k+1}^t x = 0$.

Cases 2 and 3: $s_j^{k+1} > 0$ or $s_j^{k+1} < 0$, indicating that S_j^k lies in either of the two half-spaces defined by $a_{k+1}^t x = 0$.

The construction of the extreme rays S^{k+1}, can then proceed as follows:

Case 1: if an extreme ray in S^k lies in the hyperplane (18), it is also an extreme ray of S^{k+1}, *Cases 2 and 3:* any two adjacent extreme rays, lying on either side of hyperplane (18), define an extreme face, intersecting the hyperplane. This intersection is an extreme ray of S^{k+1}: if $s_j^{k+1} < 0$ and $s_i^{k+1} > 0$ and S_j^k and S_i^k are adjacent, then $|s_j^{k+1}| \cdot S_i^k + |s_i^{k+1}| \cdot S_j^k \in S^{k+1}$. The adjacency tests are described in Theorem 2 and Corollary 2.

The GLCP algorithm is now obvious from a combination of the inductive algorithm for the solution of (12), and Theorem 3 which allows to eliminate at each stage those vertices of S^k that do not satisfy the complementarity conditions. The matrix S^k then contains at each stage the solution of the rectangular $k \times n$ GLCP, formed by the first k rows of (13).

Example:

We apply the GLCP-algorithm to the GLCP in Example 3:

$$A = \begin{bmatrix} 1 & 0 & 2 & -3 & -1 \\ -1 & 1 & 1 & -1 & -1 \end{bmatrix}$$

- $S^0 = I_5$ (Table V(a)).
- S^1 is constituted by column 2 of S^0 and by the combinations of columns 1 with 4, 1 with 5, 3 with 4 and 3 with 5 (Table V(b)).
- The following combinations of columns in S^1 are forbidden: 1 with 2, 4 with 2, and 4 with 3 (not cross-complementary). Combination of 1 with 3, 1 with 5 and 4 with 5 yields S^2 (Table V(c)).

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Joseph (Joos) Vandewalle (S'71–M'79–SM'82) received the degree of Electro-Mechanical engineer in electronics in 1971, doctorate in Applied Sciences (1976), Special doctor in applied sciences 1984 all at the Katholieke Universiteit Leuven. He was Research Associate (1976–1978) and visiting assistant professor (1978–1979) at the Department of Electrical Engineering and Computer Sciences, University of California, Berkeley. He was Assistant Professor (1980–1986) and since 1986 he is Full Professor (Gewoon Hoogleraar) at the Department of Electrical Engineering, Katholieke Universteit Leuven. Since October 1980 he is a member of the staff of the ESAT Laboratory. He is an Academic Consultant since 1984 at the VSDM group of IMEC. He has teaching assignments in Algebra, System Theory, Signal Processing, Network Theory, and Nonlinear Systems. He is the promotor of 8 completed doctoral research activities and about 13 current doctoral research activities at the KUL. His research interests are in system theory, digital signal processing and cryptography. Special attention is devoted to the numerical aspects of the algorithms and to the use of the Singular Value Decomposition. In these areas he has (co-)authored more than 100 papers.

Bart L. De Moor (S'86–M'88) received the degree in electrical engineering (control theory) in 1983 and a Ph.D. degree in applied sciences in 1988, both in the Department of Electrical Engineering, at the Katholieke Universiteit Leuven, Belgium. Since September 1988 he is a visiting research associate with the Department of Computer Science (Numerical Analysis Group) and the Department of Electrical Engineering (Information Systems Lab) of Stanford University, USA. His research interests include system theory and identification, numerical linear algebra and complementarity problems with special emphasis towards applications.

Dr. De Moor is also a member of the Society for Industrial and Applied Mathematics.

Lieven Vandenberghe (S'85–M'87) received the electrical engineering degree from the Katholieke Universiteit Leuven in 1987. Since 1987 he is working at the ESAT Laboratory as a research assistant of the Belgian National Fund for Scientific Research. His main research interests are in nonlinear circuit theory and in neural networks.