A Numerical Linear Algebra Framework for Solving Problems with Multivariate Polynomials

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September 12
2013

Outline

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How do multivariate polynomials look like?

Remember from your high school days

- $9x^2 - 5x + 2$
- $x^3 + x^2 - x$

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- $9x^2 - 5x + 2$
- $x^3 + x^2 - x$

Now with more than 1 'x'

- $x_1\, x_2^2 + x_1\, x_3^2 - 1.1\, x_1 + 1$
- $-x_1\, x_3^3 + 4\, x_2\, x_3^2\, x_4 + 4\, x_1\, x_3\, x_4^2 + 2\, x_2\, x_4^3 + 4\, x_1\, x_3 + 4\, x_3^2 - 10\, x_2\, x_4 - 10\, x_4^2 + 2$
- $5.22 x_1 x_2^4 + 3.98 x_1^3 - x_2^4 - 3 x_2^2$
- $9.124 x_1^2 x_2 - 2.22 x_1^2$
- $2 x_1 x_2^4 - x_1^3 - 2 x_2^4 + x_1^2$

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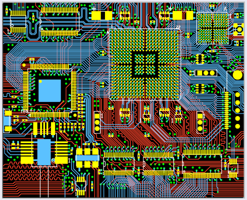
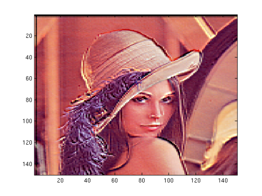
In which engineering domains do this kind of polynomials appear?

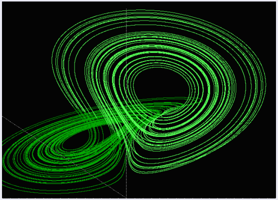
Computational Biology



Circuit Design



Signal Processing



Nonlinear Dynamical Systems

In which engineering domains do this kind of polynomials appear?

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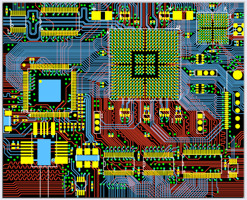

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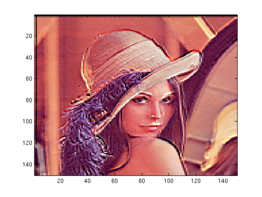
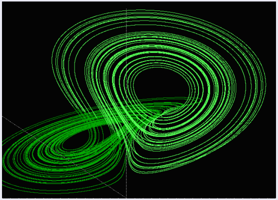


Signal Processing



Nonlinear Dynamical Systems

And many others...

What needs to be done with these multivariate polynomials?

- Find the solutions,
- Multiply and divide,
- Eliminate variables,
- Compute least common multiples and greatest common divisors,
- ...

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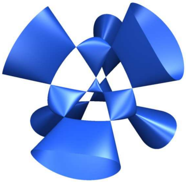
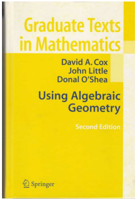
How are these problems mostly solved these days?

Algebraic Geometry

- Branch of mathematics



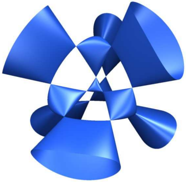


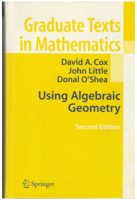
Wolfgang Gröbner
(1899-1980)

Bruno Buchberger

Algebraic Geometry

- Branch of mathematics
- Symbolic operations





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Algebraic Geometry

- Branch of mathematics
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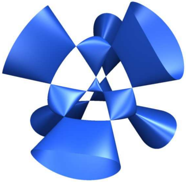

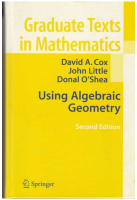

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Algebraic Geometry

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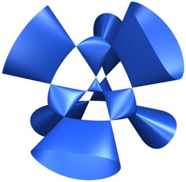

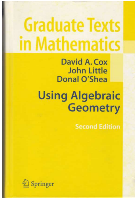

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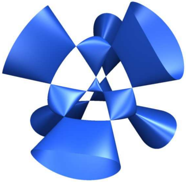
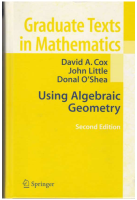
Algebraic Geometry

- Branch of mathematics
- Symbolic operations
- Computer algebra software
- Huge body of literature in Algebraic Geometry
- **Produces exact results for exact data!!**





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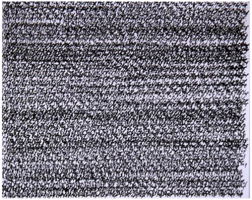
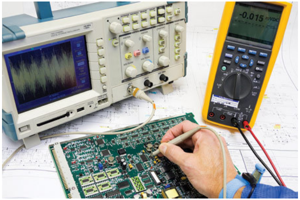
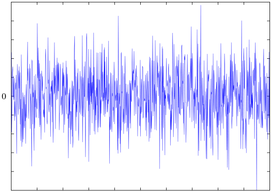
Engineers do not usually work with exact data

Uncertainties in the measurements ⇒ uncertainties in the coefficients of the multivariate polynomials







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Engineers do not need exact solutions

$$
\begin{pmatrix}
-85 & -55 & -37 & -35 & 97 & 50 & 79 & 56 & 49 & 63 \\
57 & -59 & 45 & -8 & -93 & 92 & 43 & -62 & 77 & 66 \\
54 & -5 & 99 & -61 & -50 & -12 & -18 & 31 & -26 & -62 \\
1 & -47 & -91 & -47 & -61 & 41 & -58 & -90 & 53 & -1 \\
94 & 83 & -86 & 23 & -84 & 19 & -50 & 88 & -53 & 85 \\
49 & 78 & 17 & 72 & -99 & -85 & -86 & 30 & 80 & 72 \\
66 & -29 & -91 & -53 & -19 & -47 & 68 & -72 & -87 & 79 \\
43 & -66 & -53 & -61 & -23 & -37 & 31 & -34 & -42 & 88 \\
-76 & -65 & 25 & 28 & -61 & -60 & 9 & 29 & -66 & -32 \\
78 & 39 & 94 & 68 & -17 & -98 & -36 & 40 & 22 & 5
\end{pmatrix} z = \begin{pmatrix} -88 \\ -43 \\ -73 \\ 25 \\ 4 \\ -59 \\ 62 \\ -55 \\ 25 \\ 9 \end{pmatrix}
$$

(from Numerical Polynomial Algebra - H.J. Stetter)

Engineers do not need exact solutions

$$\begin{pmatrix} -85 & -55 & -37 & -35 & 97 & 50 & 79 & 56 & 49 & 63 \\ 57 & -59 & 45 & -8 & -93 & 92 & 43 & -62 & 77 & 66 \\ 54 & -5 & 99 & -61 & -50 & -12 & -18 & 31 & -26 & -62 \\ 1 & -47 & -91 & -47 & -61 & 41 & -58 & -90 & 53 & -1 \\ 94 & 83 & -86 & 23 & -84 & 19 & -50 & 88 & -53 & 85 \\ 49 & 78 & 17 & 72 & -99 & -85 & -86 & 30 & 80 & 72 \\ 66 & -29 & -91 & -53 & -19 & -47 & 68 & -72 & -87 & 79 \\ 43 & -66 & -53 & -61 & -23 & -37 & 31 & -34 & -42 & 88 \\ -76 & -65 & 25 & 28 & -61 & -60 & 9 & 29 & -66 & -32 \\ 78 & 39 & 94 & 68 & -17 & -98 & -36 & 40 & 22 & 5 \end{pmatrix} z = \begin{pmatrix} -88 \\ -43 \\ -73 \\ 25 \\ 4 \\ -59 \\ 62 \\ -55 \\ 25 \\ 9 \end{pmatrix}$$

$$z = \left(\frac{782834491243408476131}{8714574463184678755 27}, \frac{-2485679712713251977 81}{871457446318467875527}, \frac{-114174123958691622 4104}{871457446318467875527}, \ldots\right)^T$$

(from Numerical Polynomial Algebra - H.J. Stetter)

Engineers do not need exact solutions

$$\begin{pmatrix} -85 & -55 & -37 & -35 & 97 & 50 & 79 & 56 & 49 & 63 \\ 57 & -59 & 45 & -8 & -93 & 92 & 43 & -62 & 77 & 66 \\ 54 & -5 & 99 & -61 & -50 & -12 & -18 & 31 & -26 & -62 \\ 1 & -47 & -91 & -47 & -61 & 41 & -58 & -90 & 53 & -1 \\ 94 & 83 & -86 & 23 & -84 & 19 & -50 & 88 & -53 & 85 \\ 49 & 78 & 17 & 72 & -99 & -85 & -86 & 30 & 80 & 72 \\ 66 & -29 & -91 & -53 & -19 & -47 & 68 & -72 & -87 & 79 \\ 43 & -66 & -53 & -61 & -23 & -37 & 31 & -34 & -42 & 88 \\ -76 & -65 & 25 & 28 & -61 & -60 & 9 & 29 & -66 & -32 \\ 78 & 39 & 94 & 68 & -17 & -98 & -36 & 40 & 22 & 5 \end{pmatrix} z = \begin{pmatrix} -88 \\ -43 \\ -73 \\ 25 \\ 4 \\ -59 \\ 62 \\ -55 \\ 25 \\ 9 \end{pmatrix}$$

$$z = \left(\frac{7828344912434084761 31}{8714574463184678755 27}, \frac{-2485679712713251977 81}{8714574463184678755 27}, \frac{-1141741239586916224 104}{8714574463184678755 27}, \ldots\right)^T$$

$$\tilde{z} = (.8983049, -.2852325, -1.3101515, \ldots, -1.6168161)^T$$

(from Numerical Polynomial Algebra - H.J. Stetter)

Language problem

- Algebraic Geometry not in the normal curriculum of most engineers
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Example: First sentence of the online description of the GROEBNER package of Maple 7

"The GROEBNER package is a collection of routines for doing Groebner basis calculations in skew algebras like Weyl and Ore algebras and in corresponding modules like D-modules".

Language problem

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Example: First sentence of the online description of the GROEBNER package of Maple 7

"The GROEBNER package is a collection of routines for doing Groebner basis calculations in skew algebras like Weyl and Ore algebras and in corresponding modules like D-modules".

Engineers do speak (numerical) linear algebra!

Richard Feynman

Seeing things from a Numerical Linear Algebra perspective

- Is it possible to use Numerical Linear Algebra instead?
- New insights/interpretations?
- New methods?

Richard Feynman

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This thesis:

The development of a Numerical Linear Algebra framework to solve problems with multivariate polynomials.

Outline

Building blocks of multivariate polynomials?

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Monomials!

$$1, x_1, x_2, x_3, x_1^2, x_1x_2, x_1x_3, x_2^2, x_2x_3, x_3^2, \ldots$$

Building blocks of multivariate polynomials?

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$$1, x_1, x_2, x_3, x_1^2, x_1x_2, x_1x_3, x_2^2, x_2x_3, x_3^2, \ldots$$

- ordering
- $\deg(x_1^2) = \deg(x_2x_3) = 2$

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Example

$$f_1 =$$

Building blocks of multivariate polynomials?

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Example

$$f_1 = 2.76\, x_1^2$$

Building blocks of multivariate polynomials?

Monomials!

$$1, x_1, x_2, x_3, x_1^2, x_1x_2, x_1x_3, x_2^2, x_2x_3, x_3^2, \ldots$$

- ordering
- $\deg(x_1^2) = \deg(x_2x_3) = 2$

Example

$$f_1 = 2.76\, x_1^2 - 5.51\, x_1\, x_3$$

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Building blocks of multivariate polynomials?

Monomials!

$$1, x_1, x_2, x_3, x_1^2, x_1x_2, x_1x_3, x_2^2, x_2x_3, x_3^2, \ldots$$

- ordering
- $\deg(x_1^2) = \deg(x_2x_3) = 2$

Example

$$f_1 = 2.76\, x_1^2 - 5.51\, x_1\, x_3 - 1.12\, x_1$$

Building blocks of multivariate polynomials?

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$$1, x_1, x_2, x_3, x_1^2, x_1x_2, x_1x_3, x_2^2, x_2x_3, x_3^2, \ldots$$

- ordering
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Example

$$f_1 = 2.76\, x_1^2 - 5.51\, x_1\, x_3 - 1.12\, x_1 + 1.99$$

Building blocks of multivariate polynomials?

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$$1, x_1, x_2, x_3, x_1^2, x_1x_2, x_1x_3, x_2^2, x_2x_3, x_3^2, \ldots$$

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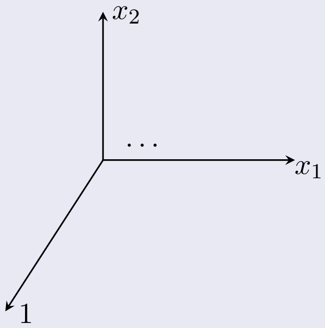
Example

$$f_1 = 2.76\, x_1^2 - 5.51\, x_1\, x_3 - 1.12\, x_1 + 1.99$$

degree of $f_1 = \deg(f_1) = 2$

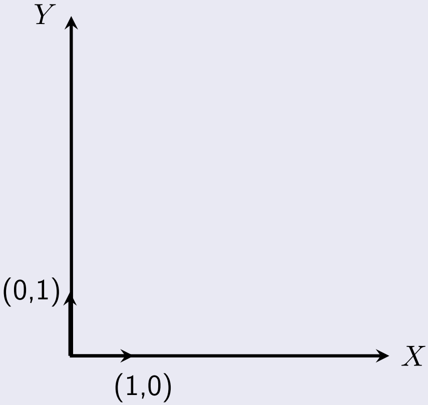
Vector Representation

Each monomial corresponds with a vector, each orthogonal with respect to all the others:



C_d^n: vector space of all polynomials in n variables with complex coefficients up to a degree d

A blast from the past



Each monomial is described by a coefficient vector:

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$$1 \sim \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \ldots \end{pmatrix}$$

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$$\begin{array}{ccc} 1 & \sim & \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots \end{pmatrix} \\ x_1 & \sim & \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \dots \end{pmatrix} \end{array}$$

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Each monomial is described by a coefficient vector:

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Each monomial is described by a coefficient vector:

$$\begin{array}{ccccccccc}
1 & \sim & (& 1 & 0 & 0 & 0 & 0 & \dots &) \\
x_1 & \sim & (& 0 & 1 & 0 & 0 & 0 & \dots &) \\
x_2 & \sim & (& 0 & 0 & 1 & 0 & 0 & \dots &) \\
x_3 & \sim & (& 0 & 0 & 0 & 1 & 0 & \dots &) \\
x_1^2 & \sim & (& 0 & 0 & 0 & 0 & 1 & \dots &) \\
\vdots & & & & & & & & &
\end{array}$$

Coefficient vector of multivariate polynomial

$$f_1 = 2.76\, x_1^2 - 5.51\, x_1\, x_3 - 1.12\, x_1 + 1.99$$

Coefficient vector of multivariate polynomial

$$
\begin{array}{rcl}
f_1 & = & 2.76\, x_1^2 - 5.51\, x_1\, x_3 - 1.12\, x_1 + 1.99 \\
& \sim & 2.76 \left(\begin{array}{cccccccccc} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{array} \right) \\
& & -5.51 \left(\begin{array}{cccccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{array} \right) \\
& & -1.12 \left(\begin{array}{cccccccccc} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right) \\
& & +1.99 \left(\begin{array}{cccccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right)
\end{array}
$$

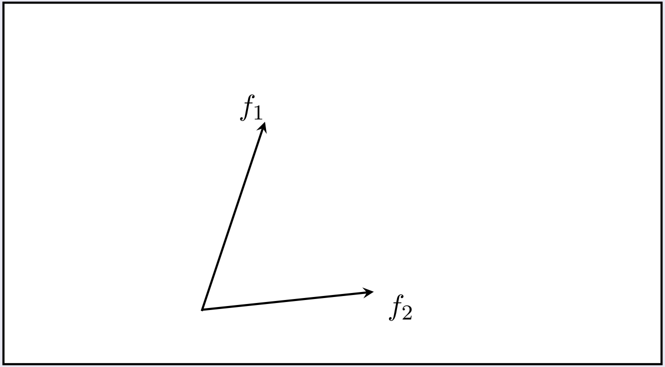
Coefficient vector of multivariate polynomial

$$
\begin{aligned}
f_1 &= 2.76\, x_1^2 - 5.51\, x_1\, x_3 - 1.12\, x_1 + 1.99 \\
&\sim 2.76 \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\
&\quad -5.51 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \\
&\quad -1.12 \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\
&\quad +1.99 \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}
\end{aligned}
$$

$$
f_1 \sim \begin{pmatrix} 1.99 & -1.12 & 0 & 0 & 2.76 & 0 & -5.51 & 0 & 0 & 0 \end{pmatrix}
$$

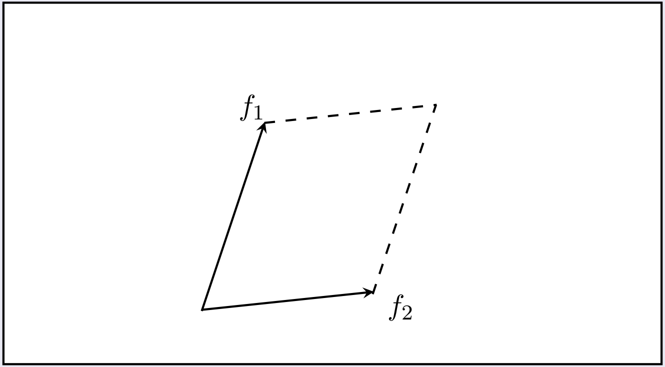
Addition of Polynomials

Addition of vectors:



Addition of Polynomials

Addition of vectors:



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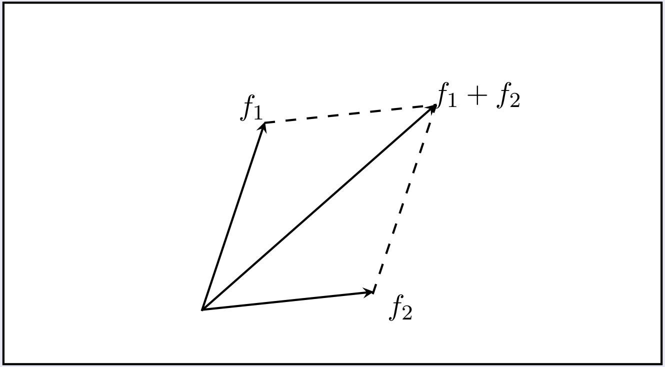
Addition of vectors:



Multiplication of 2 multivariate polynomials $h, f \in C_d^n$

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$$f \times h$$

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$$f \times h$$

$$= f \times (h_0 + h_1 x_1 + h_2 x_2 + \ldots + h_q x_n^{d_h})$$

Multiplication of 2 multivariate polynomials $h, f \in C_d^n$

$$f \times h$$

$$= f \times (h_0 + h_1 x_1 + h_2 x_2 + \ldots + h_q x_n^{d_h})$$

$$= h_0 f + h_1 x_1 f + h_2 x_2 f + \ldots + h_q x_n^{d_h} f$$

Multiplication of 2 multivariate polynomials $h, f \in C_d^n$

$$
\begin{aligned}
& f \times h \\
= \quad & f \times (h_0 + h_1 x_1 + h_2 x_2 + \ldots + h_q x_n^{d_h}) \\
= \quad & h_0 f + h_1 x_1 f + h_2 x_2 f + \ldots + h_q x_n^{d_h} f \\
\sim \quad & \begin{pmatrix} h_0 & h_1 & h_2 & \ldots & h_q \end{pmatrix} \begin{pmatrix} f \\ x_1 f \\ x_2 f \\ \vdots \\ x_n^{d_h} f \end{pmatrix}
\end{aligned}
$$

Multiplication of 2 multivariate polynomials $h, f \in C_d^n$

$$f \times h$$

$$= f \times (h_0 + h_1 x_1 + h_2 x_2 + \ldots + h_q x_n^{d_h})$$

$$= h_0 f + h_1 x_1 f + h_2 x_2 f + \ldots + h_q x_n^{d_h} f$$

$$\sim \begin{pmatrix} h_0 & h_1 & h_2 & \ldots & h_q \end{pmatrix} \begin{pmatrix} f \\ x_1 f \\ x_2 f \\ \vdots \\ x_n^{d_h} f \end{pmatrix}$$

$$\sim h \, M_f$$

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Multiplication Example

$f = x_1x_2 - x_2$ and $h = x_1^2 + 2x_2 - 9$.

$$h\ M_f = \begin{pmatrix} -9 & 0 & 2 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} f \\ x_1 f \\ x_2 f \\ x_1^2 f \\ x_1x_2 f \\ x_2^2 f \end{pmatrix}.$$

Multiplication Example

M_f =

	1	x_1	x_2	x_1^2	x_1x_2	x_2^2	x_1^3	$x_1^2x_2$	$x_1x_2^2$	x_2^3	x_1^4	$x_1^3x_2$	$x_1^2x_2^2$	$x_1x_1^3$	x_2^4
f	0	0	−1	0	1	0	0	0	0	0	0	0	0	0	0
x_1f	0	0	0	0	−1	0	0	1	0	0	0	0	0	0	0
x_2f	0	0	0	0	0	−1	0	0	1	0	0	0	0	0	0
x_1^2f	0	0	0	0	0	0	0	−1	0	0	0	1	0	0	0
x_1x_2f	0	0	0	0	0	0	0	0	−1	0	0	0	1	0	0
x_2^2f	0	0	0	0	0	0	0	0	0	−1	0	0	0	1	0

Multiplication Example

$$M_f =$$

$$
\begin{array}{c} \\ f \\ x_1 f \\ x_2 f \\ x_1^2 f \\ x_1 x_2 f \\ x_2^2 f \end{array}
\begin{array}{c}
\begin{array}{ccccccccccccccc} 1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 & x_1^3 & x_1^2x_2 & x_1x_2^2 & x_2^3 & x_1^4 & x_1^3x_2 & x_1^2x_2^2 & x_1x_1^3 & x_2^4 \end{array} \\
\begin{pmatrix}
0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}
\end{array}
$$

$$hM_f =$$

$$
\begin{array}{c}
\begin{array}{ccccccccccccccc} 1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 & x_1^3 & x_1^2x_2 & x_1x_2^2 & x_2^3 & x_1^4 & x_1^3x_2 & x_1^2x_2^2 & x_1x_1^3 & x_2^4 \end{array} \\
\begin{pmatrix}
0 & 0 & 9 & 0 & -9 & -2 & 0 & -1 & 2 & 0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}
\end{array}
$$

Multiplication Example

$$M_f \; =$$

$$\begin{array}{c} \\ f \\ x_1 f \\ x_2 f \\ x_1^2 f \\ x_1 x_2 f \\ x_2^2 f \end{array} \begin{array}{c} \begin{array}{ccccccccccccccc} 1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 & x_1^3 & x_1^2x_2 & x_1x_2^2 & x_2^3 & x_1^4 & x_1^3x_2 & x_1^2x_2^2 & x_1x_1^3 & x_2^4 \end{array} \\ \begin{pmatrix} 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

$$hM_f \; =$$

$$\begin{array}{c} \begin{array}{ccccccccccccccc} 1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 & x_1^3 & x_1^2x_2 & x_1x_2^2 & x_2^3 & x_1^4 & x_1^3x_2 & x_1^2x_2^2 & x_1x_1^3 & x_2^4 \end{array} \\ \begin{pmatrix} 0 & 0 & 9 & 0 & -9 & -2 & 0 & -1 & 2 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

$$\sim 9x_2 - 9x_1x_2 - 2x_2^2 - x_1^2x_2 + 2x_1x_2^2 + x_1^3x_2$$

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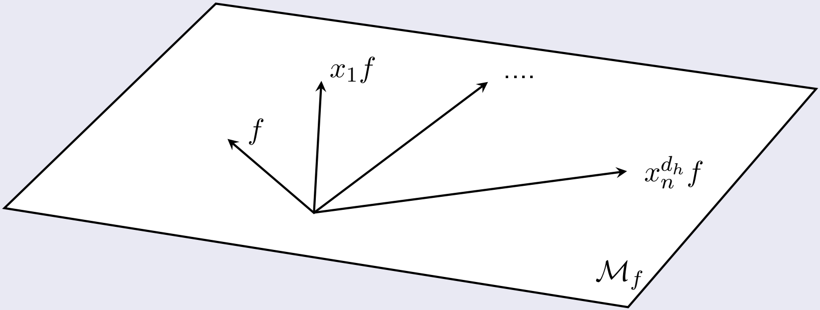
Multiplication of Polynomials

Every possible multiplication of f lies in a vector space $\mathcal{M}_f$ spanned by $f, x_1 f, x_2 f, \ldots$



Definition multivariate polynomials

Fix any monomial order $>$ on C_d^n and let $F = (f_1, \ldots, f_s)$ be a s-tuple of polynomials in C_d^n. Then every $p \in C_d^n$ can be written as

$$p \;=\; h_1 f_1 + \ldots + h_s f_s + r$$

where $h_i, r \in C_d^n$. For each $i, h_i f_i = 0$ or $\mathrm{LM}(p) \geq \mathrm{LM}(h_i f_i)$, and either $r = 0$, or r is a linear combination of monomials, none of which is divisible by any of $\mathrm{LM}(f_1), \ldots, \mathrm{LM}(f_s)$.

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Differences with division of numbers

- Remainder r depends on the way we order monomials
- Dividends $h_1, \ldots, h_s$ and remainder r depend on order of divisors $f_1, \ldots, f_s$

Describing the quotient

$$p = h_1 f_1 + \ldots + h_s f_s + r$$

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$$\begin{pmatrix} h_{10} & h_{11} & h_{12} & \ldots & h_{1q} \end{pmatrix} \begin{pmatrix} f_1 \\ x_1\, f_1 \\ x_2\, f_1 \\ \vdots \\ x_n^{d_1}\, f_1 \end{pmatrix}$$

Describing the quotient

$$p = h_1 f_1 + \ldots + h_s f_s + r$$

$$\begin{pmatrix} h_{k0} & h_{k1} & h_{k2} & \ldots & h_{kw} \end{pmatrix} \begin{pmatrix} f_k \\ x_1\, f_k \\ x_2\, f_k \\ \vdots \\ x_n^{d_k}\, f_k \end{pmatrix}$$

Describing the quotient

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Describing the quotient

$$p = h_1 f_1 + \ldots + h_s f_s + r$$

$$\begin{pmatrix} h_{10} & h_{11} & h_{12} & \ldots & h_{1q} & h_{20} & h_{21} & \ldots & h_{sv} \end{pmatrix} \begin{pmatrix} f_1 \\ x_1 \, f_1 \\ x_2 \, f_1 \\ \vdots \\ x_n^{d_1} \, f_1 \\ f_2 \\ x_1 \, f_2 \\ \vdots \\ x_n^{d_s} \, f_s \end{pmatrix}$$

Divisor Matrix D

Given a set of polynomials $f_1, \ldots, f_s \in C_d^n$, each of degree d_i $(i = 1 \ldots s)$ and a polynomial $p \in C_d^n$ of degree d then the divisor matrix D is given by

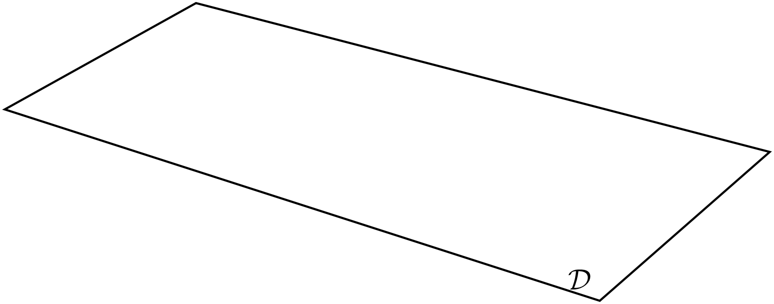
$$D = \begin{pmatrix} f_1 \\ x_1 f_1 \\ x_2 f_1 \\ \vdots \\ x_n^{d_1} f_1 \\ f_2 \\ x_1 f_2 \\ \vdots \\ x_n^{d_s} f_s \end{pmatrix}$$

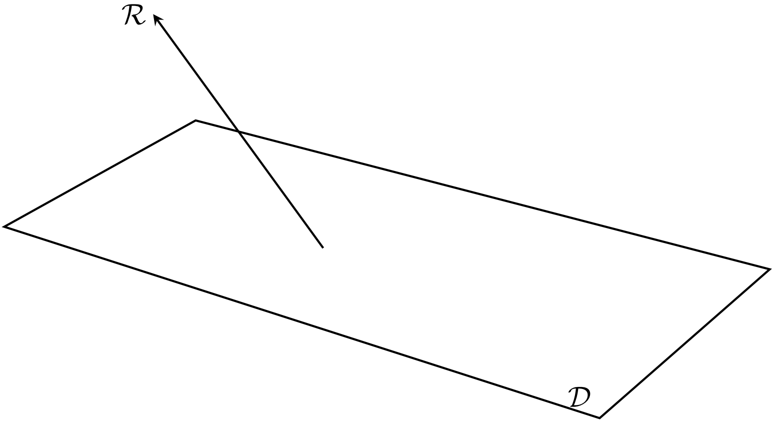
where each polynomial f_i is multiplied with all monomials x^{α_i} from degree 0 up to degree $k_i = \deg(p) - \deg(f_i)$ such that $x^{\alpha_i}\, \mathrm{LM}(f_i) \leq \mathrm{LM}(p)$.

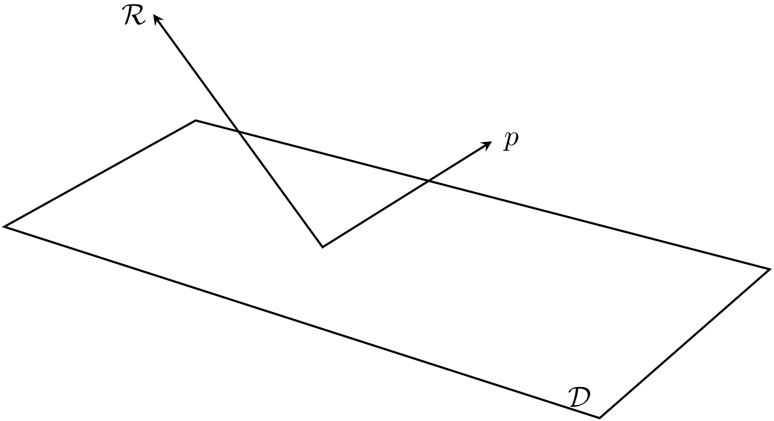
Example Divisor Matrix

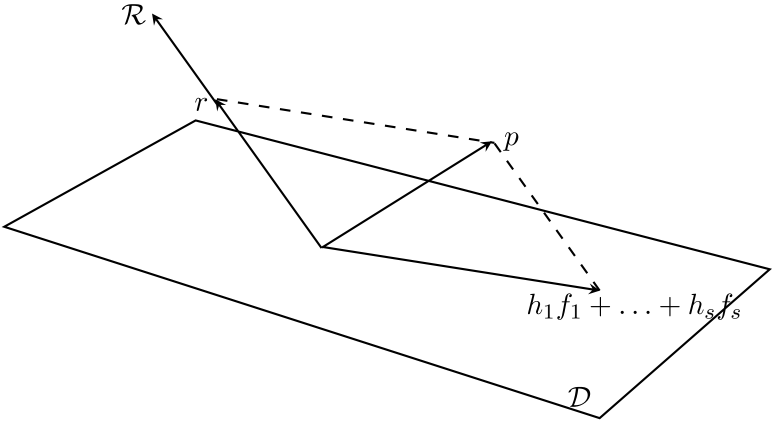
To divide $p = 4 + 5x_1 - 3x_2 - 9x_1^2 + 7x_1x_2$ by $f_1 = -2 + x_1 + x_2$, $f_2 = 3 - x_1$:

$$D = \begin{array}{c} \\ f_1 \\ x_1 f_1 \\ f_2 \\ x_1 f_2 \\ x_2 f_2 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & x_1 & x_2 & x_1^2 & x_1x_2 \end{array} \\ \begin{pmatrix} -2 & 1 & 1 & 0 & 0 \\ 0 & -2 & 0 & 1 & 1 \\ 3 & -1 & 0 & 0 & 0 \\ 0 & 3 & 0 & -1 & 0 \\ 0 & 0 & 3 & 0 & -1 \end{pmatrix} \end{array}$$


$\mathcal{D}$


$\mathcal{R}$
$\mathcal{D}$


$\mathcal{R}$
p
$\mathcal{D}$


$\mathcal{R}$
r
p
$h_1f_1 + \ldots + h_sf_s$
$\mathcal{D}$

Outline

"Advanced" operations on polynomials

- Eliminate variables
- Compute a least common multiple of 2 multivariate polynomials
- Compute a greatest common divisor of 2 multivariate polynomials

One More Key Player:

Macaulay matrix

Macaulay Matrix

Given a set of multivariate polynomials $f_1, \ldots, f_s$, each of degree $d_i (i = 1 \ldots s)$ then the Macaulay matrix of degree d is given by

$$M(d) = \begin{pmatrix} f_1 \\ x_1 f_1 \\ \vdots \\ x_n^{d-d_1} f_1 \\ f_2 \\ x_1 f_2 \\ \vdots \\ x_n^{d-d_s} f_s \end{pmatrix}$$

where each polynomial f_i is multiplied with all monomials up to degree $d - d_i$ for all $i = 1 \ldots s$.

Row space of the Macaulay matrix

$$\mathcal{M}_d = \{h_1 f_1 + h_2 f_2 + \ldots + h_s f_s \mid \text{for all possible } h_1, h_2, \ldots, h_s \text{ with degrees } d - d_1, d - d_2, \ldots, d - d_s \text{ respectively}\}$$

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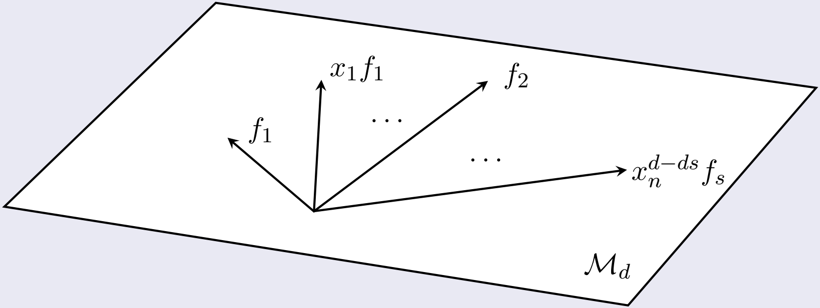

For the following polynomial system:

$$\begin{cases} f_1: & x_1x_2 - 2x_2 & = & 0 \\ f_2: & x_2 - 3 & = & 0 \end{cases}$$

the Macaulay matrix of degree 3 is

$$M(3) = \begin{array}{c} \\ f_1 \\ x_1 f_1 \\ x_2 f_1 \\ f_2 \\ x_1 f_2 \\ x_2 f_2 \\ x_1^2 f_2 \\ x_1x_2 f_2 \\ x_2^2 f_2 \end{array} \begin{array}{c} \begin{array}{cccccccccc} 1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 & x_1^3 & x_1^2x_2 & x_1x_2^2 & x_2^3 \end{array} \\ \left(\begin{array}{cccccccccc} 0 & 0 & -2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 1 & 0 \\ -3 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 1 \end{array} \right) \end{array}$$

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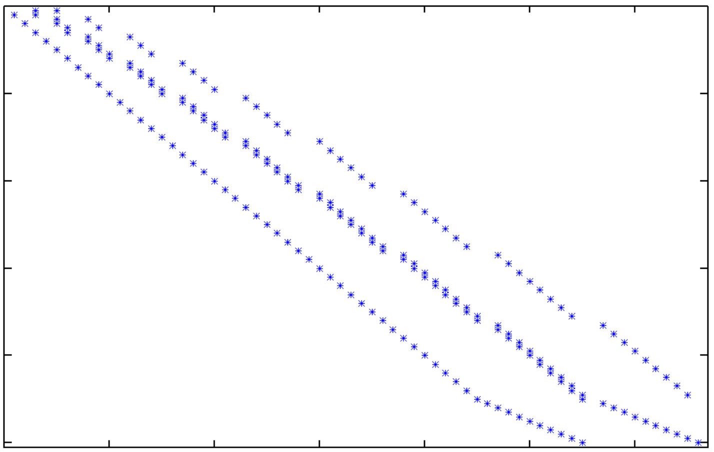
Sparsity pattern $M(10)$



Elimination Problem

Given a set of multivariate polynomials $f_1, \ldots, f_s$ and $x_e \subsetneq \{x_1, \ldots, x_n\}$. Find a polynomial $g = h_1 f_1 + \ldots + h_s f_s$ that does not contain any of the x_e variables.

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Example

From the following polynomial system in 3 variables x_1, x_2, x_3:

$$\begin{cases} f_1 & = & x_1^2 + x_2 + x_3 - 1, \\ f_2 & = & x_1 + x_2^2 + x_3 - 1, \\ f_3 & = & x_1 + x_2 + x_3^2 - 1, \end{cases}$$

we want to find a $g = h_1 f_1 + h_2 f_2 + h_3 f_3$ only in x_3.

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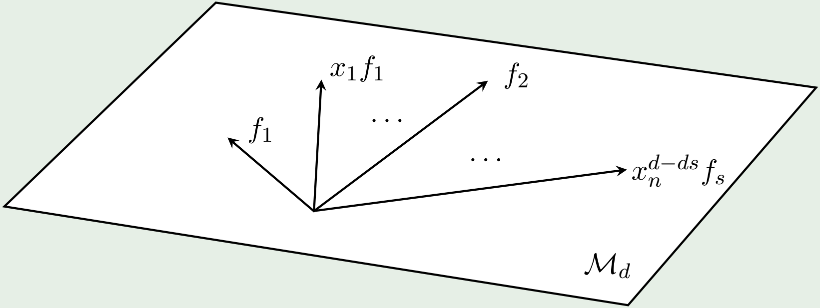
Example

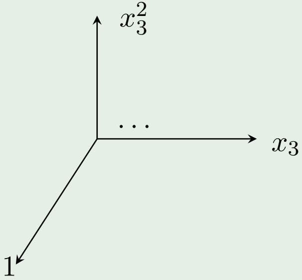
Since $g = h_1 f_1 + h_2 f_2 + h_3 f_3$, it lies in



for a certain degree d.

Example

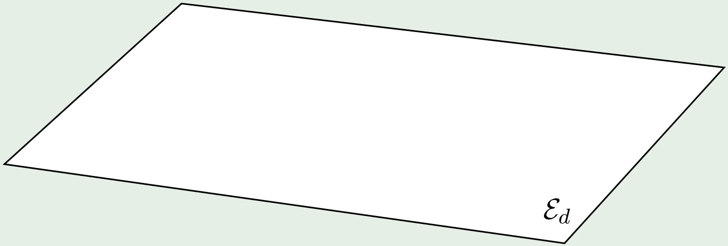
Also, since g only contains the variables x_3, it is built up from the monomial basis



up to a certain degree d.

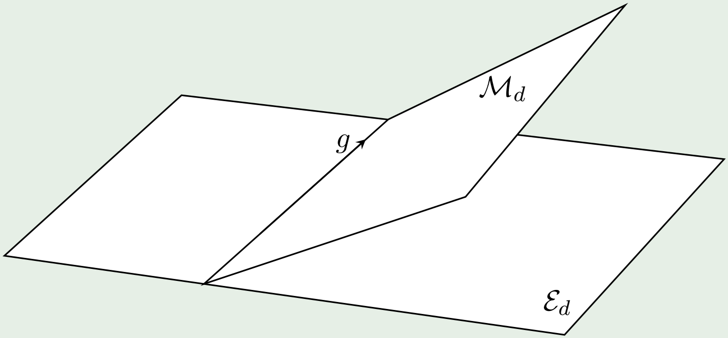
Example

We will call this vector space that is spanned by the variables x_3 $\mathcal{E}_d$:



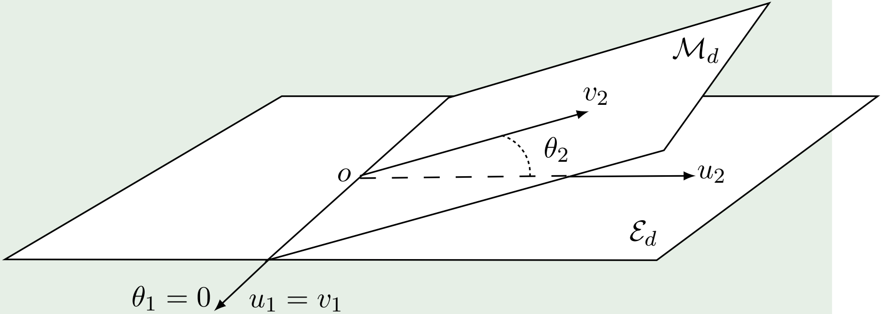
Example

$g \in \mathcal{M}_d$ and $g \in \mathcal{E}_d$; hence g lies in the intersection $\mathcal{M}_d \cap \mathcal{E}_d$:



for some particular degree d.

Finding the intersection



Example

We revisit

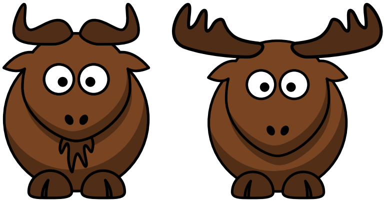
$$\begin{cases} x_1^2 + x_2 + x_3 & = & 1, \\ x_1 + x_2^2 + x_3 & = & 1, \\ x_1 + x_2 + x_3^2 & = & 1. \end{cases}$$

- we eliminate both x_1 and x_2
  - $d = 6$,
  - $g(x_3) = x_3^2 - 4x_3^3 + 4x_3^4 - x_3^6$.
- we eliminate x_2:
  - $d = 2$,
  - $g(x_1, x_3) = x_1 - x_3 - x_1^2 + x_3^2$.

Least Common Multiple

A multivariate polynomial l is called a least common multiple (LCM) of 2 multivariate polynomials f_1, f_2 if

1. f_1 divides l and f_2 divides l.
2. l divides any polynomial which both f_1 and f_2 divide.



f_1 f_2

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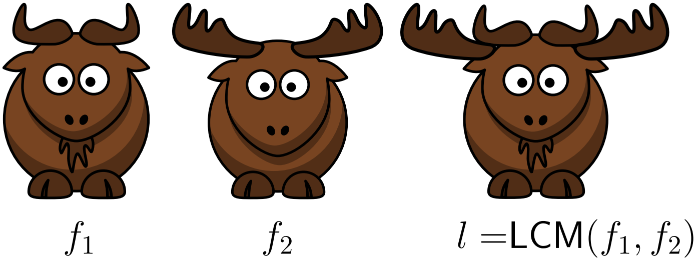

Finding the LCM

The LCM l of f_1 and f_2 satisfies:

$$\mathrm{LCM}(f_1, f_2) \triangleq l \; = \; f_1\, h_1 \; = \; f_2\, h_2$$

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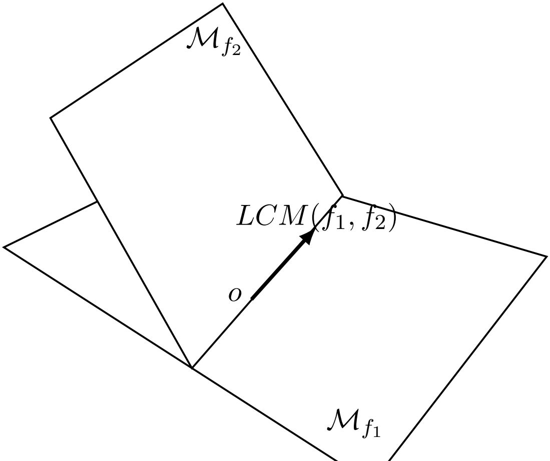

Greatest Common Divisor

A multivariate polynomial g is called a greatest common divisor of 2 multivariate polynomials f_1 and f_2 if

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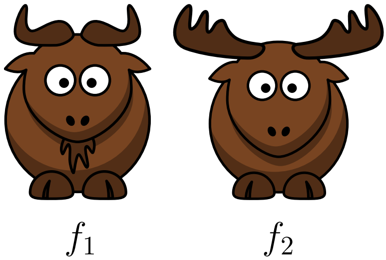

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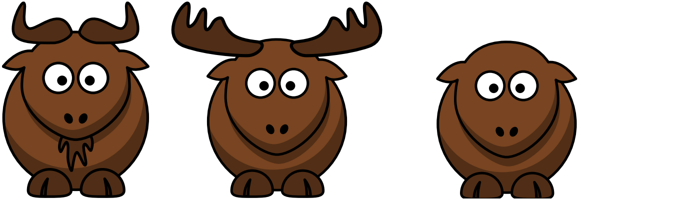

f_1 $\quad$ f_2 $\quad$ $g = \text{GCD}(f_1, f_2)$

Finding the GCD

Remember that

$$\mathrm{LCM}(f_1, f_2) \triangleq l \; = \; f_1\, h_1 \; = \; f_2\, h_2.$$

We also have that

$$f_1 \; f_2 \; = \; l \; g,$$

with $g \triangleq \mathrm{GCD}(f_1, f_2)$.

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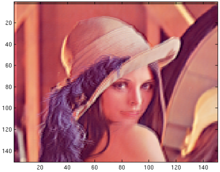
Answer:

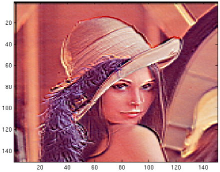
$$g = \frac{f_1 \, f_2}{l} = \frac{f_1}{h_2} = \frac{f_2}{h_1}.$$

Blind Image Deconvolution

- $F_1(z_1, z_2) = I(z_1, z_2)\, D_1(z_1, z_2) + N_1(z_1, z_2)$
- $F_2(z_1, z_2) = I(z_1, z_2)\, D_2(z_1, z_2) + N_2(z_1, z_2)$
- $I(z_1, z_2) = \tau\text{-GCD}(F_1, F_2)$



$F_1(z_1, z_2)$



$F_2(z_1, z_2)$

Blind Image Deconvolution

- $F_1(z_1, z_2) = I(z_1, z_2)\, D_1(z_1, z_2) + N_1(z_1, z_2)$
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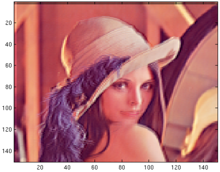
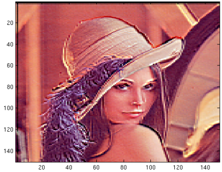


$F_1(z_1, z_2)$



$F_2(z_1, z_2)$

$\tau\text{-GCD}(F_1, F_2)$

Other Operations worked out in the thesis

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- Computing a Gröbner basis of multivariate polynomials $f_1, \ldots, f_s$

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- Removing multiplicities of solutions of $f_1, \ldots, f_s$
- Counting total number of affine solutions of $f_1, \ldots, f_s$

Other Operations worked out in the thesis

- Computing a Gröbner basis of multivariate polynomials $f_1, \ldots, f_s$
- Describing all syzygies of multivariate polynomials $f_1, \ldots, f_s$
- Removing multiplicities of solutions of $f_1, \ldots, f_s$
- Counting total number of affine solutions of $f_1, \ldots, f_s$
- Solving the ideal membership problem

Outline

Conclusions

- Numerical Linear Algebra Framework
- Addition, Multiplication
- Polynomial Division and oblique projections
- Elimination and intersection of vector spaces
- LCM and GCD's
- syzygy analysis, counting affine solutions, removing multiplicities of solutions, ...

Future Research/Work

- Exploit sparsity + structure matrices
- Numerical Analysis:
  - Polynomial division
  - Intersection of vector spaces
  - Numerical rank
- Open problems:
  - Modelling higher dimensional solution sets
  - Full understanding of roots at infinity
  - ...


THANK
YOU