Dissipative Dynamical Systems Part I: General Theory

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Communicated by C. TRUESDELL

Contents

Abstract

The first part of this two-part paper presents a general theory of dissipative dynamical systems. The mathematical model used is a state space model and dissipativeness is defined in terms of an inequality involving the storage function and the supply function. It is shown that the storage function satisfies an *a priori* inequality: it is bounded from below by the available storage and from above by the required supply. The available storage is the amount of internal storage which may be recovered from the system and the required supply is the amount of supply which has to be delivered to the system in order to transfer it from the state of minimum storage to a given state. These functions are themselves possible storage functions, *i.e.*, they satisfy the dissipation inequality. Moreover, since the class of possible storage functions forms a convex set, there is thus a continuum of possible storage functions ranging from its lower bound, the available storage, to its upper bound, the required supply. The paper then considers interconnected systems. It is shown that dissipative systems which are interconnected via a neutral interconnection constraint define a new dissipative dynamical system and that the sum of the storage functions of the individual subsystems is a storage function for the interconnected system. The stability of dissipative systems is then investigated

and it is shown that a point in the state space where the storage function attains a local minimum defines a stable equilibrium and that the storage function is a Lyapunov function for this equilibrium. These results are then applied to several examples. These concepts and results will be applied to linear dynamical systems with quadratic supply rates in the second part of this paper.

1. Introduction

Dissipative systems are of particular interest in engineering and physics. The dissipation hypothesis, which distinguishes such systems from general dynamical systems, results in a fundamental constraint on their dynamic behavior. Typical examples of dissipative systems are electrical networks in which part of the electrical energy is dissipated in the resistors in the form of heat, viscoelastic systems in which viscous friction is responsible for a similar loss in energy, and thermodynamic systems for which the second law postulates a form of dissipation leading to an increase in entropy.

In the first part of this paper we hope to provide an axiomatic foundation for a general theory of dissipative systems. In the course of doing this we examine the concepts of an internal storage function and of a dissipation function.

There will be an obvious search for generality in the theoretical discussion of the first part of this paper. This stems from a belief that in studying specialized classes of dynamical systems it is important to keep the axioms separated. Such a procedure has more than just an aesthetic appeal: it allows one to pinpoint clearly what is a consequence of what.

My interest in dissipative systems stems from their implications on the stability of control systems. One of the main results in stability theory states that a feedback system consisting of a passive dynamical system in both the forward and the feedback loop is itself passive and thus stable. Moreover, the sum of the stored "energies" in the forward loop and in the feedback loop is a Lyapunov function for the closed loop system. The existence of a stored energy function is rather simple to eastablish since it is equivalent to the passivity assumption. It was in computing this stored energy function that we encountered some difficulties. It became clear that there is no uniqueness of the stored energy function, rather that there is a range of possible stored energy functions for a system with a prescribed input/output behavior.

In this paper these concepts are studied in detail and generalized. The terminology *dissipative* will be used as a generalization of the concept of passivity and *storage function* as a generalization of the concept of stored energy or entropy.

One of the main results obtained in this paper is that the storage function is as a rule not uniquely defined by the input/output behavior. It is shown that the storage function associated with a dissipative dynamical system satisfies an *a priori* inequality: it is bounded from below by the available storage and from above by the required supply. Moreover, and possibly more important, there is a continuum of possible storage functions between these upper and lower bounds.

This situation has important consequences. To give but one example, consider the familiar area of linear viscoelasticity. This is a typical example of a situation where the internal physical mechanism which is responsible for a stress/strain relationship is admittedly not completely understood. For many applications, one is, however, satisfied with an input/output description in terms of a relaxation function which may be obtained experimentally. Such an input/output description has, in fact, become the starting point of a general approach to the description of materials with memory. Nevertheless, the literature insists on postulating the knowledge of an internal energy function. It should be realized that this destroys some of the advantages of working with an input/output description since this knowledge of an internal energy function cannot be obtained from the relaxation function but requires additional information about the physical process. (In the present example one may often circumvent this difficulty by determining the heat production as well as the stress/strain relation, but this problem remains very fundamental in the context of thermodynamic systems where it is unclear what is being dissipated while the entropy increases.)

There are several methods for further reducing the number of possible storage functions. One rather obvious method is to consider a system as an interconnection of dissipative subsystems. Another possibility is by assuming additional qualitative internal properties for the system. A typical example is by *postulating* internal symmetry conditions as the Onsager-Casimir reciprocal relations. These will be examined in the second part of the paper.

We shall use the state space formalism for representing systems with memory. This feature is felt to be essential and the absence of the state space formalism in continuum mechanics and thermodynamics is somewhat disturbing. It is indeed customary in these areas to assume that the functionals appearing in the constitutive equations of materials with memory may depend on the entire past history (see for example [1] and [2]). This approach, however, does not recognize the idea of "equivalent histories": two histories are said to be *equivalent* if they bring the system into the same state and are thus indistinguishable under future experiments. Hence, one should constrain *a priori* the constitutive relations of any internal function as, for example, the internal energy or the entropy to take on the same value for equivalent (but not necessarily identical) histories. The state space formalism is the natural way for incorporating this constraint. There has, in fact, been some recent work by ONAT [3, 4] which deals with the construction of state space models for continuum systems.

We consider this paper as a contribution to mathematical system theory. The methods employed are those which have grown out of the modern developments of control theory; some of the auxiliary results, particularly in the second part of the paper, are drawn from network synthesis and optimal control theory. The implications of the results obtained and the methods used ought to be of interest to physicists, in particular those concerned with continuum mechanics and thermodynamics. We have tried to make the paper self-contained by being as explicit as possible whenever known results are being used.

2. Dynamical Systems

A dynamical system is viewed as an abstract mathematical object which maps *inputs* (causes, excitations) into *outputs* (effects, responses) via a set of intermediate variables, the *state*, which summarizes the influence of past inputs. The following lengthy definition is concerned with continuous systems (the time-interval of definition is the real line). In order to avoid unnecessary complications mainly of a notational nature, we will restrict ourselves to stationary (*i.e.*, time-invariant, nonaging) systems. The time-varying case is briefly discussed in Section 6.

Definition 1. A (continuous stationary) dynamical system Σ is defined through the sets U, $\mathcal{U}$, Y, $\mathcal{Y}$, X and the maps ϕ and r. These satisfy the following axioms:

(i) $\mathcal{U}$ is called the *input space* and consists of a class of U-valued functions on* R. The set U is called the set of *input values*. The space $\mathcal{U}$ is assumed to be closed under the shift operator, *i.e.*, if $u \in \mathcal{U}$ then the function u_T defined by $u_T(t) = u(t+T)$ also belongs to $\mathcal{U}$ for any $T \in R$;

(ii) $\mathcal{Y}$ is called the *output space* and consists of a class of Y-valued functions on R. The set Y is called the set of *output values*. The space $\mathcal{Y}$ is assumed to be closed under the shift operator, *i.e.*, if $y \in \mathcal{Y}$ then the function y_T defined by $y_T(t) = y(t+T)$ belongs to $\mathcal{Y}$ for any $T \in R$;

(iii) X is an abstract set called the *state space*;

(iv) ϕ is called the *state transition function* and is a map from $R_2^+ \times X \times \mathcal{U}$ into X. It obeys the following axioms:

- (iv)$_a$ (*consistency*): $\phi(t_0, t_0, x_0, u) = x_0$ for all $t_0 \in R$, $x_0 \in X$, and $u \in \mathcal{U}$;
- (iv)$_b$ (*determinism*): $\phi(t_1, t_0, x_0, u_1) = \phi(t_1, t_0, x_0, u_2)$ for all $(t_1, t_0) \in R_2^+$, $x_0 \in X$, and $u_1, u_2 \in \mathcal{U}$ satisfying $u_1(t) = u_2(t)$ for $t_0 \leqq t \leqq t_1$;
- (iv)$_c$ (*semi-group property*): $\phi(t_2, t_0, x_0, u) = \phi(t_2, t_1, \phi(t_1, t_0, x_0, u), u)$ for all $t_0 \leqq t_1 \leqq t_2$, $x_0 \in X$, and $u \in \mathcal{U}$;
- (iv)$_d$ (*stationarity*): $\phi(t_1 + T, t_0 + T, x_0, u_T) = \phi(t_1, t_0, x_0, u)$ for all $(t_1, t_0) \in R_2^+$, $T \in R$, $x_0 \in X$, and $u, u_T \in \mathcal{U}$ related by $u_T(t) = u(t+T)$ for all $t \in R$;

(v) r is called the *read-out function* and is a map from $X \times U$ into Y;

(vi) the Y-valued function $r(\phi(t, t_0, x_0, u), u(t))$ defined for $t \geqq t_0$ is, for all $x_0 \in X$, $t_0 \in R$ and $u \in \mathcal{U}$, the restriction to $[t_0, \infty)$ of a function $y \in \mathcal{Y}$. This means that there exists an element $y \in \mathcal{Y}$ such that $y(t) = r(\phi(t, t_0, x_0, u), u(t))$ for $t \geqq t_0$.

A dynamical system thus generates outputs from inputs as follows: the system starts off in some initial state x_0 at time t_0 and an input u is applied to it. Then the state at time t_1 is given by $\phi(t_1, t_0, x_0, u)$. The output resulting from this experiment is given by $y(t) = r(\phi(t, t_0, x_0, u), u(t))$ and is defined for $t \geqq t_0$. It is important (for applications to systems described by partial differential equations for example) to realize that state transitions, and thus outputs, need only be defined in the forward time direction.

We call $\phi(t_1, t_0, x_0, u)$ "*the state at time t_1 reached from the initial state x_0 at time t_0 by applying the input u to the dynamical system Σ*" *and* $r(x, u)$ "*the output*

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* We are using the following notation: R = the real numbers; R^n = n-dimensional Euclidean space; R^+ = the nonnegative real numbers; R_2^+ = the causal triangular sector of R_2 defined by $R_2^+ = \{(t_2, t_1) \in R_2 \mid t_2 \geqq t_1\}$; R^e = the extended real number system $= \{-\infty\} \cup R \cup \{+\infty\}$.

due to the presence of state x and the input-value u". We will denote the function $r(\phi(t, t_0, x_0, u), u(t))$ defined for $t \geqq t_0$ unambiguously by $y(t_0, x_0, u)$.

Definition 1 is precise and yet very general. By a suitable choice of the state space, the state transition function, and the read-out function, it includes all common deterministic models used in classical physics, in circuit theory, and control theory.

The axiom of determinism is the crucial one. It expresses at the same time a fundamental property of the state and an important restriction on the class of systems which qualify for dynamical systems in the above sense. It states that the initial state summarizes the effect of past inputs in the sense that for future responses it does not matter how the system was brought into this state; it also implies that the state and thus the output before some time are not influenced by the values of the input after that time. We are hence in effect restricting our attention to systems in which future inputs do not affect past and present outputs. The idea is simple: since all experimental evidence indicates that physical systems indeed satisfy this property of causality, we require this to be preserved in the model.

It should be emphasized that the read-out function is required to be a memoryless map in the sense that the output only depends on the *present* value of the state and the input. All dynamical effects (*i.e.*, those phenomena involving memory) are required to be taken care of by the state.

The above definition is commonly used in mathematical system theory (see, for instance, references [5, 6]). Although physicists have been groping for a similar concept for a long time, it is only for systems in which the input space consists of only one element (*i.e.*, the autonomous dynamical systems of classical mechanics) that such mathematical structures have been introduced in a formal way. In the framework of Definition 1 the state at every moment completely describes the present situation. It is, however, impossible to deduce *a priori*, in physical terms, what will be the state. This, indeed, is a very difficult problem even for relatively simple systems, and it appears to be the cause for much of the reluctance of introducing this concept in physics. The approach which has been taken for describing materials with memory is to allow the outputs to be a function of the whole past history of the input. This is particularly prominent in the pioneering work of TRUESDELL, COLEMAN, and NOLL [1, 2]. Another approach is that of ONAT [3, 4] where the state is constructed in terms of observables. These two extreme points of view are particular cases of Definition 1, but we see no compelling reason to adhere to either of them. The first approach does not recognize the idea of equivalent histories, and the second approach will lead to difficulties when we consider isolated systems for example.

In view of this dichotomy, it would appear to be useful to allow some time discussing these state space concepts further. Let us take the point of view that all the information the experimenter may obtain about a system is a table of input functions in $\mathscr{U}$ versus the corresponding output functions in $\mathscr{Y}$. The so-called *problem of realization* is to define a state space X and the functions ϕ and r in such a way that the resulting dynamical system in state space form generates the given input/output pairs by a suitable choice for the initial state in each tabulated experiment. This problem has attracted a great deal of attention in the literature. Both the questions, "Does a state space realization exist?" and "What are the maps ϕ and r?", have been examined. For the first question we mention the work of ZADEH [7] and for the second question the work by YOULA [8] and, especially, KALMAN [5], among others. The existence question essentially only requires a determinism postulate on the input/output pairs. The construction of ϕ and r is understandably much more intricate but has been satisfactorily resolved for large classes of systems. In particular, there exists a very elegant solution to this problem for linear systems with a finite number of degrees of freedom. This material is considered to be of prime importance and can be found in a number of recent texts (*e.g.*, [10]).

We now consider an important particular case of this realization problem. Assume that F is a given map from $\mathscr{U}$ into $\mathscr{Y}$ satisfying the postulate of determinism which states that inputs $u_1, u_2 \in \mathscr{U}$ satisfying $u_1(t)=u_2(t)$ for $t \leqq t_0$ yield outputs $y_1 = Fu_1$ and $y_2 = Fu_2$ which similarly satisfy $y_1(t)=y_2(t)$ for $t \leqq t_0$. Assume in addition that this map is stationary *i.e.*, two inputs $u_1, u_2 \in \mathscr{U}$ related by $u_1(t)=u_2(t+T)$ yield outputs $y_1 = Fu_1$ and $y_2 = Fu_2$ which are similarly related by $y_1(t)=y_2(t+T)$. The question is to realize F by a dynamical system in state space form. The solution to this problem is by no means unique. One possibility is to consider the function $f: R^+ \to U$ defined by $f(s)=u(t-s)$ for $s \geqq 0$ as the state at time t resulting from the input u. It is clear how the state transition function and the read-out function may be defined from here [11]. This state space realization is of course completely inefficient: in trying to store sufficient information about the past inputs, we decided to store the whole past input. The most efficient and natural state space realization of F is the one obtained by considering as the state at time t the equivalence class of those inputs up to time t which yield the same output after time t regardless of how the input is continued after time t. More specifically, in this realization we start with the space of functions $f: R^+ \to U$ satisfying $f(s)=u(-s)$, $s \geqq 0$, for some $u \in \mathscr{U}$. We then group these functions into equivalence classes by letting $f_1 \sim f_2$ if $y_1 = Fu_1$, and $y_2 = Fu_2$ satisfy $y_1(t)=y_2(t)$ for $t \geqq 0$ whenever $u_1(-t)=f_1(t)$, $u_2(-t)=f_2(t)$, and $u_1(t)=u_2(t)$ for $t \geqq 0$. The latter realization is sometimes called a "*minimal* realization" and plays a central role in control theory [5, 10]. A similar idea has been proposed by ONAT [3, 4] in a restricted context.

The point of view taken in this paper is that the state space realization is given, *i.e.*, it has been inferred from previous considerations what the state space is. We do not demand minimality since, in our opinion, there is no compelling reason for doing so: minimality is very much a function of the class of experiments and observations which are allowed, is sensitive to modelling, and is not necessarily a good physical assumption. Neither do we adhere to the idea that the state is the whole past input since this point of view leads to nonsensical situations. Consider for example an electrical *RLC* network which has a given set of charges on the C's and fluxes through the L's. Does it make sense to allow the stored energy of such a system to depend on exactly how these charges and fluxes came about? The whole question of what the state space of a physical system is requires much consideration. In this paper we have taken the easy way out by assuming that this has already been decided.

3. Dissipative Dynamical Systems

In this section the concepts, which will be the basis for the further developments, are introduced. Assume that a dynamical system Σ is given together with a real-valued function w defined on $U \times Y$. This function will be called the *supply rate.* We assume that for any $(t_1, t_0) \in R_2^+$, $u \in U$, and $y \in Y$, the function $w(t) = w(u(t), y(t))$ satisfies* $\int_{t_0}^{t_1} |w(t)|\, dt < \infty$, i.e., w is locally integrable.

Definition 2. A dynamical system Σ with supply rate w is said to be *dissipative* if there exists a nonnegative function $S: X \to R^+$, called the *storage function,* such that for all $(t_1, t_0) \in R_2^+$, $x_0 \in X$, and $u \in U$,

$$S(x_0) + \int_{t_0}^{t_1} w(t)\, dt \geqq S(x_1)$$

where $x_1 = \phi(t_1, t_0, x_0, u)$ and $w(t) = w(u(t), y(t))$, with $y = y(t_0, x_0, u)$.

The above inequality will be called the *dissipation inequality.* Note that $\oint w(t)\, dt \geqq 0$ with $\oint$ indicating that the dynamical system is taken from a particular initial state to the same terminal state along some path in state space. This condition is in itself inadequate as a definition for dissipativeness but dynamical systems which are dissipative in such cyclic motions only are of independent interest.

The approach taken here proceeds from the knowledge, from physical considerations, that the dynamical system is dissipative and thus that the storage function exists. The fact that this storage function is "defined" via an inequality requires further analysis. Central in this analysis is the question: "In how far is S defined by the dissipation inequality?" (The question is not so much "Does a storage function exist?" but rather "What can it be?")

A crucial role will be played in the sequel by a quantity termed the *available storage:* it is the maximum amount of storage which may at any time have been extracted from a dynamical system. The notion of available storage is a generalization of the concept of "available energy" [11, 12, 13] studied in control theory and of "recoverable work" encountered in the theory of viscoelasticity [14, 15].

Definition 3. The *available storage*, S_a, of a dynamical system Σ with supply rate is the function from X into R^e defined by

$$S_a(x) = \sup_{\substack{x \to \\ t_1 \geqq 0}} - \int_0^{t_1} w(t)\, dt$$

where the notation $x \to$ denotes the supremum over all motions starting in state x at time 0 and where the supremum is taken over all $u \in \mathscr{U}$.

The available storage is an essential function in determining whether or not a system is dissipative. This is shown in the following theorem:

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* The shorthand notation $w(t)$ for $w(u(t), y(t))$ will be used whenever it is obvious from the context what x_0, t_0, and u are.

Theorem 1. *The available storage, S_a, is finite for all $x \in X$ if and only if Σ is dissipative. Moreover, $0 \leqq S_a \leqq S$ for dissipative dynamical systems and S_a is itself a possible storage function.*

Proof. Assume first that $S_a < \infty$: it will be shown that Σ is then dissipative. It suffices therefore to show that S_a is a possible storage function. Notice that $S_a \geqq 0$ since $S_a(x)$ is the supremum over a set of numbers which contains the zero element ($t_1 = 0$). Consider now the quantity $S_a(x_0) + \int_{t_0}^{t_1} w(t)\,dt$. We have to show that this quantity is not less than $S_a(x_1)$ whenever w is evaluated along a trajectory generated by an input u which transfers the state from x_0 at t_0 to x_1 at t_1. The proof of this is quite simple although writing out the details is somewhat laborious. The idea is the following: in extracting the available storage from Σ when it is in state x_0 we could first take Σ along the path generated by u, thus transferring Σ to x_1, and then extract the available storage with Σ in state x_1. This combined process is clearly a suboptimal procedure for extracting the storage originally present with Σ in state x_0. Formalizing this idea immediately leads to the desired dissipation inequality for S_a. Assume next that Σ is dissipative. Then

$$S(x_0) + \int_{t_0}^{t_1} w(t)\,dt \geqq S(x_1) \geqq 0$$

which shows that

$$S(x_0) \geqq \sup_{\substack{x_0 \to \\ t_1 \geqq 0}} - \int_0^{t_1} w(t)\,dt = S_a(x_0).$$

Hence $S_a < \infty$ as claimed. This ends the proof of Theorem 1. ||

Theorem 1 gives a method which in theory may be used to verify whether or not a dynamical system is dissipative and this procedure does not require knowledge of the storage functions. In this sense it is an input/output test. Note that the theorem only states that the available storage may be the storage function. Usually it will not be the actual storage function. In fact, under certain additional assumptions (*e.g.*, the Onsager-Casimir reciprocal relations) it may be shown that it will *not* be the actual storage function. This fact should be kept in mind when interpreting the results of [12, 14, 15]. A dynamical system which has the available storage as its actual storage function has the interesting (and unusual) property that all of its internal storage is available to the outside via its external terminals.

It is convenient to introduce at this point the concept of reachability. This notion is related to controllability and plays a central role in mathematical systems theory.

Definition 4. The state space of the dynamical system Σ is said to be *reachable* from x_{-1} if for any $x \in X$ there exists a $t_{-1} \leqq 0$ and $u \in \mathscr{U}$ such that

$$x = \phi(0, t_{-1}, x_{-1}, u).$$

It is said to be *controllable* to x_1 if for any $x \in X$ there exists a $t_1 \geqq 0$ and a $u \in \mathscr{U}$ such that $x_1 = \phi(t_1, 0, x, u)$.

Theorem 1 emphasizes what happens when the system starts off in a particular state. One may similarly examine what happens when the system ends up in a particular state. We will therefore introduce the concept of required supply. This is done by letting the system start in a given state and by bringing it to its present state in the most efficient manner, *i.e.*, by using no more supply from the outside than is absolutely necessary. The notion of required supply has been introduced in [11]. Although one could choose any point in state space as the initial state, it is most logical to assume that the system starts in a state of minimum storage.

Assumption. It will be assumed that there exists a point $x^* \in X$ such that $S(x^*) = \min_{x \in X} S(x)$ and that the storage function S has been normalized to $S(x^*) = 0$.

Definition 5. The required supply, S_r, of a dissipative dynamical system Σ with supply rate w is the function from X into R^e defined by

$$S_r(x) = \inf_{\substack{x^* \to x \\ t_{-1} \leqq 0}} \int_{t_{-1}}^{0} w(t)\,dt$$

where the notation $\inf_{\substack{x^* \to x \\ t_{-1} \leqq 0}}$ denotes* the infimum over all $u \in \mathscr{U}$ and $t_{-1} \leqq 0$ such that $x = \phi(0, t_{-1}, x^*, u)$.

Theorem 2. (i) *Assume that the state space of Σ is reachable from x_{-1}. Then Σ is dissipative if and only if there exists a constant K such that*

$$\inf_{\substack{x_{-1} \to x \\ t_{-1} \leqq 0}} \int_{t_{-1}}^{0} w(t)\,dt \geqq K \quad \textit{for all } x \in X.$$

Moreover,

$$S_a(x_{-1}) + \inf_{\substack{x_{-1} \to x \\ t_{-1} \leqq 0}} \int_{t_{-1}}^{0} w(t)\,dt$$

is a possible storage function.

(ii) *Let Σ be a dissipative dynamical system and assume that $S(x^*) = 0$. Then $S_r(x^*) = 0$ and $0 \leqq S_a \leqq S \leqq S_r$. Moreover, if the state space Σ is reachable from x^* then $S_r < \infty$ and the required supply S_r is a possible storage function.*

Proof. (i) By reachability and Theorem 1 we see that Σ is dissipative if and only if $S_a(x_{-1}) < \infty$. Any $K \leqq -S_a(x_{-1})$ will thus yield the inequality in part (i) of the theorem statement. It remains to be shown that

$$S_a(x_{-1}) + \inf_{\substack{x_{-1} \to x \\ t_{-1} \leqq 0}} \int_{t_{-1}}^{0} w(t)\,dt$$

is a possible storage function. This function is clearly nonnegative. To prove that it satisfies the dissipation inequality, consider the following idea: in taking the system from x_{-1} to x_1 at t_1, we can first take it to x_0 at t_0 while minimizing the supply and then take it from x_0 at t_0 to x_1 at t_1 along the path for which we are to

* This notation, along with the similar one introduced in Definition 3, will be used throughout.

demonstrate the dissipation inequality. This results in a suboptimal policy for taking the system to x_1 and the formalization of this procedure leads to the desired dissipation inequality.

(ii) That $S_r(x^*)=0$ is obvious. Moreover, any $u\in\mathscr{U}$ resulting in a transfer from x^* at t_{-1} to x at 0 satisfies $S(x)\leqq \int_{t_{-1}}^{0} w(t)\,dt$ by the dissipation inequality. The inequality $S_r(x)\geqq S(x)$ follows by taking the infimum at the right-hand side. Assume now that the state space of Σ is reachable. Then clearly $S_r\leqq\infty$. It remains to be shown that S_r is a possible storage function. This, however, follows from (i). ‖

It is an immediate consequence of the normalization $S(x^*)=0$ that for a dissipative system any motion starting in x^* at t_0 satisfies $\int_{t_0}^{t_1} w(t)\,dt\geqq 0$ for all $u\in\mathscr{U}$ and $t_1\geqq t_0$. Thus the net supply flow is into the system. This idea has been proposed [16, 17, 18, 19] as a definition of passivity. It has the advantage of being an input/output concept which does not involve introduction of state space notions. However implicit in this approach is the fact that one knows the state of minimum internal storage.

Note that the required supply is in general a function of S and x^*. Usually, however, the point of minimum storage is a unique *a priori* known equilibrium point which may thus be shown to be independent of S and this ambiguity does not arise.

Remarks. 1. Under the assumptions of reachability from x_{-1} and controllability to x_1 we always have the following inequalities for a dissipative system:

$$S(x_1)+\sup_{\substack{x\to x_1\\ t_1\geqq 0}} -\int_0^{t_1} w(t)\,dt\leqq S(x)\leqq S(x_{-1})+\inf_{\substack{x_{-1}\to x\\ t_{-1}\leqq 0}}\int_{t_{-1}}^{0} w(t)\,dt.$$

Note however that the lower bound on S thus obtained is itself in general not a possible storage function because it need not be nonnegative.

2. Often a state space model of a dynamical system is constructed on the basis of an input/output description. Particularly important realizations are the minimal realization mentioned earlier and the realization in which the state is the whole past history. It is quite simple to associate a storage function with these realizations when one has determined a storage function on a particular state space X. For example, defining $S(u_{(-\infty,0)})=S(x(0))$ leads to a storage function on a state space which keeps track of the whole past input history. The available storage function of these realizations will in fact agree on that part of the state space which is reachable along some past history. Assuming that for t sufficiently small every element of $\mathscr{U}$ is equal to a fixed constant u^* (typically the zero element of some vector space) such that $w(u^*,y^*)=0$ and that $\int_{-\infty}^{0} w(u(t),y(t))\,dt$ exists and is nonnegative (thus the state at "$t=-\infty$" is assumed to be the state of minimal storage), then we may actually also evaluate the required supply for the realization in which the state keeps track of the whole past history. This does not require any infimization and is simply equal to $\int_{-\infty}^{0} w(u(t),\ y(t))\,dt$. It may in principle

be different for every history. Moreover, the dissipation inequality holds with equality for this storage function. (This fact does not conflict with Theorem 4 since this realization will never be controllable.)

If one works with the minimal realization then one may associate a storage function by defining $S(\tilde{x}_{\min})=S(x)$ where x is a state in the equivalence class $\tilde{x}_{\min}$. After elimination of the non-reachable states, one thus divides the state space X into equivalence classes and defines the storage to be the storage of an arbitrary element in this class. The available storage functions of these realizations again agrees on that part of the state space which is reachable along some past history. The required storage may now take on more values in X than in $X_{\min}$. An interesting consequence of the above reasoning is that the notion of available storage is defined purely as an input/output concept for states which are reachable. Thus, taking equivalence classes as the state or the whole past history as the state leads to the same value for the available storage function. This reemphasizes the importance of Theorem 1 as an input/output test for dissipativeness. There is an interesting paper by DAY [33] which has used the concept of available storage (or "useful work" as it is called in [33]) in setting up an axiomatic theory of thermodynamics. Although the technical details are quite different, the ideas exploited in that paper appear to be very much along the lines of those on which Theorem 1 is based.

To summarize the above results, we have shown that the storage function of a dissipative dynamical system satisfies the *a priori* inequality $S_a \leqq S \leqq S_r$, *i.e.*, a dissipative system can supply to the outside only a fraction of what it has stored and can store only a fraction of what has been supplied to it. The available storage always satisfies the dissipation inequality, as does the required supply for systems with a state space which is reachable from a point of minimum storage. (This show that the above inequality is the best of its type.) Of course not every function bounded by this *a priori* inequality will be a possible storage function. It appears to be difficult to state other general properties of the set of possible storage functions. One interesting property is its convexity:

Theorem 3. *The set of possible storage functions of a dissipative dynamical system forms a convex set. Hence* $\alpha S_a + (1-\alpha) S_r$, $0 \leqq \alpha \leqq 1$, *is a possible storage function for a dissipative dynamical system whose state space is reachable from* x^*.

Proof. This theorem is an immediate consequence of the dissipation inequality. ||

The ultimate test for a theory of dissipative systems is whether or not there exists a (possibly idealized) "physical" system which realizes the input/output exchange process and which has the desired storage function. Such a synthesis program based on interconnecting ideal elements may in fact be carried out for linear systems with a finite number of degrees of freedom and quadratic supply functions. Some results in this direction will be indicated in Part II.

We now proceed with a few remarks regarding the evaluation of the available storage and the required supply:

(i) If the state of minimum storage x^* is an equilibrium point corresponding to the constant input $u^* \in \mathscr{U}$ (*i.e.*, $\phi(t,0,x^*,u^*)=x^*$ for all $t \geqq 0$) and if $w(u^*,y^*)=0$, then

$$S_r(x) = \lim_{t_{-1}\to-\infty} \inf_{x^*\to x} \int_{t_{-1}}^{0} w(t)\,dt;$$

(ii) If for all $x \in X$ there exists a $u \in \mathscr{U}$ such that $w(u, y) \leqq 0$ (*i.e.*, the external termination may always be adjusted so that the supply flows out of the system), then

$$S_a(x) = \lim_{t_1\to\infty} \sup_{x\to} \int_{0}^{t_1} w(t)\,dt;$$

(iii) The concept of required supply assumes that there exists a point $x^* \in X$ such that $S(x^*) = \min_{x^* \in X} S(x)$. There need however not be a point of minimum storage. One may then define $S_r(x)$ by considering a sequence of states $\{x_n\}$ with $\lim_{n\to\infty} S(x_n) = \inf_{x \in X} S(x)$ and define

$$S_r(x) = \lim_{n\to\infty} S_{r,n}(x) \quad \text{where} \quad S_{r,n}(x) = \inf_{\substack{x_n \to x \\ t_{-1} \leqq 0}} \int_{t_{-1}}^{0} w(t)\,dt.$$

We now show how to treat conservative systems as particular cases of dissipative dynamical systems.

Definition 6. A dissipative dynamical system Σ with supply rate w and storage function S is said to be *lossless* if for all $(t_1, t_0) \in R_2^+$, $x_0 \in X$, and $u \in \mathscr{U}$

$$S(x_0) + \int_{t_0}^{t_1} w(t)\,dt = S(x_1)$$

where $x_1 = \phi(t_1, t_0, x_0, u)$.

The following theorem is immediate:

Theorem 4. *Let Σ be a lossless dissipative dynamical system and assume that $S(x^*) = 0$. If the state space is reachable from x^* and controllable to x^*, then $S_a = S_r$ and thus the storage function is unique and given by*

$$S(x) = \int_{t_{-1}}^{0} w(t)\,dt$$

with any $t_{-1} \leqq 0$ and $u \in \mathscr{U}$ such that $x = \phi(0, t_{-1}, x^*, u)$, or

$$S(x) = -\int_{0}^{t_1} w(t)\,dt$$

with any $t_1 \geqq 0$ and $u \in \mathscr{U}$ such that $x^* = \phi(t_1, 0, x, u)$.

The condition $S_a = S_r$ which implies uniqueness of the storage function is in itself not sufficient to imply losslessness. We could call such systems *quasi-lossless* since they may be transferred between states without dissipation provided; however, this transfer is executed optimally. An arbitrary transfer instead is expected to involve dissipation.

An interesting property of dissipative dynamical systems is the following:

(i) For dissipative dynamical systems with $x(t_0) = x^*$, $\int_{t_0}^{t_1} w(t)\,k(t)\,dt \geqq 0$ for all bounded functions k with $k(t) \geqq 0$ and $\dot{k}(t) \leqq 0$;

(ii) For lossless dynamical systems with $x(t_0)=x^*$ and $\phi(t_1, t_0, x^*, u)=x^*$, $\int_{t_0}^{t_1} w(t)\,k(t)\,dt \geqq 0$ for all bounded functions k with $\dot{k}(t) \leqq 0$.

These inequalities formalize the idea that for a dissipative system with no initial storage the supply flows into the system before part of it is recovered whereas in addition all of it gets recovered in a lossless system. These expressions generalize similar inequalities obtained in [20, 21, 22].

We conclude this section with a discussion of the concept of a dissipation function.

Definition 7. A real-valued function $d\colon X \times U \to R$ is said to be the *dissipation rate* of a dissipative dynamical system Σ with supply rate w and storage function S if for all $(t_1, t_0) \in R_2^+$, $x_0 \in X$, and $u \in \mathscr{U}$

$$S(x_0)+\int_{t_0}^{t_1}\big(w(t)+d(t)\big)\,dt=S(x_1)$$

where $x_1=\phi(t_1, t_0, x_0, u)$.

It is clear that d being nonnegative implies dissipativeness. Moreover, since the dynamical system Σ is lossless with respect to the new supply rate $(w+d)$ it follows that the dissipation rate d uniquely determines the storage function S provided the appropriate reachability and controllability conditions are satisfied. The converse, *i.e.*, that dissipativeness implies the existence of a nonnegative dissipation rate is also the case under some technical smoothness conditions. The set of dissipation rate functions for a given dissipative system forms a convex set.

Remarks. Note that if $S(\phi(t, 0, x, u))$ is differentiable at $t=0$ for all $x \in X$ and $u \in \mathscr{U}$, then the dissipation inequality is equivalent to

$$\dot{S}(x, u) \leqq w\big(r(x, u), u\big)$$

for all $x \in X$ and $u \in U$ where $\dot{S}$ denotes $\left.\frac{d}{dt} S\big(\phi(t, 0, x, u)\big)\right|_{t=0}$. This definition is more standard but slightly less general than the one proposed here. The dissipation function d is then given by

$$d=\dot{S}-w.$$

4. Interconnected Systems

The main result obtained in the previous section yields an *a priori* bound on the storage function of a dissipative dynamical system. Moreover these bounds themselves define possible storage functions and the storage function is thus uniquely determined by the dissipation inequality if and only if the required supply equals the available storage. This situation is the exception and as a rule there are consequently many possible storage functions. If we consider the implications of this result to physical systems which dissipate energy or to thermodynamic systems, then we conclude that experiments on a physical system will usually only give bounds on the stored energy function or on the entropy function. This result is unexpected in the sense that in classical physical systems this ambiguity does not arise: we thus expect that the additional structural assumptions implicit in such systems will greatly reduce the number of possible storage functions and often render it unique.

In this section we examine one such possibility: it will be shown that by considering a given dissipative system as an interconnection of dissipative subsystems the number of possible storage functions is greatly reduced. Other qualitative assumptions (linearity, reciprocity, *etc.*) on the system will be investigated in Part II.

The idea of an interconnected system is actually quite simple, albeit somewhat difficult to formalize. We start with a collection of dynamical systems $\{\Sigma_\alpha\}$ where α ranges over some given index set A. For simplicity we will assume that A is a finite set. More general interconnections involve the introduction of a measure on A which would take us somewhat astray. Each Σ_α is determined, as in Definition 1, by a septuplet $\{U_\alpha, \mathscr{U}_\alpha, Y_\alpha, \mathscr{Y}_\alpha, X_\alpha, \phi_\alpha, r_\alpha\}$. We assume that the inputs and outputs of each dynamical system Σ_α are divided into two groups, *i.e.*,

$$U_\alpha = U_\alpha^e \times U_\alpha^i, \quad \mathscr{U}_\alpha = \mathscr{U}_\alpha^e \times \mathscr{U}_\alpha^i, \quad Y_\alpha = Y_\alpha^e \times Y_\alpha^i, \quad \text{and} \quad \mathscr{Y}_\alpha = \mathscr{Y}_\alpha^e \times \mathscr{Y}_\alpha^i$$

when the superscripts e and i stand for the adjectives *external* and *interconnecting*.

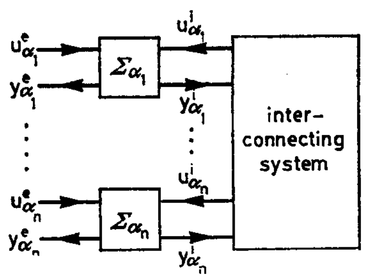

Fig. 1. Illustrating the concept of an interconnected system

Next we introduce the notion of an *interconnecting function* which is simply a function* $f: \prod_{\alpha\in A} (U_\alpha^i \times Y_\alpha^i) \to V$ where V is some vector space, and of the *interconnection constraint* which states that $f(\prod_{\alpha\in A} (U_\alpha^i \times Y_\alpha^i)) = 0$: it is thus a relation between the instantaneous values of the inputs and the outputs. The idea of an interconnected system is illustrated in Figure 1 and indicates that the external inputs u_α^e are given but that the interconnecting inputs u_α^i are to be determined implicitly. More precisely, given any $u_\alpha^e \in \mathscr{U}_\alpha^e$ and $x_\alpha \in X_\alpha$, $\alpha\in A$, we may attempt to solve the implicit equations

$$f(\prod_{\alpha\in A} (u_\alpha^i(t) \times r_\alpha^i(\phi_\alpha(t, t_0, x_\alpha(u_\alpha^e, u_\alpha^i)), (u_\alpha^e(t), u_\alpha^i(t))))) = 0, \qquad t \geqq t_0,$$

for some set of the functions $u_\alpha^i \in \mathscr{U}_\alpha$. This equation is of course not necessarily uniquely solvable and this fact needs to be assumed explicitly. Notice that only

* The notation $\prod_{\alpha\in A}$ stands for the Cartesian set product.

the values of $u_\alpha^i(t)$ for $t \geq 0$ enter in the equations. Hence we can only hope these equations determine u_α^i on the half line $t \geq t_0$. However it may be expected that under reasonable assumptions there will be a map from the set of states and external input values into the set of internal inputs and outputs which determines a solution to these equations.

We will thus assume *explicitly* that the interconnected system, denoted by $\prod_{\alpha \in A} \Sigma_\alpha/f$, is well-posed in the sense that it defines unique functions ϕ and r and

$$\psi_\alpha^i: \prod_{\alpha \in A} X_\alpha \times \prod_{\alpha \in A} U_\alpha^e \to U_\alpha^i, \qquad \alpha \in A,$$

such that:

(i) the septuplet $\{\prod_{\alpha \in A} U_\alpha^e, \prod_{\alpha \in A} \mathscr{U}_\alpha^e, \prod_{\alpha \in A} Y_\alpha^e, \prod_{\alpha \in A} \mathscr{Y}_\alpha^e, \prod_{\alpha \in A} X_\alpha, \phi, r\}$ defines a dynamical system in the sense of Definition 1;

(ii) the function $u_\alpha^i(t) = \psi_\alpha^i(\phi(t, t_0, \prod_{\alpha \in A} x_\alpha, \prod_{\alpha \in A} u_\alpha^e), \prod_{\alpha \in A} u_\alpha^e(t))$ defined for $t \geq t_0$ is the restriction to $[t_0, \infty)$ of a function in $\mathscr{U}_\alpha^i$;

(iii) $\phi(t, t_0, \prod_{\alpha \in A} x_\alpha, \prod_{\alpha \in A} u_\alpha^e) = \prod_{\alpha \in A} \phi_\alpha(t, t_0, x_\alpha, (u_\alpha^e, u_\alpha^i))$ with u_α^i as in (ii);

(iv) $r(\prod_{\alpha \in A} x_\alpha, \prod_{\alpha \in A} u_\alpha^e) = \prod_{\alpha \in A} r_\alpha^e(x_\alpha, (u_\alpha^e, \psi_\alpha^i(\prod_{\alpha \in A} x_\alpha, \prod_{\alpha \in A} u_\alpha^e)))$;

(v) $f(\prod_{\alpha \in A} (\psi_\alpha^i(\prod_{\alpha \in A} x_\alpha, \prod_{\alpha \in A} u_\alpha^e), r_\alpha^i(x_\alpha, (u_\alpha^e, \psi_\alpha^i(\prod_{\alpha \in A} x_\alpha, \prod_{\alpha \in A} u_\alpha^e))))) = 0.$

It is easy to verify that the above conditions formalize the intuitive conditions one expects a well-posed interconnected dynamical system to satisfy. Examples of interconnected systems will be given in Section 7. Note that although the interconnected system may have many state space realizations we are insisting on using the one with state space the Cartesian product of the state spaces of the individual subsystems. This is indeed a natural thing to do since the interconnection itself introduces no memory.

We now introduce the concept of dissipation in this framework. Assume therefore that each dynamical system Σ_α has associated with it an *external supply rate*, w_α^e, defined on $U_\alpha^e \times Y_\alpha^e$ and an *interconnecting supply rate*, w_α^i, defined on $U_\alpha^i \times Y_\alpha^i$.

Definition 8. Consider the dynamical systems Σ_α with interconnecting supply rates w_α^i. Then the interconnection defined by the interconnection constraint $f(\prod_{\alpha \in A} (u_\alpha^i \times y_\alpha^i)) = 0$ is said to be *neutral* if all u_i^α and y_i^α satisfying this equality yield $\sum_{\alpha \in A} w_\alpha^i(u_\alpha^i, y_\alpha^i) = 0$.

In terms of Figure 1, an interconnection is thus said to be neutral if the interconnecting system itself is lossless with respect to the supply rate $\sum_{\alpha \in A} w_\alpha^i$. Thus the mere interconnection does not introduce any new supply or dissipation. One thus expects the dissipativeness of the interconnected system to be a consequence of the dissipativeness of the individual subsystems. That this is indeed the case is shown in the following theorem:

Theorem 5. *Let Σ_α, $\alpha \in A$, be a collection of dissipative dynamical systems with supply rates $w_\alpha = w_\alpha^e + w_\alpha^i$ and storage functions S_α. Let f be a neutral interconnection* *constraint. Then the interconnected system* $\Sigma=\prod_{\alpha\in A}\Sigma_\alpha|f$ *is itself dissipative with respect to the supply rate* $w=\sum_{\alpha\in A}w_\alpha^e$ *and* $S=\sum_{\alpha\in A}S_\alpha$ *is a storage function for* Σ.

Proof. Summing both sides of the inequality

$$S_\alpha(x_\alpha(t_0))+\int_{t_0}^{t_1}(w_\alpha^e(t)+w_\alpha^i(t))\,dt\geqq S_\alpha(x_\alpha(t_1))$$

over $\alpha\in A$ and using the assumption $\sum_{\alpha\in A}w_\alpha^i(t)=0$ leads to the desired dissipation inequality for Σ. ‖

The above theorem is intuitively obvious. Note however that by considering only storage functions which are additive in the sense that $S(\prod_{\alpha\in A}x_\alpha)=\sum_{\alpha\in A}S_\alpha(x_\alpha)$ one obtains only part of the admissible storage functions for Σ. It is easy to see that since the interconnection introduces additional constraints on the inputs $u_\alpha=(u_\alpha^e,u_\alpha^i)$, we always have the inequality $S_a\leqq\sum_{\alpha\in A}S_{\alpha,a}\leqq\sum_{\alpha\in A}S_{\alpha,r}\leqq S_r$, with equality holding exceptionally. Thus one obtains a unique additive storage function for the interconnected system if and only if $S_{\alpha,a}=S_{\alpha,r}$ for each $\alpha\in A$.

In many physical systems encountered in practice, *e.g.*, in lumped electrical networks or in continuum systems, one may postulate *a priori* that the system is an interconnection of dissipative systems and use this qualitative property to describe the system in terms of "local" states, *i.e.*, to take the state space $X=\prod_{\alpha\in A}X_\alpha$ and furthermore to require that the storage function be of the type $S(\prod_{\alpha\in A}x_\alpha)=\sum_\alpha S_\alpha(x_\alpha)$. This natural requirement on the storage function of a dissipative dynamical system which consists of a family of dissipative systems interconnected by means of a neutral interconnection serves to reduce greatly the number of possible storage functions. This requirement leads to a *unique* storage function whenever it is possible to regard Σ as the interconnection of lossless systems with memory (capacitors and inductors, elastic systems) and a dissipative system without memory (resistors, friction elements). The lossless part possesses a unique storage function by Theorem 4 (under the additional hypothesis of reachability and controllability) whereas the dissipative part does not contribute to the storage since its state space is the empty set. The storage of the original system is thus given by the storage in the lossless subsystem and is consequently unique.

In concluding this section we remark that the above method of considering interconnected systems is implicit in most treatments of dissipative systems. It is based on a qualitative assumption on the system (the idea of "simple" materials) and sometimes it results in the uniqueness of the storage function. This is however by no means always the case, and typical examples of areas where this nonuniqueness remains are linear viscoelasticity and the modern treatments of materials with memory in continuum mechanics and thermodynamics where the nonuniqueness of the storage function at the elementary particle level remains. In other words, one has to make more assumptions (or, equivalently, obtain more knowledge about the physics) in order to derive the storage function (internal energy, entropy, *etc.*) from the constitutive equations.

5. Stability

In this section we examine the stability of dissipative systems. As is to be expected, only some technical conditions are required in order for dissipativeness to imply stability of an equilibrium at a local minimum of the storage function.

We shall be concerned with the stability of an equilibrium state, and in order to make this study meaningful we need to isolate the system from its environment. Moreover, since stability is concerned with convergence the concept of a distance function on the state space needs to be introduced. Assume therefore that the following assumptions hold:

(i) The system is *isolated*, *i.e.*, the input space consists of one element only. In order to preserve stationarity we assume that this element is the constant function $u(t)=u^*$;

(ii) $x^* \in X$ is an *equilibrium* point, *i.e.*, $\phi(t, t_0, x^*, u^*)=x^*$ for all $t \geqq t_0$;

(iii) X is a subset of a normed space and $\| \; \|$ denotes its norm;

(iv) $\phi(t, t_0, x_0, u^*)$ is continuous in t for $t \geqq t_0$;

(v) $w(u^*, r(x, u^*)) \leqq 0$ for all $x \in X$ in a neighborhood of x^*.

The following stability definition is a standard one in the context of Lyapunov stability theory [23]:

Definition 9. The equilibrium point x^* of Σ is said to be *stable* if given $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that $\| x_0 - x^* \| \leqq \delta$ implies that

$$\| \phi(t, t_0, x_0, u^*) - x^* \| \leqq \varepsilon \quad \text{for all } t \geqq t_0.$$

A very useful method for proving stability is by means of Lyapunov functions. The notion of a Lyapunov function is introduced in the following definition. It is a slight variation of the usual definition:

Definition 10. A real-valued function V defined on the state space X of Σ is said to be a *Lyapunov function* in the neighborhood of the equilibrium point x^* if

(i) V is *continuous* at x^*;

(ii) V attains a strong *local minimum* at x^*, *i.e.*, there exists a continuous function $\alpha: R^+ \to R^+$ with $\alpha(\sigma) > 0$ for $\sigma > 0$ such that $V(x) - V(x^*) \geqq \alpha(\| x - x^* \|)$ for all $x \in X$ in a neighborhood of x^*;

(iii) V is *monotone nonincreasing* along solutions in the neighborhood of x^*, *i.e.*, the real-valued function $V(\phi(t, t_0, x_0, u^*))$ is monotone nonincreasing at $t = t_0$ for all x_0 in a neighborhood of x^*.

It is a standard exercise in (ε, δ)-manipulations to show that an equilibrium point $x^* \in X$ is stable if there exists a Lyapunov function in the neighborhood of x^*. This leads to the following theorem:

Theorem 6. *An equilibrium point $x^* \in X$ of a dissipative dynamical system Σ is stable if the storage function S is continuous and attains a strong local minimum at x^*. Moreover S is a Lyapunov function in the neighborhood of x^*.*

Proof. It suffices to show that $S(\phi(t, t_0, x_0, u^*))$ is monotone non-increasing at $t=t_0$ if $\|x_0-x^*\|$ is sufficiently small. By the dissipation inequality $S(\phi(t, t_0, x_0, u^*))$ is indeed monotone nonincreasing for all $t \geqq t_0$ and for all x^*. ‖

Note that S attains a strong local minimum at x^* if S_a does – consequently this condition may usually be verified without explicit knowledge of S. Note also that the fact that x^* is an equilibrium point itself follows if (i), (iii), (iv), (v) are satisfied and if S is continuous and attains a strong local minimum at x^*.

The consideration of the storage function is an extremely useful tool in stability investigations and by properly choosing the supply rates one may indeed obtain an interpretation for most of the existing stability criteria. In constructing a storage function it is natural to proceed to the evaluation of either the available storage or the required supply. These however lead to variational problems and it is only in exceptional circumstances that one may solve such problems, particularly if the dynamical system Σ is nonlinear. The concept of interconnected systems becomes in fact very useful in this context: it allows one to construct storage functions which correspond to neither the available storage nor the required supply, and which may be constructed by solving variational problems for – presumably less involved – subsystems of the original dynamical system Σ. This procedure will be illustrated in Section 7.2.

One may refine the basic result of Theorem 6 in several directions. Some of these are briefly discussed below:

(i) roughly speaking local minima of the storage function define stable equilibria and vice versa;

(ii) under appropriate additional hypotheses one may conclude that all trajectories actually approach the point of minimum storage. These additional hypotheses require the system to be strongly dissipative in the sense that no trajectory (other than the equilibrium) is free of dissipation. This strong form of dissipation is studied in [11]. We note here that one will usually not obtain $\dot{S}$ to be negative definite but merely semi-definite. The so-called invariance principles [24, 25] are thus very useful in establishing asymptotic stability in this context;

(iii) local maxima of the storage function will define an unstable equilibrium if all trajectories in its neighborhood involve some dissipation;

(iv) if $w(u^*, y)=0$ for all $\mathscr{Y} \in Y$, and if the system is lossless, then local minima *and* maxima of the storage function define stable equilibra.

6. Nonstationary Dynamical Systems

All of the above theory and results have been based on the hypothesis that the dynamical system under consideration is stationary. This stationarity has been postulated on two distinct levels:

(i) it has been assumed that the dynamical system Σ is itself stationary, *i.e.*, the constitutive equations defined by the maps ϕ and r are invariant under shifts of the time axis;

(ii) the storage functions have been assumed to be time-invariant, *i.e.*, the function S did not involve an explicit time dependence.

Although it may not seem so at first glance, assumptions (i) and (ii) are separate since (i) mainly refers to input/output stationarity whereas (ii) supplements this with internal stationarity. There are important types of physical systems (*e.g.*, rotating electrical machines) which are externally stationary but internally time-varying.

We view stationarity postulates as an *a priori* qualitative assumption imposed on the mathematical model of the dynamical systems under consideration. In this section we will indicate the modifications required to extend the above definitions to time-varying systems. Once the conceptual framework is appropriately expanded, one may indeed generalize the results to the time-varying case without difficulty.

1. The following definition generalizes Definition 1. In contrast with most similar definitions which have appeared in the literature we allow for the state space itself to be time-varying.

Definition 1′. A (continuous) *dynamical system*, Σ, is defined through the sets $\{U_t, \mathscr{U}, Y_t, \mathscr{Y}, X_t\}$, $t\in R$, and the maps ϕ_{t_1, t_0}, $(t_1, t_0)\in R_2^+$, and r_t, $t\in R$. These satisfy the following axioms:

(i) $\mathscr{U}$ is called the *input space* and consists of a class of functions $u(t)$, $t\in R$, taking their values at time t in the set of *input values* U_t;

(ii) $\mathscr{Y}$ is called the *output space* and consists of a class of functions $y(t)$, $t\in R$, taking their values at time t in the set of *output values* Y_t;

(iii) X_t is called the *state space* at time $t\in R$;

(iv) ϕ_{t,t_0} is called the *state transition function* and maps $X_{t_0}\times\mathscr{U}$ into X_{t_1}. It satisfies the analogous axioms of $(\text{iv})_a$, $(\text{iv})_b$, and $(\text{iv})_c$ of Definition 1;

(v) r_t is called the *read-out function* and is a map from $X_t\times U_t$ into Y_t;

(vi) the function $r_t(\phi(t, t_0, x_0, u), u(t))$ defined for $t\geqq t_0$ is the restriction to $[t_0, \infty)$ of an element of $\mathscr{Y}$.

The solution of the problem of state space realization in terms of equivalence classes goes through unchanged.

2. A (time-varying) dynamical system with supply rate at time t $w_t: U_t\times Y_t\to R$ is said to be *dissipative* if there exists a nonnegative function S_t: $X_t\to R^+$, called the *storage function*, such that

$$S_{t_0}(x_0)+\int_{t_0}^{t_1} w_t(t)\,dt\geqq S_{t_1}(x_1).$$

The available storage is defined by

$$S_{t_0,a}(x)=\sup_{\substack{x\to\\ t_1\geqq t_0}} -\int_{t_0}^{t_1} w_t(t)\,dt,$$

whereas the definition of required supply necessitates again the notion of a point of minimal storage. Assume then that $x_t^*\in X_t$ minimizes $S_t(x)$ over $x\in X_t$ and assume in addition that $S_t(x_t^*)=0$ (this postulate now involves more than simply adjusting an additive constant). The *required supply* then becomes

$$S_{t_0,r}(x)=\inf_{\substack{x^*(t_1)\to x\\ t_{-1}\leq t_0}} \int_{t_{-1}}^{t_0} w_t(t)\,dt.$$

The results of Theorems 1 and 2 follow with the obvious modifications in notation. The available storage and the required supply are thus bounds on the storage functions and are themselves possible storage functions.

7. Applications

In this section we shall present a series of applications which serve to illustrate the previous theoretical developments.

7.1. Systems with a Finite Number of Degrees of Freedom

Consider the dynamical system described by the set of first order ordinary differential equations

$$\dot{x}=f(x,u), \qquad y=g(x,u)$$

and assume that the supply function is given by

$$w=\langle u,y\rangle=u'y \qquad \text{(prime denotes transposition).}$$

Here, $x\in R^n$, u, $y\in R^m$, and it is assumed that f and g are Lipschitz continuous in x and u jointly. It is well known that this implies that the above differential equation has a unique solution for any $x(t_0)\in R^n$ and any locally square integrable $u(t)$. Moreover the resulting functions $x(t)$ and $y(t)$ are themselves also locally square integrable.

The above differential equation thus describes a dynamical system in the sense of Definition 1 with $U=Y=R^m$, $X=R^n$, and $\mathscr{U}=\mathscr{Y}$ the locally square integrable R^m-valued functions defined on R. The differential equation itself defines the state transition map ϕ whereas the relation $y=g(x,u)$ describes the read-out function r. Note also that the supply function is locally integrable for $u\in\mathscr{U}$ and $y\in\mathscr{Y}$.

The problem at hand is (i) to determine conditions on f and g which make the dynamical system under consideration dissipative with respect to the given supply function and (ii) to discover the possible storage functions. If we restrict ourselves to sufficiently smooth storage functions then we are asking to find those functions $S: R^n\to R^+$ satisfying

$$\frac{d}{dt}S(x)=\nabla_x S(x)\cdot f(x,u)\leq\langle u,y\rangle=\langle u,g(x,u)\rangle$$

for all $x\in R^n$ and $u\in R^m$. Brockett [26] has in fact proposed this as a definition of passivity. This equivalent statement hardly solves the problem. The question of dissipativeness is by Theorem 1 equivalent to whether or not

$$\inf_{u\in\mathscr{U}_0}\int_0^\infty \langle u,g(x,u)\rangle\,dt,$$

subject to the constraints $\dot{x}=f(x, u)$; $x(0)=x_0$, is finite for all $x_0 \in R^n$. The value of this infimum (which is seen to be nonpositive by taking $u=0$) yields the negative of the available storage function. This variational problem and the analogous one involved in the computation of the required supply are standard problems in optimal control and these techniques will be used in Part II to obtain some specific answers to the above questions. At the level of generality posed here it is impossible to obtain necessary and sufficient conditions on f and g for dissipativeness, but some interesting special cases offer a great deal of further insight:

(i) Consider the particular case (corresponding to elastic systems and to capacitive networks):

$$\dot{x}=u; \quad y=g(x).$$

In this case it is convenient to derive the conditions for dissipativeness directly from the dissipation inequality. Restricting ourselves again to sufficiently smooth storage functions (the available storage will in fact be smooth as a result of the assumption on f and g made earlier), we see that the dissipation inequality demands that there exists an S: $R^n \to R^+$ such that

$$\nabla_x S(x) \cdot u \leqq \langle u, g(x) \rangle$$

for all $u \in R^m$ and $x \in R^n$. This is the case if and only if the function $g'(x)$ is the gradient of a nonnegative function. It is well known that this requires $\frac{\partial g_i(x)}{\partial x_j} = \frac{\partial g_j(x)}{\partial x_i}$. This condition may be obtained in a different manner by noticing that $\int_0^t w(t)\, dt = \int_{x_0}^{x_1} g'(x)\, dx$. The integral on the right is bounded from below for a given x_0 and x_1 only if it is path independent which in turn requires $g'(x)$ to be the gradient of a real-valued function.

The necessary and sufficient conditions for dissipativeness may thus be expressed in terms of $g(x)$ by:

(i) $\frac{\partial g_i(x)}{\partial x_j} = \frac{\partial g_j(x)}{\partial x_i}$;

(ii) the path integral $P(x) = \int_{x^*}^{x} g'(x)\, dx$ is bounded from below.

Here, x^* is arbitrary and the function P differs from S only by an additive constant. It thus follows that the system is dissipative if and only if it is lossless. The storage function is thus unique and plays the role of a potential function since it determines the dynamical equations by

$$\dot{x}=u; \quad y=\nabla_x' S(x).$$

Note also that in this case one obtains reciprocity (condition (i)) as a result of dissipativeness. This is by no means a general property of dissipative systems however.

(ii) If we add "resistive" terms to the equations of motion studied in (i) then we obtain the dynamical system

$$\dot{x}=u; \qquad y=g_1(x)+g_2(u).$$

If we assume (without loss of generality) that $g_2(0)=0$ and concentrate again on sufficiently smooth storage functions then the dissipation inequality demands that there exists an $S: R^n \to R^+$ such that

$$\nabla_x S(x)\cdot u \leqq \langle g_1(x)+g_2(u), u\rangle \quad \text{for all } x\in R^n \text{ and } u\in R^m.$$

This inequality is satisfied if and only if

$$\nabla_x S(x)=g_1'(x) \quad \text{for all } x\in R^n \quad \textit{and} \quad \langle u, g_2(u)\rangle \geqq 0 \quad \text{for all } u\in R^m.$$

The conditions for dissipativeness are then those obtained in (i) augmented by the additional requirement $\langle u, g_2(u)\rangle \geqq 0$. The storage function is again unique (although the system need *not* be lossless) and is up to an additive constant given by the path integral $\int_{x^*}^{x} g_1'(x)\,dx$. The dissipation function (also unique) is given by $d(x, u)=\langle u, g_2(u)\rangle$. Notice that as a consequence of dissipativeness, we obtain reciprocity of the "elastic" terms $(g_1(x))$ but *not* of the "resistive" terms $(g_2(u))$.

The system studied here may be considered as the interconnection of the three systems:

$$\Sigma_1: \dot{x}_1=u_1; \qquad y_1=g_1(x_1) \quad \text{with} \quad w_1=\langle u_1, y_1\rangle,$$

$$\Sigma_2: y_2=g_2(u_2) \qquad \text{with} \quad w_2=\langle u_2, y_2\rangle,$$

$$\Sigma_3: y_3=y_4=-u; \qquad y=u_3+u_4 \quad \text{with} \quad w_3=\langle u_3, y_3\rangle+\langle u_4, y_4\rangle+\langle u, y\rangle$$

with the neutral interconnection constraint

$$y_3=-u_1; \qquad y_4=-u_2; \qquad u_3=y_1; \qquad u_4=y_2.$$

The storage function (but not its uniqueness) follows directly from here. The variables with subscripts represent the interconnecting inputs, outputs, and supply rates.

(iii) Consider the system described by the equations (u_2 is scalar-valued):

$$\dot{x}_1=u_1; \qquad y_1=g_1(x_1, x_2)$$

$$\dot{x}_2=f_2(x_1, x_2, u_2); \qquad y_2=g_2(x_1, x_2, u_2).$$

A simple calculation shows that dissipativeness implies that

$$\nabla_{x_1} S(x_1, x_2)=g_1'(x_1, x_2)$$

and

$$\nabla_{x_2} S(x_1, x_2)\cdot f_2(x_1, x_2, u_2)\leqq u_2\, g_2(x_1, x_2, u_2).$$

for all x_1, x_2, and u_2. Thus only part of the dependence of the storage function on the state vector is determined by the dynamical equations.

The above dynamical system is a particular case of the one studied by COLEMAN [27] and GURTIN [28] (see [2], Chapter 3). These authors obtained in fact very similar results. It should be realized however that for dynamical systems described in this amount of generality one needs a lot more information about the physics of the situation in order to obtain a unique storage function.

7.2. Stability of Feedback Systems

Consider the dynamical systems Σ_1 and Σ_2 and assume that $U_1 = U_2 = Y_1 = Y_2$ are inner product spaces. Assume now that Σ_1 and Σ_2 are interconnected via the constraint $u_2 = y_1$ and $u_1 = -y_2$. This results in the feedback system shown in Figure 2.

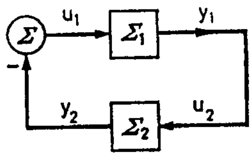

Fig. 2. Feedback system

The theory of dissipative systems discussed above offers a powerful method for investigating the stability of this feedback system. Assume that we associate the supply rate $w_1(u_1, y_1)$ with Σ_1 and the supply rate $w_2(u_2, y_2)$ with Σ_2. If w_1 and w_2 are such that $w_1(u, y) + w_2(y, -u) = 0$ for all u and y, then the feedback system may be considered as an interconnected system with the neutral interconnection constraint: $u_2 = y_1$, $u_1 = -y_2$. Thus in order to prove stability it then suffices to show that Σ_1 is dissipative with respect to w_1 and that Σ_2 is dissipative with respect to w_2 (or, equivalently, with respect to αw_2 for some $\alpha > 0$).

It may be verified that essentially all of the frequency-domain stability criteria which have recently appeared in the literature [17, 18] are based on this principle. Particularly important choices of the supply rates are $w_1 = \|u_1\|^2 - \|y_1\|^2$, $w_2 = \|u_2\|^2 - \|y_2\|^2$; $w_1 = \langle u_1, y_1 \rangle$, $w_2 = \langle u_2, y_2 \rangle$; and $w_1 = \langle u_1 + ay_1, u_1 + by_1 \rangle$, $w_2 = -ab \left\langle u_2 - \frac{1}{a} y_2, u_2 - \frac{1}{b} y_2 \right\rangle$. The stability theorems resulting from these choices of the supply rates are known as the small loop gain theorem, the positive operator theorem, and the conic operator theorem. The interpretation of these stability principles in terms of dissipative systems gives further insight in these results and unifies the existing conditions.

As an example, consider the autonomous dynamical system described by the set of first order ordinary differential equations:

$$\Sigma: \ \dot{x} = Ax - Bf(Cx)$$

where $x \in R^n$, $f: R^m \to R^m$, and A, B, and C are matrices of appropriate dimensions. We assume again that f is Lipschitz continuous. Let $f(0) = 0$; then the trajectory $x(t) = 0$ is a solution to this differential equation and the stability properties of this solution have been the subject of a number of recent papers in the control theory literature. Particularly the construction of Lyapunov functions is a matter of great practical importance. The best known result in this area is the so-called

Popov criterion [29, 30] which answered a long-standing question known as the "Lur'e problem". We will reproduce this result for a representative case using the theory of dissipative systems. In doing so we obtain a whole class of Lyapunov functions in a systematic manner and extend the results recently obtained in [11].

We begin with viewing this dynamical system as the feedback interconnection of two dynamical systems, namely:

$$\Sigma_1:\ \dot{x}_1 = Ax_1 + Bu_1;\quad y_1 = CAx_1 + CQx_1 + CBu_1$$

and

$$\Sigma_2:\ \dot{x}_2 = -Qx_2 + u_2;\quad y_2 = f(x_2)$$

with the interconnection constraint $u_1 = -y_2$; $u_2 = y_1$. The matrix Q is an arbitrary $(n\times n)$ matrix which features in the conditions for stability. It is clear that the above interconnection constraint defines a neutral interconnection with respect to the supply function $w_1 = \langle u_1, y_1\rangle$ and $w_2 = \langle u_2, y_2\rangle$. This interconnection leads to a "closed" system since we only have interconnecting, but no external, inputs and outputs. It is a simple matter involving only algebraic manipulations to show that the interconnection of Σ_1 and Σ_2 via the given interconnection constraint yield $x_1(t) = x(t)$ and $x_2(t) = Cx(t)$ provided the initial conditions are chosen as $x_1(0) = x(0)$ and $x_2(0) = Cx(0)$. The philosophy behind this equivalence is shown in Figure 3.

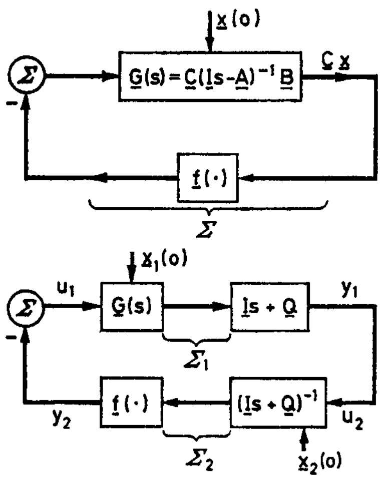

Fig. 3. Illustrating the interconnected system studied in Section 7.2

We now postulate the conditions for stability. These are:*

(i) $\{A, B, C\}$ is a minimal realization of $G(s) = C(Is-A)^{-1}B$

* Re $\lambda\{A\}$ denotes the real part of an arbitrary eigenvalue of A, the matrix inequality $P \geqq 0$ indicates that the Hermitian matrix P is nonnegative definite, and "minimal realization" is system theory jargon which will be explained in detail in Part II. Positive real functions have been studied, particularly in the context of electrical network synthesis, and will be discussed in Part II.

(ii) $(Is+Q)G(s)$ is a positive real function of s

(iii) f is the gradient of a nonnegative function, *i.e.*,

$$\frac{\partial f_i(\sigma)}{\partial \sigma_j} = \frac{\partial f_j(\sigma)}{\partial \sigma_i} \quad \text{for all } \sigma \in R^m,$$

and the path integral $\int_0^z f'(\sigma)\, d\sigma \geqq 0$ for all $z \in R^m$

(iv) $f'(\sigma)\, Q\sigma \geqq 0$ for all $\sigma \in R^m$.

This stability claim will be verified with the aid of a suitable Lyapunov function. The idea behind the above conditions is that they make both Σ_1 and Σ_2 into dissipative systems. In fact, conditions (i) and (ii) ensure that Σ_1 is dissipative with respect to $w_1 = \langle u_1, y_1 \rangle$. The available storage $x' Q_a x$ and the required supply $x' Q_r x$ are positive definite quadratic forms. These functions and the other possible storage functions which are quadratic will be the subject of study in Part II of this paper. In order to verify that Σ_2 is dissipative with respect to $w_2 = \langle u_2, y_2 \rangle$, we shall use conditions (iii) and (iv). Consider thus

$$\eta = -\inf \int_{T_1}^{T_2} \langle u_2, y_2 \rangle\, dt$$

subject to $\dot{x}_2 = -Q x_2 + u_2$; $y_2 = f(x_2)$. We may eliminate u_2 and y_2 from this integral in terms of x_2. This yields

$$\eta = -\inf \left\{ \int_{x_2(T_1)}^{x_2(T_2)} f'(x_2)\, dx_2 + \int_{T_1}^{T_2} f'(x_2(t))\, Q x_2(t)\, dt \right\}.$$

Note that the first integral is independent of path by (iii) and the second one has an integrand which is always nonnegative by (iv). This last integral may be made arbitrarily small in the evaluation of the available storage and the required supply. The dynamical system Σ_2 is thus dissipative and its storage function is uniquely given by

$$S_2(x_2) = \int_0^{x_2} f'(\sigma)\, d\sigma \geqq 0.$$

The interconnected system is hence dissipative and has $S_1(x_1) + S_2(x_2)$ as an admissible storage function. Restricted to initial conditions $x_2(0) = C x_1(0)$, this statement implies that the function $x' Q_s x + \int_0^{Cx} f'(\sigma)\, d\sigma$ is nonincreasing along solutions of Σ whenever $Q_s = Q_s'$ defines a quadratic storage function of Σ_1. Since $Q_s = Q_s' \geqq Q_a = Q_a' \geqq \varepsilon I$ for some $\varepsilon \geqq 0$ it follows that this Lyapunov function establishes the stability of the solution $x(t) = 0$. By strengthening condition (ii) to include $\operatorname{Re} \lambda \{A\} < 0$ we may in fact show, using this Lyapunov function, that all solutions approach their equilibrium solution $x = 0$ as $t \to \infty$.

7.3. Electrical Networks

Consider an electrical network with n external ports and with a number of internal nodes and branches. We shall denote the external voltages and currents by the n-vectors V and I, respectively. Assume that the network contains resistors, capacitors, inductors, and lossless memoryless elements (*e.g.*, transformers, gyrators, *etc.*) which need not be specified further. Let n_R, n_C, and n_L denote the number of resistors, capacitors, and inductors, and denote the voltage across the elements and the current into these elements by the n_R-vectors V_R, I_R, the n_C-vectors V_C, I_C, and the n_L-vectors V_L, I_L respectively. We take the sign conventions shown in Figure 4.

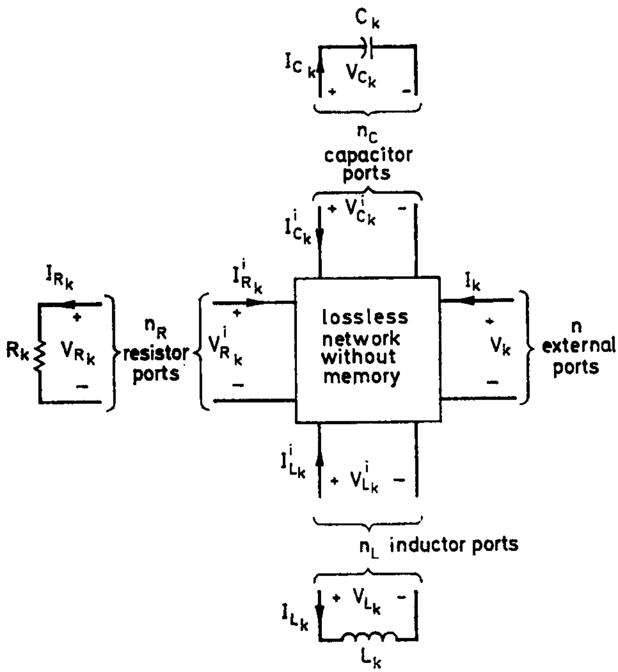

Fig. 4. The electrical n-port considered in Section 7.3

We now turn to the question of what should be considered inputs and outputs. This is a somewhat annoying question since what are the most convenient variables to work with depends on the network under consideration. In fact it has recently become apparent that the so-called scattering representation [31] (input $= v+\rho i$; output $= v-\rho i$, $\rho>0$) is by and large the most convenient model to consider. We shall consider here a somewhat simpler case and assume that [34]

(i) V is the input and I is the output;

(ii) the characteristic of the k^{th} resistor is given by

$$V_{R_k} = R_k(I_{R_k}) I_{R_k} \quad \text{with} \quad R_k \geqq 0.$$

This leads to the relation $V_R = R(I_R) I_R$.

(iii) the characteristic of the k^{th} capacitor is given by

$$I_{C_k} = C_k(V_{C_k}) \frac{dV_{C_k}}{dt} \quad \text{with} \quad C_k \geqq \varepsilon > 0.$$

This leads to the relation $I_C = C(V_C) \dfrac{dV_C}{dt}$.

(iv) the characteristic of the k^{th} inductor is given by

$$V_{L_k} = L_k(I_{L_k}) \frac{dI_{L_k}}{dt} \quad \text{with} \quad L_k \geqq \varepsilon > 0.$$

This leads to the relation $V_L = L(I_L) \dfrac{dI_L}{dt}$.

(v) the part of the network which neither involves dissipation nor memory defines an instantaneous relation from the voltages across the external ports, V, the currents into the resistor ports, I_R^i, the voltages across the capacitor ports, V_C^i, and the currents into the inductor ports, I_L^i, into the currents into the network at the external ports, I, the voltages across the resistor ports, V_R^i, the currents into the capacitor ports, V_C^i, and the voltages across the inductor parts, V_L^i. It is also assumed that $\langle V, I \rangle + \langle I_R^i, V_R^i \rangle + \langle V_C^i, I_C^i \rangle + \langle I_L^i, V_L^i \rangle = 0$.

The interconnected network may thus be considered a dynamical system with input V, output I, and state $\begin{bmatrix} V_C \\ \hdashline I_L \end{bmatrix}$. It is a neutral interconnection of dissipative systems with interconnection constraint $V_R = V_R^i$, $V_C = V_C^i$, $V_L = V_L^i$, $I_R = -I_R^i$, $I_C = -I_C^i$, and $I_L = -I_L^i$. The external supply rate is $\langle V, I \rangle$ and the internal supply rates are $\langle I_R, V_R \rangle$, $\langle V_C, I_C \rangle$, and $\langle I_L, V_L \rangle$. The stored energy in the capacitors and inductors is uniquely defined by

$$E(V_C, I_L) = \sum_1^{n_C} E_k(V_k) + \sum_1^{n_L} E_l(I_l)$$

with

$$E_k(V_{C_k}) = \int_0^{V_{C_k}} v C_k(v)\, dv \quad \text{and} \quad E(I_{L_k}) = \int_0^{I_{L_k}} i L_k(i)\, di.$$

Thus in standard electrical networks no ambiguity in the stored energy function arises. This is an immediate consequence of the fact that we are able to consider these systems as an interconnection of very simple subsystems in which the elements with memory are lossless. The individual dynamical subsystems involving memory are in fact described by first order scalar differential equations.

7.4. *Thermodynamics*

Consider a thermodynamic system at a uniform temperature. Assume that the mathematical model used for describing this system is in the form of a dynamical system in the sense of Definition 1 and that the outputs to this dynamical system Σ contain (possibly among other things) w, the rate of work done on the system, q, the rate of heat delivered to the system, and q/T where T denotes the temperature of the system. We assume that every admissible input and every initial state yield functions w, q, and q/T which are locally integrable.

The first and the second laws of thermodynamics may then be formulated by stating that a thermodynamic system is dissipative and lossless with respect to the supply rate $(w+q)$ and dissipative with respect to the supply rate $-\frac{q}{T}$. In terms of our definitions this implies the existence of two nonnegative functions E and $-S$ defined on the state space X of Σ such that every motion with $x_1 = \phi(t_1, t_0, x_0, u)$ yields

$$E(x_0)+\int_{t_0}^{t_1}\big(w(t)+q(t)\big)\,dt=E(x_1) \qquad \text{(Conservation of Energy)}$$

$$S(x_0)+\int_{t_0}^{t_1}\frac{q(t)}{T(t)}\,dt\leqq S(x_1) \qquad \text{(Clausius' inequality).*}$$

The function E is called the *internal energy* and S is called the *entropy*.

It follows from the results obtained earlier that E is uniquely defined once the equations of the dynamical system are given but that in general there will be many possible entropy functions. Two particular possibilities, S_a and S_r, may be computed *a priori* via the variational problems

$$S_a(x)=-\sup_{\substack{x\to\\ t_1\geqq 0}}\int_0^{t_1}\frac{q(t)}{T(t)}\,dt$$

and

$$S_r(x)=-\inf_{\substack{x^*\to x\\ t_{-1}\leqq 0}}-\int_{t_{-1}}^{0}\frac{q(t)}{T(t)}\,dt$$

where x^* is a point of maximal entropy normalized to $S(x^*)=0$. We may also conclude that, whatever the actual entropy may be, it satisfies the *a priori* inequality $S_r\leqq S\leqq S_a\leqq 0$. For reversible thermodynamic systems, *i.e.*, when Σ is

* DAY [35] has recently written a paper in which he shows how to replace this axiom by one involving the heat delivered and absorbed and the maximum and minimum temperature attained.

lossless with respect to $-\frac{q}{T}$ as well, the entropy is given unambiguously by $S=S_a=S_r$ provided the state space of Σ is reachable from x^* and controllable to x^*.

It should be emphasized that this ambiguity in the entropy function for irreversible thermodynamic systems is fundamental: the dynamical equations do not provide enough information to define the entropy uniquely. This difficulty has long been advertised by MEIXNER [32].

8. Conclusions

In the first part of this paper we have attempted to outline a general theory of dissipative dynamical systems. The mathematical model employed is a so-called state space model in which the map which generates outputs from inputs is viewed as the composition of a state transition map and a memoryless read-out function. This type of model is standard in control theory and dynamic estimation theory and it is argued that this model offers conceptual advantages for describing general physical systems with memory.

The definition of a dissipative dynamical system postulates the existence of a storage function which satisfies a dissipation inequality involving a given function called the supply rate. In many applications one knows from physical considerations that a storage function exists but it is often a difficult task to determine it. It is then shown that this difficulty is genuine and that the dynamical equations are insufficient to specify the storage function uniquely. However, the storage function satisfies an *a priori* bound, *i.e.*, it is bounded from below by the available storage and from above by the required supply. The available storage is the amount of internal storage which may be recovered from the system and the required supply is the amount of supply which has to be delivered to the system in order to transfer it from a state of minimum storage to a given state. Both these functions are themselves possible storage functions and their evaluation may be posed as variational problems.

These ideas were then applied to interconnected systems and it was established that for interconnected systems with interconnections which instantaneously redistribute the supply (the so-called neutral interconnections), the sum of the storage functions of the individual subsystems is a possible storage function for the interconnected system.

The stability properties of dissipative systems were then investigated and it was shown that states for which the storage function attains a local minimum are locally stable and that the storage function is a suitable Lyapunov function.

Part II of this paper will be devoted to an examination of linear systems with quadratic supply rates.

This research was supported in part by the National Science Foundation under Grant No. GK-25781 and in part by the U.K. Science Research Council. This paper was prepared while the author was a Senior Visiting Fellow at the Department of Applied Mathematics and Theoretical Physics of the University of Cambridge, Cambridge, England.

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(Received January 19, 1972)