Necessary and sufficient conditions for improving the Delsarte bound for τ-designs

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1 Introduction

Let $\mathcal{M}$ be a polynomial metric space with metric $d(x,y)$ and standard substitution $\sigma(d(x,y))$. Any finite nonempty subset C of $\mathcal{M}$ is called a code. An $(\mathcal{M},|C|,\sigma)$**-code** is a code for which $\sigma(d(x,y)) \leq \sigma(d)$, where d is the minimum distance of C. A code $C \subset \mathcal{M}$ is called a τ**-design** if $\sum_{x\in C} v(x) = 0$ for all $v(x) \in V_1 \oplus \cdots \oplus V_\tau$, where $V_1,\ldots,V_\tau$ are ordered subspaces of $\mathcal{M}$.

Any real polynomial $f(t)$ can be uniquely written in the form $f(t) = \sum_{i=0}^{k} f_i Q_i(t)$, where $\{Q_i(t)\}_{i=0}^{\infty}$ are the zonal spherical functions (ZSF) associated to $\mathcal{M}$. For each a and $b \in I\!N$, one can associate the ZSF with their *adjacent systems* of orthogonal polynomials $\{Q_i^{a,b}(t)\}_{i=0}^{\infty}$. They are orthogonal with respect to the measure $\nu^{a,b}(t)$ defined by $d\,\nu^{a,b}(t) = c^{a,b}(1-t)^a(1+t)^b\,d\nu(t)$ ($c^{a,b}$ is a constant). By $t_i^{a,b}$ we will denote the greatest roots of $Q_i^{a,b}(t)$ and by $r_i^{a,b}$ – the corresponding positive integers.

The classical upper (lower) bounds $L_{2k-1+\varepsilon}(\sigma)$ (resp. $D(\mathcal{M},\tau)$) for the cardinality of $(\mathcal{M},|C|,\sigma)$-code (resp. τ-design) can be presented in the following form [3, 2]:

$$|C| \leq L_{2k-1+\varepsilon}(\sigma) = \left(1 - \frac{Q_{k-1+\varepsilon}^{1,0}(\sigma)}{Q_k^{0,\varepsilon}(\sigma)}\right) \sum_{i=0}^{k-1+\varepsilon} r_i, \tag{1}$$

where $\varepsilon = 0$ if $t_{k-1}^{1,1} \leq \sigma < t_k^{1,0}$ and $\varepsilon = 1$ if $t_k^{1,0} \leq \sigma < t_k^{1,1}$, resp.

$$|C| \geq D(\mathcal{M},\tau) = 2^\theta c^{0,\theta} \sum_{i=0}^{k} r_i^{0,\theta}, \tag{2}$$

where $\theta \in \{0,1\}$ and $\tau = 2k+\theta$. The bound (1) can be obtained by using the polynomial $f^{(\sigma)}(t) = (t-\sigma)(t+1)^{\varepsilon}\ (T_{k-1}^{1,\varepsilon}(t,\sigma))^2$, where $T_k^{a,b}(x,y) = \sum_{i=0}^{k} r_i^{a,b} Q_i^{a,b}(x) Q_i^{a,b}(y)$, and the bound (2) can be obtained by using $f^{(\tau)}(t) = (t+1)^{\theta}((Q_k^{1,\theta}(t))^2$ in the following theorem [2].

Theorem 1.1 *Let $C \subset \mathcal{M}$ be an $(\mathcal{M}, |C|, \sigma)$-code (reps. τ-design) and let $f(t)$ be a real nonzero polynomial such that*

(A1) $f(t) \leq 0$, *for* $-1 \leq t \leq \sigma$,
(resp. **(B1)** $f(t) \geq 0$, *for* $-1 \leq t \leq 1$*),*

(A2) *the coefficients in the ZSF expansion* $f(t) = \sum_{i=0}^{k} f_i Q_i(t)$
satisfy $f_0 > 0$, $f_i \geq 0$ *for* $i = 1, \ldots, k$.
(resp. **(B2)** *the coefficients in the ZSF expansion* $f(t) = \sum_{i=0}^{k} f_i Q_i(t)$
satisfy $f_0 > 0$, $f_i \leq 0$ *for* $i = \tau+1, \ldots, k$.*)*

Then, $|C| \leq f(1)/f_0$ *(resp.* $|C| \geq f(1)/f_0$*).*

2 Test functions

In some sense, there is a duality between the linear programming bounds for codes and designs in polynomial metric spaces. We see that this is valid also for the corresponding test functions. The next theorem is special case of [3, Theorem 5.39].

Theorem 2.1 *Let α_i are the zeros of $Q_k^{1,\theta}(t)$, and $\alpha_{k+\theta} = -1$ for $\theta = 1$. Then for polynomial $f(t)$ of degree at most $\tau = 2k+\theta$ the following equality holds*

$$f_0 = \frac{f(1)}{D(\mathcal{M},\tau)} + \sum_{i=1}^{k+\theta} \rho_i^{(\tau)} f(\alpha_i), \tag{3}$$

where $\rho_i^{(\tau)}$ are positive numbers s.t. $\sum_{j=0}^{k+\theta} \rho_j^{(\tau)} = 1$.

Proof. Consider for the polynomial $g(t) = (t-1)(t+1)^{\theta} Q_k^{1,\theta}(t)$, having $k+1+\theta$ simple roots, the Lagrange polynomials $l_i(g;t)$, $i = 0,1,\ldots,k+\theta$ of degree $k+\theta$ s.t. $l_i(g;\alpha_j) = \delta_{i,j}$. Then $f(t) - \sum_{i=0}^{k+\theta} f(\alpha_i) l_i(g;t) = g(t).h(t)$, therefore $f_0 = \sum_{i=0}^{k+\theta} f(\alpha_i) \int_{-1}^{1} l_i(g;t) d\nu(t) + \int_{-1}^{1} g(t).h(t) = \sum_{i=0}^{k+\theta} \rho_i^{(\tau)} f(\alpha_i)$. $\diamond$

We consider the following linear functional

$$G_\tau(\mathcal{M}, f) = \frac{f(1)}{D(\mathcal{M},\tau)} + \sum_{i=1}^{k+\theta} \rho_i^{(\tau)} f(\alpha_i) \tag{4}$$

where $\alpha_i, \rho_i^{(\tau)}$ are as in Theorem 2.1.

By Theorem 2.1, we have $-1 \leq G_\tau(\mathcal{M}, f) \leq 1$ and $G_\tau(\mathcal{M}, f) = f_0$ for any polynomial $f(t)$ of degree at most τ and $G_\tau(\mathcal{M}, f) = f(1)/D(\mathcal{M},\tau)$ if $f(t)$ vanishes at the zeros of $f^{(\tau)}(t)$. Note also that if $Q_j(t) = g(t)q(t) + r(t)$, where $g(t)$ is from the proof of Theorem 2.1, then $G_\tau(\mathcal{M}, Q_j) = \int_{-1}^{1} r(t)d\nu(t)$. As we mentioned before analogous functions $G_\sigma(\mathcal{M}, f)$ were introduced in [1] for codes, and $G_\tau(\mathcal{M}, f)$ can be obtained from $G_\sigma(\mathcal{M}, f)$ for $\sigma = t_k^{1,\theta}$.

Theorem 2.2 *Let us denote by $B_{\mathcal{M},\tau}$ the set of polynomials satisfying* **(B1)** *and* **(B2)**. *The bound $D(\mathcal{M}, \tau)$ can be improved by a polynomial $f(t) \in B_{\mathcal{M},\tau}$ of degree at least $\tau + 1$, if and only if $G_\tau(\mathcal{M}, Q_j) < 0$ for some $j \geq \tau + 1$. Moreover, if $G_\tau(\mathcal{M}, Q_j) < 0$ for some $j \geq \tau + 1$, then $D(\mathcal{M}, \tau)$ can be improved by a polynomial $\in B_{\mathcal{M},\tau}$ of degree j.*

Proof. Suppose that $G_\tau(\mathcal{M}, Q_j) \geq 0$ for all $j \geq \tau + 1$. Let us consider polynomial $f(t) \in B_{\mathcal{M},\tau}$ of degree $m \geq \tau$ as follows

$$f(t) = \tilde{g}(t) + \sum_{i=\tau+1}^{m} f_i Q_i(t) = \tilde{g}(t) + F(t), \tag{5}$$

where $\deg(\tilde{g}) \leq \tau$. Then by Theorem 2.1 for $\tilde{g}$, (4) and (5) we have

$$\begin{aligned} f_0 = \tilde{g}_0 = G_\tau(\mathcal{M}, \tilde{g}) &= G_\tau(\mathcal{M}, f) - G_\tau(\mathcal{M}, F) \geq \\ \frac{f(1)}{D(\mathcal{M}, \tau)} - G_\tau(\mathcal{M}, F) &\geq \frac{f(1)}{D(\mathcal{M}, \tau)} \end{aligned}$$

Therefore $D(\mathcal{M}, \tau) \geq \frac{f(1)}{f_0}$ i.e. $f(t)$ does not improve the bound (2).

Conversely, let $G_\tau(\mathcal{M}, Q_j) < 0$ for some fixed $j \geq \tau+1$. Let $-Q_j(t) = f^{(\tau)}(t)a(t)+b(t)$. Consider $f(t) = f^{(\tau)}(t)(a(t)+c) = -Q_j(t)+cf^{(\tau)}(t)-b(t)$, where $c = -\min\{a(t) : t \in [-1, 1]\}$. This choice of c ensures that $f_{\tau+1} = \ldots = f_{j-1} = 0$, $f_j = -1$ and $f(t) \geq 0$. On the other hand we have $f_0 = G_\tau(\mathcal{M}, f) - G_\tau(-Q_j) < \frac{f(1)}{D(M,\tau)}$.$\diamond$

The corresponding test functions for codes were defined and investigated in [1]. In particular, the application of Theorem 2.2 and [1, Theorem 4.9] gives the following corollary.

Corollary 2.3 *For $\mathcal{M} = \mathbf{S}^{n-1}$, $n \geq 3$ fixed and τ even, $\tau > \frac{\sqrt{n-2}}{2}$, the bound $D(\mathbf{S}^{n-1}, \tau)$ can be improved with polynomial of degree $\tau + 3$.*

Theorem 2.4 *Let $\mathcal{M}$ be antipodal. If τ and $j \geq \tau$ are odd, then $G_\tau(\mathcal{M}, Q_j) = 0$.*

This conforms to the linear programming theorem for antipodal designs in antipodal spaces, where we do not pay attention to the ZSF coefficients with odd indices. Note that if $G_\tau(\mathcal{M}, Q_j) > 0$, then we can find polynomial $f(t)$ which is divisible by $f^{(\tau)}(t)$, $f_{\tau+1} = \cdots = f_{j-1} = 0$, $f_j = 1$ and $f(1)/f_0 > D(\mathcal{M}, \tau)$. So, if $G_\tau(\mathcal{M}, Q_j) = 0$ we can not expect to improve the classical bound for τ odd with polynomials of odd degree.

3 Asymptotical behavior of the test functions

Now we investigate the asymptotical behavior of the test functions.

Theorem 3.1.

a1)

$$\lim_{j\to\infty} G_\tau(\mathcal{M}, Q_j) = \begin{cases} \frac{1}{D(\mathcal{M},\tau)} & \text{for } \tau \text{ even} \\ \frac{1}{D(\mathcal{M},\tau)} + Q_k(-1)\rho_{k+1}^{(\tau)} & \text{for } \tau \text{ odd} \end{cases}$$

$$(\rho_{k+1}^{(\tau)} = \frac{1}{D(\mathcal{M},\tau)} \text{ for } \mathcal{M}\text{- antipodal})$$

a2)

$$\lim_{n\to\infty} G_\tau(\mathcal{M}, Q_j) = 0;$$

b)

$$\lim_{j\to\infty} G_\sigma(\mathcal{M}, Q_j) = \begin{cases} \frac{1}{L_{2k}(\sigma)} + Q_k(-1)\rho_0^{(\sigma)} & \text{for } \sigma \in (t_k^{1,0}, t_k^{1,1}) \\ \frac{1}{L_{2k-1}(\sigma)} & \text{for } \sigma \in (t_{k-1}^{1,1}, t_k^{1,0}) \end{cases}$$

As we can see for $\mathcal{M}$, τ fixed there exists a constant $j_0 = j_0(\mathcal{M}, \tau)$ such that $G_\tau(\mathcal{M}, Q_j) \geq 0$ for all $j \geq j_0$. That means that, for fixed $\mathcal{M}$ and τ, we can not expect to obtain better bound if we use polynomial of high degree.

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