Extremal Polynomials for Codes in Polynomial Metric Spaces

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Abstract

Let $\mathcal{M}$ be a **polynomial metric space** (PMS) [2] with metric $d(x,y)$ and standard substitution $t=\sigma(d(x,y))$. Any finite nonempty subset C of $\mathcal{M}$ is called a code. A code for which $\sigma(d(x,y)) \leq \sigma(d) \ \ (x,y \in C)$ and d is the minimum distance of C is an $(\mathcal{M},|C|,\sigma)$-code. We will give some properties of the so called test functions for codes and we will improve the Levenshtein bound with polynomials of degree $h(\sigma)+2$ and $h(\sigma)+3$.

1 Introduction

PMS are *finite* metric spaces represented by P- and Q- polynomial association schemes as well as *infinite* metric spaces, which are the real sphere, the real, complex or quaternions projective space and the Cayley projective plane. On the other hand PMS are distinguished as **antipodal** and **non-antipodal**. Any PMS is connected with a system of constants r_i, a system of orthogonal polynomials $\{Q_i(t)\}$ and **adjacent** system of polynomials $\{Q_k^{a,b}(t)\}$ with roots $-1 < t_{k,i}^{a,b} < 1$, $i=1,\ldots,k$, ordered in increasing order, $t_k^{a,b} = t_{k,k}^{a,b}$. Most of the properties of $\{Q_k^{a,b}(t)\}$ can be found in [2]. By definition $T_k^{a,b}(x,y) = \sum_{i=0}^{k} r_i^{a,b} Q_i^{a,b}(x) Q_i^{a,b}(y)$. Many bounds for the cardinality of codes and designs were obtained by using the Linear Programming Theorem [2, p.544]. If we denote by $A_{\mathcal{M},\sigma}$ the set of real polynomials which satisfy the conditions of the LP Theorem, then $|C| \leq \Omega(f)$, for $f \in A_{\mathcal{M},\sigma}$. We will investigate the Levenshtein bound $L(\mathcal{M},\sigma)$ for codes, which can be presented in the following form [2]:

$$|C| \le L(\mathcal{M},\sigma) = \Omega(f^\sigma(t)) = \Big(1 - \frac{Q^{1,0}_{k-1+\varepsilon}(\sigma)}{Q^{0,\varepsilon}_{k}(\sigma)}\Big) \sum_{i=0}^{k-1+\varepsilon} r_i, \qquad (1)$$

where $\varepsilon = 0$ if $t^{1,1}_{k-1} \le \sigma < t^{1,0}_{k}$ and $\varepsilon = 1$ if $t^{1,0}_{k} \le \sigma < t^{1,1}_{k}$, and $f^\sigma(t) = (t-\sigma)(t+1)^\varepsilon (T^{1,\varepsilon}_{k-1}(t,\sigma))^2$ of degree $h(\sigma)$.

2 Test functions and new bound

Boyvalenkov, Danev and Bumova [1] obtain necessary and sufficient conditions for the optimality of $f^\sigma(t)$ over $A_{\mathcal{M},\sigma}$, introducing the *test functions* $G_\sigma(\mathcal{M},Q_j)$. They prove that the bound (1) can be improved by a polynomial in $A_{\mathcal{M},\sigma}$ of degree j if and only if $G_\sigma(\mathcal{M},Q_j) < 0$. In [3] we define analogous test functions $G_\tau(\mathcal{M},Q_j)$ for designs.

In this section we use the connections between codes and designs and the corresponding test functions. Applying analogous approach as in [3] we investigate the properties of the test functions for codes and derive an analytical form of the polynomials, which improve the Levenshtein bound. For fixed j, $G_\sigma(\mathcal{M},Q_j)$ is a continuous function of σ and $G_\sigma(\mathcal{M},Q_j) \equiv 0$, when $h(\sigma) \ge j$. We examine the sign of $G_\sigma(\mathcal{M},Q_j)$. Let us consider the interval $I_{h(\sigma)} = [t^{1,1-\varepsilon}_{k+\varepsilon-1}, t^{1,\varepsilon}_{k})$ and denote $h(t^{1,1-\varepsilon}_{k+\varepsilon-1}) = \tau$. We have $G_\sigma(\mathcal{M},Q_{h(\sigma)+1}) > 0$.

Lemma 1 *If $G_\tau(\mathcal{M},Q_{\tau+2}) \ge 0$ then $G_\sigma(\mathcal{M},Q_{h(\sigma)+2}) > 0$ for $\sigma \in I_{h(\sigma)}$. If $G_\tau(\mathcal{M},Q_{\tau+e}) < 0$ for $e \ge 2$ then there exist $z_0 < t^{1,1-\varepsilon}_{k+\varepsilon-1}$ and $z_1 > t^{1,1-\varepsilon}_{k+\varepsilon-1}$ such that $G_\sigma(\mathcal{M},Q_{h(\sigma)+e}) < 0$ for $\sigma \in [t^{1,1-\varepsilon}_{e+\varepsilon-1}, z_1)$ and $G_\sigma(\mathcal{M},Q_{h(\sigma)+e+1}) < 0$ for $\sigma \in (z_0, t^{1,1-\varepsilon}_{k+\varepsilon-1}]$.*

In other words there exists an interval $\tilde{I}_\tau = (z_0, z_1)$ for σ, containing $t^{1,1-\varepsilon}_{k+\varepsilon-1}$ in which $G_\sigma(\mathcal{M},Q_{\tau+e})$ is negative, i.e. the Levenshtein bound can be improved in this interval using polynomial of degree $\tau+e$, $e \ge 2$.

Corollary 2 *For antipodal PMS the test function $G_\sigma(\mathcal{M},Q_{h(\sigma)+2})$ is positive.*

As a consequence of the above using our results from [3] we conclude that the smallest possible degree of the improving polynomials is $\tau+2 = h(\sigma)+2$ or $h(\sigma)+3$ for non-antipodal spaces and $\tau+3 = h(\sigma)+3$ or $h(\sigma)+4$ for antipodal PMS. Here we present the analytical form of the polynomial which improve the Levenshtein bound in the non-antipodal case.

Theorem 3 *Let $\mathcal{M}$ be non-antipodal PMS, $\tau = h(t^{1,1-\varepsilon}_{k+\varepsilon-1})$ and let us consider the interval $\tilde{I}_\tau$. Then the polynomial*

$$\begin{aligned} f^\sigma(t;\tau+2) &= (t-\sigma)(t+1)^\varepsilon \ \ [\alpha(T^{1,\varepsilon}_{k-1}(t,\sigma))^2 \\ &+ (\beta_1 T^{1,\varepsilon}_{k-2}(t,\sigma) + \beta_2 T^{0,\varepsilon}_{k-1}(t,\sigma) + T^{0,\varepsilon}_{k}(t,\sigma))^2], \end{aligned}$$

of degree $\tau+2$ belongs to $A_{\mathcal{M},\sigma}$ for constants α, β_1, β_2 satisfying certain conditions.

Now using the LP Theorem with the polynomial $f^\sigma(t;\tau+2)$ we derive new analytical bound $V(\mathcal{M},\sigma)$.

Theorem 4 *If the conditions of Theorem 3 are satisfied then*

$$|C| \leq V(\mathcal{M},\sigma) = \Omega(f^\sigma(t;\tau+2)) \leq L(\mathcal{M},\sigma).$$

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References

[1] P.G.Boyvalenkov, D.P.Danev, S.P.Bumova, Upper bounds on the minimum distance of spherical codes, *IEEE Transactions on Information Theory* 42, 1996, 1576-1581.

[2] V.I.Levenshtein, Universal bounds for codes and designs, Chapter 6 in Handbook of Coding Theory, ed. V.Pless and W.C.Huffman, 1998, Elsevier Science B.V., 449-648.

[3] S.I.Nikova, V.S.Nikov, Improvement of the Delsarte bound for τ-designs when it is not the best bound possible, submitted in *Designs Codes and Cryptography.*