On antipodal spherical 5-designs with n^2+n+4 points

Peter Boyvalenkov*
Institute of Mathematics,
Bulgarian Academy of Sciences,
8 G.Bonchev str., Sofia 1113, Bulgaria
Danyo Danev
Department of Electrical Engineering
Linköping University
Linköping 581-83, Sweden
Svetla Nikova
Department of Mathematics and Computing Sciences
Eindhoven University
P.O. Box 513, 5600 MB, Eindhoven, the Netherlands

Abstract

We obtain restrictions on the structure of n-dimensional antipodal spherical 5-designs with n^2+n+4 points.

1 Introduction

A spherical code $C \subset \mathbf{S}^{n-1}$ is called *spherical τ-design* if the formula

$$\frac{1}{\mu(\mathbf{S}^{n-1})}\int_{\mathbf{S}^{n-1}} f(x)d\mu(x) = \frac{1}{|C|}\sum_{x\in C} f(x)$$

holds for all polynomials $f(x)=f(x_1,x_2,\ldots,x_n)$ of degree at most τ. The maximal number τ for which C is a τ-design is called *strength* of C.

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A code $C \subset \mathbf{S}^{n-1}$ is called *antipodal* if $C = -C$. It follows that any antipodal $(2k)$-design must be actually $(2k+1)$-design. Thus we need to consider only antipodal designs of odd strength.
The spherical designs were inroduced by Delsarte, Goethals and Seidel [5] in 1977. If $C \subset \mathbf{S}^{n-1}$ is a $(2k+1)$-design, then [5, Theorem 5.12]

$$|C| \geq R(n, 2k+1) = 2\binom{n+k-1}{k}. \tag{1}$$

A spherical $(2k+1)$-design is called *tight* if it attains (1). Any tight $(2k+1)$-design must be antipodal [5, Theorem 5.12]. However, tight designs are quite rare (cf. [2, 3, 4]).
Reznick [8] proved that antipodal spherical 5-designs on $\mathbf{S}^{n-1}$ with $n^2 + n + 2 = R(n,5) + 2$ points do not exist. We consider the next case, i.e. antipodal 5-designs with $n^2 + n + 4 = R(n,5) + 4$ points. Note that an antipodal 5-design on $\mathbf{S}^3$ with 24 points is already known [7, Table 9.2].
We use the following equivalent characterization of spherical designs [5, 6]. Let $\{P_i^{(n)}\}_{i=0}^{\infty}$ be the Gegenbauer polynomials [1, Chapter 22], defined by

$$(i+n-2)P_{i+1}^{(n)}(t) = (2i+n-2)tP_i^{(n)}(t) - iP_{i-1}^{(n)}(t),\ i \geq 1,$$

where $P_0^{(n)}(t) = 1$ and $P_1^{(n)}(t) = t$. Any real polynomial $f(t) = \sum_{i=0}^{k} a_i t^i$ is associated with its unique Gegenbauer expansion $f(t) = \sum_{i=0}^{k} f_i P_i^{(n)}(t)$. The coefficient f_0 can be computed by the formula

$$f_0 = a_0 + \frac{a_2}{n} + \frac{3a_4}{n(n+2)} + \cdots. \tag{2}$$

If $C \subset \mathbf{S}^{n-1}$ is a τ-design and $f(t)$ is a real polynomial of degree at most τ, then for any $y \in C$

$$\sum_{x \in C} f(\langle x, y \rangle) = f_0 |C| \tag{3}$$

holds [6], where $\langle x, y \rangle$ is the usual inner product. For antipodal designs, it is enough to consider in (3) only polynomials which are even functions.

2 Restrictions on the inner products

Let $C \subset \mathbf{S}^{n-1}$ be an antipodal 5-design of cardinality $|C| = n^2 + n + 4 = 2(m+1)$. For $y \in C$ we assume that $C = \{x_0, -x_0, x_1, x_2, \ldots, x_m, -x_1, -x_2,$

$\ldots, -x_m\}$, where $x_i = (\xi_{i1}, \xi_{i2}, \ldots, \xi_{in})$. Further, let $\langle x_0, x_i\rangle = t_i \geq 0$, where $0 \leq t_1 \leq t_2 \leq \cdots \leq t_m$ and $I_2(x_0) = \{t_1^2, t_2^2, \ldots, t_m^2\}$. Equation (3) for a polynomial $f(t)$ which is an even function now becomes

$$\sum_{i=1}^{m} f(t_i) = \frac{f_0|C|}{2} - f(1). \tag{4}$$

Using a Reznick's [8, Section 4] idea, we obtain the following restriction on the numbers $t_1, t_2, \ldots, t_m$.

Theorem 1 *Let $u, v, w \in C$ and let $\langle u, v\rangle = a_1$, $\langle v, w\rangle = a_2$, $\langle w, v\rangle = a_3$, where $\{-1, 1\} \cap \{a_1, a_2, a_3\} = \phi$. Then*

$$2n^2(n+2)S_1^2 + n^2(n-4)S_1 - n^2(n+2)(n+4)S_2 + n^3(n+2)^2S_3 - (n-4)^2 = 0, \tag{5}$$

where $S_1 = a_1^2 + a_2^2 + a_3^2$, $S_2 = a_1^2a_2^2 + a_2^2a_3^2 + a_3^2a_1^2$ and $S_3 = a_1^2a_2^2a_3^2$.

Proof. Reznick [8] observed that for every antipodal 5-design C the identity

$$\sum_{k=0}^{m} (\xi_{k1}X_1 + \xi_{k2}X_2 + \cdots + \xi_{kn}X_n)^4 = \frac{3(n^2+n+4)}{2n(n+2)}(X_1^2 + \cdots + X_n^2)^2.$$

must hold for any choise of the variables $X_1, X_2, \ldots, X_n$.

After a rotation we may assume that

$$u = (1, 0, 0, \ldots, 0),\ v = (a_1, \sqrt{1-a_1^2}, 0, \ldots, 0),$$

$$w = (a_2, \frac{a_3 - a_1a_2}{\sqrt{1-a_1^2}}, \frac{\sqrt{1 + 2a_1a_2a_3 - a_1^2 - a_2^2 - a_3^2}}{\sqrt{1-a_1^2}}, \ldots, 0).$$

Using, as Reznick [8, Section 3] did, the theory of the so-called "catalecticants" for the multivariable polynomial

$$\frac{3(n^2+n+4)}{2n(n+2)}(X_1^2 + X_2^2 + \cdots + X_n^2)^2 - X_1^4 - (a_1X_1 + \sqrt{1-a_1^2}X_2)^4$$
$$-(a_2X_1 + \frac{a_3 - a_1a_2}{\sqrt{1-a_1^2}}X_2 + \frac{\sqrt{1 + 2a_1a_2a_3 - a_1^2 - a_2^2 - a_3^2}}{\sqrt{1-a_1^2}}X_3)^4,$$

we obtain a certain determinant which must be vanishing. After a routine manipulation we have to calculate a determinant of order 6 which must be zero. We now use MAPLE to obtain the necessary condition (5). $\diamond$

We point out that (5) is not so bad as it looks. Indeed, after a suitable substitution, it may become $X^2 + Y^2 + Z^2 + 2XYZ = 1$ which seems much better. Unfortunately, Theorem 1 is not strong enough to imply non-existence result as the Reznick's restriction did.

3 Further restrictions

We now employ (4) in the investigations. Set $a = \frac{n-4}{n(n+2)}$, $b = \frac{n+4}{n(n+2)}$.

Lemma 1 *For any point $y \in C$ we have $I_2(y) \subset [a,b]$ (i.e. $a \le t_1^2 \le t_2^2 \le \cdots \le t_m^2 \le b$) for $n \ge 4$ and $I_2(y) \subset [0, 7/15]$ for $n = 3$.*

Proof. Let $y = u$ in Theorem 1. Resolving (5) as a quadratic equation with respect to a_3^2, we see that the discriminat is $f(a_1)f(a_2)$, where $f(t) = (t^2 - a)(t^2 - b)$. Therefore, the numbers a_1^2 and a_2^2 must be simultaneously inside or simultaneoulsy outside the interval $[a,b]$. Suppose that $t_i^2 \notin [a,b]$ for some i. By the above argument we conclude that $t_j^2 \notin (a,b)$ for all $j \in \{1, 2, \ldots, m\}$. However, using the polynomial $f(t)$ in (4) we obtain

$$0 \le \sum_{i=1}^{m} f(t_i) = \frac{f_0|C|}{2} - f(1) = -\frac{4(n^2+n+4)}{n^2(n+2)^2} < 0,$$

which is impossible. $\diamond$

We proceed with a further close examination of Equation (5). We omit the technical proof of the next Lemma.

Lemma 2 a) *If $a_1^2 = a_2^2 = a$ or $a_1^2 = a_2^2 = b$, then $a_3^2 = a$;*
b) *If $a_1^2 = a$ and $a_2^2 = b$, then $a_3^2 = b$;*
c) *If $a_1^2 = a$ and $a_2^2 = 1/n$, then $a_3^2 = a$ or $a_3^2 = 1/n$.*

It follows from Lemma 2 that the design C can be divided into disjoint subsets,

$$C = C_1 \cup C_2 \cup \cdots \cup C_{k-1} \cup C_k,$$

such that for $u, v \in C$, $u \ne -v$, we have $\langle u, v\rangle^2 \in \{a, b\}$ if and only if $u, v \in C_i$ for some $i \in \{1, 2, ..., k-1\}$.

We now consider a particular case which seemed very promissing to give a construction. We give a scetch of the proof.

Theorem 2 *If $I_2(y) \subset [a, 1/n]$ for some point $y \in C$, then $n = 4$.*

Proof. We substitute the polynomial $f(t) = (t^2 - a)(t^2 - 1/n)$ in (4) for the point y. Since $\sum_{i \in I_2(y)} f(t_i) = f_0|C|/2 - f(1) = 0$, we obtain $I_2(y) = \{a, 1/n\}$. Then by Lemma 2c) all sets $I_2(y)$ contain a and $1/n$ only.

We now consider the derived code $C_1 = C(y, 1/\sqrt{n})$. By [5, Theorem xx] (see also [4, p. 280]), C_1 is a 2-design with four possible inner products:

$$s_1 = \frac{1}{\sqrt{n}+1}, s_2 = -\frac{1}{\sqrt{n}-1}, s_3 = \frac{\sqrt{\frac{n(n-4)}{n+2}}+1}{n+1}, s_4 = -\frac{\sqrt{\frac{n(n-4)}{n+2}}+1}{n-1}.$$

We now use (3) for the second degree polynomial $f(t) = (t - s_1)(t - s_2) = t^2 + 2t/(n-1) - 1/(n-1)$. Applying a little algebra, we obtain the equation

$$A\sqrt{\frac{n(n-4)}{n+2}} = B,$$

where A and B are rationals. Since $A = B = 0$ is impossible, this implies that $\sqrt{n(n-4)/(n+2)}$ is rational. By usual argument from number theory one sees that this happens for $n = 4$ only. $\diamond$

Corollary 1 *If $n \geq 5$, then $t_m^2 > 1/n$ for any point $y \in C$. If $n \equiv 1, 2$ (mod 4), then $t_{m-1}^2 > 1/n$ for at least one point $y \in C$.*

Proof. If $t_{m-1}^2 \leq 1/n$ for all points $y \in C$, we can divide the points of C into disjoint quadruples, which is impossible when $n^2 + n \equiv 2$ (mod 4). $\diamond$

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