# Some Characterizations of Spherical Designs with Small Cardinalities


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**Abstract**

We give two new characterizations of spherical designs with small cardinalities. This implies some restrictions on the structure of such designs.


## 1 Introduction

A spherical code $W \subset \mathbf{S}^{n-1}$ is called a spherical $\tau$-design if and only if $\sum_{x \in W} f(x) = 0$ holds for all homogeneous harmonic polynomials $f(x) = f(x_1, x_2, \ldots, x_n)$ of degree 1,2,..., $\tau$ (as usually, $(x, y)$ denotes the standard scalar product in $\mathbf{R}^n$). The spherical designs were introduced in 1977 by Delsarte-Goethals-Seidel [4].
We use the following equivalent definition. A code $W \subset \mathbf{S}^{n-1}$ is a spherical $\tau$-design if and only if

$$\sum_{x \in W} f((x,y)) = |W| f_0 \qquad (1)$$

where $y \in \mathbf{S}^{n-1}$ is an arbitrary point, $f(t)$ is a real polynomial of degree at most $\tau$, and $f_0$ is the first coefficient in the expansion of $f(t) = \sum_{i=0}^{k} f_i P_i^{(n)}(t)$ in terms of the Gegenbauer polynomials [1, Chapter 22].

In fact, we use (1) in the special case when $y$ belongs to the design. Then (1) becomes

$$\sum_{x \in W \setminus \{y\}} f((x,y)) = |W| f_0 - f(1). \tag{2}$$

We shall need some notations and results from [2, 6]. The numbers $\alpha_0 < \alpha_1 \ldots < \alpha_{k-1} = s$ $(-1 \leq \alpha_0$ and $s < 1)$ are all different zeros of certain polynomial $f^{(s)}_{2k-1}(t)$. The positive weights $\rho_i$ ,$i = 0, 1, \ldots, k$ have been defined in [6, Theorem 4.1] ($\alpha_k$=1). Then for any real polynomial $f(t)$ of degree at most $2k - 1$ one has

$$f_0 = \sum_{i=0}^{k} \rho_i f(\alpha_i). \tag{3}$$

Correspondingly, the numbers $-1 = \beta_0 < \beta_1 \ldots < \beta_k = s$ $(s < 1)$ are all different zeros of certain polynomial $f^{(s)}_{2k}(t)$ and $\gamma_i$, $i = 0, 1, \ldots, k+1$ are positive weights. We have $f_0 = \sum_{i=0}^{k+1} \gamma_i f(\beta_i)$ for any real polynomial $f(t)$ of degree at most $2k$ $(\beta_{k+1} = 1)$ [6, Theorem 4.2].
Let $A(n, s)$ denote the maximum cardinality of spherical codes on $\mathbf{S}^{n-1}$ with maximal cosine $s$. Let the numbers $\xi_k$ and $\eta_k$ are the greatest zeros of the Jacobi polynomials $P_k^{(\frac{n-1}{2}, \frac{n-1}{2})}(t)$ and $P_k^{(\frac{n-1}{2}, \frac{n-3}{2})}(t)$ respectively. Then the Levenshtein bound on $A(n, s)$ states [6, 7]

$$A(n,s) \leq \begin{cases} L_{2k-1}(n,s) = 1/\rho_k & \text{for } \xi_{k-1} \leq s \leq \eta_k, \\ L_{2k}(n,s) = 1/\gamma_{k+1} & \text{for } \eta_k \leq s \leq \xi_k. \end{cases} \tag{4}$$

In [2], test functions $Q_j(n, s)$ are introduced for checking if the bound (4) can be improved by linear programming. We have

$$Q_j(n,s) = \begin{cases} \sum_{i=0}^{k} \rho_i P_j^{(n)}(\alpha_i) & \text{for } \xi_{k-1} \leq s \leq \eta_k, \\ \sum_{i=0}^{k+1} \gamma_i P_j^{(n)}(\beta_i) & \text{for } \eta_k \leq s \leq \xi_k. \end{cases} \tag{5}$$

# 2 Two new characterizations of spherical designs

We give two new characterizations of spherical designs with relatively small cardinalities in terms of this function. In fact, we consider designs which would come after the tight spherical designs. Hardin-Sloane [5] have conjectured that such designs are very rare (see also [8]).

We consider in detail the odd strength $2k-1$ only.

**Theorem 2.1** *Let $n \geq 3$ and $s \in (\xi_{k-1}, \eta_k)$ are fixed and $W \subset \mathbf{S}^{n-1}$ is a spherical $(2k-1)$-design with cardinality $|W| = L_{2k-1}(n,s)$. Then*
(i) *$|W|Q_j(n,s) = 1 + \sum_{x\in W\setminus\{y\}} f((x,y))$ holds for any $j \geq 2k$, any polynomial $f(t)$ of degree at most $2k-1$ such that $f(\alpha_i) = P_j(\alpha_i)$ for $i = 0,1,\ldots,k-1$, and any point $y \in W$.*
(ii) *$\sum_{x\in W\setminus\{y\}} f((x,y)) = 0$ holds for any polynomial $f(t)$ of degree at most $2k-1$ such that $f(\alpha_i) = 0$ for $i = 0,1,\ldots,k-1$ and any point $y \in W$.*

*Proof.* We consider polynomials $g(t) = f(t) - P_j^{(n)}(t)$ where $deg(f) \leq 2k-1$ and $f(\alpha_i) = P_j(\alpha_i)$ for $i = 0,1,\ldots,k-1$. Then by (3) we have

$$g_0 = f_0 = \sum_{i=0}^{k} \rho_i f(\alpha_i) = \rho_k g(1) + Q_j(n,s). \tag{6}$$

(i) Let $W$ is a $(2k-1)$-design. Then by (2) and (6) we have

$$|W|Q_j(n,s) = f_0|W| - g(1) = f_0|W| - (f(1)-1) = 1 + \sum_{x\in W\setminus\{y\}} f((x,y)).$$

(ii) By (3) we have $f_0|W| = f(1)$ and then (2) gives

$$\sum_{x\in W\setminus\{y\}} f((x,y)) = 0.$$

**Corollary 2.3** *Let $n \geq 3$ and $s \in (\xi_{k-1}, \eta_k)$ are fixed and $W \subset \mathbf{S}^{n-1}$ is a spherical $(2k-1)$-design with cardinality $|W| = L_{2k-1}(n,s)$. Then for any point $y \in W$ and any $i \in \{-1,0,1,\ldots,k-1\}$ ($\alpha_{-1} = -1$ there exists point $x \in W$ such that $(x,y) \in [\alpha_i, \alpha_{i+1}]$. If for some $y \in W$ and $i$ there exists no point $x \in W$ such that $(x,y) \in (\alpha_i, \alpha_{i+1})$ then $W$ is a maximal spherical code.*

*Proof.* For $0 \leq i \leq k-2$, we apply Theorem 2.1 (iii) with

$$f(t) = (1/(t-\alpha_i)(t-\alpha_{i+1})) \prod_{j=0}^{k-1} (t-\alpha_j)^2.$$

For $i = -1$ we take $f(t) = (t-\alpha_0)\prod_{j=1}^{k-1}(t-\alpha_j)^2$, and for $i = k-1$ we take $f(t) = f(t) = (t-\alpha_{k-1})\prod_{j=0}^{k-2}(t-\alpha_j)^2 = f_{2k-1}^{(s)}(t)$.
We formulate the corresponding assertions for the even strength $2k$.

**Theorem 2.3** *Let* $n \geq 3$ *and* $s \in (\eta_k, \xi_k)$ *are fixed and* $W \subset \mathbf{S}^{n-1}$ *is a spherical* $(2k)$*-design with cardinality* $|W| = L_{2k}(n,s)$*. Then*
(i) $|W|Q_j(n,s) = 1 + \sum_{x \in W \setminus \{y\}} f((x,y))$ *holds for any* $j \geq 2k+1$*, any polynomial* $f(t)$ *of degree at most* $2k$ *such that* $f(\beta_i) = P_j(\beta_i)$ *for* $i = 0, 1, \ldots, k$*, and any point* $y \in W$*.*
(ii) $\sum_{x \in W \setminus \{y\}} f((x,y)) = 0$ *holds for any polynomial* $f(t)$ *of degree at most* $2k$ *such that* $f(\beta_i) = 0$ *for* $i = 0, 1, \ldots, k$ *and any point* $y \in W$*.*

**Corollary 2.4** *Let* $n \geq 3$ *and* $s \in (\eta_k, \xi_k)$ *are fixed and* $W \subset \mathbf{S}^{n-1}$ *is a spherical* $(2k)$*-design with cardinality* $|W| = L_{2k}(n,s)$*. Then for any point* $y \in W$ *and any* $i \in \{0, 1, \ldots, k\}$ *there exists point* $x \in W \setminus \{y\}$ *such that* $(x,y) \in (\beta_i, \beta_{i+1})$*.*

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