Improvements of the lower bounds on the size of some spherical designs

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Abstract

New lower bounds for the cardinality of some spherical designs are given. The bounds are obtained by a new method for obtaining good polynomials required in a linear programming bound due to Delsarte, Goethals and Seidel. The polynomials we have found are $B_{n,\tau}$-extremal, i.e. they are the best between the polynomials of the same or lower degrees.

1 Introduction

A non-empty finite subset W of the $n$dimensional Euclidean sphere $\mathbf{S}^{n-1}$ is called a spherical τ-design on $\mathbf{S}^{n-1}$ iff $\sum_{x\in W} f(x) = 0$ for all homogeneous harmonic polynomials $f(x) = f(x_1, x_2, ..., x_n)$ of degree $1, 2, ..., \tau$. These designs were introduced in 1977 by Delsarte, Goethals and Seidel [1]. They obtained the following lower bound for the cardinality of a τ-design:

$$|W| \geq \begin{cases} \binom{n+e-1}{n-1} + \binom{n+e-2}{n-1} & \text{if } \tau = 2e; \\ 2\binom{n+e-1}{n-1} & \text{if } \tau = 2e+1. \end{cases} \tag{1}$$

Bannai and Damerell [3, 4] proved that the bound (1) can not be attained for $\tau = 2e \geq 6$ and for $\tau = 2e + 1 \geq 9$ except for $\tau = 11$, $n = 24$ and the unique [5] design formed by the minimum norm vectors in the Leech lattice.

On the other hand, Seymour and Zaslavsky [2] have shown that spherical τ-designs on $\mathbf{S}^{n-1}$ exist for all values of n and τ.

In this paper we propose a method for obtaining new lower bounds improving (1) in some cases. In fact, we find good polynomials for the linear programming bound by Delsarte, Goethals and Seidel [1]. We give some applications for $\tau = 4$, $n = 3, 4, 5$; $\tau = 6$, $4 \leq n \leq 10$; $\tau = 7$, $n = 5, 6, 7$; $\tau = 8$, $4 \leq n \leq 17$; and $\tau = 11$, $4 \leq n \leq 24$. Our method is analogous to the method proposed in [7] (see also [8,9]) for obtaining upper bounds for spherical codes. The polynomials we have found are $B_{n,\tau}$-extremal (see Definition 2 below or [7]).

2 Linear programming bounds for spherical designs

The Gegenbauer polynomials are defined by

$$P_0^{(n)}(t) = 1, \quad P_1^{(n)}(t) = t,$$

$$(i + n - 2)P_{i+1}^{(n)}(t) = (2i + n - 2)tP_i^{(n)}(t) - iP_{i-1}^{(n)}(t) \text{ for } i \geq 1.$$

To obtain lower bounds on the size of the spherical designs we use the following theorem (Linear Programming Bound for spherical designs; Delsarte-Goethals-Seidel [1, Theorem 5.10]).

Theorem 1. *Let $f(t)$ be a real polynomial such that*
(A1) *$f(t) \geq 0$ for $-1 \leq t \leq 1$,*
and
(A2) *The coefficients in the expansion of $f(t)$ in terms of the Gegenbauer polynomials $f(t) = \sum_{i=0}^{k} f_i P_i^{(n)}(t)$ satisfy $f_{\tau+1} \leq 0, \ldots, f_k \leq 0$.*
Then the cardinality of a spherical $\tau-$design $W \subset \mathbf{S}^{n-1}$ is bounded from below by

$$\mid W \mid \geq f(1)/f_0.$$

It is to be noted that

$$f_0 = c_n \int_{-1}^{1} f(t)(1 - t^2)^{\frac{n-3}{2}} dt > 0, \tag{2}$$

where

$$c_n = (\int_{-1}^{1} (1 - t^2)^{\frac{n-3}{2}} dt)^{-1} > 0.$$

The next lemma gives us a useful expression of f_0 by the coefficients of $f(t)$.

Lemma 1. *If $f(t) = \sum_{i=0}^{k} a_i t^i = \sum_{i=0}^{k} f_i P_i^{(n)}(t)$ is a real polynomial then*

$$f_0 = a_0 + \frac{a_2}{n} + \frac{3a_4}{n(n+2)} + \cdots = a_0 + \sum_{i=1}^{[k/2]} \frac{(2i-1)!!a_{2i}}{n(n+2)\cdots(n+2i-2)} \tag{3}$$

The proof is straightforward by using (2).

Definition 1. Let $B_{n,\tau} = \{f(t) : \ f(t)$ satisfies the conditions of Theorem 1$\}$, and $L(f) = f(1)/f_0$ for $f(t) \in B_{n,\tau}$.

Definition 2. A polynomial $f(t) \in B_{n,\tau}$ of degree k is called $B_{n,\tau}-$extremal if

$$L(f) = \max\{L(g) : \ g(t) \in B_{n,\tau}, \ \deg(g) \leq k\}.$$

For example, if $f(t) \geq 0$ for $-1 \leq t \leq 1$, and $\deg(f) \leq \tau$, then $f(t) \in B_{n,\tau}$. Such (extremal) polynomials of degree τ were used in [1] for obtaining the bound (1). Here we consider $B_{n,\tau}$-extremal polynomials of higher degrees [7].

The next theorem concerns the number of the double zeros of extremal polynomials.

Theorem 2. *Let $f(t)$ be a $B_{n,\tau}$-extremal polynomial of degree $k \geq \tau + 3$.*
i) *If τ is odd or if τ is even and -1 is an even zero of $f(t)$, then $f(t)$ has at least $[\frac{\tau}{2}] + 1$ double zeros in $[-1, 1]$.*
ii) *If τ is even and -1 is an odd zero of $f(t)$, then $f(t)$ has at least $\frac{\tau}{2}$ double zeros in $[-1, 1]$.*

Proof: Let us suppose that $f(t) = A^2(t)G(t)$, where $2\deg(A) \leq \tau$, $G(t) \geq 0$ for $-1 \leq t \leq 1$, $G(1) > 0$, $A(t)$ has $\deg(A)$ zeros in $[-1, 1]$, and $G(t)$ has no double zeros in $[-1, 1]$. We shall consider two cases.

Case 1. $G(-1) > 0$.

There exists $\varepsilon > 0$ such that $G(t) \geq \varepsilon > 0$ for $t \in [-1, 1]$. Let us consider the polynomial

$$P_\varepsilon(t) = f(t) - \varepsilon A^2(t) = A^2(t)(G(t) - \varepsilon) \geq 0$$

for $-1 \leq t \leq 1$ (i.e. the condition (A1) is satisfied for $P_\varepsilon(t)$). Let $f(t) = \sum_{i=0}^{k} f_i P_i^{(n)}(t)$ and $P_\varepsilon(t) = \sum_{i=0}^{k} f_i(P_\varepsilon) P_i^{(n)}(t)$. Then we have $f_i(P_\varepsilon) = f_i$ for $i \geq 2\deg(A) + 1$. In particular, $f_i(P_\varepsilon) = f_i \leq 0$ for $i \geq \tau + 1 \geq 2\deg(A) + 1$. Thus $P_\varepsilon(t) \in B_{n,\tau}$.

Since $A^2(t) \in B_{n,\tau}$ we have $L(A^2) < L(f)$. But one can easily check that this inequality is equivalent to

$$\frac{P_\varepsilon(1)}{f_0(P_\varepsilon)} = \frac{f(1) - \varepsilon A^2(1)}{f_0 - \varepsilon f_0(A^2)} > \frac{f(1)}{f_0} \ ,$$

a contradiction. This proves the theorem in the case when -1 is even zero of $f(t)$.

Case 2. $G(-1) = 0$.

We have $f(t) = A^2(t)(t+1)G_1(t)$, where $G_1(t) > 0$ for $-1 \leq t \leq 1$ (otherwise $G_1(-1) = 0$ and -1 would be a double zero of $G(t)$). There exists $\varepsilon > 0$ such that $G_1(t) > \varepsilon > 0$ for $t \in [-1, 1]$.

If τ is odd and $B(t) = A^2(t)(t+1)$, then we have $\deg(B) = 2\deg(A) + 1 \leq \tau$. Therefore $B(t) \in B_{n,\tau}$. Going further one can obtain a contradiction as in the first case.

If τ is even and $\deg(A) \leq \tau/2 - 1$ we get a contradiction by a similar argument. This completes the proof.

We shall restrict ourselves to search for extremal polynomials of degree $\tau + 3$. These polynomials will have a form

$$f(t) = \begin{cases} A^2(t)[q(t+1) + 1 - t] & \text{if } \tau = 2e \\ A^2(t)[q(t-p)^2 + 1 - t^2] & \text{if } \tau = 2e + 1 \end{cases},$$

where $\deg(A) = e + 1$ and $0 < q < 1$, $-1 \leq p < 1$. The polynomial $A(t)$ has a leading coefficient 1 and $e + 1$ zeros in $[-1, 1]$.

In the Gegenbauer expansion $f(t) = \sum_{i=0}^{k} f_i P_i^{(n)}(t)$ we require $f_{\tau+1} = f_{\tau+2} = 0$ [7, Theorem 3.4]. Using these conditions we can express two of the coefficients of $A(t)$ as functions of the remaining parameters q and n (p, q, and n).

Next we consider $F = f(1)/f_0$ as a function of the unknown coefficients q and n (p, q, and n). Using equations obtained by equating the partial derivatives of F to zero, one can express the remaining coefficients of $A(t)$ as functions of q and n. The formula from Lemma 1 is very useful here.

It does not seem possible to use further analytical methods. So we search by a computer using a Monte Carlo method for $q \in (0, 1)$ (or for q and $p \in [-1, 1)$) in order to maximize the ratio $f(1)/f_0$. The computer calculations were made for a few seconds on a PC.

3 Some results for $4 \leq \tau \leq 8$

Below we give the application of our method for $4 \leq \tau \leq 8$ and corresponding dimensions.

Case 1. $\tau = 4$.

In this case we improve (1) in three dimensions by one. It was known that (1) can not be attained in these dimensions. Thus we have not obtained actually new bounds. However, it is easier to explain our approach in this small case.

We consider polynomials of degree 7 having the following form

$$f(t) = (t^3 + at^2 + bt + c)^2[q(t+1) + 1 - t] = \sum_{i=0}^{7} f_i P_i^{(n)}(t),$$

where $f_5 = f_6 = 0$ and $0 < q < 1$ (the last implies $f_7 < 0$). By $f_5 = f_6 = 0$ one can express

$$a = \frac{q+1}{2(1-q)},$$

$$b = \frac{3a^2}{2} - \frac{21}{2(n+10)}.$$

Using (3) we obtain

$$f_0 = f_0(c,q,n) = c^2(q+1) + \frac{1}{n}[2bc(q-1) + (b^2+2ac)(1+q)] +$$

$$\frac{3}{n(n+2)}[(2ab+2c)(q-1) + (a^2+2b)(1+q)] = \alpha c^2 + \beta c + \gamma,$$

where α, β and γ are functions of the parameters q and n. We introduce the function $F(c,q,n) = f(1)/f_0(c,q,n)$ and obtain from the equality $F'_c = 0$ the following equation

$$2f_0 - f'_{0c}(1+a+b+c) = 0. \tag{4}$$

From (4) one can easily express

$$c = \frac{2\gamma - \beta(1+a+b)}{2\alpha(1+a+b) - \beta} = c(q,n).$$

Finally, n is fixed, and we have to search for $q \in (0,1)$ maximizing $f(1)/f_0$. The best polynomials we have obtained in dimensions 3, 4 and 5 give bounds 10, 15, and 21 respectively while (1) gives 9, 14, and 21. The smallest values for which 4−designs have been found by Hardin and Sloane [10] are 12, 20, and 29 respectively.

Case 2. $\tau = 6$.

We consider polynomials of degree 9,

$$f(t) = (t^4 + at^3 + bt^2 + ct + d)^2[q(t+1) + 1 - t] = \sum_{i=0}^{9} f_i P_i^{(n)}(t),$$

where $f_7 = f_8 = 0$ and $0 < q < 1$.

Similarly to the previous case we express a, b, c and d as functions of q and n. The new bounds we have obtained are given in Table 1. Delsarte, Goethals and Seidel proved in [1, Theorem 7.7] that the bound (1) can not be attained for $\tau = 6$ and $n \geq 3$ (see also [3, Theorem 1]). Therefore, only improvements by more than 1 of the bound (1) are really of interest. We obtain such improvements in dimensions $4 \leq n \leq 10$.

Table 1. New lower bounds for the cardinality of spherical 6-designs on $\mathbf{S}^{n-1}$, $4 \leq n \leq 10$.

n	[1,Th.7.7]	New bounds
4	31	32
5	51	54
6	78	84
7	113	121
8	157	167
9	211	221
10	276	283

Case 3. $\tau = 7$.

In this case we work with polynomials of degree 10,

$$f(t) = (t^4 + at^3 + bt^2 + ct + d)^2[q(t-p)^2 + 1 - t^2] = \sum_{i=0}^{10} f_i P_i^{(n)}(t),$$

where $f_8 = f_9 = 0$, $0 < q < 1$ and $-1 \leq p \leq 1$. One can express the coefficients a, b, c and d as functions of p, q and n. Maximizing, we obtain new bounds in dimensions 5, 6 and 7.

Table 2. New lower bounds for the cardinality of spherical 7-designs on $\mathbf{S}^{n-1}$, $5 \leq n \leq 7$.

n	[4,Th.1]	New bounds
5	71	73
6	113	116
7	169	172

Case 4. $\tau = 8$.

We use polynomials of the form

$$f(t) = (t^5 + at^4 + bt^3 + ct^2 + dt + e)^2[q(t+1) + 1 - t] = \sum_{i=0}^{11} f_i P_i^{(n)}(t)$$

($f_9 = f_{10} = 0$ and $0 < q < 1$) to obtain new lower bounds improving (1) by more than 1 in dimensions $4 \leq n \leq 17$. The results are given in Table 3 below.

Table 3. New lower bounds for the cardinality of spherical 8-designs on $\mathbf{S}^{n-1}$, $4 \leq n \leq 17$.

n	[3,Th.1]	New bounds
4	56	59
5	106	115
6	183	203
7	295	332
8	451	511
9	661	750
10	936	1060
11	1288	1450
12	1730	1930
13	2276	2507
14	2941	3191
15	3741	3989
16	4630	4908
17	5815	5951

For 5−designs we have found $B_{n,5}$-extremal polynomials of degree 7 that give again the bound (1). It is to be noted that in this case only we obtain infinitely many non-proportional extremal polynomials (see [7]).

4 New bounds for 11-designs

We use polynomials of degree 14,

$$f(t) = (t^6 + at^5 + bt^4 + ct^3 + dt^2 + et + f)^2[q(t+1) + 1 - t](t+1) = \sum_{i=0}^{14} f_i P_i^{(n)}(t)$$

where $f_{12} = f_{13} = 0$ and $0 < q < 1$.

The new bounds in dimensions $4 \leq n \leq 23$ are presented in Table 4. For $n = 24$ we obtain the bound (1) again. It is attained by the 11-design formed by the minimum norm vectors in the Leech lattice [1]. This design is unique up to isometry [5]. Our result by a polynomial of degree 14 shows that the Leech lattice has an index 14 [11].

Table 4. New lower bounds on the cardinality of the spherical 11-designs on $\mathbf{S}^{n-1}$, $4 \leq n \leq 23$.

n	[4,Th.1]	New bounds
4	112	117
5	252	270
6	504	552
7	924	1035
8	1584	1808
9	2574	2985
10	4004	4701
11	6006	7117
12	8736	10413
13	12376	14790
14	17136	20464
15	23256	27664
16	31008	36623
17	40698	47574
18	52668	60744
19	67298	76344
20	85008	94566
21	106260	115577
22	131560	139514
23	161460	166483

It is to be noted that the regular polytope (3,3,5) in R^4 [12, 13] is an 11-design with 120 points [1]. So in this case our new bound seems reasonably tight.

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