On Lower Bounds on the Size of Designs in Compact Symmetric Spaces of Rank 1

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Abstract

The concept of t-designs in compact symmetric spaces of rank 1 is a generalization of the theory of classical t-designs. In this paper we obtain new lower bounds on the cardinality of designs in projective compact symmetric spaces of rank 1. With one exception our bounds are the first improvements of the classical bounds by more than one. We use the linear programming technique and follow the approach we have proposed for spherical codes and designs. Some examples are shown and compared with the classical bounds.

1 Introduction

We consider t-designs in the projective spaces $\mathbb{F}\mathrm{P}^{n-1}$ of lines through the origin in $\mathbb{F}^n$ where $\mathbb{F}$ denotes the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$, or the Cayley octonions $\mathbb{O}$. Together with the Euclidean spheres $\mathbf{S}^{n-1}$ they constitute all connected compact symmetric spaces of rank 1. For a detailed description of these spaces we refer to [10, 13]. More general references are [9, 17, 18].

The basic property of the compact symmetric spaces of rank 1 is the decomposition

$$L^2(\mathbb{F}\mathrm{P}^{n-1}) = \oplus_{i\geq 0} V_i$$

where the mutually orthogonal subspaces V_i are spaces of continuous functions. The spaces V_i are naturally ordered and V_0 is the space of constant functions.

Definition 1.1 A finite set $X \subset \mathbb{F}\mathrm{P}^{n-1}$ is called a *t-design* if and only if

$$\sum_{x\in X} f(x) = 0 \quad \text{for all} \quad f(x) \in V_1 \oplus V_2 \oplus \cdots \oplus V_t.$$

The following equivalent definition shows that t-designs can be considered as sets of nodes for Chebyshev type quadratures in the compact symmetric spaces of rank 1.

Definition 1.2 A finite set $X \subset \mathbb{F}\mathrm{P}^{n-1}$ is called a t-design if and only if

$$\frac{1}{|X|}\sum_{x\in X} f(x) = \int_{\mathbb{F}\mathrm{P}^{n-1}} f(x)d\mu(x) \quad \text{for all} \quad f(x) \in V_1 \oplus V_2 \oplus \cdots \oplus V_t$$

(the measure μ is normalized, i.e. $\mu(\mathbb{F}\mathrm{P}^{n-1}) = 1$).

Any space $\mathbb{F}\mathrm{P}^{n-1}$ is related to the so-called zonal spherical functions [10, 17], which are Jacobi polynomials [1, Chapter 22] $P_i^{(\alpha,\beta)}(z)$, $i = 0, 1, \ldots$, normalized by $P_i^{(\alpha,\beta)}(1) = 1$, where

$$(\alpha, \beta) = (mn - m - 1, m - 1), \tag{1}$$

and $2m$ is the dimension of $\mathbb{F}$ over $\mathbb{R}$ (cf. [10, Theorem 2.11]). In fact, the polynomial $P_i^{(\alpha,\beta)}(z)$ corresponds to the space V_i for any $i \geq 0$.

Any real polynomial $f(z) = \sum_{i=0}^{k} a_i z^i$ is associated with its Jacobi expansion $f(z) = \sum_{i=0}^{k} f_i P_i^{(\alpha,\beta)}(z)$ for well-defined Jacobi coefficients f_i. We are mainly interested in the coefficient f_0. It can be computed by the following formula

$$f_0 \int_{-1}^{1} (1-z)^{\alpha}(1+z)^{\beta} dz = \int_{-1}^{1} f(z)(1-z)^{\alpha}(1+z)^{\beta} dz. \tag{2}$$

In terms of zonal spherical functions, the following equivalent definition can be given.

Definition 1.3 A finite subset $X \subset \mathbb{F}\mathrm{P}^{n-1}$ is a t-design in $\mathbb{F}\mathrm{P}^{n-1}$ if and only if $\sum_{x \in X} P_k^{(\alpha,\beta)}(|(x,y)|^2) = 0$ holds for all $y \in X$, $k = 1, 2, \ldots, t$.

Seymor and Zaslavski [16] have shown the existence of t-designs in $\mathbb{F}\mathrm{P}^{n-1}$ for any t, $\mathbb{F}$, and n provided $|X|$ is sufficiently large. In this paper we are interested mainly in the lower bounds for the minimum possible size of t-designs.

Definition 1.4 $D(t, n, \mathbb{F}) = \min\{|X| : X \text{ is a } t\text{-design in } \mathbb{F}\mathrm{P}^{n-1}\}$.

Lower bounds on $D(t, n, \mathbb{F})$ are obtained usually by linear programming (cf. Dunkl [8], Hoggar [10]). The explicit form of the classical lower bounds is [2, 6, 8]

$$D(t, n, \mathbb{F}) \geq R(t, n, \mathbb{F}) = \begin{cases} \dfrac{(mn)_k \cdot (mn - m + 1)_k}{(m)_k \cdot k!} & \text{for } t = 2k, \\ \dfrac{(mn)_{k+1} \cdot (mn - m + 1)_k}{(m)_{k+1} \cdot k!} & \text{for } t = 2k + 1, \end{cases} \tag{3}$$

where m is as above and $(p)_a = p(p+1)\cdots(p+a-1)$ for $a \in \mathbb{N}$ ($(p)_0 = 1$).

Designs which attain this bound are called tight. The theory of tight designs is well developed and their classification is near to being completed. Bannai and Hoggar [2, 3, 10, 11, 12] proved that tight t-designs in the projective spaces do not exist for $t \geq 6$ and $t = 4$ in dimensions $n \geq 3$. In particular, in these cases the bound (3) is improved by one.

In this paper we apply a linear programming technique to improve (3). In many cases we are able to find good polynomials required in the so-called linear programming bound. With one exception our bounds are the first improvements of (3) by more then one.

2 A method for obtaining new lower bounds on the cardinality of designs in $\mathbb{F}\mathrm{P}^{n-1}$

2.1 Properties of extremal polynomials

The following theorem gives the so-called linear programming bound on the size of designs in the projective spaces $\mathbb{F}\mathrm{P}^{n-1}$. It can be extracted, for instance, from Delsarte-Goethals-Seidel [6], Dunkl [8], Hoggar [10], and Levenshtein [13].

Theorem 2.1 *Let $f(z)$ be a nonzero real polynomial such that*
(A1) *$f(z) \geq 0$ for $-1 \leq z \leq 1$, and*
(A2) *The coefficients in the Jacobi expansion $f(z) = \sum_{i=0}^{k} f_i P_i^{(\alpha,\beta)}(z)$ (α, β are given by (1)) satisfy $f_{t+1} \leq 0,\ldots,f_k \leq 0$.*
Then $D(t,n,\mathbb{F}) \geq \dfrac{f(1)}{f_0}$.

Remark. Our condition that $f(z)$ is a nonzero polynomial implies $f_0 > 0$ which is given as a hypothesis in some sources.

The classical bound (3) can be obtained by certain polynomials, $f_{\mathbb{F}}^{t,n}(z)$ say, of degree t (cf. [2, 3]). In general, the optimal polynomials for applying in Theorem 2.1 are unknown. However, it is known [8, 15] that the polynomials $f_{\mathbb{F}}^{t,n}(z)$ are the best among the polynomials of the same or lower degree satisfying the conditions (A1) and (A2). In this paper we propose a method for finding good polynomials of degree $t+2$. These polynomials ensure improvements of the bound (3) in some cases. Our approach follows a similar argument to the one we have applied for obtaining new bounds for some spherical designs [4, 5].

Definition 2.2 $B_{\mathbb{F}}^{t,n} = \{f(z) \in \mathbb{R}[z] : f(z)$ satisfies (A1) and (A2)$\}$.

Definition 2.3 A polynomial $f(z)$ is called $B_{\mathbb{F}}^{t,n}$-extremal if it gives the best lower bound for $D(t,n,\mathbb{F})$ among the polynomials of the same or lower degree from $B_{\mathbb{F}}^{t,n}$.

Searching for good polynomials of degree $k \geq t+2$, we require $f_{t+1} = 0$ in the Jacobi expansion of $f(z)$. A more general setting for this situation is given by the following assertion.

Theorem 2.4 *Let $j > t$ be an integer and $f_j(z)$ be a real polynomial such that*
(B1) *$f_j(z) \geq 0$ for $-1 \leq z \leq 1$, and*
(B2) *The coefficients in the Jacobi expansion $f_j(z) = \sum_{i=0}^{k} f_{i,j} P_i^{(\alpha,\beta)}(z)$ (α, β are given by (1)) satisfy $f_{i,j} \leq 0$ for $i \neq j$ and $t+1 \leq i \leq k$, and $f_{j,j} > 0$.*
Then any $B_{\mathbb{F}}^{t,n}$-extremal polynomial $f(z)$ of degree $k_1 \geq k$ such that $f(1)/f_0 < f_j(1)/f_{0,j}$ has $f_j = 0$.

Proof. Let us suppose that $f_j < 0$ under the assumptions of the theorem. There exist linear combinations $g_{a,b}(z) = af(z)+bf_j(z) = \sum_{i=0}^{k_1} g_i P_i^{(\alpha,\beta)}(z)$ with $a,\ b$ positive, such that $g_j \leq 0$. Such polynomials belong to $B_{\mathbb{F}}^{t,n}$. However, the number $g_{a,b}(1)/g_0$ lies in the interval $(f(1)/f_0, f_j(1)/f_{0,j})$, a contradiction because $f(z)$ is $B_{\mathbb{F}}^{t,n}$-extremal.

Corollary 2.5 *Let $f(z)$ be a $B_{\mathbb{F}}^{t,n}$-extremal polynomial of degree $k \geq t+2$. If $f(1)/f_0 < R(t+1,n,\mathbb{F})$ then $f_{t+1} = 0$ in the Jacobi expansion $f(z) = \sum_{i=0}^{k} f_i P_i^{(\alpha,\beta)}(z)$.*

Proof. The polynomial $f_{\mathbb{F}}^{t+1,n}(z)$ is applied in Theorem 2.4 ($j = t+1 = \deg(f_{\mathbb{F}}^{t+1,n})$).

Corollary 2.6 *There exist no extremal polynomials of degree $k \geq t+1$ which give $f(1)/f_0 < R(t+1,n,\mathbb{F})$.* Thus, looking for improvements of (3) which are less than $R(t+1,n,\mathbb{F})$, one can assume $f_{t+1} = 0$ without loss of generality.

Another important problem concerning the form of extremal polynomials asks for the number of their double zeros. The polynomials $f_{\mathbb{F}}^{t,n}(z)$ have $[t/2]$ double zeros. The following theorem shows that the higher degree polynomials must have one double zero more for t odd and again $t/2$ double zeros for t even.

Theorem 2.7 *Let $f(z)$ be a $B_{\mathbb{F}}^{t,n}$-extremal polynomial of degree $k \geq t+2$.*
a) *If t is odd then $f(z)$ has at least $(t+1)/2$ double zeros in the interval $[-1,1]$.*
b) *If t is even then $f(z)$ has at least $t/2$ double zeros in the interval $[-1,1]$.*

Proof. We can write $f(z) = A^2(z)G(z)$, where $G(z) \geq 0$ for $-1 \leq z \leq 1$, $A(z)$ has $\deg(A)$ zeros in $[-1,1]$, and $G(z)$ has no double zeros in $[-1,1]$. Thus $G(z) = 0$ is possible only for $z = -1$. We consider two cases.

Case 1. $G(-1) > 0$.

Let us suppose that $2\deg(A) \leq t-1$ contradicting the assertion. There exists $\varepsilon > 0$ such that $G(z) \geq \varepsilon > 0$ for all $z \in [-1,1]$. We consider the nonzero polynomial

$$P_\varepsilon(z) = f(z) - \varepsilon A^2(z) = A^2(z)(G(z) - \varepsilon) \geq 0$$

for $-1 \leq z \leq 1$ (i.e. the condition (A1) is satisfied for $P_\varepsilon(z)$). Let $f(z) = \sum_{i=0}^{k} f_i P_i^{(\alpha,\beta)}(z)$ and $P_\varepsilon(z) = \sum_{i=0}^{k} f_i(P_\varepsilon) P_i^{(\alpha,\beta)}(z)$. Then we have $f_i(P_\varepsilon) = f_i$ for $i \geq 2\deg(A) + 1$. In particular, $f_i(P_\varepsilon) = f_i \leq 0$ for $i \geq t+1$ (since $t \geq 2\deg(A) + 1$). Therefore $P_\varepsilon(z) \in B_{\mathbb{F}}^{t,n}$.

Since $A^2(z) \in B_{\mathbb{F}}^{t,n}$ and $\deg(A^2) \leq t-1$ we have $A^2(1)/f_0(A^2) < f(1)/f_0$ [8, 15]. But one can easily check that this inequality is equivalent to

$$\frac{P_\varepsilon(1)}{f_0(P_\varepsilon)} = \frac{f(1) - \varepsilon A^2(1)}{f_0 - \varepsilon f_0(A^2)} > \frac{f(1)}{f_0} \ ,$$

a contradiction.

Case 2. $G(-1) = 0$.

We set $f(z) = A^2(z)(z+1)G_1(z)$, where $G_1(z) > 0$ for $-1 \leq z \leq 1$. If t is odd and $2\deg(A) \leq t-1$, then we apply the same argument as in Case 1 with $A^2(z)(z+1) \in B_{\mathbb{F}}^{t,n}$ instead of $A^2(z)$. If t is even, then $2\deg(A) \leq t-2$ ensures a contradiction by our argument. This completes the proof.

Corollary 2.8 *If t is even and $f(z)$ is a $B_{\mathbb{F}}^{t,n}$-extremal polynomial of degree $t+2$ then $f(-1) = 0$.*

Proof. By Theorem 2.7 the polynomial $f(z)$ has at least $t/2$ double zeros. In fact, their number must be exactly $t/2$, otherwise $f(z)$ is a square and its leading Jacobi coefficient is positive, contradicting (A2). Therefore $f(z) = A^2(z)G(z)$ where $G(z)$ is a second degree polynomial. Obviously, $G(z)$ could vanish in $[-1,1]$ only for $z = -1$. However, the assumption $G(-1) > 0$ leads to a contradiction as in the proof of Theorem 2.7. Therefore $G(-1) = 0$.

2.2 A method for finding extremal polynomials

We now search for good polynomials of degree $t+2$. In fact, it can be proved as in [4, Section 5] that our polynomials are $B_{\mathbb{F}}^{t,n}$-extremal. Indeed, they appear as local extrema of certain functions. Any positive linear combination of two polynomials from $B_{\mathbb{F}}^{t,n}$ belongs to $B_{\mathbb{F}}^{t,n}$ again and gives a bound in the interval between the two bounds already provided by the two original polynomials. Thus, the global maximum can not exceed the greatest local maxima.

Theorem 2.7 and Corollary 2.8 determine the form of the extremal polynomials of degree $t+2$, i.e. we must take

$$f(z) = A^2(z)G(z) = (z^p + a_1 z^{p-1} + \cdots + a_{p-1}z + a_p)^2 G(z) \tag{4}$$

where $\deg(A) = p = [(t+1)/2]$, and

$$G(z) = \begin{cases} q(z+1) + 1 - z & \text{if } t \text{ is odd,} \\ [q(z+1) + 1 - z](z+1) & \text{if } t \text{ is even.} \end{cases}$$

Here $0 < q < 1$ ensures $f(z) \geq 0$ for $-1 \leq z \leq 1$ and $f_{t+2} < 0$ simultaneously. By the condition $f_{t+1} = 0$ (according to Corollary 2.5) we express the coefficient $a_1 = a_1(q,n)$ of $A(z)$ as a function of the parameters q and n.

We now consider $F = f(1)/f_0$ as a function of the unknown coefficients $a_2, \ldots, a_p$, q, and n. Using equations obtained by equating the partial derivatives of F to zero, one can express $a_2, \ldots, a_p$ as functions of q and n. The denominator f_0 is given by the formula in (2) as a function of $a_1, a_2, \ldots, a_p, q$, and n.

It does not seem possible to use further analytical methods. So we search for $q \in (0,1)$ using a computer in order to maximize the ratio $f(1)/f_0$. The computer calculations were made for a few seconds. Usually new bounds can be found in some range $n_1(t) \leq n \leq n_2(t)$.

3 Some examples with new bounds

3.1 Bounds in the complex projective space

We give as an example the case $t = 5$ in the complex projective space. We have $\alpha = n - 2$ and $\beta = 0$ by (1). Following Section 2 we have to work with polynomials of degree 7 having three double zeros:

$$f(z) = (z^3 + a_1 z^2 + a_2 z + a_3)^2 [q(z+1) + 1 - z] = \sum_{i=0}^{7} f_i P_i^{(n-2,0)}(z).$$

By the equation $f_6 = 0$ we obtain $a_1 = \frac{1}{2}(\frac{a_{7,6}}{a_{7,7}} + \frac{1+q}{1-q})$, where $a_{7,6}/a_{7,7}$ is the ratio of the first two coefficients of the Jacobi polynomial $P_7^{(n-2,0)}(z) = a_{7,7}z^7 + a_{7,6}z^6 + \cdots + a_{7,0}$.

We now examine the function

$$F(a_2, a_3, q, n) = \frac{f(1)}{f_0} = \frac{2q(1 + a_1 + a_2 + a_3)^2}{f_0(a_2, a_3, q, n)}.$$

The equalities $F'_{a_2}(a_2, a_3, q, n) = F'_{a_3}(a_2, a_3, q, n) = 0$ give (after some simplifications) a system of two equations which are linear with respect to a_2 and a_3. Thus we can express these parameters by q and n. Finally, for fixed dimension n, we search for $q \in (0,1)$ maximizing the function F. We compute F in suitable decreasing nets for q. The new bounds we have obtained are in dimensions $3 \leq n \leq 12$.

Table 1. New lower bounds on $D(5, n, \mathbb{C})$, $3 \leq n \leq 12$.

n	(3)	New bound	n	(3)	New bound
3	60	63	8	4320	4966
4	200	218	9	7425	8380
5	525	591	10	12100	13252
6	1176	1350	11	18876	19848
7	2352	2720	12	28392	28393

3.2 Bounds in the quaternionic projective space

We applied our method for 5-, 6-, and 7-designs in $\mathbb{H}P^{n-1}$. New lower bounds were found in dimensions $3 \leq n \leq 5$ for $t = 5$, $4 \leq n \leq 6$ for $t = 6$, and $3 \leq n \leq 11$ for $t = 7$. The results concerning $D(7, n, \mathbb{H})$ are shown in Table 2.

Table 2. New lower bounds on $D(7, n, \mathbb{H})$, $3 \leq n \leq 11$.

n	(3)	New bound	n	(3)	New bound
3	882	962	8	527136	678399
4	5544	6571	9	1159893	1438197
5	23595	29671	10	2355430	2760958
6	78078	101403	11	4480630	4883694
7	216580	283258			

3.3 Bounds in $\mathbb{R}P^{n-1}$

In the real projective space we use a modification of the method we have applied [4, 5] in the Euclidean sphere for estimating the size of spherical designs. Indeed, there is a one-to-one correspondence between the t-designs in $\mathbb{R}P^{n-1}$ and the antipodal spherical $(2t+1)$-designs on $\mathbf{S}^{n-1}$ [13, Theorem 9.2]. Therefore we can search for linear programming bounds for spherical designs as in [4, 5] paying attention to the antipodality. This is given by the following theorem.

Theorem 3.1 *Let $f(z)$ be a nonzero real polynomial such that*
(A1) $f(z) \geq 0$ *for* $-1 \leq z \leq 1$, *and*
(A2′) *The coefficients in the Jacobi expansion* $f(z) = \sum_{i=0}^{k} f_i P_i^{(\alpha,\beta)}(z)$ $(\alpha = \beta = (n-3)/2)$ *satisfy* $f_{2j} \leq 0$ *for* $2t+1 < 2j \leq k = \deg(f)$.
Then $D(t, n, \mathbb{R}) \geq \dfrac{f(1)}{2f_0}$.

Proof. If $W \subset \mathbb{R}P^{n-1}$ is a t-design, then its realization W' on $\mathbf{S}^{n-1}$ is an antipodal spherical $(2t+1)$-design. We have $|W'| = 2|W|$ and $|W'| \geq f(1)/f_0$ by the linear programming bound for antipodal spherical designs [7, Section 5]. This completes the proof.

We now can apply the method from [4, 5] for obtaining linear programming bounds for spherical $(2t+1)$-designs with polynomials of degree $2t+4$ without consideration of the coefficient f_{2t+3} (it can be arbitrarily chosen by Theorem 3.1). In particular, in this case the bound (3) can be written as

$$D(t, n, \mathbb{R}) \geq \binom{n+t-1}{n-1}.$$

We considered the cases $t = 3, 4, 5$. The new bounds we have obtained are in dimensions $5 \leq n \leq 7$ for $t = 3$, $4 \leq n \leq 11$ for $t = 4$, $3 \leq n \leq 23$ for $t = 5$. Examples are presented in Table 3.

Table 3. New lower bounds on $D(4, n, \mathbb{R})$, $4 \leq n \leq 11$.

n	(3)	New bound	n	(3)	New bound
4	35	37	8	330	365
5	70	75	9	445	549
6	126	137	10	715	739
7	210	231	11	1001	1094

Let us mention the result of Reznick [14, Section 4] on the nonexistence of antipodal spherical 5-designs with $n(n+1)+2$ points. This implies $D(2, n, \mathbb{R}) \geq R(2, n, \mathbb{R}) + 2 = \dfrac{n(n+1)}{2} + 2$ for $n \geq 3$. We are not aware of other improvements of (3) by more than one.

3.4 Bounds in the Cayley plane

Projective spaces over the non-associative algebra of the Cayley octonions exist only for $n = 2$ and $n = 3$ (the so-called Cayley line and plane). An appropriate model for $\mathbb{O}P^2$ and $\mathbb{O}P^3$ can be found in [10, Section 1]. In this case we obtained only one new bound, $D(7, 3, \mathbb{O}) \geq 7060$ instead of $D(7, 3, \mathbb{O}) \geq 6435$ by (3).

Acknowledgment. This research was partially supported by the Bulgarian NSF under Contract MM-502/95. The authors thank the referee for useful remarks.

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