On bounds on the size of designs in complex projective spaces

Svetla Jordanova Nikova
Department of mathematics
V.Turnovo University "St.St. Kiril and Metodii"

Abstract

We apply a linear programming technique to obtain bounds on designs in complex projective spaces. In some cases we improve the known bounds.

1 Introduction.

Let C be the field of complex numbers. Denote by C^n the set of vectors $x = (x_1, \ldots, x_n)$ over C. For an element $u \in C$ a conjugate element $\overline{u}$ and a norm $|u|$ are defined. So $|u|^2 = u\overline{u}$ and $|uv| = |u| \cdot |v|$. The inner product of vectors $x = (x_1, \ldots, x_n) \in C^n$ and $y = (y_1, \ldots, y_n) \in C^n$ is defined by $(x, y) = \sum_{i=1}^{n} \overline{x_i} y_i$.The projective space CP^{n-1} is defined as the sets of lines

$$X = \{x\lambda : \lambda \in F \backslash \{0\}\}, x \in C^n,$$

passing through the origin in C^n. The function

$$\frac{|(x, y)|}{\sqrt{|(x, x)||(y, y)|}}$$

does not depend upon the particular choice of vectors x and y on fixed lines. Thus it is a function of elements $X, Y \in CP^{n-1}$ containing x and y, respectively. This function is denoted by $\rho(X, Y)$ or $cos\varphi(X, Y)(\varphi(X, Y)$is the angle between X and Y). Then

$$d(X, Y) = \sqrt{2(1 - \rho^2(X, Y))}$$

defines a metric on CP^{n-1}.

The above model presents the complex projective space CP^{n-1} as a Delsarte space [6, 8, 9, 10]. A finite set $W \subset CP^{n-1}$ is called a τ-design if and only if

$$\sum_{X,Y\in W} P_i^{(n-2,0)}(cos^2\varphi(X,Y)) = 0$$

for $i = 1, 2, \ldots, \tau$, where $P_i^{(n-2,0)}$ are Jacobi polynomials [1, 7]. For the Delsarte space CP^{n-1} the following linear programming bound on the cardinality of the cardinality of a τ-design holds [6, 8, 9, 10]

$$|W| \geq \begin{cases} \begin{pmatrix} n+l-1 \\ l \end{pmatrix}\begin{pmatrix} n+l-2 \\ l-1 \end{pmatrix}, & if \quad \tau = 2l-1; \\ \begin{pmatrix} n+l-1 \\ l \end{pmatrix}^2, & if \quad \tau = 2l. \end{cases} \tag{1}$$

To obtain lower bounds on the cardinality of the designs one uses the following theorem ([6]):

Theorem 1 *Let $f(t)$ be a real polynomial such that*
(A1) $f(t) \geq 0$ for $-1 \leq t \leq 1$;
(A2) The coefficients in the expansion of $f(t)$ in terms of Jacoby polynomials [7, 1]

$$f(t) = \sum_{i=0}^{k} f_i P_i^{n-2,0}(t)$$

satisfy $f_{\tau+1} \leq 0, \ldots, f_k \leq 0$.
Then the cardinality of a τ-design $W \subset CP^{n-1}$ is bounded below by

$$|W| \geq f(1)/f_0.$$

The bound (1) has been obtained by polynomials of degree τ using Theorem 1 [6]. Here we improve (1) in some cases by polynomials of degree τ+2. Indeed we obtain new bounds for τ = 5,6,7 in some dimensions.

2 New lower bounds for 5-designs

Following the approach from [2, 3, 4, 5] we apply Theorem 1 with polynomials

$$f(t) = (t^3 + at^2 + bt + c)^2[q(t+1) + 1 - t] = \sum_{i=0}^{7} f_i P_i^{n-2,0}(t)$$

where $f_6 = 0$ and $0 < q < 1$ (the last ensures $f_7 < 0$).

By the equation $f_6 = 0$ we obtain

$$a = \frac{\alpha_{76}}{2\alpha_{77}} - \frac{q+1}{2(q-1)} = a(q,n)$$

($P_7^{(n-2,0)}(t) = \alpha_{77}t^7 + \alpha_{76}t^6 + \cdots$).

We determine the parameters b and c by equating to zero of the partial derivatives of $F(q,b,c) = f(1)/f_0$ (this gives a system of linear equations [5]. Finally, we find by a computer q maximizing the ratio $f(1)/f_0$.

The new bounds we have obtained are in dimensions $3 \le n \le 12$. They are presented in Table 1 below.

Table.1 New lower bounds on the size of the 5-design in CP^{n-1}, $3 \le n \le 12$

n	[6]	New bounds
3	60	63
4	200	218
5	525	591
6	1176	1350
7	2352	2720
8	4320	4966
9	7425	8380
10	12100	13252
11	18876	19848
12	28392	28393

3 New lower bounds for 6-designs

In this case we consider polynomials of degree 8 as follows:

$$f(t) = (t^3 + at^2 + bt + c)^2[q(t+1) + 1 - t](t+1) = \sum_{i=0}^{7} f_i P_i^{n-2,0}(t)$$

where $f_7 = 0$ and $0 < q < 1$

By the equation $f_7 = 0$ we express

$$a = \frac{\alpha_{87}}{2\alpha_{88}} - \frac{q}{q-1} = a(q,n)$$

($P_8^{(n-2,0)}(t) = \alpha_{88}t^8 + \alpha_{87}t^7 + \cdots$).

Next, from the partial derivatives of $F(q, b, c)$ we express b and c ((4), (see [[5],eq(4)]). Finally we fix n and search for $q \in (0, 1)$ in order to maximize the ratio $f(1)/f_0$. Again we use a computer and a method Monte Carlo. We obtain new lower bounds in dimensions $3 \leq n \leq 16$. The results are shown in Table 2.

Table.2 New lower bounds on the size of the 6-design in CP^{n-1}, $3 \leq n \leq 16$

n	[6]	New bounds
3	100	102
4	400	427
5	1225	1365
6	3136	3619
7	7056	8362
8	14400	17363
9	27225	33101
10	48400	58818
11	81796	98515
12	132496	156855
13	207025	238990
14	313600	350306
15	462400	496134
16	665856	681473

4 New lower bounds for 7-designs

Using polynomials of degree 9 and a similar method we obtain new lower bounds in dimensions $3 \leq n \leq 8$. They are presented in Table 3

Table.3 New lower bounds on the size of the 7-design in CP^{n-1}, $3 \leq n \leq 8$

n	[6]	New bounds
3	20	24
4	150	189
5	700	897
6	2450	3068
7	7056	8269
8	17640	18651

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