Connection between dynamical total least squares as a multi-parameter eigenvalue problem and as a Riemannian SVD

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STADIUS
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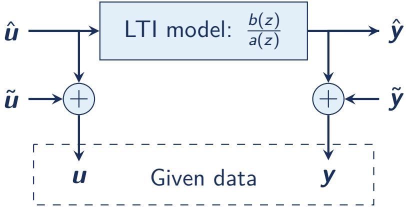
Dynamical total least squares (DTLS)

- Given input/output data: $\boldsymbol{u}, \boldsymbol{y} \in \mathbb{R}^N$
- Unknown model parameters: $\boldsymbol{a} \in \mathbb{R}^n, \boldsymbol{b} \in \mathbb{R}^{n+1}$
- Approximating data $\hat{\boldsymbol{u}}$, $\hat{\boldsymbol{y}}$
- Misfits $\tilde{\boldsymbol{u}}$, $\tilde{\boldsymbol{y}}$



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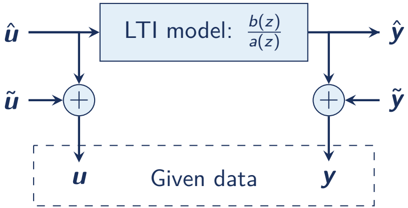

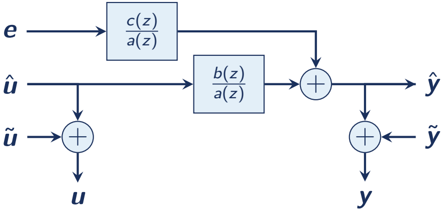

- Special case in misfit-versus-latency framework

Lemmerling et al. 2001 [6]

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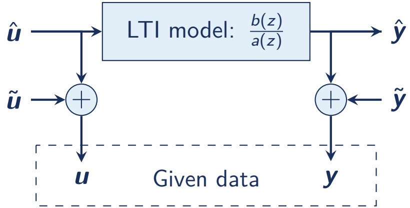

DTLS

$$\min_{\boldsymbol{a},\boldsymbol{b}} \quad \sigma^2 = \|\tilde{\boldsymbol{u}}\|_2^2 + \|\tilde{\boldsymbol{y}}\|_2^2 = \|\boldsymbol{u} - \hat{\boldsymbol{u}}\|_2^2 + \|\boldsymbol{y} - \hat{\boldsymbol{y}}\|_2^2,$$

$$\text{s.t.} \quad \hat{y}_{k+n} + \sum_{i=1}^{n} a_i \hat{y}_{k+n-i} = \sum_{i=0}^{n} b_i \hat{u}_{k+n-i} \qquad \text{for } k = 0, \ldots, N-n-1.$$

DTLS as a multi-parameter eigenvalue problem (MEVP)

DTLS

$$\min_{\boldsymbol{a},\boldsymbol{b}} \quad \sigma^2 = \|\tilde{\boldsymbol{u}}\|_2^2 + \|\tilde{\boldsymbol{y}}\|_2^2 = \|\boldsymbol{u} - \hat{\boldsymbol{u}}\|_2^2 + \|\boldsymbol{y} - \hat{\boldsymbol{y}}\|_2^2,$$

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$$\boldsymbol{w} = \begin{bmatrix} y_{N-1} & u_{N-1} & y_{N-2} & u_{N-2} & \ldots & y_0 & u_0 \end{bmatrix}^{\mathsf{T}}$$

and $\tilde{\boldsymbol{w}}$ and $\hat{\boldsymbol{w}}$ similarly.

$$\sigma^2 = \tilde{\boldsymbol{w}}^{\mathsf{T}} \tilde{\boldsymbol{w}}$$

Using a similar derivation as in De Moor 2020 [5]

DTLS as a multi-parameter eigenvalue problem (MEVP)

$$\mathbf{T}_{ab} = \begin{bmatrix} 1 & -b_0 & a_1 & -b_1 & \dots & a_n & -b_n & & & & & & & \\ & & 1 & -b_0 & \dots & a_{n-1} & -b_{n-1} & a_n & -b_n & & & & & \\ & & & \ddots & \ddots & & & & \ddots & \ddots & & & & \\ & & & & 1 & -b_0 & a_1 & -b_1 & \dots & a_n & -b_n & & & \\ & & & & & & 1 & -b_0 & \dots & a_{n-1} & -b_{n-1} & a_n & -b_n \end{bmatrix}$$

such that $\mathbf{T}_{ab}\hat{\boldsymbol{w}} = \mathbf{0}$

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such that $\mathbf{T}_{ab}\hat{\boldsymbol{w}} = \mathbf{0}$

$$\begin{aligned} \mathbf{T}_{ab}\tilde{\boldsymbol{w}} &= \mathbf{T}_{ab}\left(\boldsymbol{w} - \hat{\boldsymbol{w}}\right) \\ &= \mathbf{T}_{ab}\boldsymbol{w} \end{aligned} \quad \Rightarrow \quad \begin{aligned} \tilde{\boldsymbol{w}} &= {\mathbf{T}_{ab}}^{\dagger}\mathbf{T}_{ab}\boldsymbol{w} \\ &= \mathbf{T}_{ab}^{\top}\left(\mathbf{T}_{ab}\mathbf{T}_{ab}^{\top}\right)^{-1}\mathbf{T}_{ab}\boldsymbol{w} \end{aligned}$$

Using a similar derivation as in De Moor 2020 [5]

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$$\mathbf{T}_{ab} = \begin{bmatrix} 1 & -b_0 & a_1 & -b_1 & \dots & a_n & -b_n & & & & & & \\ & & 1 & -b_0 & \dots & a_{n-1} & -b_{n-1} & a_n & -b_n & & & & \\ & & & \ddots & \ddots & & & \ddots & \ddots & & & & \\ & & & & 1 & -b_0 & a_1 & -b_1 & \dots & a_n & -b_n & & \\ & & & & & & 1 & -b_0 & \dots & a_{n-1} & -b_{n-1} & a_n & -b_n \end{bmatrix}$$

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$$\sigma^2 = \tilde{\boldsymbol{w}}^{\mathsf{T}}\tilde{\boldsymbol{w}} = \boldsymbol{w}^{\mathsf{T}}\mathbf{T}_{ab}^{\mathsf{T}}\left(\mathbf{T}_{ab}\mathbf{T}_{ab}^{\mathsf{T}}\right)^{-1}\mathbf{T}_{ab}\boldsymbol{w}$$

Using a similar derivation as in De Moor 2020 [5]

DTLS as a multi-parameter eigenvalue problem (MEVP)

$$\sigma^2 = \tilde{\boldsymbol{w}}^\top \tilde{\boldsymbol{w}} = \boldsymbol{w}^\top \mathbf{T}_{ab}^\top \left(\mathbf{T}_{ab}\mathbf{T}_{ab}^\top\right)^{-1} \mathbf{T}_{ab}\boldsymbol{w}$$

Auxilary variables:

$$\mathbf{D}_{ab} = \mathbf{T}_{ab}\mathbf{T}_{ab}^\top$$

$$\boldsymbol{f} = \mathbf{D}_{ab}^{-1}\mathbf{T}_{ab}\boldsymbol{w}$$

$$\Rightarrow$$

$$\begin{bmatrix} \mathbf{D}_{ab} & \mathbf{T}_{ab}\boldsymbol{w} \\ \boldsymbol{w}^\top \mathbf{T}_{ab}^\top & \sigma^2 \end{bmatrix} \begin{bmatrix} \boldsymbol{f} \\ -1 \end{bmatrix} = \mathbf{0}$$

Using a similar derivation as in De Moor 2020 [5]

Multi-parameter eigenvalue problem (MEVP)

$$\begin{bmatrix} \mathbf{D}_{ab} & \mathbf{T}_{ab}\boldsymbol{w} \\ \boldsymbol{w}^\top \mathbf{T}_{ab}^\top & \sigma^2 \end{bmatrix} \begin{bmatrix} \boldsymbol{f} \\ -1 \end{bmatrix} = \mathbf{0}$$

Using a similar derivation as in De Moor 2020 [5]
De Cock and De Moor 2021 [1]

Multi-parameter eigenvalue problem (MEVP)

$$\begin{bmatrix} \mathbf{D}_{ab} & \mathbf{T}_{ab}\boldsymbol{w} \\ \boldsymbol{w}^\top \mathbf{T}_{ab}^\top & \sigma^2 \end{bmatrix} \begin{bmatrix} \boldsymbol{f} \\ -1 \end{bmatrix} = \mathbf{0}$$

$$\frac{\partial}{\partial a_i} \Downarrow \frac{\partial}{\partial b_i}$$

MEVP

(e.g., if $n = 1$)

$$\underbrace{\left(\mathbf{A}_0 + \mathbf{A}_1 a_1 + \mathbf{A}_2 a_1^2 + \mathbf{B}_{00} + \mathbf{B}_{10} b_0 + \mathbf{B}_{01} b_1 + \mathbf{B}_{20} b_0^2 + \mathbf{B}_{11} b_0 b_1 + \mathbf{B}_{02} b_1^2\right)}_{\mathcal{M}_{ab}} \boldsymbol{v} = \mathbf{0}$$

- constant coefficient matrices $\mathbf{A}$ and $\mathbf{B}$
- multivariate matrix polynomial $\mathcal{M}_{\boldsymbol{ab}}$
- $\boldsymbol{v}$ contains $\boldsymbol{f}$ and derivatives thereof
- eigenvalues given by powers of $\boldsymbol{a}$, $\boldsymbol{b}$
- solutions via rank deficiency of $\mathcal{M}_{\boldsymbol{ab}}$

Using a similar derivation as in De Moor 2020 [5]
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Dynamic TLS (DTLS) & structured TLS (STLS)

DTLS

$$\min_{\boldsymbol{a},\boldsymbol{b}} \quad \sigma^2 = \|\tilde{\boldsymbol{u}}\|_2^2 + \|\tilde{\boldsymbol{y}}\|_2^2 = \|\boldsymbol{u} - \hat{\boldsymbol{u}}\|_2^2 + \|\boldsymbol{y} - \hat{\boldsymbol{y}}\|_2^2,$$

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STLS

$$\min_{\boldsymbol{a},\boldsymbol{b}} \quad \sigma^2 = \left\|\tilde{\mathbf{H}}\right\|_F^2 = \left\|\mathbf{H} - \hat{\mathbf{H}}\right\|_F^2,$$

$$\text{s.t.} \quad \hat{\mathbf{H}} \begin{bmatrix} 1 \\ \boldsymbol{a} \\ -\boldsymbol{b} \end{bmatrix} = \mathbf{0}.$$

- $\mathbf{H} = [\mathbf{Y} \mid \mathbf{U}], \quad \hat{\mathbf{H}} = [\hat{\mathbf{Y}} \mid \hat{\mathbf{U}}], \quad \tilde{\mathbf{H}} = [\tilde{\mathbf{Y}} \mid \tilde{\mathbf{U}}]$
- $\mathbf{U}, \hat{\mathbf{U}}, \tilde{\mathbf{U}}$: Hankel matrices constructed from $\boldsymbol{u}, \hat{\boldsymbol{u}}, \tilde{\boldsymbol{u}}$
- $\mathbf{Y}, \hat{\mathbf{Y}}, \tilde{\mathbf{Y}}$: Hankel matrices constructed from $\boldsymbol{y}, \hat{\boldsymbol{y}}, \tilde{\boldsymbol{y}}$
- constraint imposes rank-deficiency of $\hat{\mathbf{H}}$

STLS & the Riemannian SVD (RiSVD)

STLS

$$\min_{\boldsymbol{a},\boldsymbol{b}} \quad \sigma^2 = \left\|\tilde{\mathbf{H}}\right\|_F^2 = \left\|\mathbf{H} - \hat{\mathbf{H}}\right\|_F^2,$$

$$\text{s.t.} \quad \hat{\mathbf{H}} \begin{bmatrix} 1 \\ \boldsymbol{a} \\ -\boldsymbol{b} \end{bmatrix} = \mathbf{0}.$$

$\Leftrightarrow$

RiSVD

$$\mathbf{H}\boldsymbol{v} = \mathbf{D}_{\boldsymbol{v}}\boldsymbol{u}\tau, \quad \boldsymbol{u}^\mathsf{T}\mathbf{D}_{\boldsymbol{v}}\boldsymbol{u} = 1,$$
$$\mathbf{H}^\mathsf{T}\boldsymbol{u} = \mathbf{D}_{\boldsymbol{u}}\boldsymbol{v}\tau, \quad \boldsymbol{v}^\mathsf{T}\mathbf{D}_{\boldsymbol{u}}\boldsymbol{v} = 1.$$

- Riemannian singular triplets $(\boldsymbol{u}, \tau, \boldsymbol{v})$
- RiSVD triplets correspond to critical points of STLS
- $\boldsymbol{v} = \begin{bmatrix}\boldsymbol{a}^\mathsf{T} & -\boldsymbol{b}^\mathsf{T}\end{bmatrix}^\mathsf{T}$
- $\mathbf{D}_{\boldsymbol{u}}$ and $\mathbf{D}_{\boldsymbol{v}}$ positive definite weighting matrices quadratic in the elements of $\boldsymbol{u}$ and $\boldsymbol{v}$
- $\tau^2 = \sigma^2$

Theorem 1 of De Moor 1994 [3]

Solving the RiSVD

- Heuristic solution methods:
  - Inverse iteration algorithm
  - Christiaan's flow (CF)
  - Dependent on initial guess
- Convergence (to global minimizer) not proven
  - "Works in most cases"
  - RiSVD triplets correspond to equilibria
  - CF: triplet with minimal τ^2 is stable

De Moor 1993 [2]
De Moor 1995 [4]

RiSVD

$$\mathbf{H}\boldsymbol{v} = \mathbf{D}_{\boldsymbol{v}}\boldsymbol{u}\tau, \quad \boldsymbol{u}^{\mathsf{T}}\mathbf{D}_{\boldsymbol{v}}\boldsymbol{u} = 1,$$
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Research questions

- Can a proof be formulated for the heuristic algorithms in the RiSVD problem setting?
- Is there any connection between both formulations?

MEVP

$$\mathcal{M}_{\boldsymbol{ab}}\boldsymbol{v} = \mathbf{0}$$

RiSVD

$$\mathbf{H}\boldsymbol{v} = \mathbf{D}_{\boldsymbol{v}}\boldsymbol{u}\tau, \quad \boldsymbol{u}^{\mathsf{T}}\mathbf{D}_{\boldsymbol{v}}\boldsymbol{u} = 1,$$
$$\mathbf{H}^{\mathsf{T}}\boldsymbol{u} = \mathbf{D}_{\boldsymbol{u}}\boldsymbol{v}\tau, \quad \boldsymbol{v}^{\mathsf{T}}\mathbf{D}_{\boldsymbol{u}}\boldsymbol{v} = 1.$$

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