About the Fundamental Subspaces of the Macaulay Matrix

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Introduction

A presentation about linear algebra in algebraic geometry

map	$\rightarrow$	matrix
ideal	$\rightarrow$	row space
normal set/standard monomials	$\rightarrow$	linearly (in)dependent rows/columns
Gröbner basis	$\rightarrow$	set of rows
Buchberger's algorithm	$\rightarrow$	singular value decomposition
roots	$\rightarrow$	eigenvalues

Batselier (2013); Dreesen (2013)

Polynomials and Macaulay Matrix

We consider **polynomials** $p_i(\boldsymbol{x}) \in \mathcal{P}^n = \mathbb{C}[x_1, \ldots, x_n]$, $\forall i = 1, \ldots, s$,

$$p_i(\boldsymbol{x}) = \sum_{\{\boldsymbol{\alpha}\}} c_{\boldsymbol{\alpha}} \boldsymbol{x}^{\boldsymbol{\alpha}}.$$

The **ideal** $\mathcal{I} \subset \mathcal{P}^n$ generated by these polynomials is defined by

$$\mathcal{I} = \langle p_1(\boldsymbol{x}), \ldots, p_s(\boldsymbol{x}) \rangle = \left\{ \sum_{i=1}^{s} h_i(\boldsymbol{x}) p_i(\boldsymbol{x}) : h_i(\boldsymbol{x}) \in \mathbb{C}[x_1, \ldots, x_n] \right\}.$$

Of course, an other important object is the **variety** of that ideal, i.e.,

$$\mathcal{V}(\mathcal{I}) = \{\boldsymbol{a} \in \mathbb{C}^n : p_i(\boldsymbol{a}) = 0, \forall i = 1, \ldots, s\}.$$

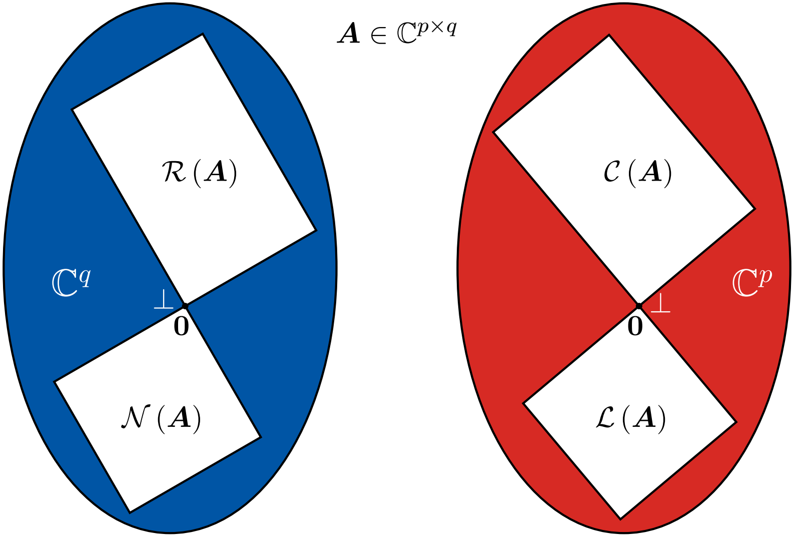
Polynomials and Macaulay matrix

$$\begin{cases} p_1(\boldsymbol{x}) = -3x_2 + 6x_1x_2 + (-1)x_2^2 + 1x_1^2x_2 + (-3)x_1x_2^2 + 1x_2^3 = 0 \\ p_2(\boldsymbol{x}) = 1 + 2x_1 + (-2)x_2 + 1x_1^2 + (-2)x_1x_2 + 1x_2^2 = 0 \\ p_3(\boldsymbol{x}) = (-4)x_2 + 1x_2^2 = 0 \end{cases}$$

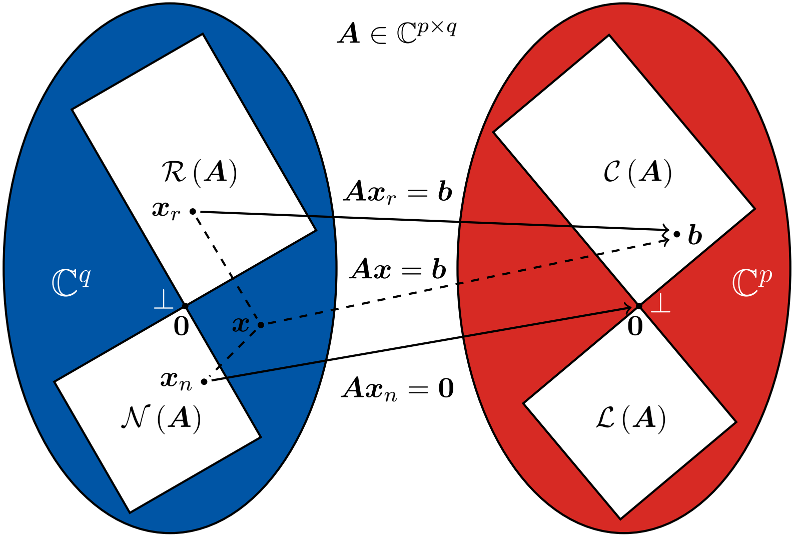
$$\Downarrow$$

$$\underbrace{\begin{array}{r|cccccccccc} & 1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 & x_1^3 & x_1^2x_2 & x_1x_2^2 & x_2^3 \\ \hline p_1(\boldsymbol{x}) & 0 & 0 & -3 & 0 & 6 & -1 & 0 & 1 & -3 & 1 \\ p_2(\boldsymbol{x}) & 1 & 2 & -2 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ x_1p_2(\boldsymbol{x}) & 0 & 1 & 0 & 2 & -2 & 0 & 1 & -2 & 1 & 0 \\ x_2p_2(\boldsymbol{x}) & 0 & 0 & 1 & 0 & 2 & -2 & 0 & 1 & -2 & 1 \\ p_3(\boldsymbol{x}) & 0 & 0 & -4 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ x_1p_3(\boldsymbol{x}) & 0 & 0 & 0 & 0 & -4 & 0 & 0 & 0 & 1 & 0 \\ x_2p_3(\boldsymbol{x}) & 0 & 0 & 0 & 0 & 0 & -4 & 0 & 0 & 0 & 1 \end{array}}_{\text{Macaulay matrix } \boldsymbol{M}_3} \begin{bmatrix} 1 \\ x_1 \\ x_2 \\ x_1^2 \\ x_1x_2 \\ x_2^2 \\ x_1^3 \\ x_1^2x_2 \\ x_1x_2^2 \\ x_2^3 \end{bmatrix} = \boldsymbol{0}$$

Fundamental subspaces of a matrix

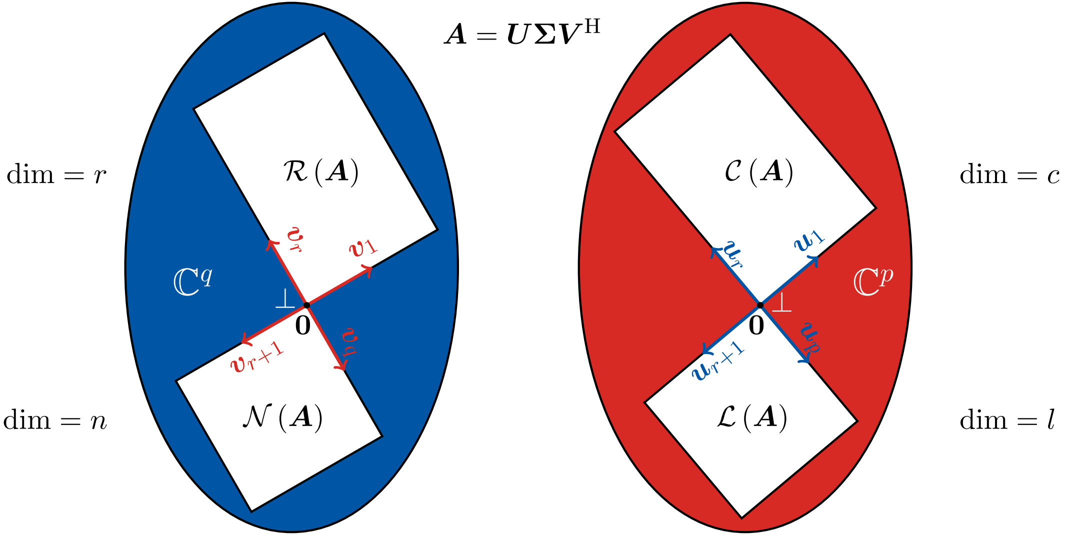


Fundamental subspaces of a matrix



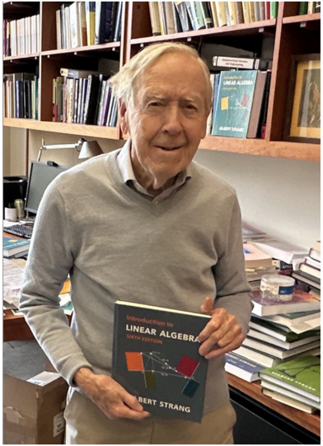
Fundamental subspaces of a matrix

$$\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^{\mathrm{H}}$$



singular value decomposition as our "work horse"!

Two important theorems



Gilbert Strang poses with his “Introduction to Linear Algebra”.

Rank-nullity theorem

$$c = \dim\left(\mathcal{C}\left(\boldsymbol{A}\right)\right) = \dim\left(\mathcal{R}\left(\boldsymbol{A}\right)\right) = r$$

Fundamental theorem of linear algebra

$$\operatorname{rank}\left(\boldsymbol{A}\right) + \operatorname{nullity}\left(\boldsymbol{A}\right) = r + n = q$$

The photo of Gilbert Strang was taken by Miller (2023).

Outline

Outline

Row space of the Macaulay matrix

What can we find in the row space?

$$\begin{array}{r} \\ p_1(\boldsymbol{x}) \\ p_2(\boldsymbol{x}) \\ x_1p_2(\boldsymbol{x}) \\ x_2p_2(\boldsymbol{x}) \\ p_3(\boldsymbol{x}) \\ x_1p_3(\boldsymbol{x}) \\ x_2p_3(\boldsymbol{x}) \end{array}
\begin{array}{c} \begin{array}{cccccccccc} 1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 & x_1^3 & x_1^2x_2 & x_1x_2^2 & x_2^3 \end{array} \\
\begin{bmatrix} 0 & 0 & -3 & 0 & 6 & -1 & 0 & 1 & -3 & 1 \\ 1 & 2 & -2 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & -2 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 & -2 & 0 & 1 & -2 & 1 \\ 0 & 0 & -4 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$$\Downarrow$$

$$\left\{ \sum_{i=0}^{s} h_i\left(\boldsymbol{x}\right) p_i\left(\boldsymbol{x}\right) : h_i\left(\boldsymbol{x}\right) \in \mathcal{P}^n_{d-d_i} \right\}$$

Ideal?

So, it is now very tempting to say that

$$\mathcal{R}\left(\boldsymbol{M}_d\right) \stackrel{?}{=} \left\langle p_1\left(\boldsymbol{x}\right), \ldots, p_s\left(\boldsymbol{x}\right)\right\rangle \cap \mathcal{P}_d^n \triangleq \left\langle p_1\left(\boldsymbol{x}\right), \ldots, p_s\left(\boldsymbol{x}\right)\right\rangle_d.$$

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However, this is not true!

Counter example

$$\begin{cases} p_1\left(\boldsymbol{x}\right) = x_1^2 + 2x_1 + 1 = 0 \\ p_2\left(\boldsymbol{x}\right) = x_1^2 + x_1 + 1 = 0 \end{cases} \qquad \boldsymbol{M}_2 = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$1 \in \langle p_1\left(\boldsymbol{x}\right), p_2\left(\boldsymbol{x}\right)\rangle$, because

$$\left(-1 - x_1\right) p_1\left(\boldsymbol{x}\right) + \left(2 + x_1\right) p_2\left(\boldsymbol{x}\right) = 1,$$

but $1 \notin \mathcal{R}\left(\boldsymbol{M}_2\right)$. In fact, the polynomial combination implies that $1 \in \mathcal{R}\left(\boldsymbol{M}_3\right)$.

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Interpretation of the row space of the Macaulay matrix

We consider the homogeneous ideal instead:

$$\left\langle p_1^h\left(\tilde{\boldsymbol{x}}\right), \ldots, p_s^h\left(\tilde{\boldsymbol{x}}\right)\right\rangle = \left\{\sum_{i=0}^{s} h_i\left(\tilde{\boldsymbol{x}}\right) p_i^h\left(\tilde{\boldsymbol{x}}\right) : h_i\left(\tilde{\boldsymbol{x}}\right) \in \mathcal{P}_{d-d_i}^n\right\}.$$

The homogeneity guarantees that all homogeneous polynomials of degree d are contained in the $\mathcal{R}\left(\boldsymbol{M}_d\right)$, which corresponds to

$$\mathcal{R}\left(\boldsymbol{M}_d\right) = \left\langle p_1^h\left(\boldsymbol{x}\right), \ldots, p_s^h\left(\boldsymbol{x}\right)\right\rangle_d.$$

An important consequence is that

$$\dim\left(\left\langle p_1^h\left(\tilde{\boldsymbol{x}}\right), \ldots, p_s^h\left(\tilde{\boldsymbol{x}}\right)\right\rangle_d\right) = r_d.$$

But, what makes $r_d \leq p_d$?

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$$\dim\left(\left\langle p_1^h\left(\tilde{\boldsymbol{x}}\right), \ldots, p_s^h\left(\tilde{\boldsymbol{x}}\right)\right\rangle_d\right) = r_d.$$

But, what makes $r_d \leq p_d$? **Linearly dependent rows or syzygies!**

(Batselier et al., 2014b)

Linearly dependent rows and syzygies

A linearly dependent row can be written as a linear combination of the previous rows:

	1	x_1	x_2	x_1^2	x_1x_2	x_2^2	x_1^3	$x_1^2x_2$	$x_1x_2^2$	x_2^3
$p_1(\boldsymbol{x})$	0	0	-3	0	6	-1	0	1	-3	1
$p_2(\boldsymbol{x})$	1	2	-2	1	-2	1	0	0	0	0
$x_1p_2(\boldsymbol{x})$	0	1	0	2	-2	0	1	-2	1	0
$x_2p_2(\boldsymbol{x})$	0	0	1	0	2	-2	0	1	-2	1
$p_3(\boldsymbol{x})$	0	0	-4	0	0	1	0	0	0	0
$x_1p_3(\boldsymbol{x})$	0	0	0	0	-4	0	0	0	1	0
$x_2p_3(\boldsymbol{x})$	0	0	0	0	0	-4	0	0	0	1

r6 = r5 + r4 - r1

In terms of the polynomials, this means that there are **syzygies**!

- basis syzygy: $\sum_{i=1}^{s} h_i(\boldsymbol{x})\, p_i(\boldsymbol{x}) = 0$
- derived syzygy: $\boldsymbol{x}^{\boldsymbol{\beta}} \sum_{i=1}^{s} h_i(\boldsymbol{x})\, p_i(\boldsymbol{x}) = 0$

Linearly dependent rows and syzygies

A linearly dependent row can be written as a linear combination of the previous rows:

$$\begin{array}{r} \\ p_1(\boldsymbol{x}) \\ p_2(\boldsymbol{x}) \\ x_1p_2(\boldsymbol{x}) \\ x_2p_2(\boldsymbol{x}) \\ p_3(\boldsymbol{x}) \\ x_1p_3(\boldsymbol{x}) \\ x_2p_3(\boldsymbol{x}) \end{array} \begin{array}{c} \begin{array}{cccccccccc} 1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 & x_1^3 & x_1^2x_2 & x_1x_2^2 & x_2^3 \end{array} \\ \begin{bmatrix} 0 & 0 & -3 & 0 & 6 & -1 & 0 & 1 & -3 & 1 \\ 1 & 2 & -2 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & -2 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 & -2 & 0 & 1 & -2 & 1 \\ 0 & 0 & -4 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

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- basis syzygy: $\sum_{i=1}^{s} h_i(\boldsymbol{x})\, p_i(\boldsymbol{x}) = 0$ $\quad (1 - x_1)\, p_3(\boldsymbol{x}) + x_2 p_2(\boldsymbol{x}) - p_1(\boldsymbol{x}) = 0$
- derived syzygy: $\boldsymbol{x}^{\boldsymbol{\beta}} \sum_{i=1}^{s} h_i(\boldsymbol{x})\, p_i(\boldsymbol{x}) = 0$

Linearly dependent rows and syzygies

A linearly dependent row can be written as a linear combination of the previous rows:

$$\begin{array}{r} \\ p_1(\boldsymbol{x}) \\ p_2(\boldsymbol{x}) \\ x_1p_2(\boldsymbol{x}) \\ x_2p_2(\boldsymbol{x}) \\ p_3(\boldsymbol{x}) \\ x_1p_3(\boldsymbol{x}) \\ x_2p_3(\boldsymbol{x}) \end{array} \begin{array}{c} \begin{array}{cccccccccc} 1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 & x_1^3 & x_1^2x_2 & x_1x_2^2 & x_2^3 \end{array} \\ \begin{bmatrix} 0 & 0 & -3 & 0 & 6 & -1 & 0 & 1 & -3 & 1 \\ 1 & 2 & -2 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & -2 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 & -2 & 0 & 1 & -2 & 1 \\ 0 & 0 & -4 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \quad \text{r6 = r5 + r4 - r1}$$

In terms of the polynomials, this means that there are **syzygies**!

- basis syzygy: $\sum_{i=1}^{s} h_i(\boldsymbol{x})\, p_i(\boldsymbol{x}) = 0$ $\quad (1 - x_1)\, p_3(\boldsymbol{x}) + x_2 p_2(\boldsymbol{x}) - p_1(\boldsymbol{x}) = 0$
- derived syzygy: $\boldsymbol{x}^{\boldsymbol{\beta}} \sum_{i=1}^{s} h_i(\boldsymbol{x})\, p_i(\boldsymbol{x}) = 0$ $\quad x_1^2\left((1 - x_1)\, p_3(\boldsymbol{x}) + x_2 p_2(\boldsymbol{x}) - p_1(\boldsymbol{x})\right) = 0$

Outline

Dimension of the left null space

The left null space contains the "coefficients" of the syzygies:

$$\mathcal{L}\left(\boldsymbol{M}_d\right) = \left\{\boldsymbol{h} \in \mathbb{C}^{1\times p} : \boldsymbol{h}\boldsymbol{M}_d = \boldsymbol{0}\right\}.$$

$$\Downarrow$$

$$\sum_{i=1}^{s} h_i\left(\boldsymbol{x}\right) p_i\left(\boldsymbol{x}\right) = 0.$$

The dimension l_d of the left null space counts the total number of syzygies in $\mathcal{R}\left(\boldsymbol{M}_d\right)$:

$$\binom{d - d_l + n}{n}$$

is added to l_d for every degree d_l basis syzygy.

(Batselier et al., 2014b)

Degree of regularity

An important consequence is the degree of regularity:

Definition

The minimal degree d^* for which the dimension l_d can be computed via the obtained syzygies at that degree is called the **degree of regularity**.

Once l_d is known, then the rank r_d and nullity n_d of the Macaulay matrix are also fully determined!

- We need higher degrees to check whether we found d^*
- There are upper bounds, but sometimes d^* is much lower (Lazard, 1983)
- Recursive algorithms are essential (Batselier et al., 2014a,b)

(Batselier et al., 2014b)

Outline

Multiplication maps

Consider an ideal $\mathcal{I} = \langle p_1(\boldsymbol{x}), \ldots, p_s(\boldsymbol{x}) \rangle \subset \mathcal{P}^n$ and zero-dimensional variety $\mathcal{V}(\mathcal{I})$

$$\Downarrow$$

The **quotient ring** $\mathcal{R} \triangleq \mathcal{P}^n / \langle p_1(\boldsymbol{x}), \ldots, p_s(\boldsymbol{x}) \rangle$ is a finite-dimensional vector space.
We can define the **multiplication maps**:

$$\boldsymbol{M}_g : \mathcal{R} \to \mathcal{R} : p(\boldsymbol{x}) + \mathcal{I} \mapsto g(\boldsymbol{x})\, p(\boldsymbol{x}) + \mathcal{I}$$

with properties

- Eigenvalues of $\boldsymbol{M}_g$ are $g(\boldsymbol{x})|_{(j)}$
- The multiplication maps commute, i.e., $\boldsymbol{M}_{g_1}\boldsymbol{M}_{g_2} = \boldsymbol{M}_{g_2}\boldsymbol{M}_{g_1}$

How do we construct these maps?

Stetter's eigenvalue problem

Stetter's approach

1. Compute a Gröbner basis G for $\mathcal{I}$
2. Derive a monomial basis for $\mathcal{R}$ from G
3. Solve the eigenvalue problems that expresses the monomial multiplication within the $\mathcal{R}$ (simultaneous)
4. Read the affine solutions from the eigenvalues/eigenvectors

Our approach

1. Compute a numerical basis matrix $\boldsymbol{Z}_d$ for the null space of $\boldsymbol{M}_d$, with $d \geq d^*$.
2. Determine the linearly independent rows of $\boldsymbol{Z}_d$
3. Solve the eigenvalue problems that expresses the monomial multiplication within the $\mathcal{R}$ (simultaneous)
4. Read the affine solutions from the eigenvalues/eigenvectors

A lot of details hiding in these steps, e.g., dealing with multiplicities and solutions at infinity (Stetter, 2004; Batselier et al., 2014b; Vermeersch and De Moor, 2021)

Dual vector space

Basis matrix of the right null space can be written in terms of the solutions ($d \geq d^*$)

- solutions can be described by the dual vector space of the quotient space $\mathcal{R}$
- from the rank-nullity theorem

$$\begin{aligned} n_d &= q_d - r_d \\ &= \dim \mathcal{P}_d^n / \left\langle p_1^h(\tilde{\boldsymbol{x}}), \ldots, p_s^h(\tilde{\boldsymbol{x}}) \right\rangle_d \end{aligned}$$

- requires dual vector space $\mathcal{C}_d^{n'}$ of $\mathcal{C}_d^n$ and differential functionals

$$\partial_{\boldsymbol{i}}(\cdot)\big|_{(j)} \triangleq \frac{1}{i_1! \ldots i_n!} \frac{\partial^{|\boldsymbol{i}|}(\cdot)}{\partial x_1^{i_1} \ldots \partial x_n^{i_n}}\bigg|_{(j)}$$

confluent multivariate Vandermonde basis matrix

$$\boldsymbol{V}_d = \left[\begin{array}{ccc} x_0^2|_{(1)} & 0 & x_0^2|_{(2)} \\ \hline x_0x_1|_{(1)} & x_0|_{(1)} & x_0x_1|_{(2)} \\ x_0x_2|_{(1)} & 0 & x_0x_2|_{(2)} \\ \hline x_1^2|_{(1)} & 2x_1|_{(1)} & x_1^2|_{(2)} \\ x_1x_2|_{(1)} & x_2|_{(1)} & x_1x_2|_{(2)} \\ x_2^2|_{(1)} & 0 & x_2^2|_{(2)} \end{array}\right]$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$\partial_{00}(\boldsymbol{v})|_{(1)} \quad \partial_{10}(\boldsymbol{v})|_{(1)} \quad \partial_{00}(\boldsymbol{v})|_{(2)}$$

(Batselier et al., 2014b)

Monomial multiplication property

one projective solution

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}}_{\boldsymbol{S}_{x_1,x_0}} \underbrace{\begin{bmatrix} x_0^2|_{(1)} \\ \hline x_0x_1|_{(1)} \\ x_0x_2|_{(1)} \\ \hline x_1^2|_{(1)} \\ x_1x_2|_{(1)} \\ x_2^2|_{(1)} \end{bmatrix}}_{\partial_{00}(\boldsymbol{v})|_{(1)}} x_1|_{(1)} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}}_{\boldsymbol{S}_{x_0,x_1}} \underbrace{\begin{bmatrix} x_0^2|_{(1)} \\ \hline x_0x_1|_{(1)} \\ x_0x_2|_{(1)} \\ \hline x_1^2|_{(1)} \\ x_1x_2|_{(1)} \\ x_2^2|_{(1)} \end{bmatrix}}_{\partial_{00}(\boldsymbol{v})|_{(1)}} x_0|_{(1)}$$

Note that we consider differential functionals in $\mathcal{C}_2^{2'}$ in this exposition

Monomial multiplication property

one projective solution

$$\boldsymbol{S}_{x_1,x_0} \underbrace{\left[\begin{array}{c} x_0^2\big|_{(1)} \\ \hline x_0x_1\big|_{(1)} \\ x_0x_2\big|_{(1)} \\ \hline x_1^2\big|_{(1)} \\ x_1x_2\big|_{(1)} \\ x_2^2\big|_{(1)} \end{array}\right]}_{\partial_{00}(\boldsymbol{v})|_{(1)}} x_1\big|_{(1)} = \boldsymbol{S}_{x_0,x_1} \underbrace{\left[\begin{array}{c} x_0^2\big|_{(1)} \\ \hline x_0x_1\big|_{(1)} \\ x_0x_2\big|_{(1)} \\ \hline x_1^2\big|_{(1)} \\ x_1x_2\big|_{(1)} \\ x_2^2\big|_{(1)} \end{array}\right]}_{\partial_{00}(\boldsymbol{v})|_{(1)}} x_0\big|_{(1)}$$

Note that we consider differential functionals in $\mathcal{C}_2^{2\prime}$ in this exposition

Monomial multiplication property

multiple simple projective solutions

$$\boldsymbol{S}_{x_1,x_0} \underbrace{\begin{bmatrix} x_0^2\big|_{(1)} & x_0^2\big|_{(2)} \\ \hline x_0x_1\big|_{(1)} & x_0x_1\big|_{(2)} \\ x_0x_2\big|_{(1)} & x_0x_2\big|_{(2)} \\ \hline x_1^2\big|_{(1)} & x_1^2\big|_{(2)} \\ x_1x_2\big|_{(1)} & x_1x_2\big|_{(2)} \\ x_2^2\big|_{(1)} & x_2^2\big|_{(2)} \end{bmatrix}}_{\boldsymbol{V}_d} \boldsymbol{D}_{x_1} = \boldsymbol{S}_{x_0,x_1} \underbrace{\begin{bmatrix} x_0^2\big|_{(1)} & x_0^2\big|_{(2)} \\ \hline x_0x_1\big|_{(1)} & x_0x_1\big|_{(2)} \\ x_0x_2\big|_{(1)} & x_0x_2\big|_{(2)} \\ \hline x_1^2\big|_{(1)} & x_1^2\big|_{(2)} \\ x_1x_2\big|_{(1)} & x_1x_2\big|_{(2)} \\ x_2^2\big|_{(1)} & x_2^2\big|_{(2)} \end{bmatrix}}_{\boldsymbol{V}_d} \boldsymbol{D}_{x_0}$$

$$\boldsymbol{D}_{x_1} = \begin{bmatrix} x_1\big|_{(1)} & 0 \\ 0 & x_1\big|_{(2)} \end{bmatrix}$$

Note that we consider differential functionals in $\mathcal{C}_2^{2'}$ in this exposition

Monomial multiplication property

multiple projective solutions with multiplicity larger than one

$$\boldsymbol{S}_{x_1,x_0}\underbrace{\left[\begin{array}{ccc} x_0^2\big|_{(1)} & 0 & x_0^2\big|_{(2)} \\ \hline x_0x_1\big|_{(1)} & x_0\big|_{(1)} & x_0x_1\big|_{(2)} \\ x_0x_2\big|_{(1)} & 0 & x_0x_2\big|_{(2)} \\ \hline x_1^2\big|_{(1)} & 2x_1\big|_{(1)} & x_1^2\big|_{(2)} \\ x_1x_2\big|_{(1)} & x_2\big|_{(1)} & x_1x_2\big|_{(2)} \\ x_2^2\big|_{(1)} & 0 & x_2^2\big|_{(2)} \end{array}\right]}_{\boldsymbol{V}_d}\boldsymbol{D}_{x_1} = \boldsymbol{S}_{x_0,x_1}\underbrace{\left[\begin{array}{ccc} x_0^2\big|_{(1)} & 0 & x_0^2\big|_{(2)} \\ \hline x_0x_1\big|_{(1)} & x_0\big|_{(1)} & x_0x_1\big|_{(2)} \\ x_0x_2\big|_{(1)} & 0 & x_0x_2\big|_{(2)} \\ \hline x_1^2\big|_{(1)} & 2x_1\big|_{(1)} & x_1^2\big|_{(2)} \\ x_1x_2\big|_{(1)} & x_2\big|_{(1)} & x_1x_2\big|_{(2)} \\ x_2^2\big|_{(1)} & 0 & x_2^2\big|_{(2)} \end{array}\right]}_{\boldsymbol{V}_d}\boldsymbol{D}_{x_0}$$

$$\boldsymbol{D}_{x_1} = \begin{bmatrix} x_1\big|_{(1)} & \times & \times \\ 0 & x_1\big|_{(1)} & \times \\ 0 & 0 & x_1\big|_{(2)} \end{bmatrix}$$

Note that we consider differential functionals in $\mathcal{C}_2^{2\,\prime}$ in this exposition

Monomial multiplication property

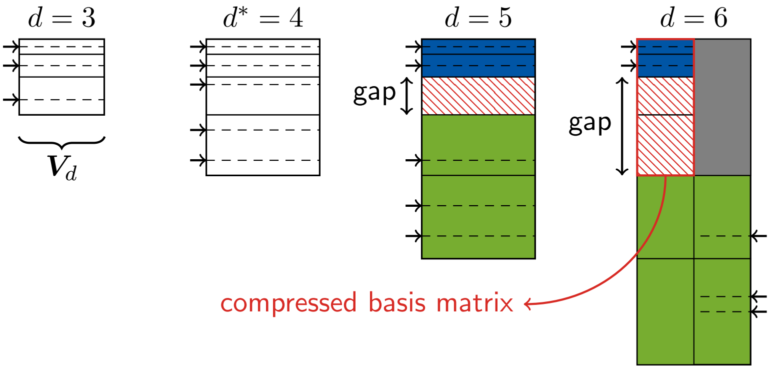
multiple affine solutions with multiplicity larger than one

$$\boldsymbol{S}_1 \underbrace{\begin{bmatrix} 1 & 0 & 1 \\ \hline x_1|_{(1)} & 1 & x_1|_{(2)} \\ x_2|_{(1)} & 0 & x_2|_{(2)} \\ \hline x_1^2|_{(1)} & 2x_1|_{(1)} & x_1^2|_{(2)} \\ x_1x_2|_{(1)} & x_2|_{(1)} & x_1x_2|_{(2)} \\ x_2^2|_{(1)} & 0 & x_2^2|_{(2)} \end{bmatrix}}_{\boldsymbol{V}_d} \boldsymbol{D}_{x_1} = \boldsymbol{S}_{x_1} \underbrace{\begin{bmatrix} 1 & 0 & 1 \\ \hline x_1|_{(1)} & 1 & x_1|_{(2)} \\ x_2|_{(1)} & 0 & x_2|_{(2)} \\ \hline x_1^2|_{(1)} & 2x_1|_{(1)} & x_1^2|_{(2)} \\ x_1x_2|_{(1)} & x_2|_{(1)} & x_1x_2|_{(2)} \\ x_2^2|_{(1)} & 0 & x_2^2|_{(2)} \end{bmatrix}}_{\boldsymbol{V}_d} \boldsymbol{I}$$

select the linearly independent rows of $\boldsymbol{V}_d$

Note that we consider differential functionals in $\mathcal{C}_2^{2'}$ in this exposition

Removing solutions at infinity



Two difficulties

$$S_1 V_d D_{x_1} = S_{x_1} V_d$$

- Solutions/confluent Vandermonde basis vectors are not known in advanced:

numerical basis matrix Z_d of the null space

$$\Downarrow V_d = Z_d T$$

$$(S_1 Z_d)\, T D_{x_1} = (S_{x_1} Z_d)\, T$$

$$T D_{x_1} T^{-1} = (S_1 Z_d)^{-1} (S_{x_1} Z_d)$$

- Not possible to numerically stable compute Jordan normal form:

numerically stable **Schur decomposition**

$$\Downarrow$$

$$Q U_{x_1} Q^{-1} = (S_1 Z_d)^{-1} (S_{x_1} Z_d)$$

(Vermeersch and De Moor, 2021; Corless et al., 1997)

Multidimensional realization theory

- It is **possible to shift with any polynomial** in the eigenvalues – for example, x_2

$$\boldsymbol{Q}\boldsymbol{U}_{x_2}\boldsymbol{Q}^{-1} = \left(\boldsymbol{S}_1\boldsymbol{Z}_d\right)^{-1}\left(\boldsymbol{S}_{x_2}\boldsymbol{Z}_d\right)$$

- This leads to the same $\boldsymbol{Q}$

- Solved via simultaneous triangularization
- Different shift polynomials can be useful
- Entire degree block rows instead of single rows can be considered

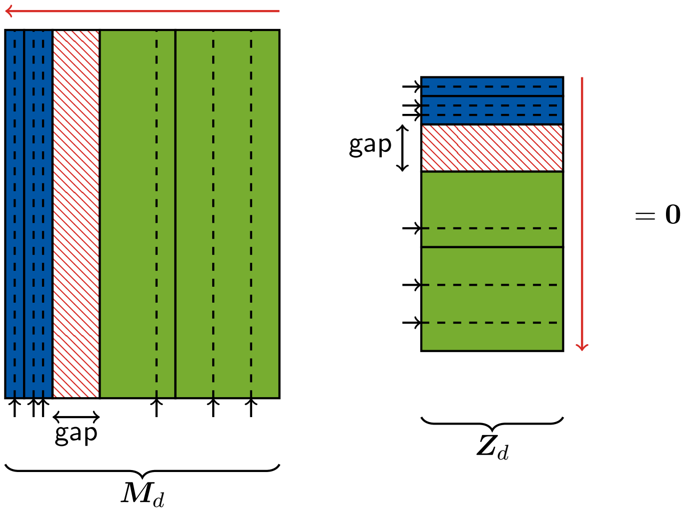
Realization theory for any shift polynomial $g\left(\boldsymbol{x}\right)$:

$$\boldsymbol{Q}\boldsymbol{U}_{g}\boldsymbol{Q}^{-1} = \left(\boldsymbol{S}_1\boldsymbol{Z}_d\right)^{\dagger}\left(\boldsymbol{S}_{g}\boldsymbol{Z}_d\right),$$

where $\boldsymbol{S}_1$ and $\boldsymbol{S}_g$ select rows from $\boldsymbol{Z}_d$

Outline

Complementarity between right null space and column space



(Dreesen, 2013)

Reordered Macaulay matrix

$$\underbrace{\begin{bmatrix} \boldsymbol{M}_1 & \boldsymbol{M}_2 & \boldsymbol{M}_3 & \boldsymbol{M}_4 \end{bmatrix}}_{\boldsymbol{N}} \begin{bmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{B} & \boldsymbol{0} \\ \boldsymbol{C} & \boldsymbol{0} \\ \boldsymbol{W}_{21} & \boldsymbol{W}_{22} \end{bmatrix} = \boldsymbol{0}$$

- $\boldsymbol{A}$: affine standard monomials
- $\boldsymbol{B}$: other affine shifted monomials
- $\boldsymbol{C}$: other affine monomials and gap
- $\boldsymbol{W}_{2\times}$: remaining rows of the basis matrix

The multiplication property

$$\boldsymbol{A}\boldsymbol{D}_g = \boldsymbol{S}_g \begin{bmatrix} \boldsymbol{A} \\ \boldsymbol{B} \end{bmatrix}$$

yields again an eigenvalue problem

$$\boldsymbol{A}\boldsymbol{D}_g\boldsymbol{A}^{-1} = \boldsymbol{S}_g \begin{bmatrix} \boldsymbol{I} \\ \boldsymbol{B}\boldsymbol{A}^{-1} \end{bmatrix}$$

(Vermeersch and De Moor, 2021)

Backward QR decomposition

The backward QR decomposition **eliminates** the dependency on **the right null space**

$$\begin{bmatrix} \boldsymbol{R}_{14} & \boldsymbol{R}_{13} & \boldsymbol{R}_{12} & \boldsymbol{R}_{11} \\ \boldsymbol{R}_{24} & \boldsymbol{R}_{23} & \boldsymbol{R}_{22} & \mathbf{0} \\ \boldsymbol{R}_{34} & \boldsymbol{R}_{33} & \mathbf{0} & \mathbf{0} \\ \boldsymbol{R}_{44} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{A} & \mathbf{0} \\ \boldsymbol{B} & \mathbf{0} \\ \boldsymbol{C} & \mathbf{0} \\ \boldsymbol{W}_{21} & \boldsymbol{W}_{22} \end{bmatrix} = \mathbf{0}$$

Backward QR decomposition

The backward QR decomposition **eliminates** the dependency on **the right null space**

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$$\boldsymbol{B}\boldsymbol{A}^{-1} = -\boldsymbol{R}_{33}^{-1}\boldsymbol{R}_{34}$$

Equivalent multidimensional realization theory

Realization theory for any shift polynomial $g(\boldsymbol{x})$:

$$\boldsymbol{A}\boldsymbol{D}_g\boldsymbol{A}^{-1} = \boldsymbol{S}_g \begin{bmatrix} \boldsymbol{I} \\ -\boldsymbol{R}_{33}^{-1}\boldsymbol{R}_{34} \end{bmatrix},$$

where $\boldsymbol{S}_g$ selects rows from $\boldsymbol{I}$ and $-\boldsymbol{R}_{33}^{-1}\boldsymbol{R}_{34}$

- a standard eigenvalue problem, with $\boldsymbol{A}$ the matrix of eigenvectors
- backward QR decomposition removes influence of solutions at infinity implicitly
- same adaptation to multiple Schur decompositions is possible

Outline

Conclusions and future work

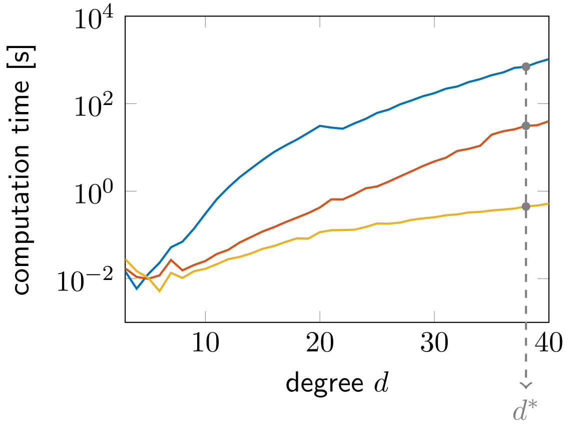
Each of the fundamental subspaces of the Macaulay matrix has a purpose:

- Many properties/questions from algebraic geometry hide in one of the fundamental subspaces of the Macaulay matrix
- A full treatment of the column space is not yet available
- Some algorithmic issues make the column space currently less useful to solve polynomial systems

Some current research efforts are:

- Investigating the properties of the column space
- Translating the interpretations to the block Macaulay matrix

A final note on the computational complexity



Comparison of the computation time to construct a numerical basis matrix of the right null space of a Macaulay matrix via the standard (—), recursive (—), and sparse (—) approach.

(Batselier et al., 2014a; Vermeersch and De Moor, 2023)

Any questions?

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A system theoretic intermezzo!

discrete-time overdetermined autonomous multidimensional systems

state space representation

$$\boldsymbol{x}[k_1+1, k_2, \ldots, k_n] = \boldsymbol{A}_1 \boldsymbol{x}[\boldsymbol{k}],$$

$$\vdots$$

$$\boldsymbol{x}[k_1, \ldots, k_{n-1}, k_n+1] = \boldsymbol{A}_n \boldsymbol{x}[\boldsymbol{k}],$$

$$y[\boldsymbol{k}] = \boldsymbol{c}^{\mathrm{T}} \boldsymbol{x}[\boldsymbol{k}],$$

where $\boldsymbol{x} \in \mathbb{R}^m$ is the state vector, $\boldsymbol{A}_i \in \mathbb{R}^{m \times m}$ define the autonomous state transitions, and $\boldsymbol{c} \in \mathbb{R}^m$ defines how the one-dimensional output y is composed from the state vector

trajectories representation

$$\boldsymbol{r}(\boldsymbol{z})\, w[k_1, \ldots, k_n] = \boldsymbol{0},$$

where $\boldsymbol{r} \in \mathbb{R}^{n \times 1}$ and $\boldsymbol{z} = (z_1, \ldots, z_n)$ denotes the multidimensional shift operator, for which holds that

$$z_i : (z_i w)[\boldsymbol{k}] = w[k_1, \ldots, k_i + 1, \ldots, k_n].$$

Via this shift operator, a difference equation can be associated with a multivariate polynomial in $\boldsymbol{x} = \boldsymbol{z}$.

(Dreesen et al., 2018; Willems, 1986)

A system theoretic intermezzo!

Consider the following set of difference equations

$$\begin{cases} \text{trajectory 1:} \; w[k_1, k_2+1] + 2w[k_1+1, k_2+1] + 3w[k_1, k_2+2] = 0, \\ \text{trajectory 2:} \; 4w[k_1, k_2] + 5w[k_1+1, k_2] + 6w[k_1, k_2+1] = 0. \end{cases}$$

By shifting these equations up to order/degree two, we find

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 2 & 3 \\ 4 & 5 & 6 & 0 & 0 & 0 \\ 0 & 4 & 0 & 5 & 6 & 0 \\ 0 & 0 & 4 & 0 & 5 & 6 \end{bmatrix} \begin{bmatrix} w[k_1, k_2] \\ \hline w[k_1+1, k_2] \\ w[k_1, k_2+1] \\ \hline w[k_1+2, k_2] \\ w[k_1+1, k_2+1] \\ w[k_1, k_2+2] \end{bmatrix} = \mathbf{0}.$$

A system theoretic intermezzo!

Consider the following system of multivariate polynomial equations

$$\begin{cases} p_1(\boldsymbol{z}) = z_2 + 2z_1z_2 + 3z_2^2 = 0, \\ p_2(\boldsymbol{z}) = 4 + 5z_1 + 6z_2 = 0. \end{cases}$$

By shifting these equations up to order/degree two, we find

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 2 & 3 \\ 4 & 5 & 6 & 0 & 0 & 0 \\ 0 & 4 & 0 & 5 & 6 & 0 \\ 0 & 0 & 4 & 0 & 5 & 6 \end{bmatrix} \left[\begin{array}{c} 1 \\ \hline z_1 \\ z_2 \\ \hline z_1^2 \\ z_1z_2 \\ z_2^2 \end{array}\right] = \mathbf{0}.$$

A system theoretic intermezzo!

A special basis matrix of the null space of the Macaulay matrix is the column echelon basis matrix

$$\boldsymbol{H}_d = \boldsymbol{Z}_d \boldsymbol{T}_Z = \boldsymbol{V}_d \boldsymbol{T}_V .$$

The multiplication property in $\boldsymbol{H}$ leads to eigenvalue problems

$$\boldsymbol{S}_0 \boldsymbol{H}_d \boldsymbol{A}_i = \boldsymbol{S}_i \boldsymbol{H}_d,$$

or

$$\boldsymbol{A}_i = \left(\boldsymbol{S}_0 \boldsymbol{H}_d\right)^\dagger \left(\boldsymbol{S}_i \boldsymbol{H}_d\right) .$$

This gives us the system matrices $\boldsymbol{A}_i$ via Ho–Kalman's shift trick; a **multidimensional realization problem!**

multidimensional extended observability matrix

$$\boldsymbol{\Gamma} = \left[\begin{array}{c} \boldsymbol{c}^{\mathrm{T}} \\ \hline \boldsymbol{c}^{\mathrm{T}} \boldsymbol{A}_1 \\ \boldsymbol{c}^{\mathrm{T}} \boldsymbol{A}_2 \\ \hline \boldsymbol{c}^{\mathrm{T}} \boldsymbol{A}_1^2 \\ \boldsymbol{c}^{\mathrm{T}} \boldsymbol{A}_1 \boldsymbol{A}_2 \\ \boldsymbol{c}^{\mathrm{T}} \boldsymbol{A}_2^2 \\ \hline \vdots \\ \hline \boldsymbol{c}^{\mathrm{T}} \boldsymbol{A}_1^d \\ \vdots \\ \boldsymbol{c}^{\mathrm{T}} \boldsymbol{A}_2^d \end{array}\right]$$

(Ho and Kalman, 1966; Dreesen et al., 2018)