About (Rectangular) Multiparameter Eigenvalue Problems

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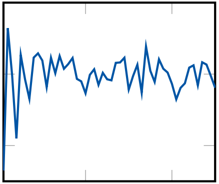
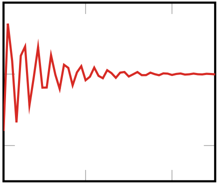
Solving the least-squares realization problem

$$\underbrace{\begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix}}_{\text{known}} \longrightarrow \underbrace{\begin{bmatrix} \hat{y}_1 \\ \vdots \\ \hat{y}_N \end{bmatrix}}_{\text{unknown}}$$

such that $\hat{\boldsymbol{y}}_k = \boldsymbol{C}\boldsymbol{A}^k\boldsymbol{x}_0$ is the output of an nth-order autonomous system





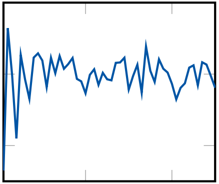
$$\min_{\hat{\boldsymbol{y}},\boldsymbol{\alpha}} \|\boldsymbol{y} - \hat{\boldsymbol{y}}\|_2^2$$
$$\text{subject to } \boldsymbol{T}_\alpha \hat{\boldsymbol{y}} = \boldsymbol{0}$$

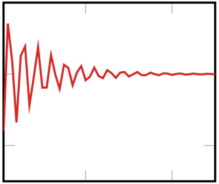
(De Moor, 2019, 2020)

Solving the least-squares realization problem

$$\underbrace{\begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix}}_{\text{known}} \longrightarrow \underbrace{\begin{bmatrix} \hat{y}_1 \\ \vdots \\ \hat{y}_N \end{bmatrix}}_{\text{unknown}}$$

$$\boldsymbol{T}_\alpha = \begin{bmatrix} \alpha_2 & \alpha_1 & 1 & 0 & 0 & 0 \\ 0 & \alpha_2 & \alpha_1 & 1 & 0 & 0 \\ 0 & 0 & \alpha_2 & \alpha_1 & 1 & 0 \\ 0 & 0 & 0 & \alpha_2 & \alpha_1 & 1 \end{bmatrix}$$





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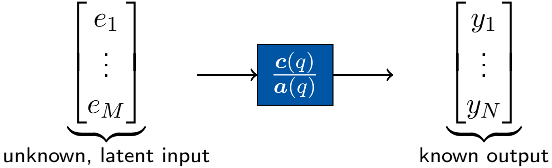
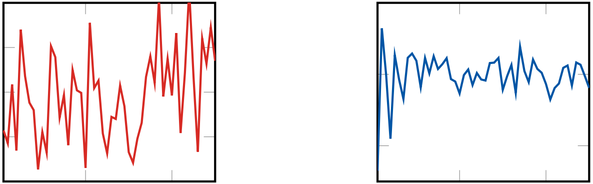
(De Moor, 2019, 2020)

Identifying the parameters of an ARMA model





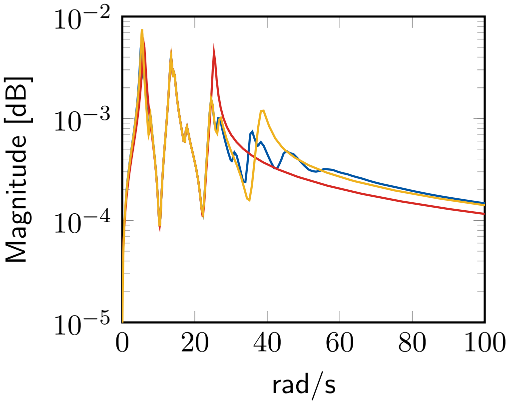
$$\min_{\boldsymbol{e},\boldsymbol{\alpha},\boldsymbol{\gamma}} \|\boldsymbol{e}\|_2^2$$

$$\text{subject to } \boldsymbol{T}_\alpha \boldsymbol{y} = \boldsymbol{T}_\gamma \boldsymbol{e}$$

such that $\sum_{i=0}^{n_a} \alpha_i y_{k-i} = \sum_{i=0}^{n_c} \gamma_i e_{k-i}$

(Vermeersch and De Moor, 2019)

Model order reduction



Transfer function of a 45th-order model (——) and two reduced-order approximations, namely $r = 18$ (——) and $r = 12$ (——)

$$\mathcal{F}_{\mathsf{H}}(s) = \frac{a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}{s^n + b_{n-1}s^{n-1} + \cdots + b_1 s + b_0}$$

$\downarrow$ with $r << n$

$$\mathcal{F}_{\mathsf{R}}(s) = \frac{\tilde{a}_{r-1}s^{r-1} + \cdots + \tilde{a}_1 s + \tilde{a}_0}{s^r + \tilde{b}_{r-1}s^{r-1} + \cdots + \tilde{b}_1 s + \tilde{b}_0}$$

$$\min_{\tilde{\boldsymbol{a}},\tilde{\boldsymbol{b}}} \|\mathcal{E}(s)\|^2_{\mathcal{H}_2}$$

$$\text{subject to } \mathcal{E}(s) = \mathcal{F}_{\mathsf{H}}(s) - \mathcal{F}_{\mathsf{R}}(s)$$

(Alsubaie, 2019; Agudelo et al., 2021)

Common problem!

These three problems have one thing in common: they are **rectangular multiparameter eigenvalue problems***!

$$\begin{bmatrix} \boldsymbol{\mathcal{M}}(\lambda_1, \ldots, \lambda_n) \end{bmatrix} \begin{bmatrix} \boldsymbol{z} \end{bmatrix} = \boldsymbol{0}$$

But, there are many other problems that also fit into this framework: vibration analysis, prediction error methods, optimization in complex variables, etc.

* . . . and in a sense also one-parameter eigenvalue problems.

(Batselier et al., 2012; Tisseur and Meerbergen, 2001; Vermeersch et al., 2023)

Outline

Outline

Multiparameter eigenvalue problem

The **rectangular multiparameter eigenvalue problem (rectangular MEP)** consists in finding all n-tuples $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ and corresponding vectors $\boldsymbol{z} \in \mathbb{C}^{l \times 1} \setminus \{\mathbf{0}\}$, so that

$$\boldsymbol{\mathcal{M}}(\boldsymbol{\lambda})\,\boldsymbol{z} = \left(\sum_{\{\boldsymbol{\omega}\}} \boldsymbol{A}_{\boldsymbol{\omega}} \boldsymbol{\lambda}^{\boldsymbol{\omega}}\right) \boldsymbol{z} = \mathbf{0}.$$

- integer multi-index $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_n)$
- rectangular coefficient matrices $\boldsymbol{A}_{\boldsymbol{\omega}} \in \mathbb{C}^{k \times l}$ (with $k \geq l + n - 1$)
- full normal column rank matrix pencil $\boldsymbol{\mathcal{M}}(\boldsymbol{\lambda})$
- eigentuples $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$ and eigenvectors $\boldsymbol{z}$ (with $\|\boldsymbol{z}\| = 1$)

(Vermeersch and De Moor, 2019, 2022)

Example: polynomial two-parameter eigenvalue problem

rectangular 3×2 coefficient matrices

$$\left(\underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 3 & 4 \end{bmatrix}}_{\boldsymbol{A}_{00}} + \underbrace{\begin{bmatrix} 2 & 1 \\ 0 & 1 \\ 1 & 3 \end{bmatrix}}_{\boldsymbol{A}_{10}} \lambda_1 + \underbrace{\begin{bmatrix} 3 & 4 \\ 2 & 1 \\ 0 & 1 \end{bmatrix}}_{\boldsymbol{A}_{11}} \lambda_1 \lambda_2 + \underbrace{\begin{bmatrix} 1 & 2 \\ 4 & 2 \\ 2 & 1 \end{bmatrix}}_{\boldsymbol{A}_{02}} \lambda_2^2 \right) \boldsymbol{z} = \boldsymbol{0}$$

(ω_1, ω_2) labels $\boldsymbol{\lambda}^{\boldsymbol{\omega}}$ and indexes $\boldsymbol{A}_{\boldsymbol{\omega}}$

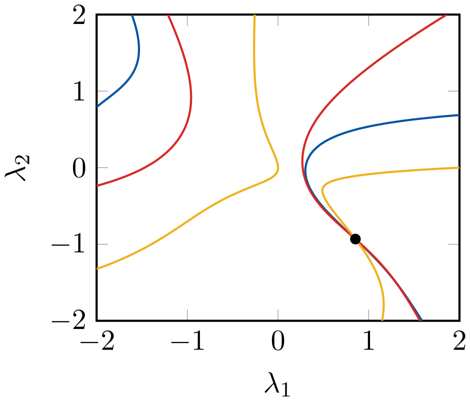
Alternative formulation

We can also phrase the rectangular MEP by considering the eigentuples for which the rank drops below the normal rank of the rectangular matrix pencil.

Given a rectangular matrix pencil $\boldsymbol{\mathcal{M}}(\boldsymbol{\lambda})$, we say that the tuple $\boldsymbol{\lambda} \in \mathbb{C}$ is an eigentuple if

$$\operatorname{rank}(\boldsymbol{\mathcal{M}}(\boldsymbol{\lambda})) < \operatorname{nrank}(\boldsymbol{\mathcal{M}}(\boldsymbol{\lambda})).$$

Example: polynomial two-parameter eigenvalue problem



Real picture of the variety of the determinants $p_1(\boldsymbol{\lambda})$ (——), $p_2(\boldsymbol{\lambda})$ (——), and $p_3(\boldsymbol{\lambda})$ (——), which contains 9 affine points (only one of which is real) and 3 points at infinity

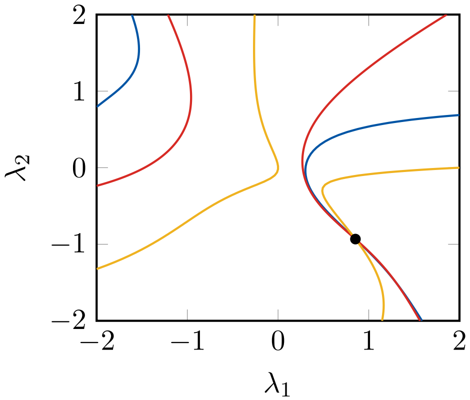
$$\boldsymbol{\mathcal{M}}(\boldsymbol{\lambda}) = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \\ m_{31} & m_{32} \end{bmatrix}$$

$\boldsymbol{D}_1$ (upper 2×2 block, rows 1–2)

$$p_i(\boldsymbol{\lambda}) = \det(\boldsymbol{D}_i) = 0$$

$$i = 1, 2, 3$$

Example: polynomial two-parameter eigenvalue problem



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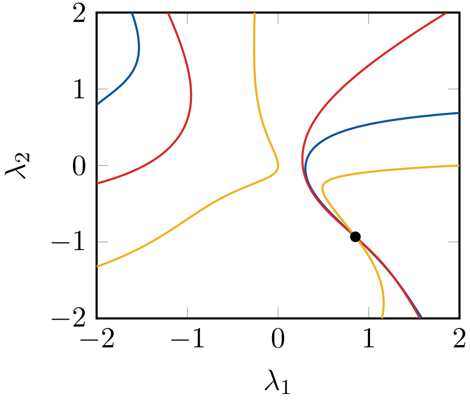
$$\boldsymbol{\mathcal{M}}(\boldsymbol{\lambda}) = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \\ m_{31} & m_{32} \end{bmatrix}$$

$\boldsymbol{D}_2$

$$p_i(\boldsymbol{\lambda}) = \det(\boldsymbol{D}_i) = 0$$

$$i = 1, 2, 3$$

Example: polynomial two-parameter eigenvalue problem



Real picture of the variety of the determinants $p_1(\boldsymbol{\lambda})$ (——), $p_2(\boldsymbol{\lambda})$ (——), and $p_3(\boldsymbol{\lambda})$ (——), which contains 9 affine points (only one of which is real) and 3 points at infinity

$$\boldsymbol{\mathcal{M}}(\boldsymbol{\lambda}) = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \\ m_{31} & m_{32} \end{bmatrix}$$

$\boldsymbol{D}_3$

$$p_i(\boldsymbol{\lambda}) = \det(\boldsymbol{D}_i) = 0$$

$$i = 1, 2, 3$$

Unifying framework for (multiparameter) eigenvalue problems

Different types of (multiparameter) eigenvalue problems*

Spectral parameter(s)	**Linear**	**Polynomial**
Eigenvalues $(n = 1)$	Type I $\{1, \lambda\}$ $\boldsymbol{A} - \boldsymbol{B}\lambda$ SEP/GEP	Type II λ^{ω} $\boldsymbol{A}_0 + \boldsymbol{A}_1\lambda + \cdots + \boldsymbol{A}_d\lambda^d$ PEP
Eigentuples $(n > 1)$ $(i = 1, \ldots, n)$	Type III λ_i $\boldsymbol{A}_{00} + \boldsymbol{A}_{10}\lambda_1 + \boldsymbol{A}_{01}\lambda_2$ linear MEP	Type IV $\boldsymbol{\lambda}^{\boldsymbol{\omega}} = \prod_{i=1}^{n} \lambda_i^{\omega_i}$ $\boldsymbol{A}_{00} + \boldsymbol{A}_{11}\lambda_1\lambda_2 + \boldsymbol{A}_{03}\lambda_2^3$ polynomial MEP

* SEP = standard eigenvalue problem – GEP = generalized eigenvalue problem – PEP = polynomial eigenvalue problem

Square problems?

Not a presentation about square multiparameter eigenvalue problems!

Volkmer's square multiparameter eigenvalue problem (square MEP):

$$\begin{cases} \left(\begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{bmatrix} \lambda_1 + \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \lambda_2 \right) \boldsymbol{x}_1 = \boldsymbol{0} \\ \left(\begin{bmatrix} 20 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & \sqrt{3} \\ \sqrt{3} & 0 \end{bmatrix} \lambda_1 + \begin{bmatrix} 7 & 0 \\ 0 & 1 \end{bmatrix} \lambda_2 \right) \boldsymbol{x}_2 = \boldsymbol{0} \end{cases}$$

There exist quite a few relations between both manifestations.

(Volkmer, 1988)

Relations between square and rectangular problems

$$\begin{cases} \boldsymbol{\mathcal{W}}_1(\boldsymbol{\lambda})\,\boldsymbol{x}_1 = \mathbf{0} \\ \boldsymbol{\mathcal{W}}_2(\boldsymbol{\lambda})\,\boldsymbol{x}_2 = \mathbf{0} \end{cases} \Leftrightarrow \begin{cases} \boldsymbol{\Delta}_1 \boldsymbol{z} = \boldsymbol{\Delta}_0 \lambda_1 \boldsymbol{z} \\ \boldsymbol{\Delta}_2 \boldsymbol{z} = \boldsymbol{\Delta}_0 \lambda_2 \boldsymbol{z} \end{cases} \Leftrightarrow \left(\begin{bmatrix} \boldsymbol{\Delta}_1 \\ \boldsymbol{\Delta}_2 \end{bmatrix} - \begin{bmatrix} \boldsymbol{\Delta}_0 \\ \mathbf{0} \end{bmatrix} \lambda_1 - \begin{bmatrix} \mathbf{0} \\ \boldsymbol{\Delta}_0 \end{bmatrix} \lambda_2 \right) \boldsymbol{z} = \mathbf{0}$$

square to rectangular

---

rectangular to square

$$\boldsymbol{\mathcal{M}}(\boldsymbol{\lambda})\,\boldsymbol{z} = \mathbf{0} \Rightarrow \begin{cases} \boldsymbol{P}_1 \boldsymbol{\mathcal{M}}(\boldsymbol{\lambda})\,\boldsymbol{x}_1 = \mathbf{0} \\ \boldsymbol{P}_2 \boldsymbol{\mathcal{M}}(\boldsymbol{\lambda})\,\boldsymbol{x}_2 = \mathbf{0} \end{cases} \Leftrightarrow \begin{cases} \boldsymbol{\Delta}_1 \boldsymbol{y} = \boldsymbol{\Delta}_0 \lambda_1 \boldsymbol{y} \\ \boldsymbol{\Delta}_2 \boldsymbol{y} = \boldsymbol{\Delta}_0 \lambda_2 \boldsymbol{y} \end{cases} \Rightarrow \begin{cases} \hat{\boldsymbol{\Delta}}_1 \boldsymbol{z} = \hat{\boldsymbol{\Delta}}_0 \lambda_1 \boldsymbol{z} \\ \hat{\boldsymbol{\Delta}}_2 \boldsymbol{z} = \hat{\boldsymbol{\Delta}}_0 \lambda_2 \boldsymbol{z} \end{cases}$$

(Vermeersch and De Moor, 2022; Hochstenbach et al., 2022)

Relations between square and rectangular problems

associated system of coupled GEPs

$$\begin{cases} \mathcal{W}_1(\boldsymbol{\lambda})\,\boldsymbol{x}_1 = \mathbf{0} \\ \mathcal{W}_2(\boldsymbol{\lambda})\,\boldsymbol{x}_2 = \mathbf{0} \end{cases} \Leftrightarrow \begin{cases} \boldsymbol{\Delta}_1 \boldsymbol{z} = \boldsymbol{\Delta}_0 \lambda_1 \boldsymbol{z} \\ \boldsymbol{\Delta}_2 \boldsymbol{z} = \boldsymbol{\Delta}_0 \lambda_2 \boldsymbol{z} \end{cases} \Leftrightarrow \left(\begin{bmatrix} \boldsymbol{\Delta}_1 \\ \boldsymbol{\Delta}_2 \end{bmatrix} - \begin{bmatrix} \boldsymbol{\Delta}_0 \\ \mathbf{0} \end{bmatrix} \lambda_1 - \begin{bmatrix} \mathbf{0} \\ \boldsymbol{\Delta}_0 \end{bmatrix} \lambda_2 \right) \boldsymbol{z} = \mathbf{0}$$

square to rectangular

---

rectangular to square

$$\mathcal{M}(\boldsymbol{\lambda})\,\boldsymbol{z} = \mathbf{0} \Rightarrow \begin{cases} \boldsymbol{P}_1 \mathcal{M}(\boldsymbol{\lambda})\,\boldsymbol{x}_1 = \mathbf{0} \\ \boldsymbol{P}_2 \mathcal{M}(\boldsymbol{\lambda})\,\boldsymbol{x}_2 = \mathbf{0} \end{cases} \Leftrightarrow \begin{cases} \boldsymbol{\Delta}_1 \boldsymbol{y} = \boldsymbol{\Delta}_0 \lambda_1 \boldsymbol{y} \\ \boldsymbol{\Delta}_2 \boldsymbol{y} = \boldsymbol{\Delta}_0 \lambda_2 \boldsymbol{y} \end{cases} \Rightarrow \begin{cases} \hat{\boldsymbol{\Delta}}_1 \boldsymbol{z} = \hat{\boldsymbol{\Delta}}_0 \lambda_1 \boldsymbol{z} \\ \hat{\boldsymbol{\Delta}}_2 \boldsymbol{z} = \hat{\boldsymbol{\Delta}}_0 \lambda_2 \boldsymbol{z} \end{cases}$$

(Vermeersch and De Moor, 2022; Hochstenbach et al., 2022)

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$$\begin{cases} \mathcal{W}_1(\boldsymbol{\lambda})\boldsymbol{x}_1 = \mathbf{0} \\ \mathcal{W}_2(\boldsymbol{\lambda})\boldsymbol{x}_2 = \mathbf{0} \end{cases} \Leftrightarrow \begin{cases} \boldsymbol{\Delta}_1\boldsymbol{z} = \boldsymbol{\Delta}_0\lambda_1\boldsymbol{z} \\ \boldsymbol{\Delta}_2\boldsymbol{z} = \boldsymbol{\Delta}_0\lambda_2\boldsymbol{z} \end{cases} \Leftrightarrow \left(\begin{bmatrix} \boldsymbol{\Delta}_1 \\ \boldsymbol{\Delta}_2 \end{bmatrix} - \begin{bmatrix} \boldsymbol{\Delta}_0 \\ \mathbf{0} \end{bmatrix}\lambda_1 - \begin{bmatrix} \mathbf{0} \\ \boldsymbol{\Delta}_0 \end{bmatrix}\lambda_2\right)\boldsymbol{z} = \mathbf{0}$$

associated system of coupled GEPs

square to rectangular

---

rectangular to square

$$\mathcal{M}(\boldsymbol{\lambda})\boldsymbol{z} = \mathbf{0} \Rightarrow \begin{cases} \boldsymbol{P}_1\mathcal{M}(\boldsymbol{\lambda})\boldsymbol{x}_1 = \mathbf{0} \\ \boldsymbol{P}_2\mathcal{M}(\boldsymbol{\lambda})\boldsymbol{x}_2 = \mathbf{0} \end{cases} \Leftrightarrow \begin{cases} \boldsymbol{\Delta}_1\boldsymbol{y} = \boldsymbol{\Delta}_0\lambda_1\boldsymbol{y} \\ \boldsymbol{\Delta}_2\boldsymbol{y} = \boldsymbol{\Delta}_0\lambda_2\boldsymbol{y} \end{cases} \Rightarrow \begin{cases} \hat{\boldsymbol{\Delta}}_1\boldsymbol{z} = \hat{\boldsymbol{\Delta}}_0\lambda_1\boldsymbol{z} \\ \hat{\boldsymbol{\Delta}}_2\boldsymbol{z} = \hat{\boldsymbol{\Delta}}_0\lambda_2\boldsymbol{z} \end{cases}$$

randomized/structured sketching

(Vermeersch and De Moor, 2022; Hochstenbach et al., 2022)

Relations between square and rectangular problems

associated system of coupled GEPs

$$\begin{cases} \mathcal{W}_1(\boldsymbol{\lambda})\,\boldsymbol{x}_1 = \mathbf{0} \\ \mathcal{W}_2(\boldsymbol{\lambda})\,\boldsymbol{x}_2 = \mathbf{0} \end{cases} \Leftrightarrow \begin{cases} \boldsymbol{\Delta}_1 \boldsymbol{z} = \boldsymbol{\Delta}_0 \lambda_1 \boldsymbol{z} \\ \boldsymbol{\Delta}_2 \boldsymbol{z} = \boldsymbol{\Delta}_0 \lambda_2 \boldsymbol{z} \end{cases} \Leftrightarrow \left(\begin{bmatrix} \boldsymbol{\Delta}_1 \\ \boldsymbol{\Delta}_2 \end{bmatrix} - \begin{bmatrix} \boldsymbol{\Delta}_0 \\ \mathbf{0} \end{bmatrix} \lambda_1 - \begin{bmatrix} \mathbf{0} \\ \boldsymbol{\Delta}_0 \end{bmatrix} \lambda_2 \right) \boldsymbol{z} = \mathbf{0}$$

square to rectangular

---

rectangular to square

$$\boldsymbol{y} = \boldsymbol{x}_1 \otimes \boldsymbol{x}_2 \neq \boldsymbol{z}$$

$$\boldsymbol{\mathcal{M}}(\boldsymbol{\lambda})\,\boldsymbol{z} = \mathbf{0} \Rightarrow \begin{cases} \boldsymbol{P}_1 \boldsymbol{\mathcal{M}}(\boldsymbol{\lambda})\,\boldsymbol{x}_1 = \mathbf{0} \\ \boldsymbol{P}_2 \boldsymbol{\mathcal{M}}(\boldsymbol{\lambda})\,\boldsymbol{x}_2 = \mathbf{0} \end{cases} \Leftrightarrow \begin{cases} \boldsymbol{\Delta}_1 \boldsymbol{y} = \boldsymbol{\Delta}_0 \lambda_1 \boldsymbol{y} \\ \boldsymbol{\Delta}_2 \boldsymbol{y} = \boldsymbol{\Delta}_0 \lambda_2 \boldsymbol{y} \end{cases} \Rightarrow \begin{cases} \hat{\boldsymbol{\Delta}}_1 \boldsymbol{z} = \hat{\boldsymbol{\Delta}}_0 \lambda_1 \boldsymbol{z} \\ \hat{\boldsymbol{\Delta}}_2 \boldsymbol{z} = \hat{\boldsymbol{\Delta}}_0 \lambda_2 \boldsymbol{z} \end{cases}$$

randomized/structured sketching

(Vermeersch and De Moor, 2022; Hochstenbach et al., 2022)

Outline

Three approaches to compute the eigentuples

Number of eigentuples?

- linear MEP

$$m_b = \binom{l+n-1}{l}$$

- polynomial MEP

$$m_b = d^n \binom{l+n-1}{l}$$

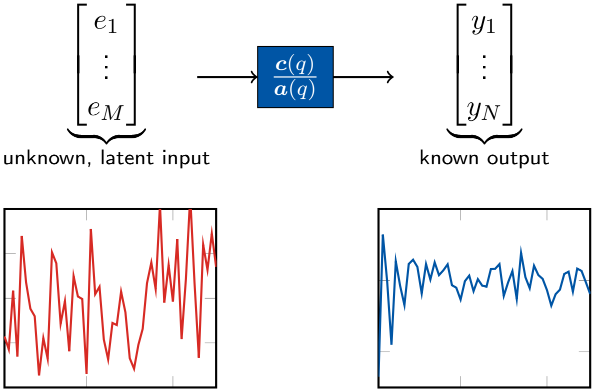
Three solution approaches exist:

- block Macaulay matrix approach – (Vermeersch and De Moor, 2022, 2023)
- reduction to one-parameter problem – (Alsubaie, 2019)
- transformation into square MEP – (Hochstenbach et al., 2022)

(Shapiro and Shapiro, 2009; Hochstenbach et al., 2022)

Outline

Identifying the parameters of an ARMA(1,1) model



such that

$$y_k + \alpha y_{k-1} = e_k + \gamma e_{k-1}$$

$$\min_{\boldsymbol{e},\alpha,\gamma} \|\boldsymbol{e}\|_2^2$$

$$\text{subject to } \boldsymbol{T}_\alpha \boldsymbol{y} = \boldsymbol{T}_\gamma \boldsymbol{e}$$

(Vermeersch and De Moor, 2019)

Identifying the parameters of an ARMA(1,1) model

$$\min_{\boldsymbol{e},\alpha,\gamma} \|\boldsymbol{e}\|_2^2$$

$$\text{subject to } \boldsymbol{T}_\alpha \boldsymbol{y} = \boldsymbol{T}_\gamma \boldsymbol{e}$$

$$\Downarrow$$

$$\begin{bmatrix} \boldsymbol{y}^{\mathrm{T}}\boldsymbol{T}_\alpha^{\mathrm{T}} & \boldsymbol{0} & \boldsymbol{y}^{\mathrm{T}}\boldsymbol{T}_\alpha^{\alpha\mathrm{T}} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{y}^{\mathrm{T}}\boldsymbol{T}_\alpha^{\mathrm{T}} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{D}_\gamma & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{T}_\alpha^\alpha \boldsymbol{y} \\ \boldsymbol{0} & \boldsymbol{D}_\gamma & \boldsymbol{D}_\gamma^\gamma & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{D}_\gamma & \boldsymbol{T}_\alpha \boldsymbol{y} \end{bmatrix} \begin{bmatrix} \boldsymbol{f}^\alpha \\ \boldsymbol{f}_1^\gamma \\ \boldsymbol{f} \\ -1 \end{bmatrix} = \boldsymbol{0}$$

Identifying the parameters of an ARMA(1,1) model

$$\min_{\boldsymbol{e},\alpha,\gamma} \|\boldsymbol{e}\|_2^2$$

$$\text{subject to } \boldsymbol{T}_\alpha \boldsymbol{y} = \boldsymbol{T}_\gamma \boldsymbol{e}$$

$$\Downarrow$$

$$\begin{bmatrix} \boldsymbol{y}^{\mathrm{T}}\boldsymbol{T}_\alpha^{\mathrm{T}} & \boldsymbol{0} & \boldsymbol{y}^{\mathrm{T}}\boldsymbol{T}_\alpha^{\alpha\mathrm{T}} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{y}^{\mathrm{T}}\boldsymbol{T}_\alpha^{\mathrm{T}} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{D}_\gamma & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{T}_\alpha^\alpha \boldsymbol{y} \\ \boldsymbol{0} & \boldsymbol{D}_\gamma & \boldsymbol{D}_\gamma^\gamma & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{D}_\gamma & \boldsymbol{T}_\alpha \boldsymbol{y} \end{bmatrix} \begin{bmatrix} \boldsymbol{f}^\alpha \\ \boldsymbol{f}_1^\gamma \\ \boldsymbol{f} \\ -1 \end{bmatrix} = \boldsymbol{0}$$

$$\Downarrow$$

$$\left(\boldsymbol{A}_{00} + \boldsymbol{A}_{10}\alpha + \boldsymbol{A}_{01}\gamma + \boldsymbol{A}_{02}\gamma^2\right)\boldsymbol{z} = \boldsymbol{0}$$

Identifying the parameters of an ARMA(1,1) model

$\boldsymbol{y} \in \mathbb{R}^8$

$\downarrow$

$\min \|\boldsymbol{e}\|_2^2$

subject to $\boldsymbol{T}_\alpha \boldsymbol{y} = \boldsymbol{T}_\gamma \boldsymbol{e}$

$\downarrow$

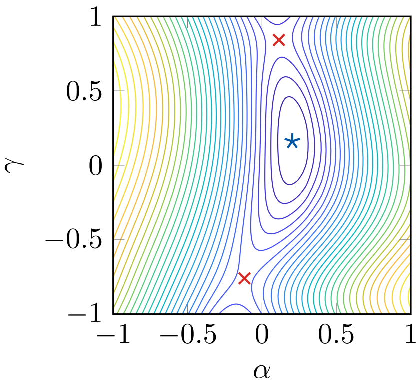
$$\left(\boldsymbol{A}_{00} + \boldsymbol{A}_{10}\alpha + \boldsymbol{A}_{01}\gamma + \boldsymbol{A}_{02}\gamma^2\right)\boldsymbol{z} = \boldsymbol{0}$$

$\downarrow$

one of the solution approaches

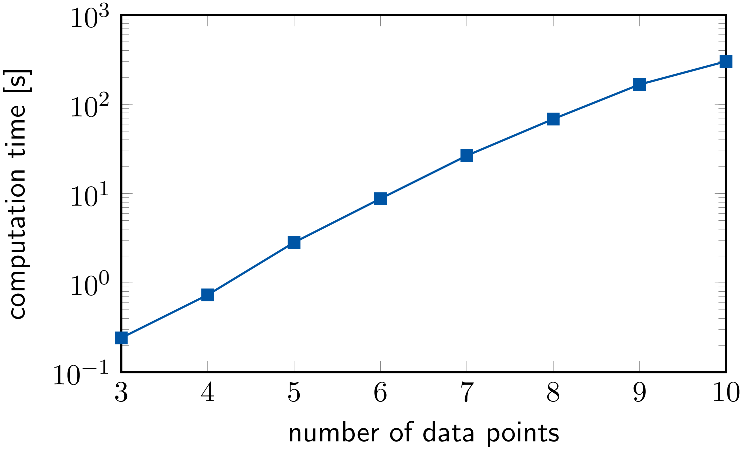
$\downarrow$

parameters α and γ



Contour plot of the cost function with one minimum (∗) and two saddle points (×)

Computational bottleneck



Computation time (—■—) of identifying the parameters of a first-order ARMA model via the MEP approach for a given number of data points

Outline

1 | Introduction

2 | Multiparameter Eigenvalue Problem

3 | Solution Approaches

4 | Numerical Example

5 | Conclusion and Future Work

Conclusion and future work

- Multiparameter eigenvalue problems are omnipresent in systems and control:
  - system identification,
  - model order reduction,
  - and partial differential equations.
- There exist relations between square and rectangular manifestations.
- Scalability remains an active research problem*!

* All algorithms are implemented in MATLAB and available at `www.macaulaylab.net`.

Any questions?

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[‡] Supported by FWO Strategic Basic Research fellowship under grant SB/1SA1319N

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