Globally optimal SISO H_2-norm model reduction

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Fonds Wetenschappelijk Onderzoek Vlaanderen
Opening new horizons

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Model reduction

- Class $\mathcal{M}$: minimal, stable, SISO LTI systems.

- Given $H(s) \in \mathcal{M}$ of order n,

$$H(s) = \frac{b(s)}{a(s)} = \frac{b_{n-1}s^{n-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}, \tag{1}$$

  find $\hat{H}(s) \in \mathcal{M}$ of order $m < n$,

$$\hat{H}(s) = \frac{\hat{b}(s)}{\hat{a}(s)} = \frac{\hat{b}_{m-1}s^{m-1} + \cdots + \hat{b}_1 s + \hat{b}_0}{s^m + \hat{a}_{m-1}s^{m-1} + \cdots + \hat{a}_1 s + \hat{a}_0}, \tag{2}$$

  so that $\hat{H}(s)$ is a 'good approximation' of $H(s)$.

- Notation: $\hat{\boldsymbol{a}} = (\hat{a}_{m-1}, \ldots, \hat{a}_0)^\mathsf{T} \in \mathbb{R}^n$, $\hat{\boldsymbol{b}} = (\hat{b}_{m-1}, \ldots, \hat{b}_0)^\mathsf{T} \in \mathbb{R}^n$.

H_2-norm model reduction

- Minimize H_2-norm of approximation error $E(s) = H(s) - \hat{H}(s)$:

$$\hat{H}(s) \in \underset{\hat{H}(s) \in \mathcal{M}}{\operatorname{argmin}} \; J^2, \tag{3}$$

where,

$$\begin{aligned} J^2 = \|E(s)\|_{H_2}^2 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} |H(j\omega) - \hat{H}(j\omega)|^2 \, \mathrm{d}\omega \\ &= \int_0^{\infty} \left(h(t) - \hat{h}(t) \right)^2 \mathrm{d}t, \end{aligned} \tag{4}$$

with $h(t)$ and $\hat{h}(t)$ the impulse responses of $H(s)$, $\hat{H}(s)$, respectively.

H_2-norm model reduction

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with $h(t)$ and $\hat{h}(t)$ the impulse responses of $H(s)$, $\hat{H}(s)$, respectively.

- Opt. problem in (3) is **non-convex** $\longrightarrow$ (many) local minimizers.

H_2-norm model reduction

- State-of-the-art solvers:
  - Interpolation-based methods (e.g., Gugercin et al., 2008; Van Dooren et al., 2010)
  - Lyapunov-based methods (e.g., Spanos et al., 1992)

  $\longrightarrow$ Strategy: solve iteratively for stationary point of (3)

[1] E.g., Agudelo et al., 2021; Ahmad et al., 2011; Alsubaie, 2019; Hanzon et al., 2007.

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- Non-convexity $\longleftrightarrow$ iterative solvers:
  - Performance depends on heuristic choice of initial point (heuristics)
  - Impossible to guarantee global optimality

  $\longrightarrow$ Suboptimal from a mathematical point of view ✗

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- Non-convexity $\longleftrightarrow$ iterative solvers:
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  - Impossible to guarantee global optimality

  $\longrightarrow$ Suboptimal from a mathematical point of view ✗
- Globally optimal approaches[1]:
  - Strategy: characterize first-order necessary conditions for optimality
  - Solve for all stationary points $\longrightarrow$ identify global minimizer(s) ✓

[1] E.g., Agudelo et al., 2021; Ahmad et al., 2011; Alsubaie, 2019; Hanzon et al., 2007.

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Interpolatory conditions for optimality

Theorem (Meier and Luenberger, 1967)

Given a stable SISO model $H(s) \in \mathcal{M}$ of order n, let $\hat{H}(s)$ of order m ($m<n$) be a stationary point of the H_2-norm model reduction problem in (3). Then for all poles p_i of $\hat{H}(s)$,

$$H(-p_i)^{(j)} = \hat{H}(-p_i)^{(j)}, \quad j = 0, \ldots, d_i,$$

where d_i is the multiplicity of the pole p_i and the superscript j denotes the jth derivative with respect to s, i.e.,

$$F(a)^{(j)} = \left.\frac{\mathrm{d}^j F(s)}{\mathrm{d}s^j}\right|_{s=a},$$

for any function $F(s)$ and $a \in \mathbb{C}$.

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Walsh's theorem

Theorem (Regalia, 1995)

Given a stable SISO model $H(s) \in \mathcal{M}$ of order n, let $\hat{H}(s)$ of order m with $m<n$ be a stationary point of the model reduction problem (3). Then for all $s \in \mathbb{C}$:

$$H(s) - \hat{H}(s) = \frac{b(s)}{a(s)} - \frac{\hat{b}(s)}{\hat{a}(s)} = \left[\frac{\hat{a}(-s)}{\hat{a}(s)}\right]^2 G(s),$$

with $G(s)$ the Laplace-transform of some real-valued, stable and causal signal, and where $a(s), b(s), \hat{a}(s)$ and $\hat{b}(s)$ are defined as in (1)–(2).

- Roots of $\hat{a}(-s)$ are $\{-p_i\}_{i=1,\ldots,m}$
- Origins within rational approximation theory (Walsh, 1960)

Walsh's theorem - revisited

Corollary

For any given stable SISO model $H(s)$ of order n and mth order approximant $\hat{H}(s)$ with $m<n$ as defined in (1)–(2), define the polynomial,

$$l(s) = b(s)\hat{a}(s) - a(s)\hat{b}(s) - [\hat{a}(-s)]^2\tilde{G}(s), \tag{3}$$

where $\tilde{G}(s)$ is a polynomial parametrized in the coefficients $\boldsymbol{g} = (g_0, \ldots, g_{n-m-1})^\mathsf{T} \in \mathbb{R}^{m-n}$:

$$\tilde{G}(s) = g_{n-m-1}s^{n-m-1} + \cdots + g_1 s + g_0.$$

Then, $\hat{H}(s)$ is a stationary point of (3) if and only if,

$$\exists\, \boldsymbol{g} \quad s.t. \quad l(s) = 0,\ \forall s \in \mathbb{C}.$$

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Walsh's theorem - revisited

Corollary

For any given stable SISO model $H(s)$ of order n and mth order approximant $\hat{H}(s)$ with $m<n$ as defined in (1)–(2)*, define the polynomial,*

$$l(s) = b(s)\hat{a}(s) - a(s)\hat{b}(s) - [\hat{a}(-s)]^2 \tilde{G}(s),$$

where $\tilde{G}(s)$ is a polynomial parametrized in the coefficients $\boldsymbol{g} = (g_0, \ldots, g_{n-m-1})^\mathsf{T} \in \mathbb{R}^{m-n}$*:*

$$\tilde{G}(s) = g_{n-m-1}s^{n-m-1} + \cdots + g_1 s + g_0.$$

Then, $\hat{H}(s)$ is a stationary point of (3) *if and only if,*

$$\exists\, \boldsymbol{g} \quad s.t. \quad l(s) = 0,\ \forall s \in \mathbb{C}.$$

$\longrightarrow$ Elegant and compact characterization of the stationary points of (3)

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System of polynomial equations

Example

- *Consider the third order model ($n = 3$) used in Agudelo et al., 2021:*

$$H(s) = \frac{s^2 + 9s - 10}{s^3 + 12s^2 + 49s + 78},$$

for which we search the optimal first-order ($m = 1$) approximation.

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- *The polynomial $I(s)$ from (3) is given as,*

$$I(s) = (\underbrace{1 - g_1 - \hat{b}_0}_{f_3})s^3 + (\underbrace{\hat{a}_0 - 12\hat{b}_0 - g_0 + 2\hat{a}_0 g_1 + 9}_{f_2})s^2$$
$$+ (\underbrace{9\hat{a}_0 - 49\hat{b}_0 + 2\hat{a}_0 g_0 - \hat{a}_0^2 g_1 - 10}_{f_1})s$$
$$+ (\underbrace{-g_0\hat{a}_0^2 - 10\hat{a}_0 - 78\hat{b}_0}_{f_0})1.$$

System of polynomial equations

Example (continued)

Strategy: find all $(\hat{\boldsymbol{a}}, \hat{\boldsymbol{b}}, \boldsymbol{g})$ *for which* $l(s) = 0, \ \forall s \in \mathbb{C}$"

$$\iff \begin{cases} 0 = f_3 = 1 - g_1 - \hat{b}_0, \\ 0 = f_2 = \hat{a}_0 - 12\hat{b}_0 - g_0 + 2\hat{a}_0 g_1 + 9, \\ 0 = f_1 = 9\hat{a}_0 - 49\hat{b}_0 + 2\hat{a}_0 g_0 - \hat{a}_0^2 g_1 - 10, \\ 0 = f_0 = -g_0 \hat{a}_0^2 - 10\hat{a}_0 - 78\hat{b}_0. \end{cases} \tag{3}$$

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System of polynomial equations

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Strategy: find all $(\hat{\boldsymbol{a}}, \hat{\boldsymbol{b}}, \boldsymbol{g})$ *for which* $l(s) = 0,\ \forall s \in \mathbb{C}$"

$$\Longleftrightarrow \begin{cases} 0 = f_3 = 1 - g_1 - \hat{b}_0, \\ 0 = f_2 = \hat{a}_0 - 12\hat{b}_0 - g_0 + 2\hat{a}_0 g_1 + 9, \\ 0 = f_1 = 9\hat{a}_0 - 49\hat{b}_0 + 2\hat{a}_0 g_0 - \hat{a}_0^2 g_1 - 10, \\ 0 = f_0 = -g_0\hat{a}_0^2 - 10\hat{a}_0 - 78\hat{b}_0. \end{cases}$$

- *Common roots of* (3) *contain all stationary points* $\hat{H}(s)$ *of* (3)*:*

J	$\hat{a}_0$	$\hat{b}_0$	g_0	g_1	*stable*
8.9403	$-4.1639 + 0.9026j$	$24.930 - 6.5393j$	$-106.84 - 18.287j$	$-23.930 + 6.5393j$	✗
8.9403	$-4.1639 - 0.9026j$	$24.930 + 6.5393j$	$-106.84 + 18.287j$	$-23.930 - 6.5393j$	✗
0.3982	0.2671	−0.0437	10.349	1.0437	✓
0.2784	0.6914	9.6796	1.2799	−2.0986	✓
0.5232	−16.618	1.9264	0.0576	−0.9264	✗

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Summary

- Let f_k be the coefficient corresponding to s^k in the polynomial,

$$l(s) = b(s)\hat{a}(s) - a(s)\hat{b}(s) - [\hat{a}(-s)]^2\tilde{G}(s),$$

and define the algebraic variety,

$$\mathcal{V}_{\mathbb{R}} = \left\{(\hat{\boldsymbol{a}}, \hat{\boldsymbol{b}}, \boldsymbol{g}) \in \mathbb{R}^{m+n} \ : \ f_k(\hat{\boldsymbol{a}}, \hat{\boldsymbol{b}}, \boldsymbol{g}) = 0, \quad \forall k = 0, \ldots, m+n-1\right\}.$$

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- Consider the subvariety $\mathcal{V}'_{\mathbb{R}} \subseteq \mathcal{V}_{\mathbb{R}}$ s.t.,

$\mathcal{V}'_{\mathbb{R}}$ contains the $(\hat{\boldsymbol{a}}, \hat{\boldsymbol{b}}, \boldsymbol{g}) \in \mathcal{V}_{\mathbb{R}}$ for which $\hat{H}(s)$ is stable.

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- Consider the subvariety $\mathcal{V}'_{\mathbb{R}} \subseteq \mathcal{V}_{\mathbb{R}}$ s.t.,

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Then, by the Corollary we know that,
$\longrightarrow$ $\mathcal{V}'_{\mathbb{R}}$ describes all mth order stationary points $\hat{H}(s)$ of (3) ✓

Summary

- Let f_k be the coefficient corresponding to s^k in the polynomial,

$$l(s) = b(s)\hat{a}(s) - a(s)\hat{b}(s) - [\hat{a}(-s)]^2\tilde{G}(s),$$

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- Consider the subvariety $\mathcal{V}'_{\mathbb{R}} \subseteq \mathcal{V}_{\mathbb{R}}$ s.t.,

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Then, by the Corollary we know that,
$\longrightarrow$ $\mathcal{V}'_{\mathbb{R}}$ describes all mth order stationary points $\hat{H}(s)$ of (3) ✓

- Computing $\mathcal{V}_{\mathbb{R}}$ is a polynomial root finding problem:
  - square system of $m+n$ polynomial equations, solve using (e.g., Breiding and Timme, 2018).

Numerical example I

We search for the globally optimal 4th order reduced model ($m = 4$) of the state-space model[2] ($n = 17$) considered in Žigić et al., 1992.

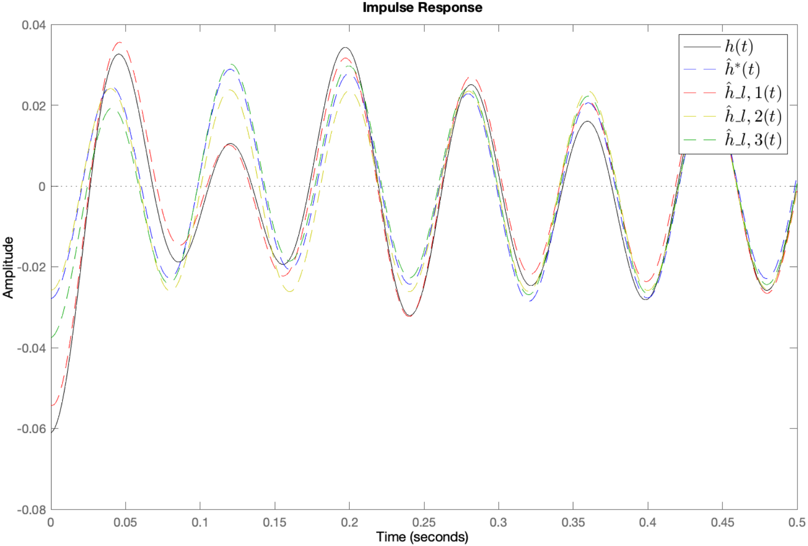
- System of 21 polynomial equations with $d_{\max} = 3$
- Takes 3h24m2s to solve using Breiding and Timme, 2018:

  $\longrightarrow$ $\mathcal{V}_{\mathbb{R}}$ contains 290 tuples, 69 remain in $\mathcal{V}'_{\mathbb{R}}$
- Four best-performing stationary points $\hat{H}(s)$:

	J	$p_{1,2}$	$p_{3,4}$
*	9.14×10^{-3}	$-0.032 \pm 78.54j$	$-0.111 \pm 15.43j$
l_1	9.22×10^{-3}	$-0.032 \pm 78.54j$	$-5.713 \pm 52.57j$
l_2	1.03×10^{-2}	$-0.032 \pm 78.54j$	$-0.023 \pm 3.842j$
l_3	1.09×10^{-2}	$-0.032 \pm 78.54j$	$-4.663 \pm 15.88j$

[2] The model describes the interaction between a torque activator and a torsional rate sensor for the ACES structure Collins et al., 1991.

Numerical example I



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Multiparameter eigenvalue problem

Example (continued)

$$\begin{cases} 0 = f_3 = 1 - g_1 - \hat{b}_0, \\ 0 = f_2 = \hat{a}_0 - 12\hat{b}_0 - g_0 + 2\hat{a}_0 g_1 + 9, \\ 0 = f_1 = 9\hat{a}_0 - 49\hat{b}_0 + 2\hat{a}_0 g_0 - \hat{a}_0^2 g_1 - 10, \\ 0 = f_0 = -g_0\hat{a}_0^2 - 10\hat{a}_0 - 78\hat{b}_0. \end{cases}$$

Multiparameter eigenvalue problem

Example (continued)

$$\begin{cases} 0 = f_3 = 1 - g_1 - \hat{b}_0, \\ 0 = f_2 = \hat{a}_0 - 12\hat{b}_0 - g_0 + 2\hat{a}_0 g_1 + 9, \\ 0 = f_1 = 9\hat{a}_0 - 49\hat{b}_0 + 2\hat{a}_0 g_0 - \hat{a}_0^2 g_1 - 10, \\ 0 = f_0 = -g_0 \hat{a}_0^2 - 10\hat{a}_0 - 78\hat{b}_0. \end{cases}$$

- Partial linear structure: $\{\hat{b}_i\}_{i=0,\ldots,m-1}$ and $\{g_i\}_{i=0,\ldots,n-m-1}$

$\longrightarrow$ quadratic m-parameter eigenvalue problem (MEVP):

$$\underbrace{\left(\sum_{\{\boldsymbol{\alpha}\}} \boldsymbol{M}_{\boldsymbol{\alpha}} \hat{\boldsymbol{a}}^{\boldsymbol{\alpha}} \right)}_{\hat{\dashv}} \begin{bmatrix} 1 \\ \hat{\boldsymbol{b}} \\ \boldsymbol{g} \end{bmatrix} = \boldsymbol{0},$$

Multiparameter eigenvalue problem

Example (continued)

For the system in (3) *this becomes:*

$$\underbrace{\begin{bmatrix} 1 & -1 & -1 & 0 \\ 9+\hat{a}_0 & -12 & 2\hat{a}_0 & -1 \\ -10+9\hat{a}_0 & -49 & -\hat{a}_0^2 & 2\hat{a}_0 \\ -10\hat{a}_0 & -78 & 0 & -\hat{a}_0^2 \end{bmatrix}}_{\mathcal{M}(\hat{a}_0)} \begin{bmatrix} 1 \\ \hat{b}_0 \\ g_1 \\ g_0 \end{bmatrix} = \mathbf{0},$$

which can be written as:

$$\left(\underbrace{\begin{bmatrix} 1 & -1 & -1 & 0 \\ 9 & -12 & 0 & -1 \\ -10 & -49 & 0 & 0 \\ 0 & -78 & 0 & 0 \end{bmatrix}}_{\boldsymbol{M}_0} + \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 9 & 0 & 0 & 2 \\ -10 & 0 & 0 & 0 \end{bmatrix}}_{\boldsymbol{M}_1} \hat{a}_0 + \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}}_{\boldsymbol{M}_2} \hat{a}_0^2 \right) \begin{bmatrix} 1 \\ \hat{b}_0 \\ g_1 \\ g_0 \end{bmatrix} = \mathbf{0}.$$

Multiparameter eigenvalue problem

Proposition

Given the above-mentioned square system of m+n equations $f_0 = \cdots = f_{n+m-1} = 0$ and the variety $\mathcal{V}_{\mathbb{R}}$, consider the corresponding quadratic m-parameter eigenvalue problem as defined in (3). Denote the set of real-valued eigentuples of this MEVP by $\mathcal{V}_{\mathbb{R}}(\hat{a})$. Then, $\mathcal{V}_{\mathbb{R}}(\hat{a})$ is equal to the projection of $\mathcal{V}_{\mathbb{R}}$ onto $\hat{\boldsymbol{a}}$.

- Computing the set of stationary points $\mathcal{V}'_{\mathbb{R}}$ is a (multiparameter) eigenvalue problem.

- Specialized MEVP solvers (e.g., Plestenjak, 2023; Vermeersch and De Moor, 2022)

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Numerical example II

- Consider the discrete-time system described in De Moor et al., 1993, and take $m = 2$:

$$H(z) = \frac{0.0448z^5 + 0.2368z^4 + 0.0013z^3 + 0.0211z^2 + 0.2250z + 0.0219}{z^6 - 1.2024z^5 + 2.3675z^4 - 2.0039z^3 + 2.2337z^2 - 1.0420z + 0.8513}$$

- Equating all coefficients of $l(z)$ to zero gives:

$$\underbrace{\left(\boldsymbol{M}_0 + \boldsymbol{M}_{10}\hat{a}_0 + \boldsymbol{M}_{01}\hat{a}_1 + \boldsymbol{M}_{20}\hat{a}_0^2 + \boldsymbol{M}_{11}\hat{a}_0\hat{a}_1 + \boldsymbol{M}_{02}\hat{a}_1^2\right)}_{\mathcal{M}(\hat{\boldsymbol{a}})} \begin{bmatrix}1 & \hat{b}_1 & \hat{b}_0 & g_3 & g_2 & g_1 & g_0\end{bmatrix}^\mathsf{T} = \boldsymbol{0},$$

where,

$$\mathcal{M}(\hat{\boldsymbol{a}}) = \left[\begin{array}{l:l:l:l:l:l:l} 0.045 & -1 & 0 & -\hat{a}_0 & 0 & 0 & 0 \\ 0.045\hat{a}_1+0.24 & 1.20 & -1 & -2\hat{a}_0\hat{a}_1 & -\hat{a}_0^2 & 0 & 0 \\ 0.045\hat{a}_0+0.24\hat{a}_1+0.001 & -2.37 & 1.20 & -\hat{a}_1^2-2\hat{a}_0 & -2\hat{a}_0\hat{a}_1 & -\hat{a}_0^2 & 0 \\ 0.24\hat{a}_0+0.001\hat{a}_1+0.02 & 2.01 & -2.37 & -2\hat{a}_1 & -\hat{a}_1^2-2\hat{a}_0 & -2\hat{a}_0\hat{a}_1 & -\hat{a}_0^2 \\ 0.001\hat{a}_0+0.02\hat{a}_1+0.22 & -2.23 & 2.01 & -1 & -2\hat{a}_1 & -\hat{a}_1^2-2\hat{a}_0 & -2\hat{a}_0\hat{a}_1 \\ 0.02\hat{a}_0+0.22\hat{a}_1+0.02 & 1.04 & -2.23 & 0 & -1 & -2\hat{a}_1 & -\hat{a}_1^2-2\hat{a}_0 \\ 0.22\hat{a}_0+0.02\hat{a}_1 & -0.85 & 1.04 & 0 & 0 & -1 & -2\hat{a}_1 \\ 0.02\hat{a}_0 & 0 & -0.85 & 0 & 0 & 0 & -1 \end{array}\right].$$

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Numerical example II

- The block Macaulay matrix (De Cock and De Moor, 2021; Vermeersch and De Moor, 2022) of degree 10 suffices to retrieve the 49 eigentuples, 11 of which real-valued.
- The variety $\mathcal{V}_{\mathbb{R}'}$ contains 5 stationary points:

$J^\dagger$	$\hat{a}_1$	$\hat{a}_0$	$\hat{b}_1$	$\hat{b}_0$
0.868	−0.293	0.941	0.139	0.266
1.076	0.505	0.930	−0.254	−0.120
1.124	0.267	0.820	−0.294	0.167
1.174	−1.423	0.969	0.069	0.028
1.254	−0.992	0.534	0.132	0.086

†the discrete-time equivalent of (4).

Questions?

An internal report describing these results is accessible via the STADIUS PubEngine.

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