Exploiting Shift-Invariant Subspaces

Poster for the EOS SeLMA closing event

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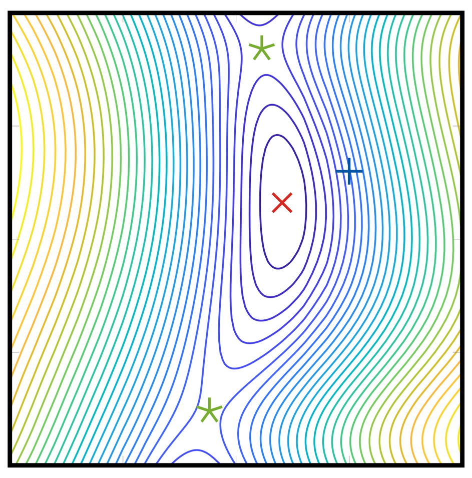
A **motivating example** from system identification [2]:

$$\min \|\boldsymbol{e}\|_2^2 \quad \text{s. t. } y_{k+1} + \alpha y_k = e_{k+1} + \gamma e_k$$
$$\Downarrow$$
$$\left(\boldsymbol{A}_{00} + \boldsymbol{A}_{10}\alpha + \boldsymbol{A}_{01}\gamma + \boldsymbol{A}_{02}\gamma^2\right)\boldsymbol{z} = \boldsymbol{0}$$

The ARMA model identification problem is a multiparameter eigenvalue problem.

Contour plot of $\|\boldsymbol{e}\|_2^2$



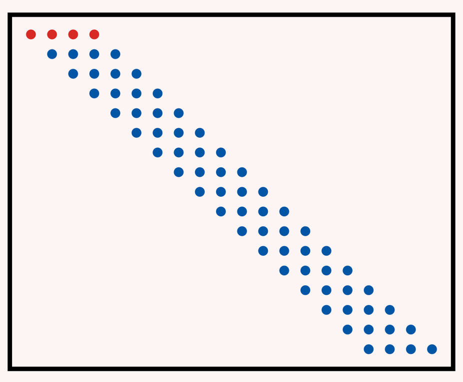
Many system identification (like the ARMA model identification problem in [2]) lead to a univariate polynomial, system of multivariate polynomial equations, polynomial eigenvalue problem (PEP), or multiparameter eigenvalue problem (MEP). These system identifcation problems generate structured matrices with a shift-invariant null space.

Problem statement

Can we exploit a shift-invariant null space to solve the system identification problems that generate them?

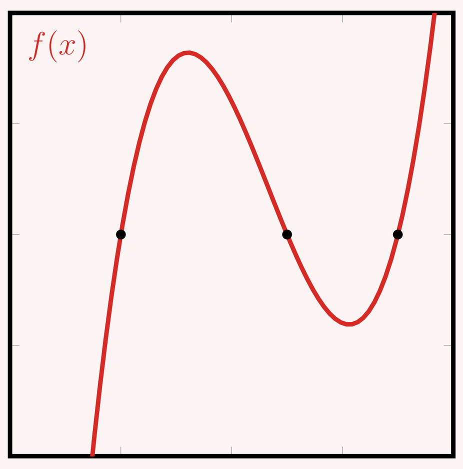
Case I: scalar single-shift-invariance

univariate polynomial: $f(x) = 6 + (-5)x + (-2)x^2 + 1x^3 = 0$



Toeplitz matrix

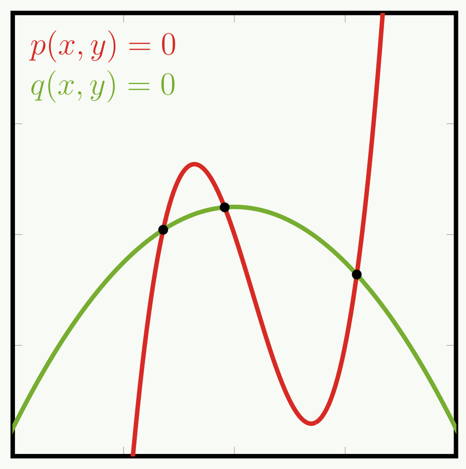
$$\begin{bmatrix} 1 \\ x \\ x^2 \\ x^3 \\ x^4 \\ x^5 \\ x^6 \\ \vdots \end{bmatrix} = \boldsymbol{0} \longleftarrow$$

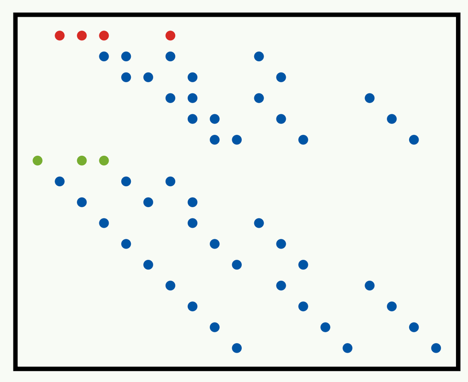


$$\begin{bmatrix} 1 \\ x \\ x^2 \\ x^3 \\ x^4 \\ x^5 \end{bmatrix} \xrightarrow{x} \begin{bmatrix} x \\ x^2 \\ x^3 \\ x^4 \\ x^5 \\ x^6 \end{bmatrix}$$

Case II: scalar multi-shift-invariance

multivariate polynomials: $\begin{cases} p(x,y) = 3x + 1y + 1x^2 + (-1)x^3 = 0 \\ q(x,y) = 2 + (-4)y + (-1)x^2 = 0 \end{cases}$





Macaulay matrix

$$\begin{bmatrix} 1 \\ x \\ y \\ x^2 \\ xy \\ y^2 \\ x^3 \\ \vdots \end{bmatrix} = \boldsymbol{0}$$

$$\begin{bmatrix} 1 \\ x \\ y \\ x^2 \\ xy \\ y^2 \end{bmatrix} \xrightarrow{x} \begin{bmatrix} x \\ x^2 \\ xy \\ x^3 \\ x^2y \\ xy^2 \end{bmatrix}$$

Exploiting a shift-invariant null space to solve four different types of problems

Every solution generates one column in the basis matrix of the null space. The shift-invariant structure allows us to set-up a standard eigenvalue problem that yields the solutions [1, 2, 3]:

for **one solution**/column:

$$\begin{bmatrix} z \\ \lambda z \\ \mu z \\ \lambda^2 z \\ \lambda\mu z \\ \mu^2 z \end{bmatrix} \xrightarrow{\lambda} \begin{bmatrix} \lambda z \\ \lambda^2 z \\ \lambda\mu z \\ \lambda^3 z \\ \lambda^2\mu z \\ \lambda\mu^2 z \end{bmatrix}$$

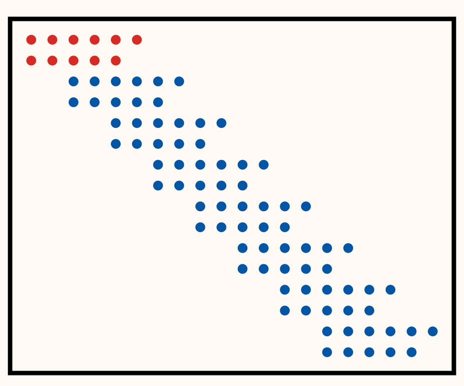
$$S_1\, v|_{(j)}\, \lambda = S_\lambda\, v|_{(j)}$$

for **all solutions**/columns:

$$S_1 V D_\lambda = S_\lambda V$$

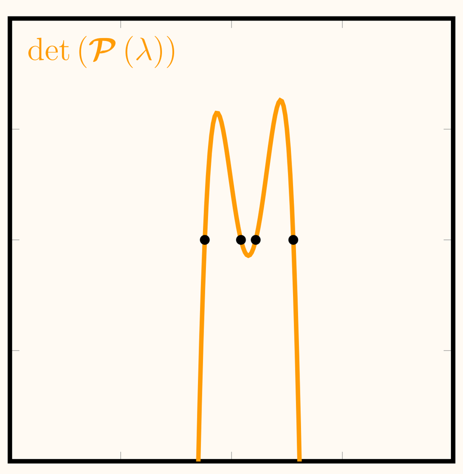
- The matrix V is not known in advance (contains solutions), so we work with a numerical basis matrix Z (with $V = ZT$):
  $$(S_1 Z)\, T D_\lambda = (S_\lambda Z)\, T$$
- We can use any shift polynomial g and the matrix D_g contains evaluations of g at its diagonal
- Multiple solutions pose no problem
- We can deflate solutions at infinity with a column compression

$$\begin{bmatrix} \boldsymbol{z} \\ \lambda\boldsymbol{z} \\ \lambda^2\boldsymbol{z} \\ \lambda^3\boldsymbol{z} \\ \lambda^4\boldsymbol{z} \\ \lambda^5\boldsymbol{z} \end{bmatrix} \xrightarrow{\lambda} \begin{bmatrix} \lambda\boldsymbol{z} \\ \lambda^2\boldsymbol{z} \\ \lambda^3\boldsymbol{z} \\ \lambda^4\boldsymbol{z} \\ \lambda^5\boldsymbol{z} \\ \lambda^6\boldsymbol{z} \end{bmatrix}$$



block Toeplitz matrix

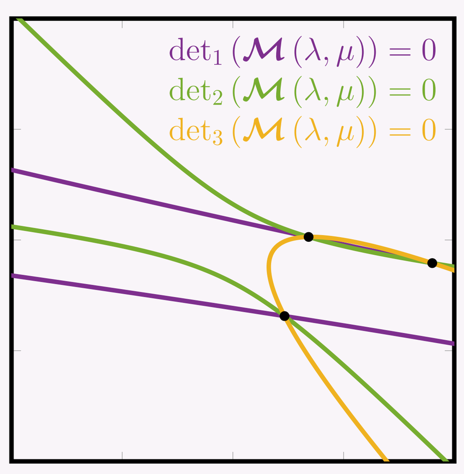
$$\begin{bmatrix} \boldsymbol{z} \\ \lambda\boldsymbol{z} \\ \lambda^2\boldsymbol{z} \\ \lambda^3\boldsymbol{z} \\ \lambda^4\boldsymbol{z} \\ \lambda^5\boldsymbol{z} \\ \lambda^6\boldsymbol{z} \\ \vdots \end{bmatrix} = \boldsymbol{0} \longleftarrow$$

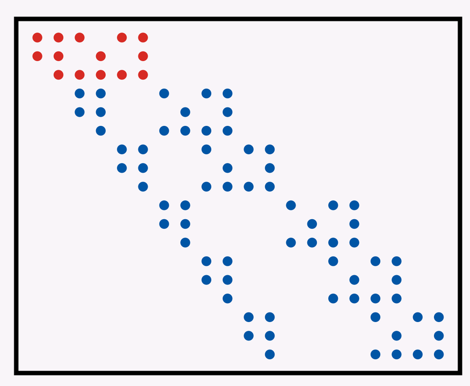


PEP: $\boldsymbol{\mathcal{P}}(\lambda)\,\boldsymbol{z} = \left(\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} -1 & 3 \\ 3 & 1 \end{bmatrix}\lambda + \begin{bmatrix} 1 & -5 \\ -5 & 0 \end{bmatrix}\lambda^2\right)\boldsymbol{z} = \boldsymbol{0}$

Case III: block single-shift-invariance

$$\begin{bmatrix} \boldsymbol{z} \\ \lambda\boldsymbol{z} \\ \mu\boldsymbol{z} \\ \lambda^2\boldsymbol{z} \\ \lambda\mu\boldsymbol{z} \\ \mu^2\boldsymbol{z} \end{bmatrix} \xrightarrow{\lambda} \begin{bmatrix} \lambda\boldsymbol{z} \\ \lambda^2\boldsymbol{z} \\ \lambda\mu\boldsymbol{z} \\ \lambda^3\boldsymbol{z} \\ \lambda^2\mu\boldsymbol{z} \\ \lambda\mu^2\boldsymbol{z} \end{bmatrix}$$





block Macaulay matrix

$$\begin{bmatrix} \boldsymbol{z} \\ \lambda\boldsymbol{z} \\ \mu\boldsymbol{z} \\ \lambda^2\boldsymbol{z} \\ \lambda\mu\boldsymbol{z} \\ \mu^2\boldsymbol{z} \\ \lambda^3\boldsymbol{z} \\ \vdots \end{bmatrix} = \boldsymbol{0}$$

MEP: $\boldsymbol{\mathcal{M}}(\lambda,\mu)\,\boldsymbol{z} = \left(\begin{bmatrix} 2 & 6 \\ 4 & 5 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}\lambda + \begin{bmatrix} 4 & 2 \\ 0 & 8 \\ 1 & 1 \end{bmatrix}\mu\right)\boldsymbol{z} = \boldsymbol{0}$

Case IV: block multi-shift-invariance

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[1] K. De Cock and B. De Moor. Multiparameter eigenvalue problems and shift-invariance. *IFAC-PapersOnLine*, 54(9):159–165, 2021. Part of special issue: 24th International Symposium on Mathematical Theory of Networks and Systems (MTNS).

[2] C. Vermeersch and B. De Moor. Globally optimal least-squares ARMA model identification is an eigenvalue problem. *IEEE Control Systems Letters*, 3(4):1062–1067, 2019.

[3] C. Vermeersch and B. De Moor. Two complementary block Macaulay matrix algorithms to solve multiparameter eigenvalue problems. Technical report, KU Leuven, Leuven, Belgium, 2022.