Positive Polynomial Constraints for POD-based Predictive Controllers

Oscar Mauricio Agudelo, Michel Baes,

Bart De Moor, Moritz Diehl
Polynomial of degree 20

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Theorem 1 (Hilbert) True only in three cases:
- univariate polynomials
- polynomials of degree 2
- polynomial of degree 4 with 2 variables
For univariate polynomials, this is a semidefinite problem

Theorem 2 (Shor, 1983) A univariate polynomial of degree $2d$ is nonnegative iff $\exists Q \in S^{d+1}_+$ such that
\[ p(t) = \pi(t)^T Q \pi(t), \]
with $\pi(t) := (1, t, \ldots, t^d)^T$

Proof: $\Rightarrow$: Suppose $p(t) \geq 0 \ \forall t \in \mathbb{R}$. By Hilbert:
\[ p(t) = \sum_{i=1}^{N} (a_i^T \pi(t))^2 = \sum_{i=1}^{N} \pi(t)^T a_i a_i^T \pi(t) = \pi(t)^T Q \pi(t), \]
with $Q := \sum_{i=1}^{N} a_i a_i^T$ positive semidefinite.

$\Leftarrow$: Suppose that $p(t) = \pi(t)^T Q \pi(t)$. As $Q$ is positive semidefinite $Q = \sum_{i=1}^{d+1} a_i a_i^T$, and
\[ p(t) = \sum_{i=1}^{d+1} (\pi(T a_i))^2 \]
Indeed, you have linear and semidefinite constraints

\[
\begin{align*}
\min_p & \langle c, p \rangle \\
\text{s.t.} & Ap = b \\
& p^T \pi(t) \geq 0 \text{ for all } t \in \mathcal{R}
\end{align*}
\]

\[
\begin{align*}
\min_{p,Q} & \langle c, p \rangle \\
\text{s.t.} & Ap = b \\
& q_{00} = p_0 \\
& q_{01} + q_{10} = p_1 \\
& q_{02} + q_{11} + q_{20} = p_2 \\
& \ldots \\
& q_{dd} = p_{2d} \\
& Q \in S^{d+1}_+
\end{align*}
\]

The dependence on \( d \) of the practical complexity is in \( O(d^3 \ln(d)) \).
Polynomials positive on an interval

Observation: $p(t) \geq 0$ for $t \in \mathbb{R}_+$ iff $p(t^2) \geq 0$ for $t \in \mathbb{R}$. 
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Generalization: Suppose $q$ is a rational function with image $S$. Then $p(t) \geq 0$ for $t \in S$ iff $p(q(t)) \geq 0$ for all $t \in \mathbb{R}$.

Application: $p(t) \geq 0$ for all $t \in [a, b]$ iff

$$(1 + z^2)^d p \left( a + \frac{(b - a)z^2}{1 + z^2} \right) \geq 0 \quad \text{for all } z \in \mathbb{R}.$$
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A real-life application:
Temperature constraints in a reactor
Modeling a tubular reactor

\[ \frac{\partial C}{\partial t} = -v \frac{\partial C}{\partial z} - k_0 C \exp\left(-\frac{E}{RT}\right) \]

\[ \frac{\partial T}{\partial t} = -v \frac{\partial T}{\partial z} - G_r C \exp\left(-\frac{E}{RT}\right) + H_r (T_w - T') \]

Control variables: \( T_{w1}, T_{w2}, T_{w3} \)
Constraints: \( T_w \in [T_{w\min}, T_{w\max}], T \leq T_{\max} \)
Modeling a tubular reactor

\[
\begin{align*}
\frac{\partial C}{\partial t} &= -v \frac{\partial C}{\partial z} - k_0 C \exp\left(-\frac{E}{RT}\right) \\
\frac{\partial T}{\partial t} &= -v \frac{\partial T}{\partial z} - G_T C \exp\left(-\frac{E}{RT}\right) + H_T (T_w - T)
\end{align*}
\]

Control variables: \( T_{w1}, T_{w2}, T_{w3} \)
Constraints: \( T_w \in [T_{w\text{min}}, T_{w\text{max}}], \quad T \leq T_{\text{max}} \)

After linearization around steady-state solution and space discretization, the model becomes:

\[
\dot{x}(t) = Ax(t) + Bu(t), \quad \text{with} \quad x(t) = (C_1, \ldots, C_N, T_1, \ldots, T_N)
\]

\[
u_{\text{min}} \leq u \leq u_{\text{max}}, \quad T_i \leq T_{\text{max},i}
\]
A huge least-square problem

\[ \dot{x}(t) = Ax(t) + Bu(t) \]

with \( x = (C_1, \ldots, T_1, \ldots) \),

\( u_{\text{min}} \leq u \leq u_{\text{max}}, \)

\( T_i \leq T_{\text{max},i} \)

**Objective:** given an initial state \( x(0) \), find \( u \) that stabilizes the reactor, i.e. that drives the state \( x \) to \( x_{\text{ref}} \):

\[
\min \int ||x(t) - x_{\text{ref}}(t)||^2 dt + \sum_k ||u(t_{k+1}) - u(t_k)||^2
\]
POD reduction doesn’t help with the number of constraints

\[ \dot{x}(t) = Ax(t) + Bu(t), \quad \text{with} \quad x(t) = (C_1, \ldots, C_N, T_1, \ldots, T_N). \]

**Reduce the number \(2N\) of states by POD:**

Given an (experimental) *snapshot matrix*

\[ S = [s(0), s(dt), \ldots, s(Mdt)] = U\Sigma V, \]

use only the first few vectors of \(U = [\phi_1, \ldots, \phi_N]:\)

\[
\begin{align*}
x(t) &= \sum_{j=1}^{2N} a_j(t) \phi_j \\ &\approx \sum_{j=1}^{n} a_j(t) \phi_j = \Phi a(t).
\end{align*}
\]
POD reduction doesn’t help with the number of constraints

\[ \dot{x}(t) = Ax(t) + Bu(t), \quad \text{with } x(t) = (C_1, \ldots, C_N, T_1, \ldots, T_N). \]

**Reduce the number \( 2N \) of states by POD:**

Given an (experimental) **snapshot matrix**

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\[ x(t) = \sum_{j=1}^{2N} a_j(t) \phi_j \approx \sum_{j=1}^{n} a_j(t) \phi_j = \Phi a(t). \]

You still have \( N \) temperature constraints:

\[ T_i(t) = (\Phi_L a(t))_i = \sum_{j=1}^{n} (a_j(t) \phi_{L,j})_i \leq T_{\text{max},i} \quad \forall i \]
But these constraints "look" polynomial

You have a large number of temperature constraints

\[ T_i(t) = (\Phi_L a(t))_i = \sum_{j=1}^{n} (a_j(t)\phi_{L,j})_i \leq T_{\text{max},i} \forall i \]

Another idea: Interpolate \( \Phi_{L,j} \) by a polynomial \( P_j(z) \), and \( T_{\text{max}} \) by a polynomial \( P_{\text{max}}(z) \)

\[ \sum_{j=1}^{n} a_j(t)P_j(z) \leq P_{\text{max}}(z) \quad \text{for} \ z \in [0, 1] \]

⇒ Reduction of number of constraints
⇒ Constraint satisfied for all \( z \)'s
How accurate is it?
### How efficient is it?

<table>
<thead>
<tr>
<th>Constraints</th>
<th>Many Linear</th>
<th>Polynomial</th>
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</thead>
<tbody>
<tr>
<td>Lin. Equ.</td>
<td>-</td>
<td>2065</td>
</tr>
<tr>
<td>Lin. Inequ.</td>
<td>24061</td>
<td>61</td>
</tr>
<tr>
<td>SOC</td>
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<td>2</td>
</tr>
<tr>
<td>LMI</td>
<td>-</td>
<td>80 times $S_+^{13}$</td>
</tr>
<tr>
<td>Memory</td>
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<td>0.67 MB</td>
</tr>
<tr>
<td>Time</td>
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<td>5.77s</td>
</tr>
</tbody>
</table>

**Note:** both problems are solved using Sedumi