1. Introduction

2. Distributed MPC descriptions
   - Game theory
   - Suboptimal MPC

3. Distributed control example

4. Coupled constraints
   - Constraint augmentation
   - Constraint manager

5. Conclusions and future work
Electrical power distribution

[Map of the United States showing electrical power distribution systems and interconnections.]
Chemical plant integration

Material flow

Energy flow
MPC at the large scale

Decentralized Control

- Traditional approach
  - Wealth of literature from the early 1970’s on improved decentralized control
  - Well-known that poor performance may result if the interconnections are not negligible

MPC at the large scale

Decentralized Control

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  - Well-known that poor performance may result if the interconnections are not negligible


Centralized Control

- Steady increase in available computational power has provided the opportunity for centralized control
- Many practitioners view centralized control of large, networked systems as impractical and unrealistic
Nomenclature: consider two interacting subsystems

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<tr>
<th>Objective functions</th>
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### Nomenclature: consider two interacting subsystems

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and  

| Objective Functions | $V(u_1, u_2) = w_1 V_1(u_1, u_2) + w_2 V_2(u_1, u_2)$ |

| Decentralized Control | $\min_{u_1 \in \Omega_1} \tilde{V}_1(u_1)$, $\min_{u_2 \in \Omega_2} \tilde{V}_2(u_2)$ |

| Noncooperative Control | $\min_{u_1 \in \Omega_1} V_1(u_1, u_2)$, $\min_{u_2 \in \Omega_2} V_2(u_1, u_2)$ |

(Nash equilibrium)
Nomenclature: consider two interacting subsystems

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<td>Cooperative Control</td>
<td>$\min_{u_1 \in \Omega_1} V(u_1, u_2) \min_{u_2 \in \Omega_2} V(u_1, u_2)$</td>
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Nomenclature: consider two interacting subsystems

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Noninteracting systems

\[ V_2(u) \]

\[ V_1(u) \]

\( b \)

\( a \)

\( n, d, p \)
Weakly interacting systems
Moderately interacting systems

\[ V_1(u) \]

\[ V_2(u) \]

Point labels: b, p, a, d, n

Axes: u_1, u_2

Stewart and Rawlings
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Strongly interacting (conflicting) systems
Strongly interacting (conflicting) systems

\[ V_2(u) \]

\[ V_1(u) \]
Geometry of cooperative vs. noncooperative MPC

\[ \begin{align*}
V_1(u) & \geq 4 \\
V_2(u) & \\
\end{align*} \]
Geometry of cooperative vs. noncooperative MPC

Stewart and Rawlings

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Cooperative Model Predictive Control

\[ V(u_1, u_2) \]

\[ u_1 \]

\[ u_2 \]

MPC 1  

MPC 2
Cooperative Model Predictive Control

\[ V(u_1, u_2) \]

\[ u_1, u_2 \]

MPC 1

\[ u^0 \]

MPC 2

\[ u^0 \]
Cooperative Model Predictive Control

\[ V(u_1, u_2) \]

\[ u_1 \]

\[ u_2 \]

\[ u_0 \]

\[ u_1^* \]

\[ u_2^* \]

\[ u_0^* \]

MPC 1

\[ u_1^* \]

MPC 2

\[ u_2^* \]
Cooperative Model Predictive Control

\[ V(u_1, u_2) \]

\[ u_1 \]

\[ u_2 \]

\[ u^*_1 \]

\[ u^*_2 \]

\[ u^0 \]

\[ u_1 \]

\[ u_2 \]

MPC 1

MPC 2

Stewart and Rawlings

Cooperative MPC with Coupled Constraints
Cooperative Model Predictive Control

\[ V(u_1, u_2) \]

\[ u_0 \]

MPC 1

\[ u^1 \]

MPC 2

\[ u^1 \]
Cooperative Model Predictive Control

\[ V(u_1, u_2) \]

\[ u_0, u_1, u_2 \]

MPC 1

\[ u^1 \]

MPC 2

\[ u^1 \]
Cooperative Model Predictive Control

\[ V(u_1, u_2) \]

MPC 1
\[ u_1^* \]

MPC 2
\[ u_2^* \]
Cooperative Model Predictive Control

$V(u_1, u_2)$

$u_0$

$u_1$

$u_2$

MPC 1

MPC 2

$u^*_1$

$u^*_2$
Cooperative Model Predictive Control

\[ V(u_1, u_2) \]

\[
\begin{align*}
V(u_1, u_2) &= \text{Objective function} \\
\text{MPC 1} &\quad u^2 \\
\text{MPC 2} &\quad u^2
\end{align*}
\]
Cooperative Model Predictive Control

\[ V(u_1, u_2) \]

MPC 1  
\[ u^2 \]

MPC 2  
\[ u^2 \]

Stewart and Rawlings
Properties established by suboptimal MPC theory

- **Stability**: Given a feasible initial condition, adding the stability constraint
  \[ \|u_i\| \leq d_i \|x_i(0)\| \]
  gives nominal closed-loop stability for any number of information exchanges

\[1\] Venkat et al. [2006a] and Venkat et al. [2006b]
Properties established by suboptimal MPC theory\textsuperscript{1}

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\textsuperscript{1}Venkat et al. [2006a] and Venkat et al. [2006b]
Cooperative control as suboptimal MPC

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- **Stability:** Given a feasible initial condition, adding the stability constraint
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- **Cost decrease:** Plant-wide objective is decreased at each iterate

- **Convergence:** Cooperative MPC produces centralized control performance at the limit of infinite iterates

\textsuperscript{1}Venkat et al. [2006a] and Venkat et al. [2006b]
Two reactors with separation and recycle

\[ \begin{align*}
D, x_{Ad}, x_{Bd} & \quad \text{MPC}_3 \\
F_r, x_{Ar}, x_{Br} & \quad A \rightarrow B \\
F_r, x_{Ar}, x_{Br} & \quad B \rightarrow C \\
F_{m}, x_{Am}, x_{Bm} & \quad A \rightarrow B \\
F_{m}, x_{Am}, x_{Bm} & \quad B \rightarrow C \\
F_{purge} & \\
H_b & \quad \text{MPC}_3
\end{align*} \]
Two reactors with separation and recycle

\begin{align*}
H_m &\quad 0 \quad 5 \quad 10 \quad 15 \quad 20 \quad 25 \quad 30 \quad 35 \quad 40 \\
H_b &\quad 0 \quad 5 \quad 10 \quad 15 \quad 20 \quad 25 \quad 30 \quad 35 \quad 40 \\
F_1 &\quad 0 \quad 5 \quad 10 \quad 15 \quad 20 \quad 25 \quad 30 \quad 35 \quad 40 \\
D &\quad 0 \quad 5 \quad 10 \quad 15 \quad 20 \quad 25 \quad 30 \quad 35 \quad 40 
\end{align*}

Stewart and Rawlings

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Two reactors with separation and recycle

Performance comparison

<table>
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<tr>
<th></th>
<th>Cost ($\times 10^{-2}$)</th>
<th>Performance loss</th>
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<tbody>
<tr>
<td>Centralized MPC</td>
<td>1.75</td>
<td>-</td>
</tr>
<tr>
<td>Decentralized MPC</td>
<td>$\infty$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>Noncooperative MPC</td>
<td>$\infty$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>Cooperative MPC (1 iterate)</td>
<td>2.2</td>
<td>25.7%</td>
</tr>
<tr>
<td>Cooperative MPC (10 iterates)</td>
<td>1.84</td>
<td>5%</td>
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</table>
Coupled constraints

- Feed distribution between MPC controllers

\[ F_0 + F_1 \leq F_T \]
Geometry of coupled constraints

Feasible region cannot be separated into Cartesian product of subspaces

$$\mathcal{F} = (\Omega_1 \times \cdots \times \Omega_M) \cap \Delta$$
Coupled constraints give suboptimal points of attraction

\[ \Phi(u_1, u_2) \]

Relaxation of coupled constraints guarantees feasibility only at convergence

\[ u^0 \]

---

\(^2\text{Cheng et al. [2007]}\)
Geometry of coupled constraints

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\[ \Phi(u_1, u_2) \]

\[ u_1 \]

\[ u_2 \]

\[ u^* \]

\[ u^0 \]

\[ \Phi(u_1, u_2) \]

---

\(^2\) Cheng et al. [2007]
Coupled constraints give suboptimal points of attraction

Relaxation of coupled constraints guarantees feasibility only at convergence\textsuperscript{2}

\textsuperscript{2}Cheng et al. [2007]
Coupled constraints

- Resource sharing between subsystems usually couples only a small subset of inputs.
- These coupled inputs can be included in every subsystem as a decision variable.

\[ \tilde{u}_i = \begin{bmatrix} u_{U,i} \\ u_C \end{bmatrix} \]

- Augmenting each subsystem’s decision variables with coupled inputs gives decoupled constraints for each subsystem.

\[
\begin{bmatrix}
D_1 & D_2 & \cdots & D_C
\end{bmatrix}
\begin{bmatrix}
u_{U,1} \\
u_{U,2} \\
\vdots \\
u_C
\end{bmatrix} \leq \begin{bmatrix} d_1 \\
d_2 \\
\vdots \\
d_C
\end{bmatrix} \rightarrow
\begin{bmatrix} D_i & D_C \end{bmatrix}
\begin{bmatrix} u_{U,i} \\ u_C \end{bmatrix} \leq \begin{bmatrix} d_i \\ d_C \end{bmatrix}
\]
Two reactors with separation and recycle

\[ \Delta H_r \]

\[ \Delta H_m \]

\[ \Delta F_0 \]

\[ \Delta F_1 \]
Two reactors with separation and recycle

When using cooperative control alone, the inputs $F_0$ and $F_1$ get stuck at a suboptimal point, leading to steady-state offset

Performance comparison

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<td>30.7%</td>
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<tr>
<td>with augmented inputs</td>
<td>15.402</td>
<td>0.071%</td>
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Coupled constraints

- Decision variable augmentation is reasonable only when constraint coupling is sparse.
- Some applications have nonsparse coupling, such as highly integrated plants with state constraints.
- In these cases, the optimization in each subsystem approaches the centralized optimization.
- Instead, we can optimize the constraint slack available to each subsystem in an auxiliary optimization.
We want to solve the optimization problem

\[
\min_u \ V(u) \\
\text{s.t.} \quad Du \leq d
\]

Instead we can solve the bilevel optimization

\[
\min_{\hat{d}} V(u(\hat{d})) \\
\text{s.t.} \quad \sum \hat{d}_i = d \\
u(\hat{d}) = \arg\{\min_u V(u) \text{ s.t. } D_i u_i \leq \hat{d}_i \quad \forall i\}
\]

The optimal solution to this optimization \( \hat{d}^* \) gives the optimal constraint partitioning so that

\[
D_i u_i^* \leq \hat{d}_i^* \quad \forall i
\]
Geometry of constraint manager

Constraint manager optimality lemma:
An optimal inner box $\hat{\Omega}^* \subseteq \Upsilon$ exists such that the cooperative MPC iterates with constraint $u \in \hat{\Omega}^*$ converge to $u^*$.
Geometry of constraint manager

Constraint manager optimality lemma:
An optimal inner box $\hat{\Omega}^* \subseteq \Upsilon$ exists such that the cooperative MPC iterates with constraint $u \in \hat{\Omega}^*$ converge to $u^*$

Feasible constraint partitioning

Constraint manager chooses sequence of inner box constraints $\{\hat{\Omega}^k\}$ forming point of attraction sequence $\{u(\hat{\Omega}^k)\}$ so that $u(\hat{\Omega}^k) \rightarrow u^*$
The constraint manager is a Stackelberg game (bilevel optimization)
- Difficult to find solutions
- Checking local optimality conditions is NP hard

Because both upper and lower optimizations have the same objective function, we perform a trick

1. Solve lower problem as suboptimal centralized MPC, giving candidate solution $\tilde{u}$
2. Choose corresponding constraints $\hat{\Omega}(\tilde{u})$ that has maximum feasible volume
Use the constraint manager iterate $\tilde{u}$ as the initial condition for cooperative MPC.

The cooperative MPC can do only better therefore $V(\tilde{u}^k)$ is a control Lyapunov function.
Two reactors with separation and recycle

\[ \Delta H_r \]

\[ \Delta H_m \]

\[ \Delta F_0 \]

\[ \Delta F_1 \]

Stewart and Rawlings

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Two reactors with separation and recycle

When using cooperative control alone, the inputs $F_0$ and $F_1$ get stuck at a suboptimal point, leading to steady-state offset.

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Conclusions and future work

- Distributed MPC can be split into two types based on game theory
  - Noncooperative MPC can produce closed-loop instability for strongly interacting systems
  - Cooperative MPC gives nominal closed-loop stability for any number of iterations
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- Distributed MPC can be split into two types based on game theory
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- Subsystem inputs can be augmented with coupled inputs to achieve plantwide optimality
- Plants with dense constraint coupling can achieve optimality via auxiliary optimization (constraint manager)
- Both schemes are stabilizing
Conclusions and future work

- Distributed MPC can be split into two types based on game theory
  - **Noncooperative MPC** can produce closed-loop instability for strongly interacting systems
  - **Cooperative MPC** gives nominal closed-loop stability for any number of iterations

- Subsystem inputs can be augmented with coupled inputs to achieve plantwide optimality

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- Additional test cases should be used to evaluate each approach
Acknowledgments

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- Collaboration with and support from Aspentech, Eastman, ExxonMobil and Shell Global Solutions.


