Towards Robust Nonlinear Constrained Model Predictive Control

P. Kühl, M. Diehl, H.G. Bock, J. Schlöder,
Tom Kraus, Andreas Schäfer, Leonhard Wirsching,
A. Milewska, E. Molga,

Overview

- Introduction to NMPC
- Numerical methods for NMPC
- A medium-scale example
- The state and parameter estimation problem
- Highlight: The Tennessee-Eastman Benchmark Process
- Robust open-loop control
- An idea for robust NMPC
Introductory Example - Van-der-Vusse Reaction scheme

For example:

Dynamic Multi-Input Multi-Output (MIMO) System

(Klatt & Engell)

Introductory Example - Van-der-Vusse Reaction scheme

Steady State gain map:

Optimal steady state!
Introductory Example - Van-der-Vusse Reaction scheme

Dynamic evolution:

Typical nonlinearities to account for in processes:
- Nonlinear gains
- Sign changes
- Directional nonlinearities

Most common industrial control approach:
PI(D) - controllers

\[
\begin{align*}
u(t) &= u_0 + k_p e(t) + k_i \int_{t_0}^{t} e(\tau) d\tau + k_d \dot{e}(t) \\
e(t) &= y_r(t) - y(t)
\end{align*}
\]

will obviously NOT work here!

If only we could see problems in advance ...
Model Predictive Control

- Online solution of a (nonlinear) constrained optimal control problem
- Only a first portion of solution, $u_0$, is applied to the process
- Results in implicit closed-loop control law $u(x_0)$

![Diagram of NMPC controller](image)
A little bit of history…

Model Predictive Control – a new thing?

“One technique for obtaining a feedback controller synthesis from knowledge of open-loop controllers is to measure the current control process state and then compute very rapidly for the open-loop control function. The first portion of this function is then used during a short time interval, after which a new measurement of the function is computed for this new measurement. The procedure is then repeated.”

(Lee and Markus, 1967)

Nonlinear Model Predictive Control

Mathematical formulation of the nonlinear constrained optimal control problem:

\[
\begin{align*}
\text{minimize} & \quad \int_0^T L(x(t), u(t)) \, dt + E(x(T)) \\
\text{s.t.} & \quad x(0) = x_0, \quad \text{(fixed initial value)} \\
& \quad \dot{x}(t) - f(x(t), z(t), u(t), p) = 0, \quad t \in [0, T], \quad \text{(DAE model)} \\
& \quad g(x(t), z(t), u(t), p) \geq 0, \quad t \in [0, T], \quad \text{(path constraints)} \\
& \quad h(x(T), z(t), u(T), p) \geq 0, \quad t \in [0, T], \quad \text{(terminal constraints)} \\
& \quad r(x(T), p) \geq 0
\end{align*}
\]
Nonlinear Model Predictive Control

A paradox:  Real-time requirements vs. complex optimization problem

\[
P(t_1) \quad \Delta t \quad P(t_2)
\]
\[t_1 \quad \Delta t \quad t_2\]

Must become small!!!

Breakthroughs to meet real-time requirements

- Fast numerical optimization
- Initial Value Embedding (M. Diehl)
- Real time iteration scheme (M. Diehl)
- Immediate feedback (M. Diehl)

Nonlinear Model Predictive Control – Numerical solution

Online solution based on Direct Multiple Shooting:

- Parameterization of states and controls
- Large but structured Nonlinear Program
- Reduction and condensing techniques to reduce size of problem
- Solved by Newton-type methods
Nonlinear Model Predictive Control – Numerical solution

Different levels of reduction depending on which set of constraints is used

LEV 1
- full-space SQP:
  - cons. conditions
  - cont. conditions
  - init. conditions
  - BCP conditions

LEV 2
- PRSQP (MUSCOD-II):
  - cont. conditions
  - init. conditions
  - BCP conditions

LEV 3
- PRSQP (MSOPT):
  - BCP conditions

Nonlinear Model Predictive Control – Initial value embedding

How to initialize subsequent optimization problems?

32. iteration

classical approach
\[ x(0) = x_0 \]
Nonlinear Model Predictive Control – Initial value embedding

How to **initialize** subsequent optimization problems?

### 2. iteration

Initial value embedding
\[ x(0) = x^* \]

Initial value constraint
\[ x(0) - x_0 = 0 \]
allowed to be violated!

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Nonlinear Model Predictive Control – Initial value embedding

Initial value embedding:
- First iteration is already tangential predictor of exact solution (for exact Hessian)
- Works with active set changes

Why do nothing and wait until convergence?
Nonlinear Model Predictive Control – Real-time iterations

Real-time iteration:
• tangential prediction after each new $x_0$
• Solution accuracy iteratively improves when $x_0$ changes little
• Iterates stay close to solution manifold

Nonlinear Model Predictive Control – Immediate feedback

Immediate feedback:
• Derivatives for QP can be calculated before $x_0$ is known
• Once $x_0$ is known, first iterate is available very fast

This allows for the following two-step procedure:
1. Preparation step (“long”):
   Linearize system at current iterate, perform partial reduction and condensing
2. Feedback step (“short”):
   Once $x_0$ is known, solve condensed QP and implement control $u_0$ immediately.
   Complete SQP iteration. Go to 1.
A little bit of theory…

- Optimality?

- Stability?

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Stability of the Infinite horizon scheme:

1. **Feasibility** at one sampling instance implies feasibility at next sampling instance for the nominal case
2. **Value function** is strictly decreasing (Lyapunov stability theory)
3. All feasible points belong to the region of attraction of the considered setpoint

Stability of the finite horizon scheme:

- Additionally impose end constraints $\bar{x}(t+T_p; x(t), t, \bar{u}) = \emptyset$ or $\bar{x}(t+T_p) \in \Omega \subseteq X$
- Add penalty term (Mayer term) to cost function to approximate upper bound of infinite horizon cost functional

**Standard assumption: Perfect optimal solution immediately available!!!**
A little bit of theory…

Stability of the finite horizon scheme including optimizer dynamics:

- Hardly any results available in literature
- For real-time iteration, contraction properties of the algorithm shown! [Diehl, Findeisen, Allgöwer, Bock, Schlöder (2005)]

Introductory Example revisited

Scenario: Sudden change in outlet flow
Comparison: PI control vs. NMPC

Scenario: Sudden change in feed flow temperature
Comparison: No control vs. NMPC
A medium-scale example

Ternary distillation with thermally coupled columns

Characterized by
- DAE model
- 106 different states
- 159 algebraic states
- 3 controls

Response to a feed flow disturbance
- Full convergence PRSQP
- Real-time iteration PRSQP
- Full convergence extended PRSQP
- Real-time iteration extended PRSQP
A medium-scale example

CPU times

Full convergence:

Real time iteration:

Moving Horizon State and Parameter Estimation

So far, we have always assumed to have

• full state information
• exact initial values
• a perfect model structure
• perfectly known model parameters

We would like to use all the nice NMPC features again…

Optimization-based estimator
Moving Horizon State and Parameter Estimation

- Idea: use past $m$ measurements in a window $[t_L, t_K]$
- Compare real measurements $y_k$ at times $t_k$ with simulated ones
- Assume Gaussian noise
- Obtain maximum likelihood estimator by minimizing weighted quadratic deviation
- Add regularization at start of horizon „arrival cost“
- Formulate bounds where necessary

Obtain $x(t_L)$ and $p$ by optimization...

Use multiple shooting with constrained Gauss-Newton Method

Adapted to real-time MHE problems:
- use shift for initialization of subsequent problems
- iterate while problem is changing (real-time iteration!)
- use EKF principle for the arrival cost (equivalence for $m=1$)

Stability of numerical scheme under investigation
A full MHE-NMPC example – Tennessee Eastman Benchmark

- Large chemical process model published by industry as a "well-suited process for a wide variety of studies" [Downs and Vogel, 1993]
- Nonlinear and unstable process
- Exothermic, irreversible reactions
- Extended variant: DAE System with 170 states [Jockenhövel et al., 2003]
- only 30 measurements: temperatures, tank levels, pressures, flowrates...

Case 1: Measurement noise

- pressure drop:
- purge change:
A full MHE-NMPC example – Tennessee Eastman Benchmark

Case 2: Measurement noise, process noise, estimated parameters

EKF-based NMPC crashed!!!

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EKF-based NMPC crashed!!!

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A full MHE-NMPC example – Tennessee Eastman Benchmark

EKF-based NMPC crashed!!!

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A full MHE-NMPC example – Tennessee Eastman Benchmark

EKF-based NMPC crashed!!!
An example that calls for robustness

What if we **know** that we do **not know** model parameters perfectly?

Esterification of 2-butanol with propionic anhydrid

- Exothermic reaction
- Autocatalytic
- Semibatch-reactor
- Goal:
  - Full **convergence** of B in a given time
  - **Avoiding critical behavior** (runaways)

$$A + B \xrightarrow{K} C + D$$
Robust control - Min-max problem formulation

A number of critical uncertain parameters should be considered explicitly

Worst-case formulation:

$$\min_{u \in U} \max_{\|p - \bar{p}\|_{2,\Sigma} \leq \gamma} J(x(u, p))$$

$$\max_{\|p - \bar{p}\|_{2,\Sigma} \leq \gamma} r(x(u, p), z(u, p)) \leq 0$$

Semi-infinite nonlinear constrained dynamic optimization problem!

Robust control - Approximated min-max formulation

Idea: Solve the min-max problem at least approximately by first-order expansion of max-problem

[Körkel et al., 2004; Diehl, Bock & Kostina, 2005]

$$\min_{u \in U} \max_{\|p - \bar{p}\|_{2,\Sigma} \leq \gamma} J(x(u, \bar{p})) + \frac{\partial}{\partial p} J(x(u, \bar{p})) (p - \bar{p})$$

$$\max_{\|p - \bar{p}\|_{2,\Sigma} \leq \gamma} r(x(u, \bar{p}), z(u, \bar{p})) + \frac{\partial}{\partial p} r(x(u, \bar{p}), z(u, \bar{p})) (p - \bar{p}) \leq 0$$

$$\min_{u \in U} J(x(u, \bar{p})) + \gamma \left| \left| \frac{\partial}{\partial p} J(x(u, \bar{p})) \right|_{2,\Sigma} \right|$$

$$r_1(x(u, \bar{p}), z(u, \bar{p})) + \gamma \left| \left| \frac{\partial}{\partial p} r_1(x(u, \bar{p}), z(u, \bar{p})) \right|_{2,\Sigma} \right| \leq 0$$

Intelligent safety margins!
Robust control - Stochastic Interpretation of min-max-formulation

$$\text{prob} \left( r(\bar{p}, u) + \frac{\partial r}{\partial \bar{p}} \Delta \bar{p} \leq 0 \right) \geq \alpha$$

$$\Delta \bar{p} \sim \mathcal{N}(0, \Sigma) \Rightarrow \frac{\partial r}{\partial \bar{p}} \Delta \bar{p} \sim \mathcal{N} \left( 0, \frac{\partial r}{\partial \bar{p}} \Sigma \frac{\partial r}{\partial \bar{p}}^T \right)$$

$$r(\bar{p}, u) + \gamma \sqrt{\frac{\partial r}{\partial \bar{p}} \Sigma \frac{\partial r}{\partial \bar{p}}^T} = r(\bar{p}, u) + \gamma \left\| \frac{\partial r}{\partial \bar{p}} \right\|_{2, \Sigma}$$

Design

Interpretation

$$\left\| p - \bar{p} \right\|_{2, \Sigma^{-1}} \leq \gamma$$

An example that calls for robustness - Robust solution

Optimal dosing profile – Robust case

Safety margin

Simulation

Experiment
Towards robust NMPC – Simulations

Use robust formulation in NMPC cost function!

- Nominal NMPC is robust against catalyst uncertainty
- Robust NMPC is “cowardish”
- Real-time iteration scheme applied
- Feedback time below 2 seconds!
- Nominal NMPC is robust against catalyst uncertainty
- Robust NMPC is “cowardish”

How to reduce conservatism of open-loop formulation???

Dank U wel

Thank you for your attention