ROBUST NONLINEAR OPTIMAL CONTROL OF DYNAMIC SYSTEMS WITH AFFINE UNCERTAINTIES

Boris Houska, Moritz Diehl

Optimization in Engineering Center (OPTEC)

K. U. Leuven
Overview

- Uncertain Linear Systems
- Robust Optimal Control
- A Numerical Example
Consider linear time-varying system \( (t \in [0, T]) \)

\[
\begin{align*}
\dot{x}(t) &= A(t)x(t) + B(t)w(t) \\
y(t) &= C(t)x(t) \\
x(0) &= B_0w_0
\end{align*}
\]

with disturbance \( \omega := (w_0, w(\cdot)) \).
• Consider linear time-varying system \((t \in [0, T])\)

\[
\begin{align*}
\dot{x}(t) &= A(t)x(t) + B(t)w(t) \\
y(t) &= C(t)x(t) \\
x(0) &= B_0w_0
\end{align*}
\]

with disturbance \(\omega := (w_0, w(\cdot))\).

• Assume that uncertainty is bounded by

\[
\omega \in \mathcal{B} := \left\{ \omega \mid w_0^T w_0 + \int_0^T w(\tau)^T w(\tau) \, d\tau \leq 1 \right\}.
\]
Linear Inequality Constraints

Question:

- Are inequality constraints of the form

$$y_i(t) \leq d_i(t).$$

for all $t \in [0, T]$ and all $\omega \in \mathcal{B}$ regarded?
Linear Inequality Constraints

Question:

• Are inequality constraints of the form

\[ y_i(t) \leq d_i(t) . \]

for all \( t \in [0, T] \) and all \( \omega \in B \) regarded?

Idea:

• Write output component \( y_i \) in the form

\[
y_i(t) = \langle H^t_i | \omega \rangle := C_i(t)G(t, 0)B_0w_0 + \int_0^T C_i(t)G(t, \tau)B(\tau)w(\tau)d\tau .
\]
Question:

• Are inequality constraints of the form

\[ y_i(t) \leq d_i(t) \]

for all \( t \in [0, T] \) and all \( \omega \in B \) regarded?

Idea:

• We can compute maximum excitation as

\[
\max_{\omega \in B} y_i(t) = \max_{\omega \in B} \langle H_t^i | \omega \rangle = \sqrt{\langle H_t^i | H_t^i \rangle}
\]
Linear Inequality Constraints

Theorem:

$$\max_{\omega \in B} y_i(t) = \sqrt{\langle H_i^t \mid H_i^t \rangle} = \sqrt{C_i(t)P(t)C_i(t)^T}$$

with $P(\cdot)$ solving a Lyapunov equation ($\tau \in [0, t]$):

$$\dot{P}(\tau) = A(\tau)P(\tau) + P(\tau)A(\tau)^T + B(\tau)B(\tau)^T$$

$$P(0) = B_0B_0^T.$$
Linear Inequality Constraints

Theorem:

\[
\max_{\omega \in B} y_i(t) = \sqrt{\langle H_i^t \mid H_i^t \rangle} = \sqrt{C_i(t)P(t)C_i(t)^T}
\]

with \( P(\cdot) \) solving a Lyapunov equation (\( \tau \in [0, t] \)):

\[
\dot{P}(\tau) = A(\tau)P(\tau) + P(\tau)A(\tau)^T + B(\tau)B(\tau)^T \\
P(0) = B_0B_0^T.
\]

Proof:

\[
\langle H_i^t \mid H_i^t \rangle = C_i(t) \left[ G(t, 0)B_0B_0^T G(t, 0)^T + \int_0^t G(t, \tau)B(\tau)B(\tau)^T G(t, \tau)^T d\tau \right] C_i(t)^T.
\]
Linear Inequality Constraints

Theorem:

\[ \max_{\omega \in \mathcal{B}} y_i(t) = \sqrt{\langle H_i^t | H_i^t \rangle} = \sqrt{C_i(t) P(t) C_i(t)^T} \]

with \( P(\cdot) \) solving a Lyapunov equation \((\tau \in [0, t]):\)

\[ \dot{P}(\tau) = A(\tau) P(\tau) + P(\tau) A(\tau)^T + B(\tau) B(\tau)^T \]

\[ P(0) = B_0 B_0^T. \]

Proof:

\[ \langle H_i^t | H_i^t \rangle = C_i(t) \left[ G(t, 0) B_0 B_0^T G(t, 0)^T \right. \]

\[ + \int_0^t G(t, \tau) B(\tau) B(\tau)^T G(t, \tau)^T d\tau \] \left. \right] C_i(t)^T. \]
Overview

- Uncertain Linear Systems
- Robust Optimal Control
- A Numerical Example
Consider uncertain optimal control problem

$$\min_{x(\cdot), v(\cdot), T} \mathcal{J}[v(\cdot), T]$$

s.t.

$$\dot{x}(t) = A(v(t))x(t) + B(v(t))w(t) + r(v(t))$$

$$0 \geq C(v(t))x(t) - d(v(t))$$

$$x(0) = r_0(v(0)) + B_0w_0,$$

$$v \in \mathbb{V}$$

which is affine in $\omega \in \mathcal{B}$. 
Robust Optimal Control

• Nonlinear behavior

\[ V = \left\{ v: T \rightarrow \mathbb{R}^{n_v} \mid \forall t \in T : \begin{align*}
0 &= F(t, \dot{v}, v) \\
0 &= G(v(0), v(T)) = 0 \\
0 &\geq H(v(t))
\end{align*} \right\}. \]
Robust Optimal Control

• Nonlinear behavior

\[ V = \left\{ v : T \rightarrow \mathbb{R}^{n_v} \right\} \quad \forall t \in T : \\
\begin{align*}
0 &= F(t, \dot{v}, v) \\
0 &= G(v(0), v(T)) = 0 \\
0 &\geq H(v(t))
\end{align*} \]

Idea:

• Transfer Lyapunov Differential Equations to formulate robust counterpart problem.
\[
\begin{align*}
\min_{x_r(\cdot), P(\cdot), v(\cdot), T} & \quad J[v(\cdot), T] \\
\dot{x}_r(t) &= A(v(t)) x_r(t) + r(v(t)) \\
x_r(0) &= r_0(v(0)) \\
\dot{P}(t) &= A(v(t)) P(t) + P(t) A(v(t))^T + B(v(t)) B(v(t))^T \\
P(0) &= B_0 B_0^T \\
0 &\geq C_i(v(t)) x_r(t) - d_i(v(t)) \\
&\quad + \sqrt{C_i(v(t)) P(t) C_i(v(t))^T} \\
v &\in \mathbb{V}
\end{align*}
\]

for all \( t \in \mathbb{T}, i \in \{1, \ldots, n_C\} \).
Advantages of the Formulation:

- infinite dimensional disturbance
- infinite number of constraints
Advantages of the Formulation:

- infinite dimensional disturbance
- infinite number of constraints
- transfer: approximate nonlinear robust optimization
Advantages of the Formulation:

- infinite dimensional disturbance
- infinite number of constraints
- transfer: approximate nonlinear robust optimization
- can be used to optimize linear closed loop systems

\[
\dot{x}(t) = [A(t) + B(t)K(t)] x(t) + D(t)w(t) + r(t)
\]
Robust Optimal Control

Advantages of the Formulation:

- infinite dimensional disturbance
- infinite number of constraints
- transfer: approximate nonlinear robust optimization
- can be used to optimize linear closed loop systems

\[
\dot{x}(t) = [A(t) + B(t)K(t)] x(t) + D(t)w(t) + r(t)
\]

- can be used to optimize stability of periodic systems
Overview

• Uncertain Linear Systems
• Robust Optimal Control
• A Numerical Example
Example: Optimal Control of a Crane

Aim: Carry mass $m$ in minimum time $T$ from one to another point.

Problem: $F$ uncertain

Equations of motion:

$$
\begin{pmatrix}
\dot{\phi} \\
\ddot{\phi}
\end{pmatrix} = A \begin{pmatrix}
\phi \\
\dot{\phi}
\end{pmatrix} + BF + r
$$

$A := \begin{pmatrix}
0 & 1 \\
-\frac{g}{L} & -\left(b + 2\frac{\dot{L}}{L}\right)
\end{pmatrix}$

$B := \begin{pmatrix}
0 \\
\frac{1}{mL}
\end{pmatrix}$

$r := \begin{pmatrix}
\ddot{x} \\
-\frac{\dot{x}}{L} - \frac{\dot{L}\ddot{x}}{L^2}
\end{pmatrix}$
Example: Optimal Control of a Crane

DifferentialState phi, dphi
DifferentialState L, dL
DifferentialState x, dx
DifferentialState P0, P1, P2
Control ddx, dL
Parameter T
DifferentialEquation f(0, T)

const double m = 100.0;
const double g = 9.8;
const double b = 0.1;

f « dot(L) == dL
f « dot(x) == dx
f « ...

OCP ocp;
ocp.minimize(T);
ocp.subjectTo(f);

ocp.subjectTo(AT_START, phi == 0.0);
ocp.subjectTo(AT_START, dphi == 0.0);
ocp.subjectTo(AT_START, x == 0.0);
ocp.subjectTo(AT_START, ...);

ocp.subjectTo(AT_END, x == 40.0);
ocp.subjectTo(AT_END, phi + sqrt(P0) <= 0.04);
ocp.subjectTo(AT_END, phi - sqrt(P0) >= -0.04);

ocp.subjectTo(phi + sqrt(P0) <= 0.05);
ocp.subjectTo(phi - sqrt(P0) >= -0.05);

OptimizationAlgorithm algorithm(ocp);
algorithm.set(KKT_TOLERANCE, 1e-10);
algorithm.set(...);
algorithm.solve();
Example: Optimal Control of a Crane

- Minimum time:
  \[ T_{\text{min}} = 24.56 \text{ s}. \]
- Without robustness:
  \[ T_{\text{min}} = 23.33 \text{ s}. \]
Summary

• Lyapunov differential equations are suitable to compute worst case excitations of linear systems.

• Exact robust counterpart formulation is possible for nonlinear system which are affine in uncertainty $\omega$.

• Approach can deal with infinite dimensional disturbances and robustified state constraints.

• Use ACADO Toolkit to solve robust and nonlinear optimal control problems.
Summary

• Lyapunov differential equations are suitable to compute worst case excitations of linear systems.
• Exact robust counterpart formulation is possible for nonlinear system which are affine in uncertainty $\omega$.
• Approach can deal with infinite dimensional disturbances and robustified state constraints.
• Use ACADO Toolkit to solve robust and nonlinear optimal control problems.

Thank you for your attention!