The Online Active Set Strategy for Fast Linear MPC (qpOASES)

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Linear MPC (or QP subproblems in NMPC)

For

- linear dynamic system
- linear constraints
- quadratic cost

only quadratic program (QP) needs to be solved:

\[
\begin{align*}
\min_{u_0, \ldots, u_{N-1}} & \quad x_N^T P x_N + \sum_{i=0}^{N-1} (x_i^T Q x_i + u_i^T R u_i) \\
\text{s.t.} & \quad x_{k+1} = A x_k + B u_k, \\
& \quad (x_0 \text{ given}), \\
& \quad c \leq C x_k \leq \bar{c}, \\
& \quad d \leq D u_k \leq \bar{d}, \\
& \quad c_T \leq C_T x_N,
\end{align*}
\]
Linear MPC = parametric QP

Eliminate states via “condensing“, obtain smaller scale quadratic program (QP) in variables \( w := (u_0^T, \ldots, u_{N-1}^T)^T \)

\[
\begin{align*}
\text{QP}(x_0) : \quad & \min_w \frac{1}{2} w^T H w + w^T F^T x_0 \\
& =: g(x_0) \\
\text{s.t.} \quad & Gw \geq \bar{b} + E x_0, \\
& =: b(x_0)
\end{align*}
\]

(assumption: \( H \) positive definite)

QP depends on \( x_0 \) via affine functions \( g(x_0) \) and \( b(x_0) \)
Karush-Kuhn-Tucker (KKT) Conditions

**Theorem**
Let $\text{QP}(x_0)$ be a strictly convex and feasible quadratic program. Then there exists a unique $w^* \in \mathbb{R}^n$ and at least one working set $A$ and a vector $y^* \in \mathbb{R}^m$ which satisfy the following conditions:

\[
Hw^* - G^T_A y_A^* = -g(x_0),
\]

\[
G_A w^* = b_A(x_0),
\]

\[
y^*_I = 0, \ (I := \{1, \ldots, m\} \setminus A),
\]

\[
G_I w^* \geq b_I(x_0),
\]

\[
y_A^* \geq 0.
\]
Define set of “feasible parameters“:

\[
\mathbb{P} := \{ x_0 \in \mathbb{R}^{n_x} \mid \text{QP}(x_0) \text{ is feasible.} \}
\]

Well known:

**THEOREM:** Set \( \mathbb{P} \) is convex, and can be partitioned into polyhedral „critical regions“ each corresponding to a different working set \( \mathbb{A} \). QP solution on each region is affine in \( x_0 \).
Sketch of Proof for Polyhedral Critical Regions

Check KKT conditions for fixed working set $\mathbb{A}$:

1. $g(x_0), b(x_0)$ affine: then $w^*, y^*$ affine, because solution of linear system:

$$Hw^* - G_A^T y_A^* = -g(x_0),$$
$$G_A w^* = b_A(x_0),$$
$$y_1^* = 0,$$

2. $w^*, y^*$ affine, therefore

$$G_1 w^* \geq b_1(x_0),$$
$$y_1^* \geq 0.$$

are linear constraints on $x_0$ that define polyhedral „critical region“ in $\mathbb{P}$.
Explicit MPC: Precalculate Everything

Idea: Compute control on all critical regions in advance (Bemporad, Borrelli, Morari, 2002).

**Pro:** MPC in microsecs. possible

**Contra:** problem size limited

Example: 50 variables, lower and upper bounds: $3^{50} = 10^{23}$ possible critical regions. Prohibitive.
Online Active Set Strategy (qpOASES)

Combine Explicit and Online MPC

- compute affine solution only on current critical region
- go on straight line from old to new problem data (\( P \) convex !)
- solve each QP on path exactly (keep primal-dual feasibility)!
- need to change working set only at boundaries of critical regions
How to compute each step?

- determine change in $g(x_0)$ and $b(x_0)$, solve KKT system
  \[
  \begin{pmatrix} H & G_T^A \\ G_A & 0 \end{pmatrix} \begin{pmatrix} \Delta w^* \\ -\Delta y_A^* \end{pmatrix} = \begin{pmatrix} -\Delta g \\ \Delta b_A \end{pmatrix}
  \]

- choose steplength $\tau_{\text{max}}$ maximal such that
  \[
  G_i^T (w^* + \tau \Delta w^*) \geq b_i(x_0) + \tau \Delta b_i
  \]
  still holds for all inactive (primal) constraints, and
  \[
  y_i^* + \tau \Delta y_i \geq 0
  \]
  for all active dual variables: Set $\tau_{\text{max}} := \min \{ 1, \tau_{\text{max}}^{\text{prim}}, \tau_{\text{max}}^{\text{dual}} \}$

with $\tau_{\text{max}}^{\text{prim}} := \min_{i \in I} \frac{b_i(x_0) - G_i^T w^*}{G_i^T \Delta w^* - \Delta b_i}$ and $\tau_{\text{max}}^{\text{dual}} := \min_{i \in A, \Delta y_i < 0} \frac{-y_i^*}{\Delta y_i}$
How to change working set?

- add or remove constraints to/from working set when crossing borders of critical regions
- use null space approach, keep QR-factorization of active constraint matrix, and Cholesky factorization of projected hessian:

\[ G_A = \begin{pmatrix} 0 & T \end{pmatrix} \begin{pmatrix} Z^T \\ Y^T \end{pmatrix}, \]

\[ Z^T H Z = R^T R, \]

- each working set change costs only \( O(n^2) \) flops, exactly as one QP iteration in efficient QP solvers!
Extra difficulty: linear independence often violated

- during homotopy, often redundant constraints become active and cause degeneracy
- example of three active constraints in 2-D:
How to deal with degeneracy?

Without linear independence, KKT system becomes unsolvable! Addition of degenerate constraints must be avoided.

Remedy [Best 1996]: before adding extra row \( G_j \) to solve auxiliary system

\[
\begin{pmatrix}
H & G_j^T \\
G_A & 0
\end{pmatrix}
\begin{pmatrix}
s \\
\xi
\end{pmatrix}
= \begin{pmatrix}
G_j \\
0
\end{pmatrix}
\]

If \( s = 0 \), linear dependence is detected, and \( \xi \) helps to find a constraint from \( A \) that can be removed.
Summary of OASES Algorithm

(1) Calculate $\Delta x_0$, $\Delta g$ and $\Delta b$
(2) Calculate primal and dual step directions $\Delta w^*$ and $\Delta y^*$
(3) Determine maximum homotopy step length $\tau_{\text{max}} := \min \left\{ 1, \tau_{\text{max}}^{\text{prim}}, \tau_{\text{max}}^{\text{dual}} \right\}$
(4) Obtain optimal solution of $\text{QP}(\tilde{x}_0)$:
   (a) $\tilde{x}_0 \leftarrow x_0 + \tau_{\text{max}} \Delta x_0$,
   (b) $\tilde{w}^* \leftarrow w^* + \tau_{\text{max}} \Delta w^*$,
   (c) $\tilde{y}^* \leftarrow y^* + \tau_{\text{max}} \Delta y^*$.
(5) if $\tau_{\text{max}} = 1$:
   Optimal solution of $\text{QP}(x_0^{\text{new}})$ found.
(6) if $\tau_{\text{max}} = \tau_{\text{max}}^{\text{dual}}$:
   Remove a dual blocking constraint $j \left( \tau_{\text{max}}^{\text{dual}} = -\frac{y_j^*}{\Delta y_j} \right)$ from working set,
   elseif $\tau_{\text{max}} = \tau_{\text{max}}^{\text{prim}}$:
   Add a primal blocking constraint $j \left( \tau_{\text{max}}^{\text{prim}} = \frac{b_j(x_0) - G^T_j w^*}{G^T_j \Delta w^* - \Delta b_j} \right)$
   ensuring linear independence
(7) Set $x_0 \leftarrow \tilde{x}_0$, $w^* \leftarrow \tilde{w}^*$, $y^* \leftarrow \tilde{y}^*$ and continue with step (1).
Limit number of active set changes per sampling time:
- lag behind, if too many changes necessary
- deliver solution of some problem between old and new
- make good for lag in later problems
**Infeasibility treatment**

\( P \) convex: QP on path infeasible \( => \) new QP infeasible

Stop at last feasible QP, wait for better posed problems
P convex: QP on path infeasible <=> new QP infeasible

Stop at last feasible QP, wait for better posed problems

Fortunately:
new QP feasible <=> full path is feasible, and strategy works again
qpOASES: Open Code by Hans Joachim Ferreau

qpOASES: open source C++ code by Hans Joachim Ferreau

http://www.kuleuven.be/optec/software/qpOASES
Application to Chain of Masses

- 10 balls connected by springs, No. 1 fixed
- 3-D velocity of ball No. 10 controlled:
  \[ \ddot{x}_{N+1} = u(t) \]
- 2nd order ODE for other balls:
  \[ \ddot{x}_i = \frac{1}{m} \left( F_{i,i+1} - F_{i-1,i} \right) + g, \quad i = 1, \ldots, N \]
- Force according to Hooke’s law strongly nonlinear:
  \[ F_{i,i+1} = D \left( 1 - \frac{L}{\|x_{i+1} - x_i\|} \right) (x_{i+1} - x_i), \]
- Together: 57 nonlinear ODEs, chaotic system
After disturbance, chain crashes into wall
MPC controller shall avoid crashing into wall

- linearize system at steady state
- choose 200 ms sampling time
- predict 80 samples: \( 3 \times 80 = 240 \) degrees of freedom
- **bounds** (up/lo): \( 2 \times 240 = 480 \)
- **state constraints** that avoid hitting the wall: \( 9 \times 80 = 720 \)

Note: large QP with \(~1\) MB data
MPC respects bounds and state constraint
Compare four QP strategies

- Standard solver (QPSOL), cold start
- QPSOL, warm start
- Online Active Set Strategy (OASES), full convergence
- OASES, real-time variant with at most 10 QP iterations

Note: Explicit MPC cannot be applied due to problem size
Number of QP Iterations (Working Set Changes)
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Performance of Online Active Set Strategy

- Number of QP working set changes 3-5 times lower than for QPSOL with warm starts
- Can limit maximum number without much suboptimality

Additionally:
- QPSOL often even needs Phase1 LP Iterations
- OASES needs no new matrix factorizations
- CPU times compare even more favourably...
CPU Time Comparison: OASES Factor 10 Faster
Time Optimal MPC: a 100 Hz Application

- Quarter car: oscillating spring damper system
- MPC Aim: settle at any new setpoint in *in minimal time*
- Two level algorithm: MIQP
  - 6 online data
  - 40 variables + one integer
  - 242 constraints (in-&output)
- use qpOASES on dSPACE
- CPU time: <10 ms

*Lieboud Van den Broeck in front of quarter car experiment*
Setpoint change without control: oscillations
With LQR control: inequalities violated
With Time Optimal MPC
Time Optimal MPC: qpOASES Optimizer Contents
Time Optimal MPC: a 60 Hz Application

- Overhead crane
- MPC Aim: settle at any new setpoint in *in minimal time*
- Two level algorithm: MIQP
  - 6 online data
  - 40 variables + one integer
  - 242 constraints (in-&output)
- use qpOASES on dSPACE
- CPU time: <10 ms
Overhead crane: multiple set point changes
qpOASES: open code, but direct industrial funding

Hoerbiger: MPC of Large Bore Gas Engines for US Pipeline Compressors

IPCOS: EfficientQP for Process Control
qpOASES running on Industrial Control Hardware (20 ms)

Project manager (Dec. 2008): “...we had NO problem at all with the qpOASES code. Your Software has throughout the whole project shown reliable and robust performance.”
Conclusions

- Linear and Nonlinear MPC need reliable QP solution
- Explicit MPC prohibitive for nontrivial problem dimensions
- Online Active Set Strategy (qpOASES) is one order of magnitude faster than conventional QP with warmstarts
- Linear MPC in kHz range realizable even for larger QPs
- Time Optimal MPC interesting alternative to tracking