Outline of the Talk

- Uncertain Nonlinear Dynamic Systems
- Computation of Robust Positive Invariant Tubes
- Robust Optimization of Dynamic Systems
- Application: Robust Control of a Tubular Reactor
- Robust Optimization of Periodic Systems
- Open-Loop Stable Orbits of an Inverted Spring Pendulum
- Conclusions and Outlook
## The Inverted Spring Pendulum

![Diagram of the inverted spring pendulum](image)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L$</td>
<td>1 m</td>
</tr>
<tr>
<td>$m$</td>
<td>0.1 kg</td>
</tr>
<tr>
<td>$D$</td>
<td>$700 , \text{N/m}$</td>
</tr>
<tr>
<td>$g$</td>
<td>$9.81 , \text{m/s}^2$</td>
</tr>
<tr>
<td>$b$</td>
<td>$5 \frac{1}{s}$</td>
</tr>
<tr>
<td>$ar{w}$</td>
<td>$0.03 , \text{N/kg}$</td>
</tr>
<tr>
<td>$ar{u}$</td>
<td>$200 , \text{N/kg}$</td>
</tr>
<tr>
<td>$v_z$</td>
<td>$3.2 , \text{m/s}$</td>
</tr>
</tbody>
</table>
Model:

- Right-hand side function is given by

\[
\begin{pmatrix}
v_x \\
v_y \\
-\frac{Dx}{m} \left( 1 - \frac{L}{\sqrt{x^2 + y^2}} \right) - bv_x + w \\
-g + u - \frac{Dy}{m} \left( 1 - \frac{L}{\sqrt{x^2 + y^2}} \right) - bv_y \\
v_z \\
u
\end{pmatrix}
\]
Optimal Control Problem

Aim:
- Find an open-loop stable periodic orbit at the “inverted” position.

Objective:
- Minimize the time-average over the maximum displacement of the mass point in $x$-direction:

\[
\mathcal{L}(\tau, u(\tau), T_e, X(\tau), W(\tau)) := \max_{\xi \in X(\tau)} \frac{(e_x^T \xi)^2}{T_e}
\]

with $e_x^T := (1, 0, \ldots, 0)^T \in \mathbb{R}^6$.

Optimization Variables:
- Set valued function $X$, control input $u$, end time $T_e$. 
Optimal Control Problem

Constraints:

• Periodic propagation of the uncertainty tube

\[ X(\tau^+) = F(\tau, u(\tau), p, X(\tau), W(\tau)) \quad \text{with} \quad X(0) = X(T_e) . \]

• Here: \( W(\tau) = \{ w \in \mathbb{R} \mid w \leq w \leq \bar{w} \} \)

• Control and State Constraint Function:

\[
H(\tau, u(\tau), X(\tau)) := \begin{pmatrix}
    u(\tau) - \bar{u} \\
    -u(\tau) + \bar{u} \\
    \max_{\xi \in X(\tau)} e_{v_z}^T \xi - \bar{v}_z \\
    \min_{\xi \in X(\tau)} -e_{v_z}^T \xi + \bar{v}_z
\end{pmatrix}.
\]
Optimization Results Based on Ellipsoidal Technique:

Optimal period time: $T_e \approx 0.79\text{ms} \ (\approx 80\text{Hz})$. 