Outline of the Talk

• Uncertain Nonlinear Dynamic Systems
• Computation of Robust Positive Invariant Tubes
• Robust Optimization of Dynamic Systems
• Application: Robust Control of a Tubular Reactor
• Robust Optimization of Periodic Systems
• Open-Loop Stable Orbits of an Inverted Spring Pendulum
• Conclusions and Outlook
Problem Formulation:

- Let us start with a robust optimal control problem of the form:

\[
\min_{u(\cdot), p, T_e, X(\cdot)} M(p, T_e, X(T_e))
\]

\[
\begin{align*}
X(\tau^+) &= F(\tau, u(\tau), p, X(\tau), W(\tau)) \\
X(0) &= X_0 \\
0 &\geq H(\tau, u(\tau), p, X(\tau), W(\tau))
\end{align*}
\]

- **Aim:** Solve the above problem in a conservative approximation.
Definition of Monotonicity

Definition:

• A function $Z : \Pi(\mathbb{R}^{n_x}) \rightarrow \mathbb{R}$ is monotonically increasing, if for any sets $X, Y \subseteq \mathbb{R}^{n_x}$ with $X \subseteq Y$, we have that $Z(X) \leq Z(Y)$. 
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**Assumptions:**

1. The Mayer term \( M(p, T_e, X(T_e)) \) is mononically increasing with respect to the variable \( X(T_e) \).
2. The constraint function \( H(\tau, u(\tau), p, X(\tau), W(\tau)) \) is componentwise mononically increasing in \( X(\tau) \).
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Assumptions:

1. The Mayer term $M(p, T_e, X(T_e))$ is mononically increasing with respect to the variable $X(T_e)$.

2. The constraint function $H(\tau, u(\tau), p, X(\tau), W(\tau))$ is componentwise mononically increasing in $X(\tau)$.

Question:

• Are the above assumption reasonable?
Robust Mayer Term:

• Assume nominal Mayer term \( m : \mathbb{R}^{n_p} \times \mathbb{R}_+ \times \mathbb{R}^{n_x} \rightarrow \mathbb{R} \) is given.

• Define \( M(p, T_e, X(T_e)) := \sup_{x \in X(T_e)} m(p, T_e, x) \).

• \( M \) is monotonically increasing in \( X(T_e) \).
Robust Mayer Term:

- Assume nominal Mayer term \( m : \mathbb{R}^{np} \times \mathbb{R}_+ \times \mathbb{R}^{nx} \rightarrow \mathbb{R} \) is given.
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- \( M \) is monotonically increasing in \( X(T_e) \).

Robust Constraint Function:

- Analogous formulation of a constraint:

\[
H_i(\tau, u(\tau), p, X(\tau), W(\tau)) := \sup_{x \in X(\tau)} \sup_{w \in W(\tau)} h_i(\tau, u(\tau), p, x, w).
\]
Example: Robustness Design Criteria

Minimize Maximum Distance of Two Points in the Terminal Set:

- $M(p, T_e, X(T_e)) := \text{diag}(X(T_e)) := \sup_{x,y \in X(T_e)} \|x - y\|.$

- $M$ is monotonically increasing in $X(T_e)$. 
Example: Robustness Design Criteria

Minimize Maximum Distance of Two Points in the Terminal Set:

- \( M(p, T_e, X(T_e)) := \text{diag}(X(T_e)) := \sup_{x,y \in X(T_e)} ||x - y||. \)
- \( M \) is monotonically increasing in \( X(T_e) \).

Minimize the Inertia of the Terminal Set:

- \( M(p, T_e, X(T_e)) := \int_{X(T_e)} \left\| x - \int_{X(T_e)} x \, dx \right\|^2 \, dx. \)
- \( M \) is monotonically increasing in \( X(T_e) \).
Minimize Maximum Distance of Two Points in the Terminal Set:

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Minimize the Inertia of the Terminal Set:

- \( M(p, T_e, X(T_e)) := \int_{X(T_e)} \left\| x - \int_{X(T_e)} x \, dx \right\|^2 \, dx \).

- \( M \) is monotonically increasing in \( X(T_e) \).

Minimize the Volume of the Terminal Set:

- \( M(p, T_e, X(T_e)) := \int_{X(T_e)} 1 \, dx \).

- \( M \) is monotonically increasing in \( X(T_e) \).
Ellipsoidal Approximation Assumption

Assumption:

• We have functions \( \varphi, \Phi, q_0, \) and \( Q_0 \) such that: for any function \( \kappa : [0, T_e] \to \mathbb{R}^{m}_{++} \), and any vector \( \kappa_0 \in \mathbb{R}^{n_0}_{++} \), which admit solutions \( q : [0, T_e] \to \mathbb{R}^{n_x} \) and \( Q : [0, T_e] \to \mathbb{S}^{n_x}_+ \) of the coupled differential equation

\[
\dot{q}(\tau) = \varphi(\tau, u(\tau), p, q(\tau), Q(\tau), \kappa(\tau)) \quad q(0) = q_0(\kappa_0)
\]

\[
\dot{Q}(\tau) = \Phi(\tau, u(\tau), p, q(\tau), Q(\tau), \kappa(\tau)) \quad Q(0) = Q_0(\kappa_0),
\]

\((\forall \tau \in [0, T_e])\) the set valued function \( \mathbb{X}(\cdot) := \mathcal{E}(Q(\cdot), q(\cdot)) \) is a robust positive invariant tube on the interval \([0, T_e]\) for which the condition \( X_0 \subseteq \mathbb{X}(0) \) is also satisfied.
Ellipsoidal Approximation Strategy

• Consider an auxiliary problem of the form

$$\inf_{\xi(\cdot), \zeta(\cdot), \pi, T_e} \ M(p, T_e, \mathcal{E}(Q(T_e), q(T_e)))$$

\[
\begin{align*}
\dot{q}(\tau) &= \varphi(\tau, u(\tau), p, q(\tau), Q(\tau), \kappa(\tau)) \quad q(0) = q_0(\kappa_0) \\
\dot{Q}(\tau) &= \Phi(\tau, u(\tau), p, q(\tau), Q(\tau), \kappa(\tau)) \quad Q(0) = Q_0(\kappa_0), \\
0 &\geq H(\tau, u(\tau), \mathcal{E}(Q(\tau), q(\tau)), W(\tau)) \quad \forall \, \tau \in [0, T_e].
\end{align*}
\]

• If $H$ is componentwise mononically increasing: every feasible input $(u, p)$ corresponds to a feasible point of the original problem.

• If additionally $M$ is mononically increasing: objective value of the above problem is an upper bound on exact objective value.