1 Problem of Hanging Chain

We want to model a chain attached to two supports and hanging in between. Let’s discretise it with \( N_p \) mass points connected by springs. Each mass \( i \) has position \((x_i, y_i), \ i = 1, \ldots, N_p\). The equilibrium point of the system minimises the potential energy. The potential energy of each spring is
\[ V_{el}^i = \frac{1}{2} D_i \left((x_i - x_{i+1})^2 + (y_i - y_{i+1})^2 \right). \]
The gravitational potential energy of each mass is
\[ V_{g}^i = m_i g y_i. \]
The total potential energy is thus given by:
\[
V_{\text{chain}}(x, y) = \frac{1}{2} \sum_{i=1}^{N_p-1} D_i \left((x_i - x_{i+1})^2 + (y_i - y_{i+1})^2 \right) + g \sum_{i=1}^{N_p} m_i y_i, \tag{1}
\]
where \( x = [x_1, \ldots, x_{N_p}]^T \) and \( y = [y_1, \ldots, y_{N_p}]^T \).

The problem we want to solve is then defined as:
\[
\min_{x,y} V_{\text{chain}}(x, y). \tag{2}
\]

The problem we want to solve is relatively simple; this gives us the possibility to easily analyse the behaviour of the numerical methods we will use. The problem can be made a bit more involved by adding inequality constraints, modelling a plane that the chain might touch.

Formulate the problem in a standard form of quadratic programming
\[
\min_z \frac{1}{2} z^T H z + f^T z \\
\text{s.t. } A z \leq b \\
C z = d \\
l_z \leq z \leq u_z,
\]
where \( z = [x_1, y_1, \ldots, x_{N_p}, y_{N_p}]^T \). Formulate the problem using \( N_p = 40 \), \( m_i = 40/N_p \) kg, \( D_i = 70 N_p \) N/m, \( g = 9.81 \text{ m}/\text{s}^2 \) with the first and last mass point fixed to \((-2, 1)\) and \((2, 1)\), respectively.
Tasks:

1. **Before starting to program, think well and write down all the matrices and vectors you need:**

2. solve the problem using `quadprog` and visualize the solution by plotting $(x, y)$

3. Introduce a linear ground constraint: $y_i \geq 0.5$. Solve your QP again and plot the result. Compare the result with the previous one.

4. **Teaser:** what would happen if you add instead of the linear ground constraints, the nonlinear ground constraints $y_i \geq x_i^2$ to your problem? The resulting problem is no longer a QP, but is it convex?

5. **Teaser:** what would happen if you add instead the nonlinear ground constraints $y_i \geq -x_i^2$ to your problem? Is the problem convex?

2 Nonlinear Hanging Chain

The problem defined in Sec. 1 uses the assumption that the springs have a rest length $L = 0$. This is not very realistic, thus a more realistic model includes the rest length $L$ in the potential energy:

$$V_{\text{chain}}(x, y) = \frac{1}{2} \sum_{i=1}^{N_p-1} D_i \left( \sqrt{(x_i - x_{i+1})^2 + (y_i - y_{i+1})^2} - \frac{L}{N_p} \right)^2 + mg \sum_{i=1}^{N_p} y_i. \quad (3)$$

Note that with $L = 0$ we obtain the same expression as in (1).

The minimization problem (2) is no longer a QP, thus it cannot be solved by using `quadprog`. Instead we will solve it with `fmincon`. Please use $N_p = 30$ for this problem.
Tasks:

1. Write down the optimization problem in the form of
\[
\begin{align*}
\min_z & \quad f(z) \\
\text{s.t} & \quad Az \leq b \\
& \quad A_{eq}z = b_{eq} \\
& \quad l_z \leq z \leq u_z,
\end{align*}
\]

2. Using \( L = 1, N_p = 30 \), solve the problem using \texttt{fmincon} and visualize the solution by plotting \((x, y)\).

3. Introduce a linear ground constraint: \( y_i \geq 0.5 \). Solve the problem again and plot the result. Compare the result with the previous one.

3 OPTIONAL: Convex Hanging Chain

Some chain materials (like string) have an asymmetric force. They can exhibit tension but buckle under compression.

\[
\Delta_i = \sqrt{(x_i - x_{i+1})^2 + (y_i - y_{i+1})^2} - \frac{L}{N_p}
\]

\[
V_{\text{chain}}(x, y) = mg \sum_{i=1}^{N_p} y_i + \frac{1}{2} \sum_{i=1}^{N_p-1} D_i \max(0, \Delta_i)^2
\]
The objective is not differentiable, so we have to reformulate this using slack variables.

\[
\begin{align*}
\min_{x,y,s} & \quad \sum_{j=1}^{N_p-1} \frac{1}{2} s_j^2 + mg \sum_{j=1}^{N_p} y_j \\
\text{s.t.} & \quad \sqrt{(x_i - x_{i+1})^2 + (y_i - y_{i+1})^2} - \frac{L}{N_p} \leq s_i \\
& \quad 0 \leq s_i \\
& \quad A_x x = b_x \\
& \quad A_y y = b_y
\end{align*}
\]

Tasks:

1. Google “SDPT3” and install the SDPT3 solver in MATLAB.

2. Using YALMIP, solve the problem of minimizing the potential energy of the one-sided hanging chain using the data from the previous exercises and \( L = 1 \). To use the SDPT3 solver, set the yalmip option `options = sdpsettings('solver', 'sdpt3');` Plot the solution as before and check that it makes sense.