Exercise 4

Consider a DC motor connected to a flywheel modeled as in the figure below.

The equations which give the back electromotive force $e(t)$ (expressed in V) and motor torque $\tau(t)$ (expressed in N·m) are

$$e(t) = k_e \omega(t) \quad (1)$$
$$\tau(t) = k_t i_a(t) \quad (2)$$

where

- $k_e = 0.4 \text{ [V s rad}^{-1}]$ is the electric constant;
- $k_t = 0.4 \text{ [N m A}^{-1}]$ is the torque constant;
- $i_a(t) \text{ [A]}$ is the armature current;
- $\omega(t) \text{ [rad s}^{-1}]$ is the motor angular velocity.

The differential equations characterizing the system are

$$v_{in}(t) = R_a i_a(t) + L_a \frac{di_a(t)}{dt} + e(t) \quad (3)$$
$$(J_m + J_f) \frac{d\omega(t)}{dt} = \tau(t) - c_v \omega(t) \quad (4)$$

where

- $R_a = 4 \text{ [Ω]}$ armature coil resistance;
- $L_a = 1 \times 10^{-4} \text{ [H]}$ armature coil inductance;
- $J_m = 0.08 \text{ [kg m}^2\text{]}$ motor inertia;
- $c_v = 1 \times 10^{-2} \text{ [kg m}^2\text{s}^{-1}]$ motor viscous friction;
- $J_f = 1 \text{ [kg m}^2\text{]}$ flywheel inertia.
Model and simulate the system

1. Write the system dynamics in the form

\[ d_2 \ddot{y}(t) + d_1 \dot{y}(t) + d_0 y(t) = n_0 u(t) \]  

where \( y(t) = \omega(t) \) and \( u(t) = v_{in}(t) \).

Hint: Use equations (2) and (4) to find a closed form expression of the current and then substitute it into equation (3). Use equation (1) to eliminate the back electromotive force.

2. Using the Laplace transform and equation (5) we obtain the transfer function of the system

\[ G(s) = \frac{Y(s)}{U(s)} = \frac{n_0}{d_2 s^2 + d_1 s + d_0} \]

where \( Y(s) = \mathcal{L}_t\{y(t)\} \) and \( U(s) = \mathcal{L}_t\{u(t)\} \) (\( \mathcal{L}_t \) represents the Laplace transform operator).

Define in Matlab the vectors `num` and `den` containing the transfer function coefficients.

3. Define the continuous time system `sys` using the command `tf`.

4. Using the input voltage defined by

\[ v_{in}(t) = \begin{cases} 0 & \text{if } t < 4 \\ 100 & \text{if } t \geq 4 \end{cases} \]

simulate the system dynamics using the sampling time \( T = 2 \) seconds and \( N = 200 \) samples. To simulate the presence of equation error add Gaussian noise (zero mean and standard deviation equal to 1) to the input \( v_{in} \).

5. Generate pseudo-data for the measured input and output vectors by adding Gaussian noise to the input and output vectors simulation. Name these two vectors \( v_{in\text{meas}} \) and \( \omega_{meas} \). The input noise should have zero mean and standard deviation equal to 2, while the output noise should have zero mean and standard deviation equal to 10. Plot the values of \( \omega_{meas} \) w.r.t. time.

Identify the system using the Prediction Error Method

6. Using the function `arx_pem` created for the solution of Exercise 3 identify an ARX model of order 1\(^1\). Use \( v_{in\text{meas}} \) and \( \omega_{meas} \) as input and output vectors, respectively.

7. Using the Matlab function `tf` define the transfer function of the system identified in the previous task.

8. Simulate the dynamics of the identified system using \( v_{in\text{meas}} \) as input vector and plot the values of the output w.r.t. time (add the graph to the plot created in Task 5).

Identify the system using Maximum Likelihood Estimation

9. In the Maximum Likelihood Estimation procedure we will use the model

\[ y(k) + \sum_{i=1}^{n} a_i y(k - i) = \sum_{i=1}^{n} b_i u(k - i) \]

and we will consider three types of error:

- equation error

\[ \epsilon_{eq}(k) = y(k) + \sum_{i=1}^{n} a_i y(k - i) - \sum_{i=1}^{n} b_i u(k - i); \]

- input measurement error

\[ \epsilon_{in}(k) = u_{in}(k) - u(k) \]

\( (u_{in}(k) \) is the measured input);\n
\(^1\)The DC motor model considered corresponds to a second order system, but the time constants are far apart from each other (this is usually the case). Therefore, we use a first order model as an approximation.
• output measurement error  
\( \epsilon_{\text{out}}(k) = y_m(k) - y(k) \)  

\((y_m(k) \text{ is the measured output})\).

Write a Matlab function which calculates the maximum likelihood residual vector \( r \in \mathbb{R}^{3N-n} \) for the cost

\[
\frac{1}{2} r(x)^T r(x) = \frac{1}{2} \sum_{k=n+1}^{N} \left( \frac{y(k) + \sum_{i=1}^{n} a_i y(k-i) - \sum_{i=1}^{n} b_i u(k-i)}{\sigma_{\text{eq}}} \right)^2 + \\
\frac{1}{2} \sum_{k=1}^{N} \left( \frac{u_m(k) - u(k)}{\sigma_{\text{in}}} \right)^2 + \frac{1}{2} \sum_{k=1}^{N} \left( \frac{y_m(k) - y(k)}{\sigma_{\text{out}}} \right)^2
\]

where

• \( x = [a^T \ b^T \ y^T \ u^T]^T \);
• \( y \in \mathbb{R}^N \), \( u \in \mathbb{R}^N \), \( a \in \mathbb{R}^n \) and \( b \in \mathbb{R}^n \) are the unknowns to be found in the maximum likelihood estimation procedure;
• \( \sigma_{\text{eq}}, \sigma_{\text{in}} \) and \( \sigma_{\text{out}} \) are weights that express our confidence in the different measurements.

The function should have the following prototype

\[
r = \text{siso_residual}(x, y_m, u_m, n, \sigma_{\text{eq}}, \sigma_{\text{in}}, \sigma_{\text{out}})
\]

where \( y_m= y_m, u_m= u_m, \sigma_{\text{eq}}= \sigma_{\text{eq}}, \sigma_{\text{in}}= \sigma_{\text{in}}, \sigma_{\text{out}}= \sigma_{\text{out}}.\)

10. Identify the system dynamics with the maximum likelihood estimation method. Use the Matlab command \texttt{lsqnonlin} with the syntax

\[
x = \text{lsqnonlin}(\emptyset(x)\text{siso_residual}(x, \omega_{\text{meas}}, v_{\text{in meas}}, n, ... \ 
\text{sigma}_{\text{eq}}, \text{sigma}_{\text{in}}, \text{sigma}_{\text{out}}), x0);
\]

where \( n=1, \sigma_{\text{eq}}=1, \sigma_{\text{in}}=2, \sigma_{\text{out}}=10. \) Choose a starting point \( x0 \) for the optimization procedure using the coefficients found in Task 6 and the input and output measurements (assuming the model coefficient vectors found with the prediction error method are \( a_{\text{pem}} \) and \( b_{\text{pem}} \), set \( x0 = [a_{\text{pem}}; b_{\text{pem}}; \omega_{\text{meas}}; v_{\text{in meas}}]. \))

11. Using the Matlab function \texttt{tf} define the transfer function of the system identified in the previous task.

12. Simulate the dynamics of the identified system using \( v_{\text{in meas}} \) as input vector and plot the values of the output w.r.t. time (add the graph to the plot created in Task 5).

13. Change the values of \( \sigma_{\text{eq}}, \sigma_{\text{in}} \) and \( \sigma_{\text{out}} \) and observe the different behaviors of the estimated model (e.g. try \( \sigma_{\text{eq}}=100, \sigma_{\text{in}}=2, \sigma_{\text{out}}=10 \) or \( \sigma_{\text{eq}}=1, \sigma_{\text{in}}=200, \sigma_{\text{out}}=10 \)). Why do you observe such behavior?