Equivalence of State Representations for Hidden Markov Models

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Outline

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Introduction

- Hidden Markov Models (HMM) are frequently used in many engineering aplications: speech processing, image analysis, bioinformatics.
- Many open problems.
- The realization problem:
  - Realizability
  - Realization algorithms
  - Equivalence problem

Outline

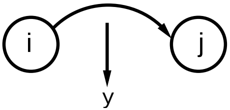
Hidden Markov models (1)

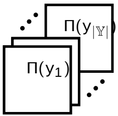
- Moore and Mealy type

  State and output process take values from finite sets

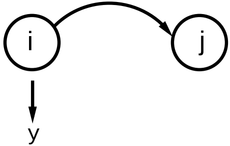
  $\mathbb{X} = \{1, 2, \ldots, |\mathbb{X}|\}$ and $\mathbb{Y} = \{y_1, y_2, \ldots, y_{|\mathbb{Y}|}\}$.

  - Mealy model $(\mathbb{X}, \mathbb{Y}, \Pi, \pi(1))$
    - $\Pi(y)_{ij} = P(x(t+1) = j, y(t) = y | x(t) = i)$,
    - $\pi_i = P(x(1) = i)$.





  - Moore model $(\mathbb{X}, \mathbb{Y}, \Pi_{\mathbb{X}}, \beta, \pi(1))$
    - $(\Pi_{\mathbb{X}})_{ij} = P(x(t+1) = j | x(t) = i)$,
    - $\beta(y)_i = P(y(t) = y | x(t) = i)$ or $B := \left[\begin{array}{ccc} \beta(y_1) & \ldots & \beta(y_{|\mathbb{Y}|}) \end{array} \right]$,
    - $\pi_i = P(x(1) = i)$.



Hidden Markov models (2)

- **String probabilities**
  for Mealy models $(\mathbb{X}, \mathbb{Y}, \Pi, \pi(1))$

$$\mathcal{P}(y_1 y_2 \ldots y_{|y|}) = \pi \Pi(y_1) \Pi(y_2) \ldots \Pi(y_{|y|}) e,$$

with $e := \begin{bmatrix} 1 & 1 & \ldots & 1 \end{bmatrix}$. for Moore models $(\mathbb{X}, \mathbb{Y}, \Pi_{\mathbb{X}}, \beta, \pi(1))$

$$\mathcal{P}(y_1 y_2 \ldots y_{|y|}) = \pi \,\mathrm{diag}(\beta(y_1)) \Pi_{\mathbb{X}} \,\mathrm{diag}(\beta(y_2)) \Pi_{\mathbb{X}} \ldots \mathrm{diag}(\beta(y_{|y|})) \Pi_{\mathbb{X}} e.$$

- **Realization problem**
  - Mealy realization problem:
    Given output string probabilities $\mathcal{P}$, find a Mealy HMM $(\mathbb{X}, \mathbb{Y}, \Pi, \pi(1))$ that realizes $\mathcal{P}$, i.e. such that

$$\mathcal{P}(y) = \pi(1) \Pi(y_1)) \Pi(y_2) \ldots \Pi(y_{|y|}) e, \qquad \forall y = y_1 y_2 \ldots y_{|y|} \in \mathbb{Y}^*.$$

  - Realizations problem very hard due to nonnegativity constraints.
  - Typically one (first) solves the quasi realization problem.

Hidden Markov models (3)

- Quasi Hidden Markov model
  - Analogous to HMM but no nonnegativity constraints on the system vectors and matrices.
    - ★ Mealy type: $(\mathbb{X}_q, \mathbb{Y}, \Pi_q, \pi_q(1), e_q)$

$$\mathcal{P}(\mathrm{y}_1\mathrm{y}_2\ldots\mathrm{y}_{|\mathrm{y}|}) = \pi_q \Pi_q(\mathrm{y}_1)\Pi_q(\mathrm{y}_2)\ldots\Pi_q(\mathrm{y}_{|\mathrm{y}|})e_q,$$

    - ★ Moore type: $(\mathbb{X}_q, \mathbb{Y}, \Pi_{\mathbb{X},q}, \beta_q, \pi_q(1), e_q)$

$$\mathcal{P}(\mathrm{y}_1\mathrm{y}_2\ldots\mathrm{y}_{|\mathrm{y}|}) = \pi_q \,\mathrm{diag}(\beta_q(\mathrm{y}_1))\Pi_{\mathbb{X},q}\ldots\mathrm{diag}(\beta_q(\mathrm{y}_{|\mathrm{y}|}))\Pi_{\mathbb{X},q}e_q.$$

  - In some applications (filtering problems) it suffices to have a quasi realization.

Analogy: linear Gaussian models

- Output $y(t) \in \mathbb{R}^p$ generated by stochastic system (A, C, Q, R, S) of order n (assumed *minimal*)

$$
\begin{aligned}
x(t+1) &= Ax(t) + w(t) \\
y(t) &= Cx(t) + v(t)
\end{aligned}
$$

  with $w(t)$ and $v(t)$ zero mean, white, Gaussian vector sequences with joint covariance matrix:

$$
E \begin{bmatrix} w(t') \\ v(t') \end{bmatrix} \begin{bmatrix} w^\top(t'') & v^\top(t'') \end{bmatrix} = \begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix} \delta_{t't''}.
$$

- General: Mealy; S=0: Moore.

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Equivalence of (Mealy) linear Gaussian models

- (A, C, Q, R, S) gives rise to the autocovariance sequence $\Lambda(\tau) = E(y(t+\tau)y(t)^\top)$ through $P := E(x(t)x(t)^\top)$ and $G := E(x(t+1)y(t)^\top))$, computed as

$$\begin{aligned} P &= APA^\top + Q & \qquad G &= APC^\top + S \\ \Lambda(0) &= CPC^\top + R & \qquad \Lambda(t) &= CA^{t-1}G \end{aligned}$$

- Equivalent models :$\Leftrightarrow$ same autocovariance sequence.
- First type of equivalence:
  Performing a basis change in the state space gives rise to an equivalent model $(TAT^{-1}, CT^{-1}, TQT^\top, R, TS)$.
- Second type of equivalence:
  Every possible $P > 0$ which fullfills

$$\left[\begin{array}{c|c} P - APA^\top & G - APC^\top \\ \hline G^\top - CPA^\top & \Lambda(0) - CPC^\top \end{array}\right] \geq 0,$$

  gives also rise to an equivalent model $(A, C, P - APA^\top, \Lambda(0) - CPC^\top, G - APC^\top)$.

Equivalence of Mealy HMMs (1)

- Equivalent models $:\Leftrightarrow$ same string probabilities.
- Permuting states gives rise to equivalent model ($\approx$ basis change).
- Many more equivalent representations

Theorem (Vidyasagar)

Two minimal quasi Mealy models $(\mathbb{X}_q, \mathbb{Y}, \Pi_q, \pi_q(1), e_q)$ *and* $(\mathbb{X}'_q, \mathbb{Y}, \Pi'_q, \pi'_q(1), e'_q)$ *are equivalent, if and only if there exists a nonsingular matrix S, such that*

$$\begin{aligned} \pi_q &= \pi'_q S^{-1} \\ \Pi_q(y) &= S \Pi'_q(y) S^{-1} \quad \forall y \in \mathbb{Y} \\ e_q &= S e'_q \end{aligned}$$

Equivalence of Mealy HMMs (2)

Theorem

For a (positive) Mealy hidden Markov model $(\mathbb{X}, \mathbb{Y}, \Pi, \pi(1))$ *the following holds. If there exists a nonsingular matrix* T *such that*

- $Te = e$,
- $\pi(1)T^{-1}$ *and* $T\Pi(y)T^{-1} \quad \forall y \in \mathbb{Y}$ *are non-negative*

then $(\mathbb{X}, \mathbb{Y}, T\Pi T^{-1}, \pi(1)T^{-1})$ *is a positive hidden Markov model which is equivalent to the given HMM* $(\mathbb{X}, \mathbb{Y}, \Pi, \pi(1))$.

- If $(\mathbb{X}, \mathbb{Y}, \Pi, \pi(1))$ is minimal as a quasi model, these conditions are necessary and sufficient. In that case the theorem gives a complete description of the set of equivalent models.
- Simulation: Plot π of all possible HMMs equivalent to the given HMM (minimal as quasi HMM).

$$\Pi(0) = \begin{bmatrix} 0.05 & 0.4 & 0.1 \\ 0.2 & 0.15 & 0.2 \\ 0.35 & 0.15 & 0.2 \end{bmatrix}, \quad \Pi(1) = \begin{bmatrix} 0.15 & 0.1 & 0.2 \\ 0.1 & 0.05 & 0.3 \\ 0.05 & 0.15 & 0.1 \end{bmatrix}, \quad \pi = [\ 0.3060 \quad 0.3284 \quad 0.3657\ ].$$

Equivalence of Mealy HMMs (3)

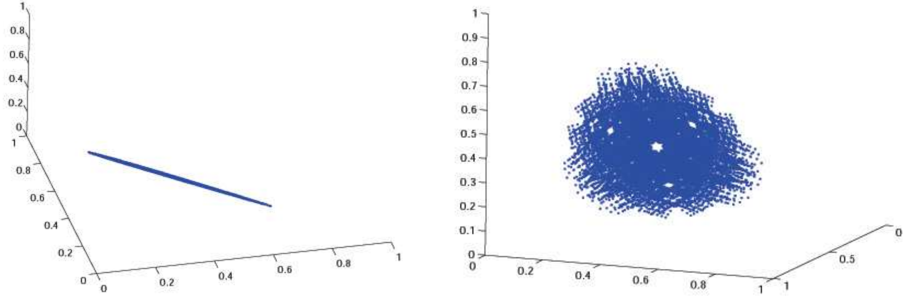

Figure: Plot of π of all possible HMMs which are equivalent to a given HMM (the two subplots give different views of the same three-dimensional plot).

- Problem: Give an accurate description of this set.

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Equivalence of Moore HMMs (1)

Theorem

Given a quasi Moore HMM $(\mathbb{X}, \mathbb{Y}, \Pi_{\mathbb{X},q}, B_q, \pi_q(1), e_q)$ *which is minimal as a quasi Mealy model then the following holds.*

If all states of the Moore model have a different output distribution

and if the state transition matrix $\Pi_{\mathbb{X},q}$ *has full rank,*

then the only equivalent minimal quasi Moore models are obtained by permuting the states of the original model.

Equivalence of Moore HMMs (2)

- A quasi Mealy model with two output symbols can be converted, under general conditions, into an equivalent quasi Moore model with the same number of states.
- Every minimal quasi Moore model with two outputs is minimal as a quasi Mealy model $\rightarrow$ previous theorem can be applied.
- Also useful for HMMs with more than two outputs. According to the theorem, the Moore model below (3 outputs) has no Moore equivalents (except trivial ones).

$$\Pi_{\mathbb{X}} = \begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0.3 & 0.3 & 0.4 \\ 0.2 & 0.2 & 0.6 \end{bmatrix}, \quad B = \begin{bmatrix} 0.3 & 0.3 & 0.4 \\ 0.1 & 0.1 & 0.8 \\ 0.2 & 0.6 & 0.2 \end{bmatrix}, \quad \pi(1) = \begin{bmatrix} 0.5405 & 0.1622 & 0.2973 \end{bmatrix}.$$

Equivalence of Moore linear Gaussian models

Theorem

Given a minimal Moore Gauss Markov model $(A, C, Q, R, 0)$ *then the following holds:*

If C has full column rank

and A has full rank,

then the only equivalent minimal Moore models are obtained by performing a base change in the state space.

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Summary

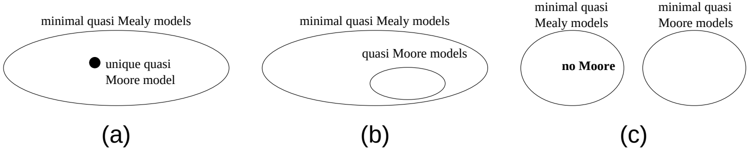

Figure: The three main cases for equivalence classes of HMMs.

a Quasi Moore model minimal as a quasi Mealy model and conditions of Theorem [equivalence of Moore models] fulfilled.
b Quasi Moore model minimal as a quasi Mealy model and conditions of Theorem [equivalence of Moore models] not fulfilled.
c Quasi Moore model not minimal as a quasi Mealy model.

Summary

- We considered the question: Given a (quasi) HMM, what can be said about the set of all equivalent (quasi) HMMs?
- For quasi Mealy HMM, a necessary and sufficient condition for two models to be equivalent was already proven in literature.
- We have proven a sufficient condition for two positive Mealy models to be equivalent.
- We have proven that under certain conditions, the set of equivalent minimal quasi Moore models consists of only one element.
- The situation is very analogous to the situation for Gauss Markov stochastic systems.