Open Problem Session

When is a linear system optimal?

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Plagiarism is the greatest form of compliment
Consider the QDF

\[ \omega \in C^\infty(\mathbb{R}, \mathbb{R}^w) \mapsto \sum_{k, \ell} \left( \frac{d^k}{dt^k} \omega \right)^\top \Phi_{k,\ell} \left( \frac{d^k}{dt^\ell} \omega \right) \]

\[ \Phi_{k,\ell} = \Phi_{\ell,k} \in \mathbb{R}^{w \times w}. \text{ Introduce} \]

\[ \Phi(\zeta, \eta) := \sum_{k, \ell} \Phi_{k,\ell} \zeta^k \eta^\ell, \]

and denote the QDF by \( Q_\Phi(\omega) \).

\( Q_\Phi \) is like a Lagrangian.
\( w \in C^\infty(\mathbb{R}, \mathbb{R}^w) \) is an \textit{optimal trajectory} if

\[
\int_{-\infty}^{+\infty} \left( Q_\Phi(w + \Delta) - Q_\Phi(w) \right) dt \geq 0
\]

for all \( \Delta \in C^\infty(\mathbb{R}, \mathbb{R}^w) \) with compact support.
Stationarity: \[ \Leftrightarrow \]
\[
\Phi(-\frac{d}{dt}, \frac{d}{dt})w = 0
\]

Minimality: \[ \Leftrightarrow \text{ in addition} \]
\[
\Phi(-i\omega, i\omega) \geq 0 \text{ for all } \omega \in \mathbb{R}
\]

Note \[ \Phi(-\frac{d}{dt}, \frac{d}{dt}) = \Phi^\top\left(\frac{d}{dt}, -\frac{d}{dt}\right) \]
Opens up the possibility of describing a behavior very effectively by a single function $Q_\Phi$:

The behavior consists of the trajectories $w : \mathbb{R} \to \mathbb{R}^w$ than minimize, or render stationary, $\int_{-\infty}^{+\infty} Q_\Phi(w) \, dt$. 
Consider

\[ R\left( \frac{d}{dt}\right)w = 0. \]

Denote its *behavior* by \( \mathcal{B} \).

**Open Problem:** *When is \( \mathcal{B} \) an optimal behavior?*

i.e., Given \( \mathcal{B} \), \( \mathcal{C} \ni \Phi \), with

\[ \Phi(-i\omega, i\omega) \geq 0 \quad \text{for all} \quad \omega \in \mathbb{R}, \text{such that} \]

\[ \Phi\left(-\frac{d}{dt}, \frac{d}{dt}\right)w = 0 \]

has also the given behavior \( \mathcal{B} \)?
Consider

\[ R\left(\frac{d}{dt}\right) w = 0. \]

Denote its \textit{behavior} by \( \mathcal{B} \).

\textbf{Open Problem:} \textit{When is \( \mathcal{B} \) an optimal behavior?}

\textbf{Sufficient:}

\[ [R(\xi) = R^\top(-\xi)] \land [R(i\omega) \geq 0 \text{ for all } \omega \in \mathbb{R}] \]

\textbf{Necessary (autonomous case) and Sufficient:}

Given \( \exists \ U \) unimodular, such that \( UR \) has these properties.

But, we are looking for conditions on \( \mathcal{B} \)!
The scalar case $\tilde{w} = 1$ is easy, but not uninteresting.

\[
R\left(\frac{d}{dt}\right)\tilde{w} = 0, \quad R \in \mathbb{R}[\xi]
\]

is stationary iff $R(\xi) = R(-\xi)$, i.e., $R$ is even.
The scalar case $\omega = 1$ is easy, but not uninteresting.

$$R\left(\frac{d}{dt}\right)w = 0, \quad R \in \mathbb{R}[\xi]$$

is stationary iff $R(\xi) = R(-\xi)$, i.e., $R$ is even.

Equivalently, iff $\mathcal{B}$ is

1. time-reversible $:=[w(t) \in \mathcal{B}] \Leftrightarrow [w(-t) \in \mathcal{B}]$
2. and of even dimension.

Root pattern:
The scalar case $w = 1$ is easy, but not uninteresting.

$$R \left( \frac{d}{dt} \right) w = 0, \quad R \in \mathbb{R}[\xi]$$

is stationary iff $R(\xi) = R(-\xi)$, i.e., $R$ is even.

**optimal behavior** iff imaginary roots even multiplicity.

Root pattern:

Time-reversible, even dimension, non-oscillatory.
Optimality $\Rightarrow$ non-constant trajectories unbounded.

If, as young Leibniz claimed, 

*ours is the best of all possible worlds,*

there was a Big Bang, and it will end as a Supernova...
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What $\mathcal{W} \in \mathcal{L}^W$ are optimal?

Please hand in solutions by noon on Thursday!
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Thank you