DISSIPATIVE SYSTEMS, STORAGE FUNCTIONS, and POLYNOMIAL FACTORIZATION

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A dissipative system absorbs supply, ‘globally’, over time + space.

Can this be expressed ‘locally’, as

rate of change in storage + spatial flux ≤ supply rate
A **dissipative system** absorbs supply, ‘**globally**’, over time + space. 
¿¿ Can this be expressed ‘**locally**’, as

\[
\text{rate of change in storage + spatial flux} \leq \text{supply rate}
\]

rate of change in storage + spatial flux
\[
= \text{supply rate} + (\text{non-negative}) \text{ dissipation rate}
\]
OUTLINE

- Lyapunov theory and dissipative dynamical systems
- Linear differential systems
- Dissipative distributed systems
- Local dissipation law
- The factorization equation
Consider the classical dynamical system, the ‘flow’

\[ \Sigma : \frac{d}{dt} x = f(x) \]

with \( x \in X = \mathbb{R}^n \), the state space, and \( f : X \rightarrow X \).

Denote the set of solutions \( x : \mathbb{R} \rightarrow X \) by \( \mathcal{B} \), the ‘behavior’.
LYAPUNOV FUNCTIONS

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is said to be a Lyapunov function for \( \Sigma \) if along \( x \in \mathcal{B} \)

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is said to be a Lyapunov function for $\Sigma$ if along $x \in \mathcal{B}$

$$\frac{d}{dt} V(x(\cdot)) \leq 0$$

Equivalently, if $V^\Sigma := \nabla V \cdot f \leq 0$. 
Typical Lyapunov ‘theorem’:

\[ V(x) > 0 \text{ and } \dot{V}(x) < 0 \text{ for } 0 \neq x \in X \]

\[ \Rightarrow \]

\[ \forall x \in \mathcal{B}, \text{ there holds } x(t) \to 0 \text{ for } t \to \infty \text{ ‘global stability’} \]
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Aleksandr Mikhailovich Lyapunov (1857-1918)

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\[ \sum : \frac{d}{dt} x = f(x, u), \quad y = h(x, u). \]

\( u \in U = \mathbb{R}^m, \quad y \in Y = \mathbb{R}^p, \quad x \in X = \mathbb{R}^n: \) input, output, state.

Behavior \( \mathcal{B} = \) all sol’ns \( (u, y, x) : \mathbb{R} \rightarrow U \times Y \times X. \)
Let \( s : \mathbb{U} \times \mathbb{Y} \rightarrow \mathbb{R} \) be a function, called the \textit{supply rate}.

\( \Sigma \) is said to be \textit{dissipative} w.r.t. the supply rate \( s \) if \( \exists \)

\[ V : \mathbb{X} \rightarrow \mathbb{R}, \]

called the \textit{storage function}, such that

\[ \frac{d}{dt} V(x(\cdot)) \leq s(u(\cdot), y(\cdot)) \]

along input/output/state trajectories \((\forall \ (u(\cdot), y(\cdot), x(\cdot)) \in \mathcal{B})\).
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This inequality is called the \textit{dissipation inequality}.

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This inequality is called the \textit{dissipation inequality}.
Equivalent to

\[ \dot{V}^\Sigma(x, u) := \nabla V(x) \cdot f(x, u) \leq s(u, h(x, u)) \]

for all \((u, x) \in U \times X\).
Equivalent to

$$\dot{V}^\Sigma(x, u) := \nabla V(x) \cdot f(x, u) \leq s(u, h(x, u))$$

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If equality holds: ‘conservative’ system.
$s(u, y)$ models something like the power delivered to the system when the input value is $u$ and output value is $y$.

$V(x)$ then models the internally stored energy.

Dissipativity $\iff$ rate of increase of internal energy $\leq$ power delivered.
Special case: ‘closed’ system: \( s = 0 \) then

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\text{dissipativeness} \leftrightarrow V \text{ is a Lyapunov function.}
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Dissipativity is the natural generalization to open systems of Lyapunov theory.
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\text{Stability for closed systems } \sim \text{ Dissipativity for open systems.}
\]
Basic question:

Given (a representation of) $\Sigma$, the dynamics, and given $s$, the supply rate, is the system dissipative w.r.t. $s$, i.e., does there exist a storage function $V$ such that the dissipation inequality holds?
THE CONSTRUCTION OF STORAGE FUNCTIONS

Basic question:

Given (a representation of) $\Sigma$, the dynamics, and given $s$, the supply rate, is the system dissipative w.r.t. $s$, i.e., does there exist a storage function $V$ such that the dissipation inequality holds?

Assume $s$ ‘power’, known dynamics, what is the internal stored energy?
The construction of storage f’ns is very well understood, particularly for finite dimensional linear systems and quadratic supply rates.
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Leads to the KYP-lemma, LMI’s, ARIneq, ARE, semi-definite programming, spectral factorization, Lyapunov functions, $\mathcal{H}_\infty$ and robust control, positive and bounded real functions, electrical circuit synthesis, stochastic realization theory.
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The storage function $V$ is in general far from unique. There are two ‘canonical’ storage functions: the available storage and the required supply.

For conservative systems, $V$ is unique.
Dissipative systems and storage f’ns play a remarkably central role in the field.
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The construction of storage functions is the question which we shall discuss today for systems described by PDE’s.
Lyapunov theory and dissipative dynamical systems

Linear differential systems: Systems described by linear constant coefficient PDE’s

Dissipative distributed systems

Local dissipation law

The factorization equation
PDE’s: polynomial notation

Consider, for example, the PDE:

\[ w_1(x_1, x_2) + \frac{\partial^2}{\partial x_2^2} w_1(x_1, x_2) + \frac{\partial}{\partial x_1} w_2(x_1, x_2) = 0 \]

\[ w_2(x_1, x_2) + \frac{\partial^3}{\partial x_2^3} w_1(x_1, x_2) + \frac{\partial^4}{\partial x_1^4} w_2(x_1, x_2) = 0 \]
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Notation:

\[
\begin{align*}
    \xi_1 \leftrightarrow \frac{\partial}{\partial x_1}, \quad \xi_2 \leftrightarrow \frac{\partial}{\partial x_2}, \quad w &= \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \quad R(\xi_1, \xi_2) = \begin{bmatrix} 1 + \xi_2^2 & \xi_1 \\ \xi_2^3 & 1 + \xi_1^4 \end{bmatrix}. 
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\[ R(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})w = 0. \]
LINEAR DIFFERENTIAL SYSTEMS

\[ T = \mathbb{R}^n, \text{ the set of independent variables, typically } n = 4, \]
\[ W = \mathbb{R}^w, \text{ the set of dependent variables,} \]
\[ \mathcal{W} = \text{the solutions of a linear constant coefficient system of PDE's.} \]
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Let \( R \in \mathbb{R}^\bullet \times \mathbb{R}^{\bullet \times w}[\xi_1, \ldots, \xi_n], \) and consider

\[ R(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})w = 0. \quad (\ast) \]

Define the associated behavior

\[ \mathcal{V} = \{ w \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^w) \mid (\ast) \text{ holds} \}. \]
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**Notation** for \( n \)-D linear differential systems:

\( (\mathbb{R}^n, \mathbb{R}^w, \mathcal{B}) \in \mathcal{L}_n^w \), or \( \mathcal{B} \in \mathcal{L}_n^w \).
Examples: Maxwell’s eq’ns, diffusion eq’n, wave eq’n, . . .

\[
\begin{align*}
\nabla \cdot \vec{E} & = \frac{1}{\varepsilon_0} \rho , \\
\nabla \times \vec{E} & = -\frac{\partial}{\partial t} \vec{B} , \\
\nabla \cdot \vec{B} & = 0 , \\
c^2 \nabla \times \vec{B} & = \frac{1}{\varepsilon_0} \vec{j} + \frac{\partial}{\partial t} \vec{E} .
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\end{align*}
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\( T = \mathbb{R} \times \mathbb{R}^3 \) (time and space) \( n = 4, \)
\( w = (\vec{E}, \vec{B}, \vec{j}, \rho) \)
(electric field, magnetic field, current density, charge density),
\( \mathcal{W} = \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}, \ w = 10, \)
\( \mathcal{B} = \text{set of solutions to these PDE’s.} \)
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\[ \nabla \cdot \vec{E} = \frac{1}{\varepsilon_0} \rho, \]

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Note: 10 variables, 8 equations! \( \Rightarrow \) \exists free variables.
\[ R \left( \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right) w = 0 \]

is called a **kernel representation** of the associated \( \mathcal{B} \in \mathcal{L}_n^w \).
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Another representation: **image representation**

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w = M\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right)\ell.
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‘Elimination’ thm $\Rightarrow \text{im}(M(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})) \in \mathcal{L}_n^w$!
is called a **kernel representation** of the associated $\mathcal{B} \in \mathcal{L}^w_n$.

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‘Elimination’ thm  \implies  $\text{im}(M(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})) \in \mathcal{L}^w_n$ !

Do all linear differential systems admit an image representation???
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`Elimination` thm $\Rightarrow$ im($M\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right)$) $\in \mathcal{L}_n^w$ !

Do all linear differential systems admit an image representation???

$\mathcal{B} \in \mathcal{L}_n^w$ admits an image representation iff it is `controllable`.
Controllability def’n in pictures:

\[ w_1, w_2 \in \mathcal{B}. \]
$w \in \mathcal{B}$ ‘patches’ $w_1, w_2 \in \mathcal{B}$. 
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Controllability $\iff$ ‘patch-ability’.
Are Maxwell’s equations controllable?
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The following equations in the **scalar potential** $\phi : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ and the **vector potential** $\vec{A} : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$, generate exactly the solutions to Maxwell’s equations:

\[
\begin{align*}
\vec{E} &= -\frac{\partial}{\partial t} \vec{A} - \nabla \phi, \\
\vec{B} &= \nabla \times \vec{A}, \\
\vec{j} &= \varepsilon_0 \frac{\partial^2}{\partial t^2} \vec{A} - \varepsilon_0 c^2 \nabla^2 \vec{A} + \varepsilon_0 c^2 \nabla(\nabla \cdot \vec{A}) + \varepsilon_0 \frac{\partial}{\partial t} \nabla \phi, \\
\rho &= -\varepsilon_0 \frac{\partial}{\partial t} \nabla \cdot \vec{A} - \varepsilon_0 \nabla^2 \phi.
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Proves controllability.
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Proves controllability. Illustrates the interesting connection

\begin{center}
controllability $\iff \exists \text{ potential!}$
\end{center}
Observability of the image representation

\[ w = M \left( \frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n} \right) \ell \]

is defined as: \( \ell \) can be deduced from \( w \),

i.e., \( M \left( \frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n} \right) \) should be injective.
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Not all controllable systems admit an observable image representation. For \( n = 1 \), they do. For \( n > 1 \), exceptionally so. The latent variable in an image representation \( \ell \) may be ‘hidden’.
OBSERVABILITY

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The latent variable in an image representation \( \ell \) may be ‘hidden’.

**Example:** Maxwell’s equations do not allow a potential representation that is *observable*. 
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NOTATION

Multi-index notation:

\[ x = (x_1, \ldots, x_n), \]
\[ k = (k_1, \ldots, k_n), \ell = (\ell_1, \ldots, \ell_n), \]
\[ \xi = (\xi_1, \ldots, \xi_n), \zeta = (\zeta_1, \ldots, \zeta_n), \eta = (\eta_1, \ldots, \eta_n), \]
\[ \frac{d}{dx} = \left( \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right), \quad \frac{d^k}{dx^k} = \left( \frac{\partial^{k_1}}{\partial x_1^{k_1}}, \ldots, \frac{\partial^{k_n}}{\partial x_n^{k_n}} \right), \]
\[ dx = dx_1 dx_2 \ldots dx_n, \]
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\[ \text{etc.} \]

\[ \nabla \cdot := \frac{\partial}{\partial x_1} + \cdots + \frac{\partial}{\partial x_n}. \]
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etc.

For simplicity of notation, and for concreteness, we often take \( n = 4, \) independent variables, \( t, \) time and \( x, y, z, \) space.
**NOTATION**

**Multi-index notation:**

\[ x = (x_1, \ldots, x_n), \]
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\[ \text{etc.} \]

\[ \nabla \cdot := \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}, \quad \text{‘spatial flux’} \]
The quadratic map acting on \( w : \mathbb{R}^n \rightarrow \mathbb{R}^w \) and its derivatives, defined by

\[
\sum_{k,\ell} \left( \frac{d^k}{dx^k} w \right) \Phi_{k,\ell} \left( \frac{d^\ell}{dx^\ell} w \right)
\]

is called \textit{quadratic differential form} (QDF) on \( C^\infty(\mathbb{R}^n, \mathbb{R}^w) \). 
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The quadratic map acting on \( w : \mathbb{R}^n \rightarrow \mathbb{R}^w \) and its derivatives, defined by

\[
    w \mapsto \sum_{k,\ell} \left( \frac{d^k}{dx^k} w \right)^\top \Phi_{k,\ell} \left( \frac{d^\ell}{dx^\ell} w \right)
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is called \textit{quadratic differential form} (QDF) on \( \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^w) \).

\( \Phi_{k,\ell} \in \mathbb{R}^{w \times w} \); \ WLOG: \( \Phi_{k,\ell} = \Phi_{\ell,k}^\top \).
QDF’s

The quadratic map acting on \( w : \mathbb{R}^n \rightarrow \mathbb{R}^w \) and its derivatives, defined by

\[
\begin{aligned}
    w \mapsto \sum_{k,\ell} \left( \frac{d^k}{dx^k} w \right) \Phi_{k,\ell} \left( \frac{d^\ell}{dx^\ell} w \right)
\end{aligned}
\]

is called \textit{quadratic differential form} (QDF) on \( C^\infty(\mathbb{R}^n, \mathbb{R}^w) \).
\( \Phi_{k,\ell} \in \mathbb{R}^{w \times w} ; \ WLOG: \Phi_{k,\ell} = \Phi_{\ell,k}^\top \).

Introduce the \( 2n \)-variable polynomial matrix \( \Phi \)

\[
\Phi(\zeta, \eta) = \sum_{k,\ell} \Phi_{k,\ell} \zeta^k \eta^\ell.
\]

Denote the QDF as \( Q_\Phi \). QDF’s are parametrized by \( \mathbb{R}^{\bullet \times \bullet}[\zeta, \eta] \).
We consider only controllable linear differential systems and QDF’s for supply rates.
We consider only controllable linear differential systems and QDF’s for supply rates.

**Definition**: $\mathcal{B} \in \mathcal{L}_n^w$, controllable, is said to be **dissipative** with respect to the supply rate $Q_\Phi$ (a QDF) if

$$\int_{\mathbb{R}^n} Q_\Phi(w) \, dx \geq 0$$

for all $w \in \mathcal{B}$ of compact support, i.e., for all $w \in \mathcal{B} \cap \mathcal{D}$.

$\mathcal{D} := \mathcal{C}^\infty$ and ‘compact support’.
Assume $n = 4$:

independent variables $x, y, z; t$: space and time.
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**Idea:** $Q_\Phi(w)(x, y, z; t) \ dx\ dy\ dz\ dt$

‘energy’ supplied to the system in the space-cube $[x, x + dx] \times [y, y + dy] \times [z, z + dz]$ during the time-interval $[t, t + dt]$. 
Assume $n = 4$:

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**Idea:** $Q_{\Phi}(w)(x, y, z; t) \ dx \ dy \ dz \ dt$ :

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during the time-interval $[t, t + dt]$.

**Dissipativity** $\iff$ 

$$\int_{\mathbb{R}} \left( \int_{\mathbb{R}^3} Q_{\Phi}(w)(x, y, z, t) \ dx \ dy \ dz \right) \ dt \geq 0 \quad \text{for all } w \in \mathbb{W} \cup \mathbb{D}.$$  

A dissipative system absorbs net energy.
Example: Maxwell’s eq’ns:

dissipative (in fact, conservative) w.r.t. the QDF $- \vec{E} \cdot \vec{j}$.
Example: Maxwell’s eq’ns:

dissipative (in fact, conservative) w.r.t. the QDF \(- \vec{E} \cdot \vec{j}\).

In other words, if \(\vec{E}, \vec{j}\) is of compact support and satisfies

\[ \varepsilon_0 \frac{\partial}{\partial t} \nabla \cdot \vec{E} + \nabla \cdot \vec{j} = 0, \]
\[ \varepsilon_0 \frac{\partial^2}{\partial t^2} \vec{E} + \varepsilon_0 c^2 \nabla \times \nabla \times \vec{E} + \frac{\partial}{\partial t} \vec{j} = 0, \]

then

\[ \int_{\mathbb{R}} \left( \int_{\mathbb{R}^3} (-\vec{E} \cdot \vec{j}) \, dx dy dz \right) \, dt = 0. \]
OUTLINE

- Lyapunov theory and dissipative dynamical systems
- Linear differential systems
- Dissipative distributed systems
- Local dissipation law
- The factorization equation
LOCAL DISSIPATION LAW

Dissipativity : $\iff$

$$\int_{\mathbb{R}} \left( \int_{\mathbb{R}^3} Q_{\Phi}(w) \, dx \, dy \, dz \right) \, dt \geq 0 \quad \text{for all } w \in \mathfrak{V} \cap \mathfrak{D}.$$
Dissipativity: $\iff$

$$\int_\mathbb{R} \left( \int_{\mathbb{R}^3} Q_\Phi(w) \, dx \, dy \, dz \right) \, dt \geq 0 \quad \text{for all } w \in \mathcal{B} \cap \mathcal{D}.$$ 

Can this be reinterpreted as: As the system evolves, energy is locally stored, and dissipated or redistributed over time and space?
!! Invent storage and flux, locally defined in time and space, such that in every spatial domain there holds:

$$\frac{d}{dt} \text{ Storage + Spatial flux } \leq \text{ Supply.}$$
!! Invent storage and flux, locally defined in time and space, such that in every spatial domain there holds:

\[ \frac{d}{dt} \text{Storage} + \text{Spatial flux} \leq \text{Supply}. \]
!! Invent **storage and flux**, locally defined in time and space, such that in every spatial domain there holds:

\[
\frac{d}{dt} \text{Storage + Spatial flux} \leq \text{Supply.}
\]

**Supply** = partly stored + partly radiated + partly dissipated.
!! Invent storage and flux, locally defined in time and space, such that in every spatial domain there holds:

\[
\frac{d}{dt} \text{ Storage + Spatial flux } \leq \text{ Supply.}
\]

!! Construct internal energy, internal entropy as a local function !!
MAIN RESULT (stated for $n = 4$)

**Theorem:** Assume $n = 4$: independent variables $x, y, z; t$: space and time. Let $\mathcal{B} \in \mathcal{L}_4^w$ be controllable. Then

$$\int_{\mathbb{R}} \left( \int_{\mathbb{R}^3} Q_\Phi(w) \, dx \, dy \, dz \right) \, dt \geq 0 \quad \text{for all } w \in \mathcal{B} \cap \mathcal{D}.$$
MAIN RESULT (stated for $n = 4$)

Theorem: Assume $n = 4$: independent variables $x, y, z; t$; space and time. Let $\mathcal{B} \in \mathcal{L}_4$ be controllable. Then

$$\int \int_{\mathbb{R}} \int_{\mathbb{R}^3} Q_\Phi(w) \, dx \, dy \, dz \, dt \geq 0 \quad \text{for all } w \in \mathcal{B} \cap \mathcal{D}.$$

if and only if

$\exists$ an image representation $w = \mathcal{M}(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial t}) \ell$ of $\mathcal{B}$, and QDF's $S$, the storage, and $F_x, F_y, F_z$, the flux,
MAIN RESULT (stated for $n = 4$)

Theorem: Assume $n = 4$: independent variables $x, y, z; t$: space and time. Let $\mathcal{B} \in \mathcal{L}_4$ be controllable. Then

$$\int \int_{\mathbb{R}^3} Q_\Phi(w) \, dx \, dy \, dz \, dt \geq 0 \quad \text{for all } w \in \mathcal{B} \cap \mathcal{D}.$$ 

if and only if

$\exists$ an image representation $w = M\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial t}\right)\ell$ of $\mathcal{B}$, and QDF's $S$, the storage, and $F_x, F_y, F_z$, the flux, such that the local dissipation law

$$\frac{\partial}{\partial t} S(\ell) + \frac{\partial}{\partial x} F_x(\ell) + \frac{\partial}{\partial y} F_y(\ell) + \frac{\partial}{\partial z} F_z(\ell) \leq Q_\Phi(w)$$

holds for all $(w, \ell)$ that satisfy $w = M\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial t}\right)\ell$. 


DISSIPATIVE SYSTEMS, STORAGE FUNCTIONS, and POLYNOMIAL FACTORIZATION – p.33/47
Note:

the local law involves (possibly unobservable, - i.e., hidden!) latent variables (the $\ell$’s).
Maxwell’s equations are dissipative (in fact, conservative) with respect to $- \mathbf{E} \cdot \mathbf{j}$, the rate of energy supplied.
Maxwell’s equations are dissipative (in fact, conservative) with respect to \(- \vec{E} \cdot \vec{j}\), the rate of energy supplied.

Introduce the **stored energy density**, \(S\), and
the **energy flux density** (the Poynting vector), \(\vec{F}\),

\[
S(\vec{E}, \vec{B}) := \frac{\varepsilon_0}{2} \vec{E} \cdot \vec{E} + \frac{\varepsilon_0 c^2}{2} \vec{B} \cdot \vec{B},
\]

\[
\vec{F}(\vec{E}, \vec{B}) := \varepsilon_0 c^2 \vec{E} \times \vec{B}.
\]
Maxwell’s equations are dissipative (in fact, conservative) with respect to $-\vec{E} \cdot \vec{j}$, the rate of energy supplied.

Introduce the **stored energy density**, $S$, and the **energy flux density (the Poynting vector)**, $\vec{F}$,

$$ S(\vec{E}, \vec{B}) := \frac{\varepsilon_0}{2} \vec{E} \cdot \vec{E} + \frac{\varepsilon_0 c^2}{2} \vec{B} \cdot \vec{B}, $$

$$ \vec{F}(\vec{E}, \vec{B}) := \varepsilon_0 c^2 \vec{E} \times \vec{B}. $$

The following is a local conservation law for Maxwell’s equations:

$$ \frac{\partial}{\partial t} S(\vec{E}, \vec{B}) + \nabla \cdot \vec{F}(\vec{E}, \vec{B}) = -\vec{E} \cdot \vec{j}. $$
EXAMPLE: ENERGY STORED IN EM FIELDS

Maxwell’s equations are dissipative (in fact, conservative) with respect to $-\vec{E} \cdot \vec{j}$, the rate of energy supplied.

Introduce the stored energy density, $S$, and the energy flux density (the Poynting vector), $\vec{F}$,

$$S(\vec{E}, \vec{B}) := \frac{\varepsilon_0}{2} \vec{E} \cdot \vec{E} + \frac{\varepsilon_0 c^2}{2} \vec{B} \cdot \vec{B},$$

$$\vec{F}(\vec{E}, \vec{B}) := \varepsilon_0 c^2 \vec{E} \times \vec{B}.$$

The following is a local conservation law for Maxwell’s equations:

$$\frac{\partial}{\partial t} S(\vec{E}, \vec{B}) + \nabla \cdot \vec{F}(\vec{E}, \vec{B}) = -\vec{E} \cdot \vec{j}.$$

Involves $\vec{B}$, unobservable from the energy variables $\vec{E}$ and $\vec{j}$. 
Using **controllability** and **image representations**, we may assume, WLOG:

\[ \mathcal{V} = C^\infty(\mathbb{R}^n, \mathbb{R}^w) \]
\[
\int_{\mathbb{R}^n} Q_{\Phi}(w) \geq 0 \text{ for all } w \in \mathcal{D}
\]

\[\Downarrow \quad \text{ (Parseval)}\]

\[\Phi(-i\omega, i\omega) \geq 0 \text{ for all } \omega \in \mathbb{R}^n\]
\[ \Phi(-i\omega, i\omega) \geq 0 \text{ for all } \omega \in \mathbb{R}^n \]

\[ \Leftrightarrow \text{(Factorization equation)} \]

\[ \exists D : \quad \Phi(-\xi, \xi) = D^\top (-\xi) D(\xi) \]
\[ \exists D : \Phi(-\xi, \xi) = D^\top (-\xi) D(\xi) \]
\[ \Downarrow \text{(easy)} \]

\[ \exists \Psi : (\zeta + \eta)^\top \Psi(\zeta, \eta) = \Phi(\zeta, \eta) - D^\top (\zeta) D(\eta) \]
\[ \exists \Psi : \ (\zeta + \eta)^T \Psi(\zeta, \eta) = \Phi(\zeta, \eta) - D^T(\zeta)D(\eta) \]

\[ \updownarrow \text{ (clearly)} \]

\[ \exists \Psi : \ \nabla \cdot Q(\omega) \leq Q(\Phi(\omega)) \text{ for all } \omega \in \mathcal{C}^\infty \]
Assuming factorizability:

**Global dissipation** \( \iff \)

\[
\int_{\mathbb{R}^n} Q_{\Phi}(w) \geq 0 \text{ for all } w \in \mathcal{D}
\]

\[\iff\]

\[\exists \Psi : \nabla \cdot Q_{\Psi}(w) \leq Q_{\Phi}(w) \text{ for all } w \in \mathcal{C}^\infty\]

\(\iff\): **Local dissipation**
Assuming factorizability:

Global dissipation: \( \iff \)

\[
\int_{\mathbb{R}^n} Q_{\Phi}(w) \geq 0 \text{ for all } w \in \mathcal{D}
\]

\( \downarrow \)

\[
\exists \, \Psi : \quad \nabla \cdot Q_{\Psi}(w) \leq Q_{\Phi}(w) \text{ for all } w \in \mathcal{C}^\infty
\]

\( \iff \): Local dissipation

This argument is valid for \( n = 1 \ldots \)
Consider

\[ X^\top ( -\xi ) X(\xi) = Y(\xi) \]

with \( Y \in \mathbb{R}^{\cdot \times \cdot}[\xi] \) given, and \( X \) the unknown. Solvable??
Consider

\[ X^\top (-\xi) X(\xi) = Y(\xi) \]

with \( Y \in \mathbb{R}^{n \times n}[\xi] \) given, and \( X \) the unknown. Solvable??

\[ X^\top (\xi) X(\xi) = Y(\xi) \]

with \( Y \in \mathbb{R}^{n \times n}[\xi] \) given, and \( X \) the unknown.

Under what conditions on \( Y \) does there exist a solution \( X \)?
THE FACTORIZATION EQ’N

Consider

\[ X^T (-\xi) X(\xi) = Y(\xi) \]

with \( Y \in \mathbb{R}^{n \times n}[\xi] \) given, and \( X \) the unknown. Solvable??

Under what conditions on \( Y \) does there exist a solution \( X \)?

Scalar case: !! write the real polynomial \( Y \) as a sum of squares

\[ Y = x_1^2 + x_2^2 + \cdots + x_k^2. \]
$X^T(\xi)X(\xi) = Y(\xi)$

$Y$ is a given polynomial matrix; $X$ is the unknown.
$X^T(\xi)X(\xi) = Y(\xi)$

\(Y\) is a given polynomial matrix; \(X\) is the unknown.

For \(n = 1\) and \(Y \in \mathbb{R}[\xi]\), solvable (for \(X \in \mathbb{R}^2[\xi]\)) iff

\[Y(\alpha) \geq 0 \quad \text{for all } \alpha \in \mathbb{R}.\]
\( X^\top(\xi)X(\xi) = Y(\xi) \)

\( Y \) is a given polynomial matrix; \( X \) is the unknown.

For \( n = 1 \), and \( Y \in \mathbb{R}^{n\times n}[\xi] \), it is well-known (but non-trivial) that this factorization equation is solvable (with \( X \in \mathbb{R}^{n\times n}[\xi] \)) iff

\[
Y(\alpha) = Y^\top(\alpha) \geq 0 \quad \text{for all } \alpha \in \mathbb{R}.
\]
\[ X^\top(\xi) X(\xi) = Y(\xi) \]

\( Y \) is a given polynomial matrix; \( X \) is the unknown.

For \( n > 1 \), and under this obvious symmetry and positivity requirement,

\[ Y(\alpha) = Y^\top(\alpha) \geq 0 \quad \text{for all } \alpha \in \mathbb{R}^n, \]

this equation \textbf{can nevertheless} in general \textbf{not} be solved over the polynomial matrices, for \( X \in \mathbb{R}^{\times \times}[\xi], \)
\[ X^\top(\xi) X(\xi) = Y(\xi) \]

\( Y \) is a given polynomial matrix; \( X \) is the unknown.

For \( n > 1 \), and under this obvious symmetry and positivity requirement,

\[ Y(\alpha) = Y^\top(\alpha) \geq 0 \quad \text{for all } \alpha \in \mathbb{R}^n, \]

this equation can nevertheless in general not be solved over the polynomial matrices, for \( X \in \mathbb{R}^{\times\times}[\xi] \), but it can be solved over the matrices of rational functions, i.e., for \( X \in \mathbb{R}^{\times\times}(\xi) \).
This factorizability is a simple consequence of Hilbert’s 17-th pbm!

!! Solve \( p = p_1^2 + p_2^2 + \cdots + p_k^2, \ p \ \text{given} \)
This factorizability is a simple consequence of Hilbert’s 17-th pbm!

!! Solve \( p = p_1^2 + p_2^2 + \cdots + p_k^2, \ p \) given

A polynomial \( p \in \mathbb{R}[^\xi_1, \ldots, \xi_n] \), with \( p(\alpha_1, \ldots, \alpha_n) \geq 0 \) for all \( (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n \) can in general not be expressed as a sum of squares of polynomials, with the \( p_i \)'s \( \in \mathbb{R}[^\xi_1, \ldots, \xi_n] \).
This factorizability is a simple consequence of Hilbert's 17-th pbm!

But a rational function (and hence a polynomial)
\[ p \in \mathbb{R}(\xi_1, \ldots, \xi_n), \text{ with } p(\alpha_1, \ldots, \alpha_n) \geq 0, \text{ for all } (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n, \text{ can be expressed as a sum of squares of } (k = 2^n) \text{ rational functions, with the } p_i \text{'s } \in \mathbb{R}(\xi_1, \ldots, \xi_n). \]
⇒ solvability of the factorization eq’n

\[ \Phi(-i\omega, i\omega) \geq 0 \text{ for all } \omega \in \mathbb{R}^n \]

\[ \uparrow \quad \text{(Factorization equation)} \]

\[ \exists \ D : \quad \Phi(-\xi, \xi) = D^\top (-\xi) D(\xi) \]

over the rational functions, i.e., with \( D \) a matrix with elements in \( \mathbb{R}(\xi_1, \cdots, \xi_n) \).
⇒ solvability of the factorization eq’n

\[ \Phi(-i\omega, i\omega) \geq 0 \text{ for all } \omega \in \mathbb{R}^n \]

\[ \exists \quad \Phi(-\xi, \xi) = D^\top (-\xi) D(\xi) \]

over the rational functions, i.e., with \( D \) a matrix with elements in \( \mathbb{R}(\xi_1, \cdots, \xi_n) \).

The need to introduce rational functions in this factorization and an image representation of \( \mathcal{B} \) (to reduce the pbm to \( C^\infty \)) are the causes of the unavoidable presence of (possibly unobservable, i.e., ‘hidden’) latent variables in the local dissipation law.
UNIQUENESS

Non-uniqueness of the storage function stems from 3 sources
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2. The non-uniqueness of $D$ in the factorization equation

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UNIQUENESS

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1. The non-uniqueness of the latent variable $\ell$ in various (non-observable) image representations.

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$$\Phi(-\xi, \xi) = D^\top(-\xi) D(\xi)$$

3. The non-uniqueness (in the case $n > 1$) of the solution $\Psi$ of

$$\left(\zeta + \eta\right)^\top \Psi(\zeta, \eta) = \Phi(\zeta, \eta) - D^\top(\zeta) D(\eta)$$
UNIQUENESS

Non-uniqueness of the storage function stems from 3 sources

1. The non-uniqueness of the latent variable $\ell$ in various (non-observable) image representations.

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3. The non-uniqueness (in the case $n > 1$) of the solution $\Psi$ of

$$(\zeta + \eta)^T \Psi(\zeta, \eta) = \Phi(\zeta, \eta) - D^T(\zeta)D(\eta)$$

For conservative systems, $\Phi(-\xi, \xi) = 0$, whence $D = 0$, but, when $n > 1$, the third source of non-uniqueness remains.
The non-uniqueness is very real, even for EM fields.
The non-uniqueness is very real, even for EM fields. Cfr.

*The ambiguity of the field energy*

... There are, in fact, an infinite number of different possibilities for \( u \) [the internal energy] and \( S \) [the flux] ... It is sometimes claimed that this problem can be resolved using the theory of gravitation ... as yet nobody has done such a delicate experiment ... So we will follow the rest of the world - besides, we believe that it [our choice] is probably perfectly right.

The theory of dissipative systems centers around the construction of the storage function.
SUMMARY

The theory of dissipative systems centers around the construction of the storage function.

- **global dissipation** ⇔ **∃** local dissipation law
SUMMARY

- The theory of dissipative systems centers around the construction of the storage function

- **global dissipation ↔ ∃ local dissipation law**

- Involves *hidden* latent variables (e.g. $\vec{B}$ in Maxwell’s eq’ns)
The theory of dissipative systems centers around the construction of the storage function.

- **global dissipation** ↔ ∃ local dissipation law

- Involves hidden latent variables (e.g. $\vec{B}$ in Maxwell’s eq’ns)

- The proof $\cong$ Hilbert’s 17-th problem
The theory of dissipative systems centers around the construction of the storage function

\[ \text{global dissipation} \iff \exists \text{ local dissipation law} \]

- Involves hidden latent variables (e.g. \( \mathbf{\dot{B}} \) in Maxwell’s eq’ns)
- The proof \( \simeq \) Hilbert’s 17-th problem
- Neither controllability nor observability are good generic system theoretic assumptions for physical models

The manuscript & copies of the lecture frames will be available from/at

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Thank you for your attention!