A Behavioral Approach to Estimation and Dead-Beat Observer Design With Applications to State–Space Models

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***Abstract*—The observer design problem is investigated in the context of linear left shift invariant discrete behaviors, whose trajectories have supports on $\mathbb{Z}_+$. Necessary and sufficient conditions for the existence of a dead-beat observer of some relevant variables from some measured ones, in the presence of some unmeasured (and irrelevant) variables, are introduced, and a complete parametrization of all dead-beat observers is given. Equivalent conditions for the existence of causal dead-beat observers are then derived. Finally, several classical problems addressed for state–space models, like state estimation, the design of unknown input observers or the design of fault detectors and identifiers (possibly in the presence of disturbances), are cast in this general framework, and the aforementioned equivalent conditions and parametrizations are specialized to these cases.**

***Index Terms*—Behaviors, fault detection and isolation (FDI), nilpotent autonomous systems, observability, observers, reconstructibility, unknown input observers (UIO).**

I. INTRODUCTION

THE original theory of state observers was concerned with the problem of estimating the state from the corresponding inputs and outputs. This problem has been later generalized in various ways, and in relatively recent years there has been a great deal of research aiming at designing state observers in the presence of unknown inputs (disturbances) [11], [12], [19].

Another research issue, which originated in the 1980s and flourished in the 1990s [4], [5], [10], but still represents a very lively research topic [3], [6] is the fault detection and isolation (FDI) problem. The problem of detecting and identifying the faults affecting the functioning of the system (possibly in the presence of disturbances) can be stated in a natural way and addressed as an estimation problem.

In the last few years, we have witnessed a renewed interest in these two issues. In some recent papers, estimation problems and observer synthesis, in a deterministic context, have been investigated for wider classes of dynamic systems, described either in a behavioral setting or by means of polynomial/rational models, thus obtaining interesting connections between the problem solutions obtained via different approaches [7], [8], [21], [22].

This paper aims to extend the analysis started in [21] and [22], thus producing a powerful setting, where all classical estimation problems for (discrete-time) state–space models can be cast. Specifically, in the first part of this paper we explore the observer design problem for linear time-invariant (discrete-time) dynamic systems that are described in behavioral terms by means of a set of difference equations. Moreover, the concept of (nonconsistent) dead-beat observer (DBO) is introduced, several new equivalent conditions for the existence of a consistent/nonconsistent DBO are given, and a complete parametrization of consistent/nonconsistent DBOs is also provided.

In the second part of the paper, these general results are applied to state–space models for formalizing, and hence solving, a wide variety of classical estimation problems (state estimation, state estimation in the presence of disturbances, fault detection and isolation, etc.). Comparisons with previous results, specifically obtained for state–space models, are also presented. A preliminary version of this paper results can be found in [1].

We remark that the choice of dealing with dead-beat observers instead of asymptotic observers (possibly under some additional robustness constraint, which may confine the system zeros within some open circle $\{z \in \mathbb{C} : |z| < r\}, 0 < r < 1)$ is just motivated by the sake of simplicity. Indeed, the analysis carried on here could be easily adjusted to deal with the asymptotic case, by simply replacing everywhere in the paper the right monomicity property with the full column rank property in every point $z \in \mathbb{C}$ with $|z| \geq 1$ (with $|z| \geq r$ in the robust case). All the results could be immediately extended to this setting, but the proofs and the details would be a little more tedious.

Also, we would like to underline that the analysis would not change at all if we assumed that all the system trajectories take values on any (possibly finite) field. In this way, the results could be immediately used in other contexts, like convolutional coding (see [18]). In convolutional coding, the dead-beat estimation problem is of higher relevance with respect to asymptotic estimation.

Before entering the main part of this paper, we introduce some notation. We consider here polynomial matrices with entries in $\mathbb{R}[z]$ and, occasionally, Laurent polynomial (L-polynomial, for short) matrices, having entries in $\mathbb{R}[z, z^{-1}]$. A polynomial matrix $M(z) \in \mathbb{R}[z]^{p\times q}$ is *right monomic* [7], [9] if rank

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$M(\lambda) = q$ for every $\lambda \in \mathbb{C} \setminus \{0\}$. This means that $M(z)$ is of full column rank and the GCD of its maximal order minors is a monomial. $M(z)$ is right monomic if and only if it admits a Laurent polynomial left inverse or, equivalently, the diophantine equation $X(z)M(z) = z^N I_q$, in the unknown polynomial matrix $X(z)$, is solvable for some nonnegative integer N.

$M(z) \in \mathbb{R}[z]^{p\times q}$ is *right prime* if rank $M(\lambda) = q$ for every $\lambda \in \mathbb{C}$. *Right prime* matrices are special cases of right monomic matrices. Actually, right primeness characterizations can be obtained by simply replacing in the previous equivalent conditions the word "monomial" with "unit" and the integer N by zero. *Left monomic* and *left prime* matrices are similarly defined and characterized.

The concepts of *left annihilator* and, in particular, of *minimal left annihilator* (MLA, for short) of a given polynomial matrix $M(z)$ have been originally introduced in [16] and can be summarized as follows: If $M(z)$ is a $p \times q$ polynomial matrix of rank r, a polynomial matrix $H(z)$ is a left annihilator of $M(z)$ if $H(z)M(z) = 0$. A left annihilator $H_m(z)$ of $M(z)$ is an MLA if it is of full row rank and for any other left annihilator $H(z)$ of $M(z)$ we have $H(z) = P(z)H_m(z)$ for some polynomial matrix $P(z)$. It can be easily proved that, unless $M(z)$ is of full row rank, an MLA always exists (if $M(z)$ is of full row rank, its left annihilators are zero matrices with an arbitrary number of rows), it is a $(p - r) \times p$ left prime matrix and is uniquely determined modulo a unimodular left factor. *Right annihilators* and *minimal right annihilators* (MRAs) can be similarly defined and enjoy analogous properties.

In the following, for the sake of simplicity, the size of any vector will be denoted by means of the same typewritten letter that is used for denoting the vector itself. In other words, $\mathtt{w}_m := \dim(\mathbf{w}_m)$, $\mathtt{w}_r := \dim(\mathbf{w}_r)$, $\mathtt{u} := \dim(\mathbf{u})$, $\mathtt{x} := \dim(\mathbf{x})$, etc.

II. Basic Results About Behaviors With Trajectories in $(\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}_+}$

In this paper, all trajectories will be assumed defined on the time set $\mathbb{Z}_+$ of nonnegative integers. The left (backward) shift operator on $(\mathbb{R}^{\mathtt{v}})^{\mathbb{Z}_+}$, the set of trajectories defined on $\mathbb{Z}_+$ and taking values in $\mathbb{R}^{\mathtt{v}}$, is defined as

$$\sigma : (\mathbb{R}^{\mathtt{v}})^{\mathbb{Z}_+} \to (\mathbb{R}^{\mathtt{v}})^{\mathbb{Z}_+} : (\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \ldots) \mapsto (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots).$$

If $M(z) = \sum_{i=0}^{L} M_i z^i \in \mathbb{R}[z]^{p\times q}$ is a polynomial matrix, we associate with it the polynomial matrix operator $M(\sigma) = \sum_{i=0}^{L} M_i \sigma^i$. Results about polynomial matrix operators acting on $(\mathbb{R}^q)^{\mathbb{Z}_+}$ can be found in [23], where these results have been derived with (and compared to) those about the more common setup of polynomial matrix operators acting on $(\mathbb{R}^q)^{\mathbb{Z}}$. Further comparisons between these two settings have been later carried on in [18] and in [20], where the few differences between the two settings have been pointed out. In this section, we only recall a few basic results. In particular, it can be proved that $M(\sigma)$ describes an injective map from $(\mathbb{R}^q)^{\mathbb{Z}_+}$ to $(\mathbb{R}^p)^{\mathbb{Z}_+}$ if and only if $M(z)$ is a right prime matrix, and a surjective map if and only if $M(z)$ is of full row rank.

In this paper, by a *behavior* $\mathfrak{B} \subseteq (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}_+}$ we mean the *linear* and *left shift invariant* set of solutions $\mathbf{w} = \{\mathbf{w}(t)\}_{t\in\mathbb{Z}_+}$ of a system of difference equations

$$R_0\mathbf{w}(t) + R_1\mathbf{w}(t+1) + \cdots + R_L\mathbf{w}(t+L) = 0, \; t \in \mathbb{Z}_+ \quad (1)$$

with $R_i \in \mathbb{R}^{p\times \mathtt{w}}$. This system is equivalently described as

$$R(\sigma)\mathbf{w} = 0 \quad (2)$$

where $R(z) := \sum_{i=0}^{L} R_i z^i$ belongs to $\mathbb{R}[z]^{p\times \mathtt{w}}$, and this leads to the short-hand notation $\mathfrak{B} = \ker R(\sigma)$. It has been shown in [23] that $\ker R_1(\sigma) \subseteq \ker R_2(\sigma)$ if and only if $R_2(z) = P(z)R_1(z)$ for some polynomial matrix $P(z)$.

A behavior $\mathfrak{B} = \ker R(\sigma) \subseteq (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}_+}$, with $R(z) \in \mathbb{R}[z]^{p\times \mathtt{w}}$, is said to be *autonomous* if it is a finite dimensional vector subspace of $(\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}_+}$, and this happens if and only if $R(z)$ is of full column rank [21], [23]. Every autonomous behavior in $(\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}_+}$ can be expressed as $\ker R(\sigma)$ for some nonsingular square polynomial matrix $R(z)$. Autonomous behaviors for which there exists some $N \in \mathbb{Z}_+$ such that (s.t.) all their trajectories have (compact) supports included in $[0, N - 1]$ are called *nilpotent autonomous* and they are kernels of polynomial matrix operators $R(\sigma)$ corresponding to right monomic matrices [21]. In particular, if $R(z)$ is nonsingular square, $\ker R(\sigma)$ is nilpotent if and only if $\det R(z) = c \cdot z^N$, for some $c \in \mathbb{R} \setminus \{0\}$ and some $N \in \mathbb{Z}_+$. If an autonomous behavior is not nilpotent, it includes at least one infinite support trajectory. It is worthwhile to remark that when dealing with behaviors defined on $\mathbb{Z}$, nilpotency cannot arise [21]. In fact, the only finite support trajectory of an autonomous behavior defined on $\mathbb{Z}$ is the zero one, and the kernel (on $\mathbb{Z}$) of a monomic matrix coincides with the zero behavior.

A behavior described as $\mathfrak{B} = \ker R(\sigma)$, for some left prime polynomial matrix $R(z) \in \mathbb{R}[z]^{p\times \mathtt{w}}$, also admits an *image representation*. Indeed, for every polynomial matrix $M(z) \in \mathbb{R}[z]^{\mathtt{w}\times m}$ of rank $\mathtt{w} - p$ which is a right annihilator of $R(z)$ (or, equivalently, having $R(z)$ as an MLA), one gets $\ker R(\sigma) = \mathrm{im}\, M(\sigma)$. This type of behaviors is called *controllable* and admits several different characterizations [23], [24]. Our interest here, however, is only in the mathematical relationship between kernel and image representations, which will turn out to be useful in the sequel.

III. Observability and Reconstructibility

Consider a dynamic system $\Sigma = (\mathbb{Z}_+, \mathbb{R}^{\mathtt{w}}, \mathfrak{B})$, whose behavior $\mathfrak{B}$ is described as in (2), for some polynomial matrix $R(z)$. Independently of the physical meaning of the system variables which are grouped together in the vector $\mathbf{w}$, when dealing with any type of estimation problem a first natural distinction is introduced between measured variables, denoted by $\mathbf{w}_m$, and unmeasured variables. These latter, in turn, may be naturally split into the subvector of all system variables which are (unmeasured and) the target of our estimation problem (the "relevant" variables for the specific estimation problem), $\mathbf{w}_r$, and the subvector of all variables which are both unmeasured (for instance because they represent disturbances or modeling errors) and "irrelevant" for our estimation problem. We refer to such a subvector as $\mathbf{w}_i$. As a consequence, the vector $\mathbf{w}$ naturally splits as

$$\mathbf{w}(t) = \begin{bmatrix} \mathbf{w}_r(t) \\ \mathbf{w}_m(t) \\ \mathbf{w}_i(t) \end{bmatrix}.$$

The polynomial matrix $R(z)$ can be accordingly block-partitioned, thus leading to the following description of the behavior trajectories:

$$[\, R_r(\sigma) \quad -R_m(\sigma) \quad -R_i(\sigma)\,] \begin{bmatrix} \mathbf{w}_r(t) \\ \mathbf{w}_m(t) \\ \mathbf{w}_i(t) \end{bmatrix} = 0,\ t \in \mathbb{Z}_+ \quad (3)$$

or, equivalently

$$R_r(\sigma)\mathbf{w}_r(t) = R_m(\sigma)\mathbf{w}_m(t) + R_i(\sigma)\mathbf{w}_i(t),\ t \in \mathbb{Z}_+. \quad (4)$$

With respect to this partition of the system variables, the notions of observability and reconstructibility are easily introduced as follows.

Definition 1: [21], [22] Given a dynamic system $\Sigma = (\mathbb{Z}_+, \mathbb{R}^{\mathtt{w}}, \mathfrak{B})$ whose behavior $\mathfrak{B}$ is described as in (4), we say that $\mathbf{w}_r$ is reconstructible from $\mathbf{w}_m$, if $(\mathbf{w}_r, \mathbf{w}_m, \mathbf{w}_i), (\bar{\mathbf{w}}_r,\ \mathbf{w}_m, \bar{\mathbf{w}}_i) \in \mathfrak{B}$ implies that there exists $N \in \mathbb{Z}_+$ such that $\mathbf{w}_r(t) - \bar{\mathbf{w}}_r(t) = 0, \forall\, t \geq N$. In particular, when $N = 0$, $\mathbf{w}_r$ is said to be observable from $\mathbf{w}_m$.

Σ is said to be reconstructible (observable) if every trajectory $\mathbf{w}_r$ is reconstructible (observable) from the corresponding $\mathbf{w}_m$.

Characterizations of reconstructibility and observability have been obtained in [21]. It is worthwhile to remark that, when a system is reconstructible, a common nonnegative integer N can be found such that all relevant trajectories can be exactly evaluated (from the corresponding measured trajectories) after N steps. So, the index N does not depend on the specific pair $(\mathbf{w}_r, \mathbf{w}_m)$, but represents a system property.

Consider the dynamic system Σ described by (4), with $\mathbf{w}_m$ the measured variable, $\mathbf{w}_r$ the to-be-estimated variable and $\mathbf{w}_i$ the irrelevant one. A DBO of $\mathbf{w}_r$ from $\mathbf{w}_m$ is a system that, corresponding to every trajectory $(\mathbf{w}_r, \mathbf{w}_m, \mathbf{w}_i)$ in $\mathfrak{B}$, produces an estimate $\hat{\mathbf{w}}_r$ of the trajectory $\mathbf{w}_r$ (based on the measured variable $\mathbf{w}_m$ alone), that coincides with the sequence $\mathbf{w}_r$ except, possibly, in a finite number of initial time instants. In particular, a dead-beat observer for Σ which produces an estimate $\hat{\mathbf{w}}_r$ of $\mathbf{w}_r$ which coincides with $\mathbf{w}_r$ at each time instant $t \in \mathbb{Z}_+$ (and hence is not affected by any "estimation error") is an "exact" observer.

Definition 2: [21] Consider the dynamic system Σ, whose behavior $\mathfrak{B}$ is described as in (4). The system represented by the difference equation

$$Q(\sigma)\hat{\mathbf{w}}_r = P(\sigma)\mathbf{w}_m \quad (5)$$

with $P(z)$ and $Q(z)$ polynomial matrices of suitable dimensions, is said to be

- a DBO of $\mathbf{w}_r$ from $\mathbf{w}_m$ for Σ if
  a) for every $(\mathbf{w}_r, \mathbf{w}_m, \mathbf{w}_i) \in \mathfrak{B}$ there exists $\hat{\mathbf{w}}_r$ such that $(\hat{\mathbf{w}}_r, \mathbf{w}_m)$ satisfies (5);
  b) there exists $N \in \mathbb{Z}_+$ such that for every $(\mathbf{w}_r, \mathbf{w}_m, \mathbf{w}_i)$ in $\mathfrak{B}$ and $(\hat{\mathbf{w}}_r, \mathbf{w}_m)$ satisfying (5), we have $\mathbf{w}_r(t) - \hat{\mathbf{w}}_r(t) = 0$ for every $t \geq N$.
- A consistent DBO (cDBO) of $\mathbf{w}_r$ from $\mathbf{w}_m$ for Σ if it is a dead-beat observer and for every $(\mathbf{w}_r, \mathbf{w}_m, \mathbf{w}_i)$ in $\mathfrak{B}$ the trajectory $(\mathbf{w}_r, \mathbf{w}_m)$ always satisfies (5).
- An exact observer (EO) of $\mathbf{w}_r$ from $\mathbf{w}_m$ for Σ if a) holds, and b) holds for $N = 0$.

Remarks:

i) For an observer described by (5), the difference variable $\mathbf{e} := \mathbf{w}_r - \hat{\mathbf{w}}_r$ represents the *estimation error*. So, the previous definitions can be paraphrased by saying that an observer is dead-beat (exact) if the estimation error trajectories belong to a nilpotent autonomous behavior (to the zero behavior).

ii) The concept of consistent DBO may sound somewhat strange and redundant. Simple examples prove that this is not the case. In fact, consider the simple system

$$\begin{bmatrix} \sigma \\ 1 \end{bmatrix} w_r(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} w_m(t). \quad (6)$$

It is easily seen that $w_r(t) = 0, \forall t \geq 1$, and hence $\hat{w}_r(t) = 0, t \geq 0$, represents a DBO for the system. However, it is not consistent, since all the trajectories $(w_r(t), w_m(t))$ which are identically zero for $t \geq 1$, but such that $(w_r(0), w_m(0)) = (a, a), a \neq 0$, belong to the system behavior but do not satisfy the observer equations. As we will see, however, if a DBO exists then also a cDBO may be found. Of course, this distinction does not make sense when dealing with exact observers, which are by definition consistent.

The following theorem provides an extensive characterization of those systems which admit DBOs, thus significantly extending the results obtained in [21] and [22].

Theorem 3: Consider a dynamic system, whose behavior $\mathfrak{B}$ is described as in (4), and let $H_i(z)$ denote an MLA of $R_i(z)$. The following facts are equivalent:

ia) there exists a consistent DBO for Σ;

ib) there exists a DBO for Σ;

ii) $\mathfrak{B}$ is reconstructible.

iii) $\Gamma(z) := H_i(z)R_r(z)$ is right monomic;

iv) there exist $N \in \mathbb{Z}_+$ and a polynomial matrix $L(z)$ s.t.

$$L(z)\,[\, R_r(z) \quad -R_i(z)\,] = [\, z^N I_{\mathtt{w}_r} \quad 0\,]. \quad (7)$$

Proof: ia) $\Rightarrow$ ib) Obvious.

ib) $\Rightarrow$ ii) If $\mathfrak{B}$ were not reconstructible, there would be two trajectories $(\mathbf{w}_r, \mathbf{w}_m, \mathbf{w}_i)$, $(\bar{\mathbf{w}}_r,\ \mathbf{w}_m, \bar{\mathbf{w}}_i)$ in $\mathfrak{B}$ such that $\mathbf{w}_r(t) - \bar{\mathbf{w}}_r(t)$ is an infinite support sequence. If $(\hat{\mathbf{w}}_r, \mathbf{w}_m)$ is any pair satisfying (5), by condition (b) of a DBO, the trajectory $\hat{\mathbf{w}}_r$ should differ in a finite number of time instants both from $\mathbf{w}_r$ and from $\bar{\mathbf{w}}_r$. This is clearly impossible.

ii) $\Rightarrow$ iii) If $\Gamma(z)$ were not right monomic, there would be an infinite support trajectory $\mathbf{w}_r \in \ker \Gamma(\sigma)$. Consequently, by the definition of $\Gamma(z)$ and the relationship between kernel and image representations previously recalled,

$R_r(\sigma)\mathbf{w}_r \in \ker H_i(\sigma) = \mathrm{im}\, R_i(\sigma)$. So, there would be $\mathbf{w}_i$ such that $R_r(\sigma)\mathbf{w}_r = R_i(\sigma)\mathbf{w}_i$. This would imply that both (0, 0, 0) and $(\mathbf{w}_r, 0, \mathbf{w}_i)$ belong to $\mathfrak{B}$, thus contradicting the reconstructibility assumption.

iii) $\Rightarrow$ iv) If $\Gamma(z)$ is right monomic, there exists a polynomial matrix $S(z)$ such that $S(z)\Gamma(z) = z^N I_{\mathtt{w}_r}$ for some $N \in \mathbb{Z}_+$. So, the matrix $L(z) := S(z)H_i(z)$ satisfies (7).

iv) $\Rightarrow$ ia) Let $L(z)$ be a polynomial matrix satisfying (7). We aim to show that by assuming $(Q(z), P(z)) = (L(z)R_r(z), L(z)R_m(z)) = (z^N I_{\mathtt{w}_r}, L(z)R_m(z))$ we get a cDBO. If $(\mathbf{w}_r(t), \mathbf{w}_m(t), \mathbf{w}_i(t))$ is any trajectory in $\mathfrak{B}$, and hence satisfying (4), premultiplication by $L(\sigma)$ leads to

$$\sigma^N \mathbf{w}_r(t) - L(\sigma)R_m(\sigma)\mathbf{w}_m(t) = 0. \tag{8}$$

So, condition a) is satisfied by simply choosing $\hat{\mathbf{w}}_r = \mathbf{w}_r$ (this also ensures consistency). On the other hand, for any other $\hat{\mathbf{w}}_r$ satisfying (8) we have $\sigma^N [\mathbf{w}_r(t) - \hat{\mathbf{w}}_r(t)] = 0$, thus proving condition b). ■

Corollary 4 is easily proved along the same lines of the previous theorem.

Corollary 4: Consider a dynamic system, whose behavior $\mathfrak{B}$ is described as in (4), and let $H_i(z)$ denote an MLA of $R_i(z)$. The following facts are equivalent:

i) there exists an EO for Σ;
ii) $\mathfrak{B}$ is observable;
iii) $\Gamma(z)$ is right prime;
iv) there exists a polynomial matrix $L(z)$ such that $L(z)\,[\,R_r(z) \quad -R_i(z)\,] = [\,I_{\mathtt{w}_r} \quad 0\,]$.

Remark: It is worth enlightening two limit cases of the previous results.

1) When no irrelevant variables are involved in the behavior description (i.e., there is no R_i), then H_i reduces to the identity matrix and hence the existence of a DBO (EO) is equivalent to the right monomicity (primeness) of R_r.
2) When R_i is of full row rank, then H_i is not defined. When so, Theorem 3 (and henceforth Corollary 4) can be read in a negative sense, since none of the equivalent conditions can be satisfied.

IV. A Parametrization of All Dead-Beat (Exact) Observers

Given a DBO for $\mathfrak{B}$, its behavior $\hat{\mathfrak{B}}$ is the set of all solutions $(\hat{\mathbf{w}}_r, \mathbf{w}_m)$ of the difference equation (5). Among all the trajectories of $\hat{\mathfrak{B}}$, however, we are interested only in those produced corresponding to the trajectories of $\mathfrak{B}$, namely in the set $\{(\hat{\mathbf{w}}_r, \mathbf{w}_m) \in \hat{\mathfrak{B}} : \mathbf{w}_m \in \mathcal{P}_m\mathfrak{B}\}$, where $\mathcal{P}_m\mathfrak{B} := \{\mathbf{w}_m : \exists\, \mathbf{w}_r, \mathbf{w}_i \text{ s.t. } (\mathbf{w}_r, \mathbf{w}_m, \mathbf{w}_i) \in \mathfrak{B}\}$. So, by assuming this point of view, it is reasonable to regard as *equivalent* two observers (5), for the same system, not if their behaviors $\hat{\mathfrak{B}}_1$ and $\hat{\mathfrak{B}}_2$ coincide, but if they produce the same estimates corresponding to all measured variable trajectories $\mathbf{w}_m$ of $\mathfrak{B}$, i.e.,

$$\begin{aligned}\{(\hat{\mathbf{w}}_r, \mathbf{w}_m) \in \hat{\mathfrak{B}}_1 : \mathbf{w}_m \in \mathcal{P}_m\mathfrak{B}\}\\ = \{(\hat{\mathbf{w}}_r, \mathbf{w}_m) \in \hat{\mathfrak{B}}_2 : \mathbf{w}_m \in \mathcal{P}_m\mathfrak{B}\}.\end{aligned}$$

Of course, two equivalent observers are either both consistent or both nonconsistent. We can now introduce the following result about equivalent observers.

Lemma 5: [22] If $Q(\sigma)\hat{\mathbf{w}}_r = P(\sigma)\mathbf{w}_m$ is a DBO (an EO) for Σ, there exists an equivalent DBO (EO) $\bar{Q}(\sigma)\hat{\mathbf{w}}_r = \bar{P}(\sigma)\mathbf{w}_m$ with $\bar{Q}$ square monomic (unimodular).

Thanks to this lemma, we may now focus on the parametrization of all those observers whose matrix $Q(z)$ is square monomic. Aiming at this goal, it is convenient to reduce the original behavior description to a more suitable one. Assume, without loss of generality, that the behavior $\mathfrak{B}$ is described as in (3) with $[\,R_r(z) \quad -R_m(z) \quad -R_i(z)\,]$ of full row rank p. If $\mathfrak{B}$ satisfies any of the equivalent conditions of Theorem 3, and we let $S(z)$ be a (left prime) polynomial matrix such that $U(z) = \begin{bmatrix} S(z) \\ H_i(z) \end{bmatrix}$ is unimodular, then $\mathfrak{B}$ can be equivalently described as

$$\begin{bmatrix} S(\sigma)R_r(\sigma) \\ \Gamma(\sigma) \end{bmatrix} \mathbf{w}_r = \begin{bmatrix} S(\sigma)R_m(\sigma) \\ \Phi(\sigma) \end{bmatrix} \mathbf{w}_m + \begin{bmatrix} S(\sigma)R_i(\sigma) \\ 0 \end{bmatrix} \mathbf{w}_i \tag{9}$$

where $S(z)R_i(z)$ is (easily proved to be) of full row rank, $\Gamma(z) = H_i(z)R_r(z)$ and $\Phi(z) := H_i(z)R_m(z)$. If $V(z)$ is a unimodular matrix such that

$$V(z)\Gamma(z) = \begin{bmatrix} \Delta(z) \\ 0 \end{bmatrix}$$

with $\Delta(z)$ square monomic (unimodular), we can conformably partition $V(z)\Phi(z)$ as

$$V(z)\Phi(z) = \begin{bmatrix} L_1(z) \\ L_0(z) \end{bmatrix}.$$

The behavior $\mathfrak{B}$ can then be equivalently described as follows:

$$\begin{bmatrix} S(\sigma)R_r(\sigma) \\ \Delta(\sigma) \\ 0 \end{bmatrix} \mathbf{w}_r = \begin{bmatrix} S(\sigma)R_m(\sigma) \\ L_1(\sigma) \\ L_0(\sigma) \end{bmatrix} \mathbf{w}_m + \begin{bmatrix} S(\sigma)R_i(\sigma) \\ 0 \\ 0 \end{bmatrix} \mathbf{w}_i. \tag{10}$$

Since $S(\sigma)R_i(\sigma)$ defines a surjective map, then

$$\begin{aligned}\mathcal{P}_{r,m}\mathfrak{B} &:= \{(\mathbf{w}_r, \mathbf{w}_m) : \exists\, \mathbf{w}_i \text{ s.t. } (\mathbf{w}_r, \mathbf{w}_m, \mathbf{w}_i) \in \mathfrak{B}\}\\ &= \ker\,[\,\Gamma(\sigma) \quad -\Phi(\sigma)\,] = \ker Z(\sigma)\end{aligned}$$

where

$$Z(z) := \begin{bmatrix} \Delta(z) & -L_1(z) \\ 0 & -L_0(z) \end{bmatrix}.$$

Notice that both $[\,\Gamma(z) \quad -\Phi(z)\,]$ and $Z(z)$ are of full row rank, by the full row rank assumption on the initial system description (3). Once we have singled out $\mathcal{P}_{r,m}\mathfrak{B}$, by keeping in mind that the DBOs (EOs) do not involve $\mathbf{w}_i$, we may resort to [21, Th. 5.4], thus obtaining the following parametrization of all consistent DBOs (EOs)[1].

Theorem 6: [21] Consider a system Σ whose behavior $\mathfrak{B}$ is described as in (10), with $S(z)R_i(z)$ of full row rank and $\Delta(z)$ square monomic (unimodular). If P and Q are polynomial

[1]It is worthwhile to remark that in [21], [22] the possibility of resorting to nonconsistent DBOs had not been contemplated. So, all results and parametrizations appearing there implicitly assume consistency.

matrices, with Q nonsingular square, then $Q(\sigma)\hat{\mathbf{w}}_r = P(\sigma)\mathbf{w}_m$ is a consistent dead-beat (exact) observer for Σ if and only if

$$[\, Q(z) \quad -P(z)\,] = [\, Y(z) \quad X(z)\,]\, Z(z) \tag{11}$$

with $Y(z)$ a monomic (unimodular) polynomial matrix and $X(z)$ a polynomial matrix.

We can now provide an extension of the previous parametrization to the whole class of DBOs, thus including also nonconsistent DBOs.

Theorem 7: Consider a system Σ whose behavior $\mathfrak{B}$ is described as in (10), with $S(z)R_i(z)$ of full row rank and $\Delta(z)$ square monomic. If P and Q are polynomial matrices, with Q nonsingular square, then $Q(\sigma)\hat{\mathbf{w}}_r = P(\sigma)\mathbf{w}_m$ is a DBO for Σ if and only if

$$[\, Q(z) \quad -P(z)\,] = [\, Y(z,z^{-1}) \quad X(z,z^{-1})\,]\, Z(z) \tag{12}$$

with $Y(z,z^{-1})$ and $X(z,z^{-1})$ L-polynomial matrices such that $Q(z) = Y(z,z^{-1})\Delta(z)$ is (square polynomial and) monomic.

Proof: Assume first that the polynomial pair $(Q(z),P(z))$ satisfies (12) and $Q(z)$ is square monomic, and let $(\mathbf{w}_r,\mathbf{w}_m,\mathbf{w}_i)$ be any trajectory in $\mathfrak{B}$. Clearly, $Q(z)$ defines a surjective map and, hence, corresponding to the assigned $\mathbf{w}_m$, there exists $\hat{\mathbf{w}}_r$ such that $Q(\sigma)\hat{\mathbf{w}}_r = P(\sigma)\mathbf{w}_m$. We aim, now, to show that condition b) holds. To this end, let k be a nonnegative integer s.t. $\bar{Y}(z) := z^k \cdot Y(z,z^{-1})$ and $\bar{X}(z) := z^k \cdot X(z,z^{-1})$ are both polynomial matrices. Clearly, any such $\hat{\mathbf{w}}_r$ satisfies the difference equation $\sigma^k Q(\sigma)\hat{\mathbf{w}}_r = \sigma^k P(\sigma)\mathbf{w}_m$, which defines, by Theorem 6, a consistent DBO. Consequently, $\hat{\mathbf{w}}_r$ coincides with $\mathbf{w}_r$ after a finite number of steps.

Conversely, suppose that the polynomial pair $(Q(z),P(z))$ defines a DBO and, according to Definition 2, let N be a nonnegative integer such that $\mathbf{w}_r(t) - \hat{\mathbf{w}}_r(t) = 0, \forall\, t \geq N$, or, equivalently, $\sigma^N(\mathbf{w}_r(t) - \hat{\mathbf{w}}_r(t)) = 0, \forall\, t \geq 0$. Clearly, each trajectory $(\hat{\mathbf{w}}_r,\mathbf{w}_m)$ satisfying $Q(\sigma)\hat{\mathbf{w}}_r = P(\sigma)\mathbf{w}_m$, also satisfies

$$\sigma^N Q(\sigma)\hat{\mathbf{w}}_r = \sigma^N P(\sigma)\mathbf{w}_m \tag{13}$$

thus ensuring $\sigma^N Q(\sigma)\mathbf{w}_r = \sigma^N P(\sigma)\mathbf{w}_m$. So, (13) represents a consistent DBO and this implies, by Theorem 6, that polynomial matrices $\bar{Y}(z)$ and $\bar{X}(z)$ can be found such that

$$[\, z^N \cdot Q(z) \quad -z^N \cdot P(z)\,] = [\, \bar{Y}(z) \quad \bar{X}(z)\,]\, Z(z).$$

Consequently, (12) holds for $Y(z,z^{-1}) = z^{-N} \cdot \bar{Y}(z)$ and $X(z,z^{-1}) = z^{-N} \cdot \bar{X}(z)$. ■

Remarks:

i) Since $Z(z)$ is of full row rank, (12) establishes a bijective correspondence between polynomial pairs $(Q(z),P(z))$ and the corresponding pairs $(Y,X) \in \mathbb{R}[z,z^{-1}]^{\mathtt{w}_r \times \mathtt{w}_r} \times \mathbb{R}[z,z^{-1}]^{\mathtt{w}_r \times \ell}$ in (12), ℓ denoting the number of rows of $L_0(z)$.

ii) An equivalent parametrization of all DBOs can be easily obtained by referring to the behavior description (9). Indeed, the polynomial pair $(Q(z),P(z))$, with Q nonsingular square, defines a DBO (5) for Σ if and only if

$$[\, Q(z) \quad -P(z)\,] = Y(z,z^{-1})\,[\, \Gamma(z) \quad -\Phi(z)\,] \tag{14}$$

with $Y(z,z^{-1})$ an L-polynomial matrix such that $Y(z,z^{-1})\Gamma(z)$ is square polynomial and monomic, while $Y(z,z^{-1})\Phi(z)$ is polynomial. On the other hand, if we are interested in consistent DBOs, the above parametrization is still true, provided that Y is, in addition, strictly polynomial.

Of course, one may wonder when the class of DBOs parametrized in Theorem 7 coincides with the class of cDBOs described in Theorem 6, namely when a behavior $\mathfrak{B}$, described as in (10), admits only consistent DBOs.

Theorem 8: Consider a system Σ whose behavior $\mathfrak{B}$ is described as in (10), with $S(z)R_i(z)$ of full row rank and $\Delta(z)$ square monomic.The following facts are equivalent:

i) the class of DBOs coincides with the class of consistent DBOs;

ii) $Z(0)$ is of full row rank;

iii) $[\, \Gamma(0) \quad -\Phi(0)\,]$ is of full row rank.

Proof: i) $\Rightarrow$ ii) If $Z(0)$ were not of full row rank, (the full row rank matrix) $Z(z)$ could be expressed as $Z(z) = T(z)\bar{Z}(z)$, for some square monomic (but not unimodular) $T(z)$ and some polynomial matrix $\bar{Z}(z)$ such that $\bar{Z}(0)$ is of full row rank. It is a matter of simple calculations to show that we can assume w.l.o.g.

$$\bar{Z}(z) = \begin{bmatrix} \bar{\Delta}(z) & -\bar{L}_1(z) \\ 0 & -\bar{L}_0(z) \end{bmatrix} \text{ and } T(z) = \begin{bmatrix} T_{11}(z) & T_{12}(z) \\ 0 & T_{22}(z) \end{bmatrix}$$

with $T_{11}(z)$, $T_{22}(z)$ and $\bar{\Delta}(z)$ square monomic. If $T_{11}(0)$ is singular, corresponding to the strictly L-polynomial pair

$$\begin{aligned}[\, Y(z,z^{-1}) \quad X(z,z^{-1})\,] \\ = [\, T_{11}(z)^{-1} \quad -T_{11}(z)^{-1}T_{12}(z)T_{22}(z)^{-1}\,]\end{aligned}$$

we get a nonconsistent DBO (12). On the other hand, if $T_{11}(0)$ is nonsingular, then $T_{22}(0)$ is singular. So, a nonconsistent DBO is obtained corresponding to the strictly L-polynomial pair $[\, Y(z,z^{-1}) \quad X(z,z^{-1})\,] = [\, I_{\mathtt{w}_r} \quad -T_{22}(z)^{-1}\,]$.

ii) $\Rightarrow$ i) If $Z(0)$ is of full row rank, it admits a right inverse, say $Z_R(0)$. Then for every L-polynomial pair (Y,X) in $\mathbb{R}[z,z^{-1}]^{\mathtt{w}_r \times \mathtt{w}_r} \times \mathbb{R}[z,z^{-1}]^{\mathtt{w}_r \times \ell}$ s.t. the corresponding pair

$$[\, Q(z) \quad -P(z)\,] = [\, Y(z,z^{-1}) \quad X(z,z^{-1})\,]\, Z(z)$$

is polynomial, with $Q(z)$ monomic, we get

$$[\, Q(0) \quad -P(0)\,]\, Z_R(0) = [\, Y(z,z^{-1}) \quad X(z,z^{-1})\,]|_{z=0}\,.$$

As the left-hand side is finite, so is the right-hand side. Thus, $(Y,X) \in \mathbb{R}[z]^{\mathtt{w}_r \times \mathtt{w}_r} \times \mathbb{R}[z]^{\mathtt{w}_r \times \ell}$.

ii) $\Leftrightarrow$ iii) Obvious. ■

Remarks:

i) For the example described by (6), provided in Section III, it was

$$[\,\Gamma(z) \quad -\Phi(z)\,] = \begin{bmatrix} z & 0 \\ 1 & -1 \end{bmatrix} \Rightarrow [\,\Gamma(0) \quad -\Phi(0)\,] = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}.$$

So, condition iii) of the previous Theorem is not satisfied and in fact, as already seen, the system admits a nonconsistent DBO.

ii) One may wonder why we are interested in nonconsistent DBOs when, under the same conditions, we can always resort to consistent ones. The only reason that may lead to choose this solution is lower complexity. Indeed, by choosing L-polynomial matrices X and Y, instead of polynomial ones, we may reduce the degree of the polynomial matrices Q and P, and this leads to an autoregressive model of lower complexity. This fact is enlightened, for instance, by the simple example (6). Further examples, supporting this claim, will be provided in Section VI.

Theorem 8 shows that, when $Z(0)$ is not of full row rank, cDBOs constitute a proper subclass of DBOs. Even in that case, however, the class of cDBO transfer matrices coincides with the class of DBO transfer matrices, as one may always obtain (cfr. the proof of Theorem 7) from any DBO a consistent DBO endowed with the same transfer matrix. So, the DBO transfer matrices may be parametrized, according to (11) (for instance), as

$$\begin{aligned}\hat{W}(z) &= Q(z)^{-1}P(z) \\ &= \Delta^{-1}(z)L_1(z) + \Delta^{-1}(z)Y^{-1}(z)X(z)L_0(z)\end{aligned}$$

as $Y(z)$ and $X(z)$ vary over the set of all polynomial matrices of suitable sizes (under the constraint that $Y(z)$, and hence $Q(z) = Y(z)\Delta(z)$, is square monomic). Upon setting $\hat{W}_0(z) := \Delta^{-1}(z)L_1(z)$, which can be seen as a "particular" (L-polynomial) transfer matrix, and noting that $T(z,z^{-1}) := \Delta^{-1}(z)\,Y^{-1}(z)X(z)$ is an arbitrary Laurent polynomial matrix[2] (by the monomicity of $Y(z)\Delta(z)$), the previous parametrization becomes

$$\hat{W}(z) = \hat{W}_0(z) + T(z,z^{-1})L_0(z),\ \ T \in \mathbb{R}[z,z^{-1}]^{\mathtt{w}_r \times \ell}. \quad (15)$$

Notice that $\hat{W}(z)$ is always an L-polynomial matrix. Similarly, if we refer to the DBO parametrization (14) and assume w.l.o.g. that the matrix Y appearing in (14) is polynomial, we obtain the following parametrization of the DBO transfer matrices:

$$\hat{W}(z) = [Y(z)\Gamma(z)]^{-1}\,[Y(z)\Phi(z)] \quad (16)$$

with $Y(z)$ a polynomial matrix such that $Y(z)\Gamma(z)$ is square and monomic.

V. Causal Dead-Beat Observers

If the task we have in mind is simply that of obtaining a "behavioral approach" to the solution of various types of estimation problems and a parametric (kernel or transfer matrix) description of all available solutions, the results of the previous sections already provide satisfactory answers. If we aim at applying the previous general results to the state–space setting, however, it is extremely important to investigate the existence of a DBO which admits a state–space realization. This requires the observer L-polynomial transfer matrix $\hat{W}(z) := Q^{-1}(z)P(z)$ to be proper and this is the case if and only if it is a polynomial matrix in the negative powers of z (i.e., an F.I.R. filter). If we assume the behavior description (9), $\mathcal{P}_{r,m}\mathfrak{B}$ is described by

$$\Gamma(\sigma)\mathbf{w}_r = \Phi(\sigma)\mathbf{w}_m \quad (17)$$

and we may resort to the parametrization of the observer transfer matrices given in (16), where $Y(z)$ is any polynomial matrix such that $Y(z)\Gamma(z)$ is square and monomic.

The characterization of those behaviors which admit a (w.l.o.g. consistent) DBO endowed with a proper transfer matrix, obtained in [22], can be easily adjusted to the case when irrelevant variables are involved in the behavior description, thus leading to the following result.

Theorem 9: [22] Consider a dynamic system Σ with behavior $\mathfrak{B}$ described as in (9), with $\mathbf{w}_r$ reconstructible from $\mathbf{w}_m$. Suppose without loss of generality, that

$$[\,\Gamma(z) \quad -\Phi(z)\,] \in \mathbb{R}[z]^{(\mathtt{w}_r+\ell)\times(\mathtt{w}_r+\mathtt{w}_m)} \quad (18)$$

is row reduced [14] with row degrees $\mu_1, \mu_2, \ldots, \mu_{\mathtt{w}_r+\ell}$, so that

$$[\,\Gamma(z) \quad -\Phi(z)\,] = \begin{bmatrix} z^{\mu_1} & & & \\ & z^{\mu_2} & & \\ & & \ddots & \\ & & & z^{\mu_{\mathtt{w}_r+\ell}} \end{bmatrix} [\,\Gamma_{hr} \quad -\Phi_{hr}\,] + [\,\Gamma_{lr}(z) \quad -\Phi_{lr}(z)\,] \quad (19)$$

where $[\,\Gamma_{hr} \quad -\Phi_{hr}\,]$ is a full row rank constant matrix and $[\,\Gamma_{lr}(z) \quad -\Phi_{lr}(z)\,]$ is a polynomial matrix whose entries in the ith row have degrees smaller than μ_i, $i = 1, 2, \ldots, \mathtt{w}_r + \ell$. A necessary and sufficient condition for the existence of a consistent DBO endowed with a proper transfer matrix $\hat{W}(z)$ is that Γ_{hr} is of full column rank.

Remark: It is worthwhile remarking (see [22] for the details) that the assumption that the polynomial matrix (18) is row reduced plays a role only in the necessity part of the proof of the previous theorem. Actually, if we start with a representation corresponding to a polynomial matrix (18) which is not row reduced, but Γ_{hr} is of full column rank, then a causal DBO exists. Notice that since the proof is a constructive one, it is easy to explicitly obtain such a DBO. Clearly, if Γ_{hr} is not of full column rank in a row reduced description, it cannot exhibit this property in any other representation.

VI. Applications to State–Space Models

In this section, we will show how the observer theory, here developed within the behavioral approach, allows to treat in a homogeneous way several classical estimation problems for state–space systems. To this end we will consider the most general expression of a state–space model (in a deterministic setting), including not only the usual state, input and output

[2]Indeed, if Δ is monomic and T is an arbitrary L-polynomial matrix, then we can always find polynomial matrices X and Y, with Y square monomic, such that $\Delta(z)T(z,z^{-1}) = Y^{-1}(z)X(z)$. Consequently, the corresponding Q and P are polynomial matrices with Q square monomic. The converse is obvious.

variables, but also disturbances and additive faults. Additive faults are typically adopted in the literature for modeling abrupt changes in the system functioning, like changes in the entries of the system matrices, sensor and/or actuator failures, etc. [2]–[5]. Once we will cast the state–space model in the behavioral framework, by differently choosing the measured, the relevant and the irrelevant variables, we will be able to formalize the following traditional problems:

1) the state estimation when neither disturbances nor faults affect the system;
2) the state estimation when only disturbances affect the system. This leads to the well-known concept of *unknown input observer* (UIO);
3) the fault detection and isolation when no disturbance affects the system (but faults, of course, do) (FDI);
4) the fault detection and isolation in the presence of disturbances (dFDI).

A general state–space model is described by the following equations:

$$\mathbf{x}(t+1) = A\mathbf{x}(t) + B_u\mathbf{u}(t) + B_d\mathbf{d}(t) + B_f\mathbf{f}(t) \quad (20)$$

$$\mathbf{y}(t) = C\mathbf{x}(t) + D_u\mathbf{u}(t) + D_d\mathbf{d}(t) + D_f\mathbf{f}(t) \quad (21)$$

where $t \in \mathbb{Z}_+$, $\mathbf{x}$ denotes the state, $\mathbf{u}$ the controlled input, $\mathbf{y}$ the measured output, $\mathbf{d}$ the disturbance (i.e., the uncontrollable input) and $\mathbf{f}$ the fault. The state–space model (20)–(21) can be rewritten in behavioral form as

$$\begin{bmatrix} \sigma I_{\mathrm{x}} - A & 0 & -B_u & -B_d & -B_f \\ C & -I_{\mathrm{y}} & D_u & D_d & D_f \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{y}(t) \\ \mathbf{u}(t) \\ \mathbf{d}(t) \\ \mathbf{f}(t) \end{bmatrix} = 0 \quad (22)$$

with $t \in \mathbb{Z}_+$. It is worthwhile to remark that the polynomial matrix in (22) is always of full row rank.

Before proceeding, an algorithm for obtaining a DBO (an EO), possibly described by means of a standard state–space model, may be fruitfully sketched:

1) Check whether $\Gamma(z)$ is right monomic (right prime). If not, a DBO (an EO) is not available.

2) If the answer is positive, put the polynomial matrix (18) in row reduced form and evaluate the column rank of Γ_{hr}.

3) If Γ_{hr} is of full column rank, the transfer matrix of a causal DBO (EO) can be obtained (see [22]), and this transfer function can be realized by means of a finite memory system of the form

$$\mathbf{v}(t+1) = F\mathbf{v}(t) + G\mathbf{w}_m(t), \;\; \hat{\mathbf{w}}_r(t) = H\mathbf{v}(t) + J\mathbf{w}_m(t).$$

4) When causal DBOs are not available, by resorting to the parametrization of the DBO transfer matrices given in (15), we can obtain some transfer matrix $\hat{W}(z) = z^i \cdot \tilde{W}(z^{-1})$, with $\tilde{W}(z^{-1})$ a polynomial matrix in the variable z^{-1} and i a positive integer. By realizing $\tilde{W}(z^{-1})$ by means of a state–space model, we obtain a "delayed" DBO, as the DBO output is $\hat{\mathbf{w}}_r(t-i)$, instead of $\hat{\mathbf{w}}_r(t)$. In other words, the estimation is performed with a fixed delay of i steps.

A. Standard State Estimation

If neither faults $\mathbf{f}$ nor disturbances $\mathbf{d}$ affect the system, we are reduced to the case of plain state estimation from the controlled input and the measured output. When so, the relevant variable is $\mathbf{w}_r = \mathbf{x}$, the available measurements are $\mathbf{w}_m = \begin{bmatrix}\mathbf{y}^T \; \mathbf{u}^T\end{bmatrix}^T$, and there are no irrelevant variables $\mathbf{w}_i$. The behavioral equation takes the form

$$[\, R_r(\sigma) \quad -R_m(\sigma) \,] \begin{bmatrix} \mathbf{x} \\ - \\ \mathbf{y} \\ \mathbf{u} \end{bmatrix} =$$

$$\left[\begin{array}{c|cc} \sigma I_{\mathrm{x}} - A & 0 & -B_u \\ C & -I_{\mathrm{y}} & D_u \end{array}\right] \begin{bmatrix} \mathbf{x} \\ - \\ \mathbf{y} \\ \mathbf{u} \end{bmatrix} = 0. \quad (23)$$

In this case, there is no $R_i(z)$ and hence $H_i(z) = I_{\mathrm{x+y}}$, while $R_r(z) \equiv \mathcal{O}(z)$, the PBH observability matrix. So, reconstructibility (observability), and hence the existence of a dead-beat (an exact) state observer, corresponds to the right monomicity (primeness) of $\mathcal{O}(z)$, a well-known result [14], [15], [17]. When so, both causal and noncausal DBOs (EOs) can be constructed. Indeed, the polynomial matrix $[\, \Gamma(z) \quad -\Phi(z) \,] = [\, R_r(z) \quad -R_m(z) \,]$ is row reduced and the constant matrix $\Gamma_{hr} = \begin{bmatrix} I_{\mathrm{x}} \\ C \end{bmatrix}$ is of full column rank. Consequently, DBOs endowed with a proper transfer matrix always exist. A subclass of all cDBOs endowed with a proper transfer matrix is represented by Luenberger (full-order) observers, which are obtained by assuming in the parametrization (14) $Y(z) = [\, I_{\mathrm{x}} \quad -L \,]$ for some suitable L such that $A + LC$ is nilpotent (equivalently, $zI_{\mathrm{x}} - A - LC$ is square monomic).

We may wonder whether nonconsistent DBOs exist. Since $[\, \Gamma(0) \quad -\Phi(0) \,]$ is of full row rank if and only if $[\, A \quad B \,]$ is, nonconsistent DBOs exist if and only if the state $\mathbf{x}$ can be partitioned (possibly after a change of basis) as $\mathbf{x} = [\, \mathbf{x}_1^T \quad \mathbf{x}_2^T \,]^T$, where the evolution of the first subvector $\mathbf{x}_1$ is independent of $\mathbf{u}$ and it vanishes in a finite number of steps. Indeed, in this case, the choice $\hat{\mathbf{x}}_1(t) = 0$, together with a DBO for $\mathbf{x}_2(t)$ alone, allow to implement a nonconsistent DBO of lower complexity w.r.t. the complexity of any consistent DBO. In particular, when A is a nilpotent matrix and $B = 0$, $\hat{\mathbf{x}}(t) = 0$ represents a (static) nonconsistent DBO of minimal complexity (see Remarks in Section IV). Clearly, this result finds no counterpart in the classical Luenberger observer design.

Example 1: Consider a state–space model (23) with $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, $C = [\, 0 \quad 1 \,]$, and assume that no controlled input acts on the system. As $H_i(z) = I_3$, it follows that

$$[\, \Gamma(z) \quad -\Phi(z) \,] = [\, R_r(z) \quad -R_m(z) \,] = \left[\begin{array}{c|c} zI_2 - A & 0 \\ C & I_1 \end{array}\right]$$

$$= \left[\begin{array}{cc|c} z & -1 & 0 \\ -1 & z-1 & 0 \\ 0 & 1 & 1 \end{array}\right].$$

By applying the unimodular matrix

$$U(z) = \begin{bmatrix} 0 & -1 & z-1 \\ 0 & 0 & 1 \\ 1 & z & 1+z-z^2 \end{bmatrix}$$

one may obtain the behavior description (10) with

$$\Delta(z)=\begin{bmatrix}1&0\\0&1\end{bmatrix}, L_1(z)=\begin{bmatrix}z-1\\1\end{bmatrix}, L_0(z)=1+z-z^2.$$

Notice that the constraint $L_0(\sigma)\mathbf{y}(t)=0$, namely $\mathbf{y}(t+2)-\mathbf{y}(t+1)-\mathbf{y}(t)=0$, $t\geq 0$, is just the autoregressive equation satisfied by the free output evolution. The DBO transfer matrix parametrization leads to

$$\hat{W}(z)=\begin{bmatrix}z-1\\1\end{bmatrix}+\begin{bmatrix}p(z,z^{-1})\\q(z,z^{-1})\end{bmatrix}[1+z-z^2]$$

where $p(z,z^{-1})$ and $q(z,z^{-1})$ are arbitrary Laurent polynomials. The causality condition is satisfied (as it may be seen by direct inspection) if and only if $p(z,z^{-1})=z^{-1}[1+z^{-1}\bar{p}(z^{-1})]$ and $q(z,z^{-1})=z^{-2}\bar{q}(z^{-1})$, with $\bar{p}$, $\bar{q}$ arbitrary polynomials in the variable z^{-1} alone. As interesting special cases, it is worth mentioning the following.

1) When $(p,q)=(0,0)$, then $\hat{W}(z)=\begin{bmatrix}z-1\\1\end{bmatrix}$. Correspondingly, we obtain the noncausal EO $\hat{\mathbf{x}}(t)=\begin{bmatrix}\mathbf{y}(t+1)-\mathbf{y}(t)\\\mathbf{y}(t)\end{bmatrix}$;
2) When $(p,q)=(z^{-1},0)$, then $\hat{W}(z)=\begin{bmatrix}z^{-1}\\1\end{bmatrix}$. Correspondingly, we obtain the causal DBO (coinciding with the classical reduced order dead-beat observer) $\hat{\mathbf{x}}(t+1)=\begin{bmatrix}\mathbf{y}(t)\\\mathbf{y}(t+1)\end{bmatrix}$.
3) When $(p,q)=(z^{-1},z^{-2})$, then $\hat{W}(z)=\begin{bmatrix}z^{-1}\\z^{-1}+z^{-2}\end{bmatrix}$. Correspondingly, we obtain the causal DBO (coinciding with the classical Luenberger DBO of gain matrix $L^T=[-1\quad -1]$) described by $\hat{\mathbf{x}}(t+2)=\begin{bmatrix}\mathbf{y}(t+1)\\\mathbf{y}(t+1)+\mathbf{y}(t)\end{bmatrix}$.

B. *Unknown Input Observers (UIOs)*

When faults $\mathbf{f}$ are not contemplated, but disturbances $\mathbf{d}$ affect the system dynamics, we are reduced to the problem of designing an UIO: the relevant variable is $\mathbf{w}_r=\mathbf{x}$, while the available measurements are $\mathbf{w}_m=\begin{bmatrix}\mathbf{y}^T\ \mathbf{u}^T\end{bmatrix}^T$. The irrelevant variables are of course represented by the disturbances $\mathbf{w}_i=\mathbf{d}$. The behavioral equations can be block-partitioned in the following form:

$$[R_r(\sigma)\quad -R_m(\sigma)\quad -R_i(\sigma)]\begin{bmatrix}\mathbf{x}\\-\\\mathbf{y}\\\mathbf{u}\\-\\\mathbf{d}\end{bmatrix}=$$

$$\left[\begin{array}{c|cc|c}\sigma I_{\mathtt{x}}-A & 0 & -B_u & -B_d\\ C & -I_{\mathtt{y}} & D_u & D_d\end{array}\right]\begin{bmatrix}\mathbf{x}\\-\\\mathbf{y}\\\mathbf{u}\\-\\\mathbf{d}\end{bmatrix}=0. \quad (24)$$

Upon introducing an MLA of $R_i(z)=\begin{bmatrix}B_d\\-D_d\end{bmatrix}$, which can always be assumed to be a constant matrix so that $H_i(z)=[H_{iB}\quad H_{iD}]$, a dead-beat (an exact) UIO exists if and only if the polynomial matrix $[H_{iB}\quad H_{iD}]R_r(z)=H_{iB}(zI_{\mathtt{x}}-A)+H_{iD}C=:\Gamma_x(z)$ is right monomic (prime). In this case

$$[\Gamma(z)\quad -\Phi(z)]=[H_{iB}\quad H_{iD}]\left[\begin{array}{c|cc}zI_{\mathtt{x}}-A & 0 & -B_u\\ C & -I_{\mathtt{y}} & D_u\end{array}\right]$$

is not necessarily row reduced. Moreover, causal (dead-beat or exact) UIOs may not exist, as shown in the following example.

Example 2: Consider a state–space model (24) with

$$A=\begin{bmatrix}0&0\\1&0\end{bmatrix}, C=\begin{bmatrix}0&0\\0&1\end{bmatrix}, B_d=\begin{bmatrix}1&0\\0&1\end{bmatrix}, D_d=\begin{bmatrix}0&-1\\0&0\end{bmatrix}$$

which represents an observable system devoid of controlled inputs but affected by disturbances. In this case

$$R_r(z)=\begin{bmatrix}zI_2-A\\C\end{bmatrix}=\begin{bmatrix}z&0\\-1&z\\0&0\\0&1\end{bmatrix}, R_m(z)=\begin{bmatrix}0\\I_2\end{bmatrix}=\begin{bmatrix}0&0\\0&0\\1&0\\0&1\end{bmatrix}$$

$$R_i(z)=\begin{bmatrix}B_d\\-D_d\end{bmatrix}=\begin{bmatrix}1&0\\0&1\\0&1\\0&0\end{bmatrix}, H_i(z)=\begin{bmatrix}0&1&-1&0\\0&0&0&1\end{bmatrix}.$$

Since $\Gamma(z)=\begin{bmatrix}-1&z\\0&1\end{bmatrix}$ is unimodular, $\Delta(z)=\Gamma(z)$, $L_1(z)=\Phi(z)=\begin{bmatrix}-1&0\\0&1\end{bmatrix}$, while $L_0(z)$ does not exist. The DBO transfer matrix $\hat{W}(z)=\Delta^{-1}(z)L_1(z)=\begin{bmatrix}1&z\\0&1\end{bmatrix}$ is uniquely determined and is not a proper rational matrix, so a corresponding DBO $\hat{\mathbf{x}}(t)=\begin{bmatrix}1&\sigma\\0&1\end{bmatrix}\mathbf{y}(t)$ is a noncausal EO (actually, the only one available).

A comparison with the well-known Hautus characterization [12] (see also [13], [19], where equivalent conditions were obtained by means of slightly different techniques) of the existence of UIOs for a given state–space model, suitably adjusted for the dead-beat case, seems now appropriate. In [12], it was proved that a dead-beat UIO, described by a proper state–space model, exists if and only if

1) rank $\begin{bmatrix}D_d&CB_d\\0&D_d\end{bmatrix}=$ rank D_d+ rank $\begin{bmatrix}-B_d\\D_d\end{bmatrix}$, and
2) rank $\begin{bmatrix}zI_{\mathtt{x}}-A&-B_d\\C&D_d\end{bmatrix}=n+$ rank $\begin{bmatrix}-B_d\\D_d\end{bmatrix}$ for every $z\in\mathbb{C}$, $z\neq 0$.

Since $\begin{bmatrix}-B_d\\D_d\end{bmatrix}$ may be assumed of full column rank, w.l.o.g., the previous conditions become

a) rank $\begin{bmatrix}D_d&CB_d\\0&D_d\end{bmatrix}=$ rank $D_d+\mathtt{d}$, and
b) $\begin{bmatrix}zI_{\mathtt{x}}-A&-B_d\\C&D_d\end{bmatrix}$ is of full column rank for every $z\in\mathbb{C}$, $z\neq 0$.

Let U be a nonsingular square matrix such that $U\begin{bmatrix}-B_d\\ D_d\end{bmatrix} = \begin{bmatrix}I_{\mathrm{d}}\\ 0\end{bmatrix}$. Then, we may always assume $U = \begin{bmatrix}U_1 & U_2\\ H_{iB} & H_{iD}\end{bmatrix}$ and correspondingly get

$$U\begin{bmatrix}zI_{\mathrm{x}} - A & -B_d\\ C & D_d\end{bmatrix} = \begin{bmatrix}U_1(zI_{\mathrm{x}} - A) + U_2C & I_{\mathrm{d}}\\ \Gamma_x(z) & 0\end{bmatrix}$$

which clearly enlightens that condition b) holds if and only if $\Gamma_x(z)$ is right monomic.

Similarly, but we skip the technical details which are rather long, it is possible to prove that condition a) is equivalent to the causality property of the dead-beat UIO and hence to the condition for the existence of proper UIOs derived in Section V. Indeed, it can be proved that the polynomial matrix $[\Gamma_x(z) \quad -\Phi(z)]$ thus obtained is row reduced, and condition a) holds if and only if the corresponding Γ_{hr} is of full column rank.

Another interesting problem, even though less explored in the literature, is that of obtaining estimates both for the state and for the disturbance: In this case, the relevant variable is $\mathbf{w}_r = [\mathbf{x}^T\ \mathbf{d}^T]^T$, the measured variable is $\mathbf{w}_m = \begin{bmatrix}\mathbf{y}^T\ \mathbf{u}^T\end{bmatrix}^T$ and no irrelevant variables are involved in the system description. This situation coincides, as a matter of fact, with the first FDI problem analyzed in Section VI-C provided that the disturbance $\mathbf{d}(t)$ is regarded as a fault.

C. Fault Detection and Isolation (FDI)

Suppose, firstly, that disturbances $\mathbf{d}$ may be neglected. When so, we may face to two interesting problems: the first problem is the design of an *observer-based FDI*, which corresponds to assuming as relevant variables both $\mathbf{x}$ and $\mathbf{f}$, i.e., $\mathbf{w}_r = \begin{bmatrix}\mathbf{x}^T\ \mathbf{f}^T\end{bmatrix}^T$, while using as measurements $\mathbf{w}_m = \begin{bmatrix}\mathbf{y}^T\ \mathbf{u}^T\end{bmatrix}^T$. If so, no irrelevant variables appear in the system description and $H_i(z) = I_{\mathrm{x+y}}$. The behavioral description can be block-partitioned as follows:

$$[R_r(\sigma) \quad -R_m(\sigma)]\begin{bmatrix}\mathbf{x}\\ \mathbf{f}\\ -\\ \mathbf{y}\\ \mathbf{u}\end{bmatrix} =$$

$$\left[\begin{array}{cc|cc}\sigma I_{\mathrm{x}} - A & -B_f & 0 & -B_u\\ C & D_f & -I_{\mathrm{y}} & D_u\end{array}\right]\begin{bmatrix}\mathbf{x}\\ \mathbf{f}\\ -\\ \mathbf{y}\\ \mathbf{u}\end{bmatrix} = 0$$

and a dead-beat (exact) FDI exists if and only if the system matrix [17]

$$R_r(z) = \begin{bmatrix}zI_{\mathrm{x}} - A & -B_f\\ C & D_f\end{bmatrix} =: \Gamma_{x,f}(z)$$

is right monomic (prime).

The second problem one may want to address is the design of an FDI which allows to estimate just the faults, disregarding the state evolution (*standard FDI*). In this case $\mathbf{f}$ becomes the only relevant variable $\mathbf{w}_r$, while $\mathbf{x}$ becomes the irrelevant variable $\mathbf{w}_i$

$$[R_r(\sigma) \quad -R_m(\sigma) \quad -R_i(\sigma)]\begin{bmatrix}\mathbf{f}\\ -\\ \mathbf{y}\\ \mathbf{u}\\ -\\ \mathbf{x}\end{bmatrix} =$$

$$\left[\begin{array}{c|cc|c}-B_f & 0 & -B_u & \sigma I_{\mathrm{x}} - A\\ D_f & -I_{\mathrm{y}} & D_u & C\end{array}\right]\begin{bmatrix}\mathbf{f}\\ -\\ \mathbf{y}\\ \mathbf{u}\\ -\\ \mathbf{x}\end{bmatrix} = 0.$$

Now, $R_i(z)$ is just the PBH observability matrix and once we select any left coprime matrix fraction description $D_L(z)^{-1}N_L(z)$ of the state to output transfer matrix $C(zI_{\mathrm{x}} - A)^{-1}$, we get $H_i(z) = [-N_L(z) \quad D_L(z)]$ as an MLA of $R_i(z)$. Consequently, a dead-beat (exact) FDI exists if and only if $H_i(z)\begin{bmatrix}-B_f\\ D_f\end{bmatrix} = N_L(z)B_f + D_L(z)D_f =: \Gamma_f(z)$ is right monomic (prime).

This characterization may be easily compared with the analogous one derived in [4]. Actually, by suitably tailoring the result of [4] to the dead-beat discrete-time case, we can say that Ding and Frank proved that a causal dead-beat FDI exists if and only if the fault-to-output transfer matrix

$$\begin{aligned}G_f(z) &= C(zI_{\mathrm{x}} - A)^{-1}B_f + D_f\\ &= D_L^{-1}(z)\,[D_L(z)B_f + N_L(z)D_f]\end{aligned}$$

admits a left inverse which is an FIR filter (and, hence, is described by a polynomial matrix in the negative powers of z). This condition, however, pertains the forced evolution alone, while disregarding the free evolution. As a consequence, the condition obtained by Ding and Frank works effectively only when the original system is of finite memory. By explicitly introducing this assumption, it may be proved that the two conditions we derived for the existence of a dead-beat FDI, realized by a proper state–space model, are more powerful, since they show that such an FDI exists even when the conditions derived in [4] are not satisfied.

D. Fault Detection and Isolation in Presence of Disturbances (dFDI)

Similarly to the previous subsection, two different FDI problems in the presence of disturbances may be considered: one may be interested in estimating both $\mathbf{x}$ and $\mathbf{f}$ (*observer-based dFDI* problem), i.e., $\mathbf{w}_r = \begin{bmatrix}\mathbf{x}^T\ \mathbf{f}^T\end{bmatrix}^T$, making use of the measurements $\mathbf{w}_m = \begin{bmatrix}\mathbf{y}^T\ \mathbf{u}^T\end{bmatrix}^T$, and disregarding $\mathbf{w}_i = \mathbf{d}$. When so, the behavioral equation takes the form

$$[R_r(\sigma) \quad -R_m(\sigma) \quad -R_i(\sigma)]\begin{bmatrix}\mathbf{x}\\ \mathbf{f}\\ -\\ \mathbf{y}\\ \mathbf{u}\\ -\\ \mathbf{d}\end{bmatrix} =$$

$$\left[\begin{array}{cc|cc|c}\sigma I_{\mathrm{x}} - A & -B_f & 0 & -B_u & -B_d\\ C & D_f & -I_{\mathrm{y}} & D_u & D_d\end{array}\right]\begin{bmatrix}\mathbf{x}\\ \mathbf{f}\\ -\\ \mathbf{y}\\ \mathbf{u}\\ -\\ \mathbf{d}\end{bmatrix} = 0.$$

Upon denoting by $H_i(z) = [\,H_{iB} \quad H_{iD}\,]$ (a constant matrix) an MLA of $R_i(z)$, the existence of an observer-based FDI which produces exact estimates of both the state and the fault after a finite number of steps (after 0 steps) corresponds to the right monomicity (primeness) of

$$[\,H_{iB} \quad H_{iD}\,]\begin{bmatrix} zI_{\mathrm{x}} - A & -B_f \\ C & D_f \end{bmatrix} =: \Gamma_{x,f}(z).$$

The other case corresponds to the problem of estimating the faults, from the input and output measurements, by neglecting the state dynamics and the disturbances (*standard dFDI* problem). In this case $\mathbf{w}_r = \mathbf{f}$, $\mathbf{w}_m = \left[\mathbf{y}^T\ \mathbf{u}^T\right]^T$ and $\mathbf{w}_i = \left[\mathbf{x}^T\ \mathbf{d}^T\right]^T$. Consequently

$$[\,R_r(\sigma) \quad -R_m(\sigma) \quad -R_i(\sigma)\,]\begin{bmatrix} \mathbf{f} \\ - \\ \mathbf{y} \\ \mathbf{u} \\ - \\ \mathbf{x} \\ \mathbf{d} \end{bmatrix} =$$

$$\left[\begin{array}{c|cc|cc} -B_f & 0 & -B_u & \sigma I_{\mathrm{x}} - A & -B_d \\ D_f & -I_{\mathrm{y}} & D_u & C & D_d \end{array}\right]\begin{bmatrix} \mathbf{f} \\ - \\ \mathbf{y} \\ \mathbf{u} \\ - \\ \mathbf{x} \\ \mathbf{d} \end{bmatrix} = 0.$$

The polynomial matrix $H_i(z)$ represents, in this case, an MLA of the system matrix $-R_i(z)$, and the existence of a (nonobserver based) dead-beat (exact) FDI in the presence of disturbances is equivalent to the right monomicity (primeness) of $H_i(z)R_r(z) =: \Gamma_f(z)$.

The results of this section may be easily compared with those obtained in [5], [10]. The parity relation approach and the factorization approach lead to quite similar results as far as (non observer-based) dFDI is concerned. The main differences relying in the following facts: a) all conditions are expressed in terms of the rational transfer matrices from the disturbance and from the fault to the system output, $G_d(z)$ and $G_f(z)$, respectively; and b) such conditions are not translated, as we did in this paper, into a single algebraic condition to be tested, but always reduce to "check the existence of an L-polynomial matrix $R(z)$ such that $R(z)G_d(z) = 0$ and $R(z)G_f(z)$ is a square matrix in the negative powers of z." Again, in [5], [10] the free evolution is not explicitly addressed, so the obtained conditions may work only when the original system is of finite memory. Also, in this case, it can be shown that the conditions we derived are less restrictive and hence more powerful.

In order to better enlighten various aspects of the FDI and dFDI problems (both in their observer-based and in their standard versions), which can be obtained in this behavioral framework, let us consider the following concluding example.

Example 3: Consider a state–space model (22) with $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $C = [0 \quad 1]$, $B_d = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $D_d = [0]$, $B_f = \begin{bmatrix} 0 \\ a \end{bmatrix}$, and $D_f = [1 - a]$, $a \in \mathbb{R}$, and assume that no controlled input acts on the system. Let us firstly consider the case when disturbances may be neglected (and hence there are no B_d and D_d). For determining whether an observer-based FDI exists, we evaluate

$$\Gamma_{x,f}(z) = \begin{bmatrix} z & -1 & 0 \\ 0 & z & -a \\ 0 & 1 & 1-a \end{bmatrix}, \ \Phi(z) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and since $\det \Gamma_{x,f}(z) = z[z(1-a) + a]$, $\Gamma_{x,f}(z)$ is monomic (and, hence, the problem is solvable) if and only if $a = 0$ or $a = 1$. Notice, however, that for $a = 0, 1$, $\Gamma_{x,f}$ is square monomic but not unimodular, and hence EOs are not available. Also, $\Delta(z) = \Gamma_{x,f}(z)$, $L_1(z) = \Phi(z)$, $L_0(z) = \emptyset$. So, the DBO transfer matrix is uniquely determined as $\hat{W}(z) = \Gamma_{x,f}^{-1}(z)\Phi(z)$.

If $a = 0$, then $\hat{W}(z) = [0 \quad 0 \quad 1]^T$, which corresponds to $\hat{\mathbf{x}}(t) = 0$, $\hat{\mathbf{f}}(t) = \mathbf{y}(t)$. This is a causal DBO, and in fact $[\,\Gamma_{x,f}(z) \quad -\Phi(z)\,]$ is row reduced, with $\Gamma_{hr} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ of full column rank. On the other hand, for $a = 1$, $\hat{W}(z) = [z^{-1} \quad 1 \quad z]^T$, $\hat{\mathbf{x}}(t) = \begin{bmatrix} \mathbf{y}(t-1) \\ \mathbf{y}(t) \end{bmatrix}$, $\hat{\mathbf{f}}(t) = \mathbf{y}(t+1)$, which represents a noncausal DBO, in agreement with the fact that $\Gamma_{hr} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ is now not of full column rank.

If we are interested in estimating the fault $\mathbf{f}$ alone (namely, we search for a standard FDI), we can choose as a left coprime matrix fraction description $D_L(z)^{-1}N_L(z)$ of $C(zI_{\mathrm{x}} - A)^{-1}$ the one associated with $D_L(z) = [\,z\,]$ and $N_L(z) = [0 \quad 1]$. Consequently, $[\,\Gamma_f(z) \quad -\Phi(z)\,] = [\,z(1-a) + a \quad -z\,]$. As earlier, a necessary condition for the problem solvability is that the real parameter a takes only the values 0 or 1.

If $a = 0$, then $\hat{W}(z) = 1$, i.e., $\hat{\mathbf{f}}(t) = \mathbf{y}(t)$, which represents a causal DBO (but not an EO). In fact, $\Gamma_{hr} = [1]$ is trivially of full column rank. On the other hand, if $a = 1$, then $\hat{W}(z) = z$ and $\hat{\mathbf{f}}(t) = \mathbf{y}(t+1)$, which is a noncausal EO (indeed, in this case, $\Gamma_{hr} = [0]$).

In this specific example, therefore, estimating $(\mathbf{x}, \mathbf{f})$ or $\mathbf{f}$ alone lead to the same result for $\hat{\mathbf{f}}(t)$, but in general the case can occur that $(\mathbf{x}, \mathbf{f})$ cannot be estimated (for instance, if the pair (A, C) does not correspond to a reconstructible system) while $\mathbf{f}$ can (see, also, the example regarding dFDI).

Finally, we consider the disturbed FDI problem. For the observer-based dFDI, we have $[\,H_{iB} \quad H_{iD}\,] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, so that $\Gamma_{x,f}(z) = \begin{bmatrix} 0 & z & -a \\ 0 & 1 & 1-a \end{bmatrix}$. As this matrix is not of full column rank, the estimation problem for the pair $(\mathbf{x}, \mathbf{f})$ is not solvable.

We may now try to estimate $\mathbf{f}$ alone. This requires to determine an MLA $H_i(z)$ of the polynomial matrix $R_i(z) = \begin{bmatrix} z & -1 & 1 \\ 0 & z & 0 \\ 0 & 1 & 0 \end{bmatrix}$. A possible choice is $H_i(z) = [0 \quad 1 \quad -z]$. Correspondingly, we get $[\,\Gamma_f(z) \quad -\Phi(z)\,] = [\,-[a + z(1-a)] \quad -z\,]$. Therefore, the problem is solvable, again, only for $a = 0, 1$.

If $a = 0$, $\Gamma_{hr} = [-1]$ and, in fact, $\hat{W}(z) = 1$ represents a causal DBO (but not an EO). For $a = 1$, $\Gamma_{hr} = [0]$, a causal DBO does not exist, however $\hat{W}(z) = z$ represents a noncausal EO.

Remark: To conclude, it is worthwhile noticing that all the characterizations provided in this section never involve the two constant matrices B_u and D_u which weight the controlled input contribution to the system dynamics. This result is well-known and very intuitive, as the effect of the controlled input can always be compensated when trying to estimate the other variables.

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