

ON QUADRATIC DIFFERENTIAL FORMS*

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Abstract. This paper develops a theory around the notion of quadratic differential forms in the context of linear differential systems. In many applications, we need to not only understand the behavior of the system variables but also the behavior of certain functionals of these variables. The obvious cases where such functionals are important are in Lyapunov theory and in LQ and H_∞ optimal control. With some exceptions, these theories have almost invariably concentrated on first order models and state representations. In this paper, we develop a theory for linear time-invariant differential systems and quadratic functionals. We argue that in the context of systems described by *one-variable* polynomial matrices, the appropriate tool to express quadratic functionals of the system variables are *two-variable* polynomial matrices. The main achievement of this paper is a description of the interaction of one- and two-variable polynomial matrices for the analysis of functionals and for the application of higher order Lyapunov functionals.

Key words. quadratic differential forms, linear systems, polynomial matrices, two-variable polynomial matrices, Lyapunov theory, positivity, spectral factorization, dissipativeness, storage functions

AMS subject classifications. 93A10, 93A30, 93D05, 93D20, 93D30, 93C05, 93C45

PII. S0363012996303062

1. Introduction. In the theory of models for dynamical systems, it has been customary to consider both external input/output as well as state space models. Also, there is a well developed theory for passing from one type of model to another. Thus, there are efficient algorithms for passing from a convolution, to a transfer function, to a state model, and back. Even for stochastic and nonlinear systems, there are methods for associating a first order state representation to a high order model.

However, in addition to understanding the interaction between system variables, we need in many applications to understand also the behavior of certain functionals of these variables. The obvious cases where such functionals are crucial are in Lyapunov theory, in the theory of dissipative systems, and in optimal control. In these contexts it is remarkable to observe that the theory of dynamics has almost invariably concentrated on first order models and state representations. Thus, in studying system stability using Lyapunov methods, we are constrained to consider state representations, and optimal control problems invariably assume that the cost is an integral of a function of the state and the input. The question thus occurs of whether it is possible to develop an external theory—for example, Lyapunov theory—for systems and functionals so that analysis of stability and passivity, for instance, could proceed on the basis of a *first principles* model instead of first having to find a state representation. In this paper, we consider models that are not in state form (even though some proofs use state representations). Our models are externally specified yet they are not completely general *first principles* models in that we concentrate on models in kernel or in image representation.

It is the purpose of this paper to develop such a theory. We do not, however, set our aims too high and start with a very well-understood class of systems and functionals: linear time-invariant differential systems and quadratic functionals in the

*Received by the editors May 6, 1996; accepted for publication (in revised form) September 9, 1997; published electronically June 22, 1998.

<http://www.siam.org/journals/sicon/36-5/30306.html>

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system variables and their derivatives. We shall see that *one-variable* polynomials are the appropriate tool in which to parametrize the model (see also, among others, [16], [17]) and *two-variable* polynomials are the appropriate tool for parametrizing the functionals. Thus, the paper presents an interesting interplay between one- and two-variable polynomial matrices. Two-variable polynomials turn out to be a very effective tool for analyzing linear systems with quadratic functionals.

This paper consists of a series of general concepts and questions, combined with some specific results concerned with Lyapunov stability and with dissipativity, i.e., with positivity of (integrals of) quadratic differential forms. As such, the paper aims at making a contribution to the development of the very useful and subtle notions of dissipative and lossless (conservative) systems.

In companion papers, these ideas will be applied to LQ and H_∞ problems. The main achievement of this paper—the interaction of one- and two-variable polynomial matrices for the analysis of functionals and application in higher order Lyapunov functions—appears to be new. However, seeds of this have appeared previously in the literature. We mention especially Brockett's early work on *path integrals* [7], [8] in addition to classical work on Routh–Hurwitz-type conditions (see, for example, [6]), and early work by Kalman [13], [14].

2. Review. In order to make this paper reasonably self-contained, we first introduce some notation and some basic facts from the behavioral approach to linear dynamical systems. References in which more details can be found include [31], [32], and [33].

We will deal exclusively with continuous-time real linear time-invariant differential dynamical systems. Thus, the time axis is \mathbb{R} , the signal space is \mathbb{R}^q (the number of variables q , of course, depends on the case at hand), and the behavior \mathfrak{B} is the solution set of a system of linear constant coefficient differential equations

$$(2.1) \quad R \left(\frac{d}{dt} \right) w = 0$$

in the real variables w_1, w_2, \dots, w_q , arranged as the column vector w ; R is a real polynomial matrix with, of course, q columns. The number of rows of R depends, as do its coefficients, on the particular dynamical system described by (2.1). Hence we denote this as $R \in \mathbb{R}^{\bullet \times q}[\xi]$, where ξ denotes the indeterminate. Thus, if $R(\xi) = R_0 + R_1\xi + \dots + R_N\xi^N$, then (2.1) denotes the system of differential equations

$$(2.2) \quad R_0 w + R_1 \frac{dw}{dt} + \dots + R_N \frac{d^N w}{dt^N} = 0.$$

For the behavior, i.e., for the solution set of (2.1) or (2.2), it is usually advisable to consider locally integrable w 's as candidate solutions and to interpret the differential equation in the sense of distributions. However, it is our explicit intention to avoid mathematical technicalities as much as possible in this paper. In keeping with this, we assume that the solution set consists of infinitely differentiable functions, even though many of the results are valid without this assumption. Hence the behavior of (2.1) is defined as

$$(2.3) \quad \mathfrak{B} = \left\{ w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^q) \mid R \left(\frac{d}{dt} \right) w = 0 \right\}.$$

We denote the family of dynamical systems obtained this way by \mathfrak{L}^q . Hence elements of \mathfrak{L}^q are dynamical systems $\Sigma = (\mathbb{R}, \mathbb{R}^q, \mathfrak{B})$ with time axis \mathbb{R} , signal space \mathbb{R}^q , and

behavior \mathfrak{B} described through some $R \in \mathbb{R}^{\bullet \times q}[\xi]$ by (2.3). Note that instead of writing $\Sigma \in \mathfrak{L}^q$ we may as well write $\mathfrak{B} \in \mathfrak{L}^q$, and we prefer to use this notation in this paper.

As explained in the previous paragraphs, each $R \in \mathbb{R}^{\bullet \times q}[\xi]$ unambiguously defines a system $\mathfrak{B} \in \mathfrak{L}^q$. However, there are always many R 's defining the same $\mathfrak{B} \in \mathfrak{L}^q$. For example, if U is any unimodular polynomial matrix such that the product UR makes sense, then R and UR induce the same element of \mathfrak{L}^q . Also, there are many other ways of specifying a given $\mathfrak{B} \in \mathfrak{L}^q$. Note that (2.1) describes \mathfrak{B} as $\mathfrak{B} = \ker(R(\frac{d}{dt}))$ with $R(\frac{d}{dt})$ viewed as a map from $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^q)$ into $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\text{rowdim}(R)})$. For obvious reasons we hence refer to (2.1) as a *kernel representation* of $\mathfrak{B} \in \mathfrak{L}^q$. We will meet other representations, in particular image, latent variable, input/output, state, and input/state/output representations. These are now briefly introduced.

A system $\mathfrak{B} \in \mathfrak{L}^q$ is said to be *controllable* if for each $w_1, w_2 \in \mathfrak{B}$ there exists a $w \in \mathfrak{B}$ and a $t' \geq 0$ such that $w(t) = w_1(t)$ for $t < 0$ and $w(t) = w_2(t - t')$ for $t \geq t'$. It can be shown that \mathfrak{B} is controllable iff its kernel representation satisfies $\text{rank}(R(\lambda)) = \text{rank}(R)$ for all $\lambda \in \mathbb{C}$. Here, $\text{rank}(R)$ is defined as the rank of R considered as a matrix with elements in the field $\mathbb{R}(\xi)$ of real rational functions. On the other hand, for a given $\lambda \in \mathbb{C}$, $R(\lambda)$ is a matrix with elements in \mathbb{C} . Accordingly, $\text{rank}(R(\lambda))$ denotes the rank of the complex matrix $R(\lambda)$. It is easy to see that $\text{rank}(R) = \max_{\lambda \in \mathbb{C}} \text{rank}(R(\lambda))$.

Controllable systems are exactly those that admit image representations. More concretely, $\mathfrak{B} \in \mathfrak{L}^q$ is controllable iff there exists an $M \in \mathbb{R}^{q \times \bullet}[\xi]$ such that $\mathfrak{B} = \text{im}(M(\frac{d}{dt}))$, with $M(\frac{d}{dt})$ viewed as a mapping from $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\text{col dim}(M)})$ into $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^q)$. The resulting representation

$$(2.4) \quad w = M \left(\frac{d}{dt} \right) \ell$$

is called an *image representation* of \mathfrak{B} .

An image representation is a special case of what we call a latent variable representation of \mathfrak{B} . The system of differential equations

$$(2.5) \quad R \left(\frac{d}{dt} \right) w = M \left(\frac{d}{dt} \right) \ell$$

is said to be a *latent variable representation* of $\mathfrak{B} \in \mathfrak{L}^q$ if

$$\mathfrak{B} = \{w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^q) \mid \exists \ell \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^\bullet) \text{ such that (2.5) holds}\}.$$

A latent variable representation is said to be *observable* if $(R(\frac{d}{dt})w = M(\frac{d}{dt})\ell_1$ and $R(\frac{d}{dt})w = M(\frac{d}{dt})\ell_2)$ implies $(\ell_1 = \ell_2)$. Observability is equivalent to the condition that $M(\lambda)$ is of full column rank for all $\lambda \in \mathbb{C}$. A controllable system, it turns out, always allows an observable image representation.

Of special interest in section 4 will be the observability of the system

$$(2.6) \quad A \left(\frac{d}{dt} \right) \ell = 0, \quad w = C \left(\frac{d}{dt} \right) \ell.$$

Of course, the definition of observability applies to (2.6). If this is the case, then we call the pair of polynomial matrices (A, C) with the same number of columns an *observable pair*. Hence (A, C) is an observable pair iff

$$(2.7) \quad \begin{bmatrix} A(\lambda) \\ C(\lambda) \end{bmatrix}$$

is of full column rank for all $\lambda \in \mathbb{C}$.

Systems in \mathfrak{L}^q admit many other useful representations. We already encountered kernel and image representations. Next, we introduce state and input/output representations.

In [22] the notion of state models and their construction has been discussed in detail. Here we limit ourselves to the bare essentials. Let $\mathfrak{B} \in \mathfrak{L}^q$. A latent variable representation (with the latent variable denoted by x this time) of the form (2.5) is said to be a *state model* if, whenever (w_1, x_1) and (w_2, x_2) are \mathfrak{C}^∞ -solutions of (2.5) with $x_1(0) = x_2(0)$, then the concatenation $(w_1, x_1) \wedge (w_2, x_2)$ also satisfies (2.5). Since this concatenation need not be in \mathfrak{C}^∞ , it need only be a weak solution of (2.5), that is, a solution in the sense of distributions. State models are governed by equations of the form (2.5) with special structure. In fact, (2.5) is a state model iff there exist matrices E, F , and G such that

$$(2.8) \quad Gw + Fx + E \frac{dx}{dt} = 0$$

is equivalent to (2.5) in the case of state models. Thus (2.8) is called a *state representation* of the behavior \mathfrak{B} if

$$\mathfrak{B} = \{w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^q) \mid \exists x \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^n) \text{ such that (2.8) is satisfied}\}.$$

Here n denotes the dimension of the vector x . The important feature of (2.8) is that it is an (implicit) differential equation containing derivatives of order at most one in x and zero in w . We call a state representation *state minimal* if among all state representations of \mathfrak{B} , n is as small as possible. It is possible to prove that (2.8) is state minimal iff it is *state trim* (meaning that for all $a \in \mathbb{R}^n$ there exists $(w, x) \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{q+n})$ such that $x(0) = a$) and observable. The dimension of the state space of a state minimal representation of $\mathfrak{B} \in \mathfrak{L}^q$ is called the *McMillan degree* of \mathfrak{B} . The notion of McMillan degree usually refers to properties of polynomial matrices. Actually, for the case at hand this correspondence holds in terms of full row rank kernel representation matrices R or observable image representation matrices M , but we do not need this correspondence in this paper.

Every system $\mathfrak{B} \in \mathfrak{L}^q$ also admits an input/output representation. By reordering the components of the vector w , if need be, we can decompose w into

$$(2.9) \quad w = \begin{bmatrix} u \\ y \end{bmatrix}$$

with, in terms of R , $\text{rank}(R)$ components for y and $q - \text{rank}(R)$ components for u , such that $\mathfrak{B} \in \mathfrak{L}^q$ admits the special kernel representation

$$(2.10) \quad P \left(\frac{d}{dt} \right) y = Q \left(\frac{d}{dt} \right) u,$$

with P square, $\det P \neq 0$, and $P^{-1}Q$ a matrix of proper rational functions. Thus, in (2.10) u has the usual properties of input and y those of output. Therefore (2.10) is called an *input/output representation*.

Actually, for controllable systems, we can also recover the input/output structure in terms of the image representation. Thus the image representation

$$(2.11) \quad \begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} U \left(\frac{d}{dt} \right) \\ Y \left(\frac{d}{dt} \right) \end{bmatrix} \ell$$

is an input/output representation if U is square, $\det U \neq 0$, and YU^{-1} is a matrix of proper rational functions. The number of input components of a system in image representation (2.4) equals $\text{rank}(M)$.

It is possible to combine the above, leading to the familiar input/state/output representation

$$(2.12) \quad \frac{dx}{dt} = Ax + Bu, \quad y = Cx + Du.$$

This representation is state minimal iff it is observable, i.e., iff (A, C) is an observable pair of matrices (not to be confused with an observable pair of polynomial matrices).

Summarizing, given any $w \in \mathfrak{B}$, we may partition the components of w into inputs and outputs. Also, there exists an $X \in \mathbb{R}^{\bullet \times q}[\xi]$ such that

$$(2.13) \quad x = X \left(\frac{d}{dt} \right) w$$

is a (minimal) state map for \mathfrak{B} . For a system in image representation (2.4) this leads to a state representation of the form

$$(2.14) \quad x = X' \left(\frac{d}{dt} \right) \ell.$$

The resulting relation between u and y is as in (2.10); that between w and x is as in (2.8); and that between u, y , and x is as in (2.12).

We need a few more details about the state construction for systems in image representation (2.4). Assume that M is of full column rank. Then after permutation of the components of w (i.e., of the rows of M), if need be, M is of the form

$$M = \begin{bmatrix} U \\ Y \end{bmatrix},$$

with U square, $\det(U) \neq 0$, and YU^{-1} a matrix of proper rational functions. The resulting system

$$\begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} U(\frac{d}{dt}) \\ Y(\frac{d}{dt}) \end{bmatrix} \ell$$

is then an input/output representation. Consider all polynomial row vectors $F \in \mathbb{R}^{1 \times \bullet}[\xi]$ such that FU^{-1} is strictly proper. It can be shown (see [22]) that this set is a vector space. Now

$$(2.15) \quad x = X \left(\frac{d}{dt} \right) \ell$$

is a state map for (2.4) iff the rows of X span this vector space. It is a minimal state map iff the rows of X form a basis for this vector space.

Next, consider associated with (2.4) the variable v governed by

$$v = L \left(\frac{d}{dt} \right) \ell.$$

Then it follows from the above that there exist matrices P and Q such that

$$v = Px + Qu$$

iff LU^{-1} is proper. Moreover, Q is zero iff LU^{-1} is strictly proper, and Q is invertible iff LU^{-1} is biproper.

3. Quadratic differential forms. Differential equations and one-variable polynomial matrices play an essential role in describing the dynamics of systems, as we have seen in section 2 and the references given therein. When studying functions of the dynamical variables, as in Lyapunov theory, studying dissipation and passivity, or specifying performance criteria in optimal control, we invariably encounter quadratic expressions in the variables and their derivatives. As we shall see, two-variable polynomial matrices are the proper mathematical tool to express these quadratic functionals. We aim to illustrate throughout this paper that linear dynamical equations expressed through one-variable polynomial matrices, and quadratic functionals expressed through two-variable polynomial matrices fit as a glove fits a hand.

Let $\mathbb{R}^{q_1 \times q_2}[\zeta, \eta]$ denote the set of real polynomial matrices in the (commuting) indeterminates ζ and η . Explicitly, an element $\Phi \in \mathbb{R}^{q_1 \times q_2}[\zeta, \eta]$ is thus given by

$$(3.1) \quad \Phi(\zeta, \eta) = \sum_{k, \ell} \Phi_{k\ell} \zeta^k \eta^\ell.$$

The sum in (3.1) ranges over the nonnegative integers and is assumed to be finite, and $\Phi_{k\ell} \in \mathbb{R}^{q_1 \times q_2}$. Such a Φ induces a *bilinear differential form* (BLDF), that is, the map

$$(3.2) \quad L_\Phi : \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^{q_1}) \times \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^{q_2}) \rightarrow \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$$

defined by

$$(3.3) \quad (L_\Phi(v, w))(t) := \sum_{k, \ell} \left(\frac{d^k v}{dt^k}(t) \right)^T \Phi_{k\ell} \left(\frac{d^\ell w}{dt^\ell}(t) \right).$$

If $q_1 = q_2 (= q)$, then Φ induces a *quadratic differential form* (QDF)

$$(3.4) \quad Q_\Phi : \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^q) \rightarrow \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$$

defined by

$$(3.5) \quad Q_\Phi(w) := L_\Phi(w, w).$$

Define the asterisk operator $*$ by

$$(3.6) \quad * : \mathbb{R}^{q_1 \times q_2}[\zeta, \eta] \rightarrow \mathbb{R}^{q_2 \times q_1}[\zeta, \eta]; \quad \Phi^*(\zeta, \eta) := \Phi^T(\eta, \zeta),$$

where T denotes transposition. Obviously $L_\Phi(v, w) = L_{\Phi^*}(w, v)$. If $\Phi \in \mathbb{R}^{q \times q}[\zeta, \eta]$ satisfies $\Phi = \Phi^*$, then Φ is called *symmetric*. The symmetric elements of $\mathbb{R}^{q \times q}[\zeta, \eta]$ are denoted by $\mathbb{R}_s^{q \times q}[\zeta, \eta]$. Clearly

$$(3.7) \quad Q_\Phi = Q_{\Phi^*} = Q_{\frac{1}{2}(\Phi + \Phi^*)}$$

This shows that when considering quadratic differential forms, we can hence in principle restrict our attention to Φ 's in $\mathbb{R}_s^{q \times q}[\zeta, \eta]$. However, both bilinear and quadratic forms are of interest to us.

Associated with $\Phi \in \mathbb{R}^{q_1 \times q_2}[\zeta, \eta]$, we can form the matrix

$$(3.8) \quad \tilde{\Phi} = \begin{bmatrix} \Phi_{00} & \Phi_{01} & \cdots & \cdot & \cdots \\ \Phi_{10} & \Phi_{11} & \cdots & \cdot & \cdots \\ \vdots & \vdots & & \vdots & \\ \cdot & \cdot & \cdots & \Phi_{k\ell} & \cdots \\ \vdots & \vdots & & \vdots & \end{bmatrix}.$$

Note that, although $\tilde{\Phi}$ is an infinite matrix, all but a finite number of its elements are zero. We can factor $\tilde{\Phi}$ as $\tilde{\Phi} = \tilde{N}^T \tilde{M}$, with \tilde{N} and \tilde{M} infinite matrices having a finite number of rows and all but a finite number of elements equal to zero. This decomposition leads, after premultiplication by $[I_{q_1} \ I_{q_1} \zeta \ I_{q_1} \zeta^2 \ \cdots]$ and postmultiplication by $\text{col}[I_{q_2} \ I_{q_2} \eta \ I_{q_2} \eta^2 \ \cdots]$, to the following factorization of Φ :

$$(3.9) \quad \Phi(\zeta, \eta) = N^T(\zeta)M(\eta).$$

This decomposition is not unique, but if we take \tilde{N} and \tilde{M} surjective, then their number of rows is equal to the rank of $\tilde{\Phi}$. The factorization (3.9) is then called a *canonical factorization* of Φ . Associated with (3.9), we obtain the following expression for the BLDF L_Φ :

$$(3.10) \quad L_\Phi(w_1, w_2) = \left(N \left(\frac{d}{dt} \right) w_1 \right)^T M \left(\frac{d}{dt} \right) w_2.$$

Next we discuss the case that Φ is symmetric. Clearly $\Phi = \Phi^*$ iff $\tilde{\Phi}$ is symmetric. In that case, it can be factored as $\tilde{\Phi} = \tilde{M}^T \Sigma_M \tilde{M}$ with \tilde{M} an infinite matrix having a finite number of rows and all but a finite number of elements equal to zero, and Σ_M a signature matrix, i.e., a matrix of the form

$$\begin{bmatrix} I_{r_+} & 0 \\ 0 & -I_{r_-} \end{bmatrix}.$$

This decomposition leads to the following decomposition of Φ :

$$(3.11) \quad \Phi(\zeta, \eta) = M^T(\zeta)\Sigma_M M(\eta).$$

Also, this decomposition is not unique but if we take \tilde{M} surjective, then Σ_M is unique. We denote this Σ_M as Σ_Φ and the resulting pair (r_-, r_+) by (ϕ_-, ϕ_+) . This pair is called the *inertia* of Φ . The resulting factorization

$$(3.12) \quad \Phi(\zeta, \eta) = M^T(\zeta)\Sigma_\Phi M(\eta)$$

is called a *symmetric canonical factorization* of Φ . Of course, a symmetric canonical factorization is not unique. However, they can all be obtained from one by replacing $M(\xi)$ by $UM(\xi)$ with $U \in \mathbb{R}^{\text{rank}(\tilde{\Phi}) \times \text{rank}(\tilde{\Phi})}$ such that $U^T \Sigma_\Phi U = \Sigma_\Phi$.

Associated with (3.11), we obtain the following decomposition of Q_Φ into a sum of positive and negative squares:

$$(3.13) \quad Q_\Phi(w) = \|P \left(\frac{d}{dt} \right) w\|^2 - \|N \left(\frac{d}{dt} \right) w\|^2,$$

where $N, P \in \mathbb{R}^{\bullet \times q}[\zeta]$ are obtained by partitioning \tilde{M} conform Σ_M as:

$$(3.14) \quad \tilde{M} = \begin{bmatrix} \tilde{P} \\ \tilde{N} \end{bmatrix}.$$

For a given symmetric $\Phi(\zeta, \eta)$ we are also interested in the symmetric two-variable polynomial matrix $|\Phi|(\zeta, \eta)$, the *absolute value* of Φ , which we define as follows. For a given real symmetric matrix $A \in \mathbb{R}^{n \times n}$ define its absolute value, $|A| \in \mathbb{R}^{n \times n}$, as the unique symmetric nonnegative definite matrix $X \in \mathbb{R}^{n \times n}$ such that $X^2 = A^2$. This

matrix $|A|$ can be computed as follows. Factor $A = \tilde{U}^T \Lambda \tilde{U}$, where Λ is the diagonal matrix with, on its diagonal, the nonzero eigenvalues of A in decreasing order, and where $\tilde{U} \tilde{U}^T = I$, and define $|A| := \tilde{U}^T |\Lambda| \tilde{U}$ with $|\Lambda|$ defined in the obvious way. Let $\tilde{\Phi}$ be the symmetric matrix associated with $\Phi(\zeta, \eta)$. Let $|\tilde{\Phi}|$ be the absolute value of $\tilde{\Phi}$. Next, define $|\Phi|(\zeta, \eta)$ as the symmetric two-variable polynomial matrix associated with $|\tilde{\Phi}|$:

$$|\Phi|(\zeta, \eta) := \begin{bmatrix} I \\ \zeta I \\ \zeta^2 I \\ \vdots \end{bmatrix}^T |\tilde{\Phi}| \begin{bmatrix} I \\ \eta I \\ \eta^2 I \\ \vdots \end{bmatrix}.$$

Note that a factorization $\tilde{\Phi} = \tilde{U}^T \Lambda \tilde{U}$ immediately yields a symmetric canonical factorization of $\Phi(\zeta, \eta)$. Indeed, define $\tilde{M}_c := \sqrt{|\Lambda|} \tilde{U}$. We then have

$$(3.15) \quad \tilde{\Phi} = \tilde{U}^T \sqrt{|\Lambda|} \Sigma_{\Phi} \sqrt{|\Lambda|} \tilde{U} = \tilde{M}_c^T \Sigma_{\Phi} \tilde{M}_c,$$

with \tilde{M}_c surjective. The corresponding $M_c(\xi) := \tilde{M}_c \operatorname{col} [I \quad \xi I \quad \xi^2 I \quad \dots]$ then yields a canonical factorization $\Phi(\zeta, \eta) = M_c^T(\zeta) \Sigma_{\Phi} M_c(\eta)$. This particular canonical factorization has the property that

$$(3.16) \quad |\Phi|(\zeta, \eta) = M_c^T(\zeta) M_c(\eta).$$

In general, if $M(\xi)$ is any canonical factor of Φ , then we have $UM(\xi) = M_c(\xi)$ with U satisfying $U^T \Sigma_{\Phi} U = \Sigma_{\Phi}$, and hence $|\Phi|(\zeta, \eta) = M^T(\zeta) U^T U M(\eta)$.

One of the conveniences of identifying BLDFs and QDFs with two-variable polynomial matrices is that they allow a very convenient calculus. One instance of this is differentiation. Obviously if L_{Φ} is a BLDF, so is $\frac{d}{dt} L_{\Phi}$, and if Q_{Φ} is a QDF, so is $\frac{d}{dt} Q_{\Phi}$. The result of differentiation is easily expressed in terms of the two-variable polynomial matrices and leads to the dot operator \bullet defined as

$$(3.17) \quad \bullet : \mathbb{R}^{q_1 \times q_2}[\zeta, \eta] \rightarrow \mathbb{R}^{q_1 \times q_2}[\zeta, \eta]; \quad \Phi \bullet (\zeta, \eta) := (\zeta + \eta) \Phi(\zeta, \eta).$$

It is easily calculated that

$$(3.18) \quad \frac{d}{dt} L_{\Phi} = L_{\Phi \bullet} \quad \text{and} \quad \frac{d}{dt} Q_{\Phi} = Q_{\Phi \bullet}$$

In the following, an important role is played by certain one-variable polynomial matrices obtained from two-variable polynomial matrices by means of the delta operator ∂ , defined as

$$\partial : \mathbb{R}^{q_1 \times q_2}[\zeta, \eta] \rightarrow \mathbb{R}^{q_1 \times q_2}[\xi]; \quad \partial \Phi(\xi) := \Phi(-\xi, \xi).$$

Note that, among other things, this allows one to associate a differential operator $\Phi(-\frac{d}{dt}, \frac{d}{dt})$ with a QDF—this is one of the key ingredients in LQ—and variational problems.

Introduce the star operator \star acting on matrix polynomials by

$$\star : \mathbb{R}^{q_1 \times q_2}[\xi] \rightarrow \mathbb{R}^{q_2 \times q_1}[\xi]; \quad R^{\star}(\xi) := R^T(-\xi).$$

The importance of this operation stems from the fact that $M(\frac{d}{dt})$ and $M^{\star}(\frac{d}{dt})$ are formal adjoints as differential operators. A polynomial matrix $M \in \mathbb{R}^{q \times q}[\xi]$ is called

para-Hermitian if $M = M^*$. Note that $(\partial\Phi)^* = \partial(\Phi^*)$. Hence if $\Phi \in \mathbb{R}_s^{q \times q}[\zeta, \eta]$, then $\partial\Phi$ is para-Hermitian.

In addition to studying BLDFs and QDFs as maps to $\mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$, we are interested in their integrals. In order to make sure that those integrals exist, we assume in this case that the arguments have compact support. As is common, we denote by $\mathfrak{D}(\mathbb{R}, \mathbb{R}^q) := \{w \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^q) \mid w \text{ has compact support}\}$. Let $\Phi \in \mathbb{R}^{q_1 \times q_2}[\zeta, \eta]$. Then obviously $L_\Phi : \mathfrak{D}(\mathbb{R}, \mathbb{R}^{q_1}) \times \mathfrak{D}(\mathbb{R}, \mathbb{R}^{q_2}) \rightarrow \mathfrak{D}(\mathbb{R}, \mathbb{R})$. Consider the integral

$$(3.19) \quad \int L_\Phi : \mathfrak{D}(\mathbb{R}, \mathbb{R}^{q_1}) \times \mathfrak{D}(\mathbb{R}, \mathbb{R}^{q_2}) \rightarrow \mathbb{R}$$

defined as

$$(3.20) \quad \int L_\Phi(v, w) := \int_{-\infty}^{+\infty} L_\Phi(v, w) dt.$$

The notation $\int Q_\Phi$ follows readily from this. Furthermore, consider the same integral over a finite interval $[t_1, t_2]$

$$(3.21) \quad \int_{t_1}^{t_2} L_\Phi(v, w) dt$$

denoted as $\int_{t_1}^{t_2} L_\Phi$. We call this integral *independent of path* if for any t_1 and t_2 the result of the integral (3.21) depends only on the values of v and w and (a finite number of) their derivatives at $t = t_1$ and $t = t_2$ but not on the intermediate path used to connect these endpoints, assuming, of course that $v \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^{q_1})$ and $w \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^{q_2})$.

The questions of when the map $\int L_\Phi$ is zero and when path independence holds are studied next.

THEOREM 3.1. *Let $\Phi \in \mathbb{R}^{q_1 \times q_2}[\zeta, \eta]$. Then the following statements are equivalent:*

1. $\int L_\Phi = 0$, equivalently $\int_{t_1}^{t_2} L_\Phi$ is independent of path.
2. There exists a $\Psi \in \mathbb{R}^{q_1 \times q_2}[\zeta, \eta]$ such that $\Phi = \dot{\Psi}$, equivalently, such that $L_\Phi = \frac{d}{dt} L_\Psi$. Obviously Ψ is given by

$$(3.22) \quad \Psi(\zeta, \eta) = \frac{\Phi(\zeta, \eta)}{\zeta + \eta}.$$

3. $\partial\Phi = 0$, i.e., $\Phi(-\xi, \xi) = 0$.

The same equivalence holds for QDFs. Simply assume $\Phi \in \mathbb{R}_s^{q \times q}[\zeta, \eta]$ and replace the L 's by Q 's in 1 and 2.

Proof. For the proof, see the appendix.

The importance of this theorem is that condition (3) gives a very convenient way of checking (1) or (2). Path integrals and path independence featured prominently in Brockett's work in the sixties (see [7], [8]), and indeed some of our results can be viewed as streamlined versions of this work. Another potentially interesting connection of the above theorem and our paper with the existing literature is [3] where, in our notation, $\Phi(\zeta, \eta) = R(\zeta)M(-\eta)$ is studied, with R and M associated with a kernel representation (2.1) and an image representation (2.4) of a controllable system. This Φ defines an intriguing path independent BLDF that can be associated with any controllable \mathfrak{B} .

In this paper we also study the behavior of QDFs evaluated along a differential behavior $\mathfrak{B} \in \mathcal{L}^q$. In order to do so, it is convenient to introduce an equivalence relation on both the one- and two-variable polynomial matrices modulo a given $\mathfrak{B} \in \mathcal{L}^q$.

Let $D_1, D_2 \in \mathbb{R}^{\bullet \times q}[\xi]$. Define $(D_1 \stackrel{\mathfrak{B}}{=} D_2) :\Leftrightarrow (D_1(\frac{d}{dt}) - D_2(\frac{d}{dt}))\mathfrak{B} = 0$. Let $\Phi_1, \Phi_2 \in \mathbb{R}_s^{q \times q}[\zeta, \eta]$. Define $(\Phi_1 \stackrel{\mathfrak{B}}{=} \Phi_2) :\Leftrightarrow (Q_{\Phi_1}(w) = Q_{\Phi_2}(w) \text{ for all } w \in \mathfrak{B})$. These equivalencies are easily expressed in terms of a kernel or an image representation of \mathfrak{B} .

PROPOSITION 3.2. *Let $R \in \mathbb{R}^{\bullet \times q}[\xi]$ define a kernel representation of $\mathfrak{B} \in \mathcal{L}^q$. Then $D_1 \stackrel{\mathfrak{B}}{=} D_2$ iff*

$$(3.23) \quad D_1 - D_2 = FR$$

for some $F \in \mathbb{R}^{\bullet \times \bullet}[\xi]$ and $\Phi_1 \stackrel{\mathfrak{B}}{=} \Phi_2$ iff

$$(3.24) \quad \Phi_2(\zeta, \eta) = \Phi_1(\zeta, \eta) + R^T(\zeta)F(\zeta, \eta) + F^*(\zeta, \eta)R(\eta)$$

for some $F \in \mathbb{R}^{\bullet \times q}[\zeta, \eta]$. Let $M \in \mathbb{R}^{q \times \bullet}[\zeta, \eta]$ define an image representation of $\mathfrak{B} \in \mathcal{L}^q$. Then $D_1 \stackrel{\mathfrak{B}}{=} D_2$ iff

$$D_1M = D_2M$$

and $\Phi_1 \stackrel{\mathfrak{B}}{=} \Phi_2$ iff

$$M^T(\zeta)\Phi_1(\zeta, \eta)M(\eta) = M^T(\zeta)\Phi_2(\zeta, \eta)M(\eta)$$

Proof. For the proof, see the appendix.

The first equivalence in the above proposition was already proven in [23], with an account of the history of the result, which goes back to 1895. We will return to the second equivalence at the end of section 4.

We now briefly discuss positivity of QDFs. This will be a major issue in the following; here we restrict our attention to the basic definitions.

DEFINITION 3.3. *Let $\Phi \in \mathbb{R}_s^{q \times q}[\zeta, \eta]$. We call the QDF Q_Φ nonnegative, denoted $\Phi \geq 0$, if $Q_\Phi(w) \geq 0$ for all $w \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^q)$, and positive, denoted $\Phi > 0$, if $\Phi \geq 0$ and if the only $w \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^q)$ for which $Q_\Phi(w) = 0$ is $w = 0$.*

Using the matrix representation of Φ , it is easy to see that $\Phi \geq 0$ iff there exists $D \in \mathbb{R}^{\bullet \times q}[\xi]$ such that $\Phi(\zeta, \eta) = D^T(\zeta)D(\eta)$. Simply factor Φ as $\tilde{\Phi} = \tilde{D}^T \tilde{D}$ and take $D(\xi) = \tilde{D} \text{ col } [I_q \ I_q \xi \ I_q \xi^2 \ \dots]$. Moreover $\Phi > 0$ iff this D has the property that $D(\lambda)$ is of rank q for all $\lambda \in \mathbb{C}$; in other words, iff the image representation $w = D(\frac{d}{dt})\ell$ defined by D is observable. Note that, for $\Phi \in \mathbb{R}_s^{q \times q}[\zeta, \eta]$, we always have $|\Phi| \geq 0$.

We are also interested in QDFs which are zero or positive along a behavior $\mathfrak{B} \in \mathcal{L}^q$.

DEFINITION 3.4. *We call Φ zero along \mathfrak{B} , denoted $\Phi \stackrel{\mathfrak{B}}{=} 0$, if $Q_\Phi(w) = 0$ for all $w \in \mathfrak{B}$. The notions of nonnegative ($\stackrel{\mathfrak{B}}{\geq}$) and positive ($\stackrel{\mathfrak{B}}{>}$) along \mathfrak{B} follow readily.*

Note that it immediately follows from Proposition 3.2 that, if $R(\frac{d}{dt})w = 0$ is a kernel representation of \mathfrak{B} , then $\Phi \stackrel{\mathfrak{B}}{=} 0$ iff it can be written as $\Phi(\zeta, \eta) = F^*(\zeta, \eta)R(\eta) + R^T(\zeta)F(\zeta, \eta)$. A similar result holds for positivity as follows.

PROPOSITION 3.5. *Let $\Phi \in \mathbb{R}_s^{q \times q}[\zeta, \eta]$, $\mathfrak{B} \in \mathcal{L}^q$, and $R \in \mathbb{R}^{\bullet \times q}[\xi]$ induce a kernel representation of \mathfrak{B} . Then*

- (i) $\Phi \stackrel{\mathfrak{B}}{\geq} 0$ iff there exists $\Phi' \in \mathbb{R}_s^{q \times q}[\zeta, \eta]$ with $\Phi \stackrel{\mathfrak{B}}{=} \Phi'$ and $\Phi' \geq 0$;

- (ii) $\Phi \stackrel{\mathfrak{B}}{>} 0$ iff there exists $\Phi' \in \mathbb{R}_s^{q \times q}[\zeta, \eta]$ with $\Phi \stackrel{\mathfrak{B}}{=} \Phi'$ and $\Phi'(\zeta, \eta) = D^T(\zeta)D(\eta)$, with (R, D) an observable pair.

Proof. For the proof, see the appendix.

4. Lyapunov theory. Lyapunov theory is a firmly established and very useful technique for establishing stability. It pertains to systems described by explicit first order differential equations. However, as argued in [33], models obtained from first principles are seldomly in first order form, will contain latent variables, and may contain high order derivatives. Writing them in explicit first order form without introducing spurious solutions may not be an easy matter. Moreover, stability considerations do not require systems to be in first order form. In fact, historically the very first stability questions and results, as Maxwell’s statement of the stability problem and the Routh–Hurwitz conditions, pertain to high order differential equations. Oddly enough, to our knowledge, no attempts seem to have been made to establish Lyapunov theory for high order differential equations. This is the purpose of this section. We limit our attention, however, to linear differential systems and to Lyapunov functions that are quadratic differential forms, but we recognize the urgency of generalizing this work to nonlinear systems. We should remark that in this section for stability, we only consider systems in which the latent variables have been eliminated, although latent variables do not cause essential difficulties in the context of stability.

First, we introduce the notion of stability. We say that a system $\mathfrak{B} \in \mathcal{L}^q$ is *asymptotically stable* if $(w \in \mathfrak{B}) \Rightarrow (w(t) \xrightarrow[t \rightarrow \infty]{} 0)$ and *stable* if $(w \in \mathfrak{B}) \Rightarrow (w$ is bounded on the half-line $[0, \infty))$. For a system $\mathfrak{B} \in \mathcal{L}^q$ to be (asymptotically) stable it has to be autonomous. A system $\mathfrak{B} \in \mathcal{L}^q$ is said to be *autonomous* if $(w_1, w_2 \in \mathfrak{B})$ and $(w_1(t) = w_2(t) \text{ for } t < 0)$ imply $(w_1 = w_2)$. It is easy to see that the system with kernel representation $R(\frac{d}{dt})w = 0$ is autonomous iff $\text{rank}(R) = q$; in particular, if R is square and $\det(R) \neq 0$.

DEFINITION 4.1. Let $R \in \mathbb{R}^{\bullet \times q}[\xi]$. The complex number $\lambda \in \mathbb{C}$ is said to be a singularity of R if $\text{rank}(R(\lambda)) < \text{rank}(R)$; R is said to be Hurwitz if $\text{rank}(R) = q$ and if R has all its singularities in the open left half of the complex plane.

Thus a square $R \in \mathbb{R}^{q \times q}[\xi]$ is Hurwitz iff $\det(R)$ is a Hurwitz polynomial, i.e., a nonzero polynomial with its roots in the open left half-plane. We record the following classical result for easy reference.

PROPOSITION 4.2. The system with kernel representation (2.1) is asymptotically stable iff R is Hurwitz.

Our most basic Lyapunov theorem regarding high order systems is the following.

THEOREM 4.3. Let $\mathfrak{B} \in \mathcal{L}^q$. Then \mathfrak{B} is asymptotically stable iff there exists $\Psi \in \mathbb{R}_s^{q \times q}[\zeta, \eta]$ such that $\Psi \stackrel{\mathfrak{B}}{\geq} 0$ and $\dot{\Psi} \stackrel{\mathfrak{B}}{<} 0$.

Proof. For the proof, see the appendix.

Example 4.4. Consider the scalar system described by $w + \frac{dw}{dt} + \frac{d^2w}{dt^2} = 0$. Consider the QDF $w^2 + (\frac{dw}{dt})^2$. Its derivative QDF is $2(w + \frac{d^2w}{dt^2})\frac{dw}{dt}$. Since $w \in \mathfrak{B}$ iff $w + \frac{d^2w}{dt^2} = -\frac{dw}{dt}$, we see that this QDF is \mathfrak{B} -equivalent to the QDF $-2(\frac{dw}{dt})^2$. Finally observe that $(1 + \xi + \xi^2, \sqrt{2}\xi)$ is an observable pair. Hence $-2(\frac{dw}{dt})^2$ is negative on \mathfrak{B} . Theorem 4.3 establishes asymptotic stability (a rather trivial matter for the case at hand). Our aim was to show the use of Lyapunov theory without getting involved with state representations (admittedly also a trivial matter).

Example 4.5. Consider the multivariable system

$$(4.1) \quad Kw + D \frac{dw}{dt} + M \frac{d^2w}{dt^2} = 0,$$

with $K, D, M \in \mathbb{R}^{q \times q}$, $K = K^T \geq 0$, $D + D^T \geq 0$, and $M = M^T \geq 0$. Such second order equations occur frequently as models of (visco-)elastic mechanical systems. Take $\Psi(\zeta, \eta) = K + M\zeta\eta$. Then $\dot{\Psi}(\zeta, \eta) = K(\zeta + \eta) + M(\zeta^2\eta + \zeta\eta^2)$ which is obviously \mathfrak{B} -equivalent to $-(D + D^T)\zeta\eta$. Thus, asymptotic stability follows if

$$(4.2) \quad \left(K + D\xi + M\xi^2, \sqrt{(D + D^T)\xi} \right)$$

is an observable pair. This is the case, for example, if $\{0\} = \ker(K) \subset \ker(D + D^T) \subset \ker(M)$. Indeed, under this condition

$$\begin{bmatrix} K + D\lambda + M\lambda^2 \\ \sqrt{(D + D^T)\lambda} \end{bmatrix}$$

has full column rank for all $\lambda \in \mathbb{C}$.

State representations of autonomous systems take a very special form. Indeed, it is easy to see that $\mathfrak{B} \in \mathcal{L}^q$ is autonomous iff it admits a state representation of the form $\frac{dx}{dt} = Ax$, $w = Cx$. Such state representations are automatically state trim. If (A, C) is observable, then they are state minimal. It also follows that for every $D \in \mathbb{R}^{\bullet \times q}[\xi]$ there exists a matrix $H \in \mathbb{R}^{\bullet \times n}$ such that $D(\frac{d}{dt})w \stackrel{\mathfrak{B}}{=} Hx$, i.e., every linear differential operator acting on an autonomous $\mathfrak{B} \in \mathcal{L}^q$ is \mathfrak{B} -equivalent to an instantaneous function of the state. An analogous statement holds, of course, for QDFs.

Viewed from this perspective, one can regard Theorem 4.3 as being about state systems and in this sense not very different from classical Lyapunov theorems. The point of Theorem 4.3 is twofold:

1. It avoids the state construction which algorithmically (and conceptually) is not always easy in the multivariable case; and
2. It has the usual Lyapunov theory as a special case by applying it to systems in first order form and using memoryless QDFs. For the sake of completeness, we record this as a corollary.

COROLLARY 4.6. *Let \mathfrak{B} be the behavior of $\frac{dw}{dt} = Aw$. Let $\Psi(\zeta, \eta) = \Psi_0$ with $\Psi_0 \in \mathbb{R}^{q \times q}$, $\Psi_0 = \Psi_0^T \geq 0$. Then $\dot{\Psi}(\zeta, \eta) \stackrel{\mathfrak{B}}{=} A\Psi_0 + \Psi_0A^T =: \Delta_0$. Whence, if $\Delta_0 = \Delta_0^T \leq 0$ and if (A, Δ_0) is an observable pair of matrices, \mathfrak{B} is asymptotically stable.*

Proof. For the proof, see the appendix.

In section 3, we discussed \mathfrak{B} -positive QDFs. When \mathfrak{B} is autonomous, it is useful to consider also a stronger concept. Let $\mathfrak{B} \in \mathcal{L}^q$ and $\Phi \in \mathbb{R}_s^{q \times q}[\zeta, \eta]$; we call Φ *strongly \mathfrak{B} -positive* (denoted $\Phi \gg_{\mathfrak{B}} 0$) if $\Phi \geq 0$ and if $(w \in \mathfrak{B} \text{ and } Q_{\Phi}(w)(0) = 0)$ imply $(w = 0)$. It is easy to see that $\Phi \gg_{\mathfrak{B}} 0$ implies $\Phi \succ_{\mathfrak{B}} 0$ and that in order for $\Phi \gg_{\mathfrak{B}} 0$, \mathfrak{B} must be autonomous. In fact, $((Q_{\Phi}(w)(0) = 0) \Rightarrow (w = 0))$ by itself already implies $(\Phi \gg_{\mathfrak{B}} 0 \text{ or } \Phi \ll_{\mathfrak{B}} 0)$. Using this notion we arrive at the following refinement of theorem 4.3.

PROPOSITION 4.7. *If $\Psi \geq 0$ and $\dot{\Psi} \ll_{\mathfrak{B}} 0$, then \mathfrak{B} is asymptotically stable and $\Psi \gg_{\mathfrak{B}} 0$.*

Proof. For the proof, see the appendix.

We now formulate a stronger version of the “only if” part of Theorem 4.3.

THEOREM 4.8. *Assume that $\mathfrak{B} \in \mathcal{L}^q$ is asymptotically stable. Then for any $\Phi \in \mathbb{R}_s^{q \times q}[\zeta, \eta]$ there exists a $\Psi \in \mathbb{R}_s^{q \times q}[\zeta, \eta]$ such that $\dot{\Psi} \stackrel{\mathfrak{B}}{=} \Phi$; Ψ is unique up to \mathfrak{B} -equivalence in the sense that, if $\dot{\Psi}_1 \stackrel{\mathfrak{B}}{=} \Phi$ and $\dot{\Psi}_2 \stackrel{\mathfrak{B}}{=} \Phi$, then $\Psi_1 \stackrel{\mathfrak{B}}{=} \Psi_2$. If $\Phi \stackrel{\mathfrak{B}}{\leq} 0$, then $\Psi \stackrel{\mathfrak{B}}{\geq} 0$, and if $\Phi \stackrel{\mathfrak{B}}{<} 0$, then $\Psi \stackrel{\mathfrak{B}}{\gg} 0$.*

In order to compute Ψ from Φ , the following algorithm may be used. Let $R \in \mathbb{R}^{\bullet \times q}[\xi]$ induce a kernel representation of \mathfrak{B} . Consider the polynomial matrix equation

$$(4.3) \quad X^T(-\xi)R(\xi) + R^T(-\xi)X(\xi) = \Phi(-\xi, \xi)$$

in the unknown $X \in \mathbb{R}^{\bullet \times q}[\xi]$. Then (4.3) has a solution. Let X_0 be a solution. If R is square, then all its solutions can be obtained from this one as

$$(4.4) \quad X(\xi) = X_0(\xi) + F(\xi)R(\xi),$$

where F ranges over all polynomial matrices of appropriate size satisfying

$$(4.5) \quad F^T(-\xi) = -F(\xi).$$

Consider any $Y \in \mathbb{R}^{\bullet \times q}[\zeta, \eta]$ such that

$$(4.6) \quad Y(-\xi, \xi) = X(\xi)$$

and compute

$$(4.7) \quad \Psi(\zeta, \eta) = \frac{\Phi(\zeta, \eta) - Y^*(\zeta, \eta)R(\eta) - R^T(\zeta)Y(\zeta, \eta)}{\zeta + \eta}.$$

Then $\dot{\Psi} \stackrel{\mathfrak{B}}{=} \Phi$. Since any two Ψ_1, Ψ_2 such that $\dot{\Psi}_1 \stackrel{\mathfrak{B}}{=} \Phi$ and $\dot{\Psi}_2 \stackrel{\mathfrak{B}}{=} \Phi$ satisfy $\Psi_1 \stackrel{\mathfrak{B}}{=} \Psi_2$, any other solutions of (4.3) and/or (4.6) yield Ψ 's in (4.7) that are \mathfrak{B} -equivalent.

Proof. For the proof, see the appendix.

Theorem 4.8 is more than a mouthful and so we illustrate it for ordinary state space systems $\frac{dw}{dt} = Aw$. Let $\Phi = \Phi^T \in \mathbb{R}^{n \times n}$. Then (4.3) becomes

$$(4.8) \quad X^T(-\xi)(A - I\xi) + (A^T + I\xi)X(\xi) = \Phi.$$

This equation has a constant solution which must be symmetric, $X_0 = X_0^T$, the solution of the ordinary Lyapunov equation

$$(4.9) \quad X_0A + A_0^T X = \Phi.$$

Choose $Y = X_0$ and verify that (4.7) reduces to $\Psi = X_0$, whence the Lyapunov function is $Q_\Psi(w) = w^T X_0 w$ and for $w \in \mathfrak{B}$ its derivative is $Q_\Phi(w) = w^T \Phi w$. Because of this analogy, we refer to (4.3) as the polynomial matrix *Lyapunov equation*.

The above shows that it seems to suffice to consider Lyapunov functions Ψ and their derivatives Φ that are of lower degree than that of R . That is, in fact, a general feature of the equations in Theorem 4.8. However, in order to formalize this, we return first to the notion of \mathfrak{B} -equivalence of differential operators in the case that $\mathfrak{B} \in \mathcal{L}^q$ is autonomous.

Let $\mathfrak{B} \in \mathfrak{L}^q$ be autonomous. Then there always exists a square kernel representation for it. Let $R \in \mathbb{R}^{q \times q}[\xi]$ be such that $\mathfrak{B} = \ker(R(\frac{d}{dt}))$. We assume in the remainder of this section that R is square.

Let $D \in \mathbb{R}^{\bullet \times q}[\xi]$. We call D *R-canonical* if DR^{-1} is a matrix of strictly proper rational functions. Let $\Phi \in \mathbb{R}_s^{q \times q}[\zeta, \eta]$. We call Φ *R-canonical* if $(R^T(\zeta))^{-1}\Phi(\zeta, \eta)(R(\eta))^{-1}$ is a matrix of strictly proper two-variable rational functions. (Note that there is no ambiguity about what “strictly proper” means for these two-variable rational functions.) Since for autonomous systems all differential operators can be seen as instantaneous functions of the state, it is clear that for any D there exists a canonical D' that is R -equivalent to D . The aim of the next result is to derive this also for QDFs.

PROPOSITION 4.9. *Let $D \in \mathbb{R}^{\bullet \times q}[\xi]$. Among all differential operators \mathfrak{B} -equivalent to D , there is exactly one, D' , which is R -canonical. This D' can be computed as follows. Compute $DR^{-1} \in \mathbb{R}^{\bullet \times q}(\xi)$ and write it as $DR^{-1} = P + S$, with P the polynomial part and S the strictly proper rational part of DR^{-1} . Then $D' = D - PR$. Let $\Phi \in \mathbb{R}_s^{q \times q}[\zeta, \eta]$. Among all QDFs \mathfrak{B} -equivalent to Φ there is exactly one, Φ' , which is R -canonical. This Φ' can be computed as follows. Write Φ as $\Phi(\zeta, \eta) = M^T(\zeta)N(\eta)$. Compute the R -canonical representatives M' of M and N' of N . Then $\Phi'(\zeta, \eta) = M'^T(\zeta)N'(\eta)$.*

Proof. For the proof, see the appendix.

The following proposition shows that \mathfrak{B} -positivity reduces to positivity of the \mathfrak{B} -canonical representative.

PROPOSITION 4.10. *If Ψ is R -canonical, then we have*

- (i) $(\Psi \stackrel{\mathfrak{B}}{=} 0) \Leftrightarrow (\Psi = 0)$,
- (ii) $(\Psi \stackrel{\mathfrak{B}}{\geq} 0) \Leftrightarrow (\Psi \geq 0) \Leftrightarrow (\Psi(\zeta, \eta) = D^T(\zeta)D(\eta) \text{ with } D \text{ } R\text{-canonical})$,
- (iii) $(\Psi \stackrel{\mathfrak{B}}{>} 0) \Leftrightarrow (\Psi > 0 \text{ and } \Psi(\zeta, \eta) = D^T(\zeta)D(\eta) \text{ with } (R, D) \text{ observable}) \Leftrightarrow (\Psi(\zeta, \eta) = D^T(\zeta)D(\eta) \text{ with } (R, D) \text{ observable and } D \text{ } R\text{-canonical})$.

Proof. For the proof, see the appendix.

We immediately obtain the following consequence of Theorem 4.3.

COROLLARY 4.11. *$\mathfrak{B} \in \mathfrak{L}^q$ is asymptotically stable iff there exists a $\Psi \in \mathbb{R}_s^{q \times q}[\zeta, \eta]$, $\Psi \geq 0$, such that the R -canonical representative of $(\zeta + \eta)\Psi(\zeta, \eta)$, computed as in Proposition 4.9, is ≤ 0 and factors as $-D^T(\zeta)D(\eta)$ with (R, D) observable.*

Our next result is perhaps the most useful of all. It shows how to walk through the algorithm of Theorem 4.8 and preserve canonicity.

THEOREM 4.12. *Assume that $\mathfrak{B} \in \mathfrak{L}^q$ is asymptotically stable and has kernel representation (2.1) with R square. Assume that Φ is R -canonical. Then the polynomial matrix Lyapunov equation (4.3) has a unique R -canonical solution. Denote it by X' . Then*

$$(4.10) \quad \Psi(\zeta, \eta) = \frac{\Phi(\zeta, \eta) - X'^T(\zeta)R(\eta) - R^T(\zeta)X'(\eta)}{\zeta + \eta}$$

is the unique R -canonical Ψ such that $\Psi \stackrel{\bullet}{=} \mathfrak{B} \Phi$. Hence if $\Phi \leq 0$, then $\Psi \geq 0$, and if, in addition, $\Phi(\zeta, \eta) = -D^T(\zeta)D(\eta)$ with (R, D) observable, then $\Psi \stackrel{\mathfrak{B}}{\gg} 0$.

Proof. For the proof, see the appendix.

We make a short comment relating these results to state representations. The state maps (2.13) associating a minimal state to \mathfrak{B} are uniquely defined up to \mathfrak{B} -equivalence. There is, consequently, a minimal state map (unique up to premultiplication by a nonsingular matrix that is R -canonical, say, $x = X(\frac{d}{dt})w$). An R -canonical

$\Phi \in \mathbb{R}_s^{q \times q}[\zeta, \eta]$ is of the form $Q_\Phi(w) = x^T \Gamma x$, i.e., $\Phi(\zeta, \eta) = X^T(\zeta) \Gamma X(\eta)$, with $\Gamma = \Gamma^T$ an $(n \times n)$ matrix. Of course for this Φ there holds $(\Phi \stackrel{\mathfrak{B}}{\geq} 0) \Leftrightarrow (\Phi \geq 0) \Leftrightarrow (\Gamma \geq 0)$; furthermore, $(\Phi \stackrel{\mathfrak{B}}{>} 0) \Leftrightarrow (\Gamma \geq 0 \text{ and observability of the pair of matrices } (A, \Gamma) \text{ (with } A \text{ associated with the state } x))$, and finally $(\Phi \stackrel{\mathfrak{B}}{\gg} 0) \Leftrightarrow (\Gamma > 0)$.

The above results allow generalizations to unstable systems. Let us briefly mention a few. We have seen that $(\mathfrak{B} \text{ asymptotically stable}) \Leftrightarrow (\exists \Psi(\zeta, \eta) \text{ such that } (\Psi \stackrel{\mathfrak{B}}{\geq} 0 \text{ and } \dot{\Psi} \stackrel{\mathfrak{B}}{<} 0))$. There also holds $(\mathfrak{B} \text{ stable}) \Leftrightarrow (\exists \Psi(\zeta, \eta) \text{ such that } (\Psi \stackrel{\mathfrak{B}}{>} 0 \text{ and } \dot{\Psi} \stackrel{\mathfrak{B}}{\leq} 0))$ and $(\text{an autonomous } \mathfrak{B} \text{ is not stable}) \Leftrightarrow (\exists \Psi(\zeta, \eta) \text{ such that } (\Psi \not\stackrel{\mathfrak{B}}{\geq} 0 \text{ and } \dot{\Psi} \stackrel{\mathfrak{B}}{<} 0))$. Furthermore, the result that a Lyapunov function Ψ can be constructed so that it has a given derivative Φ (Theorem 4.8) can be generalized to autonomous systems, as long as they have the property that if λ is a singularity of R then $-\lambda$ will not be a singularity of R . As such this theorem extends in this sense to a large class of unstable systems.

We close this section with two extensive examples.

Example 4.13. In this first example we use Theorem 4.8 in order to give a Lyapunov proof of the Routh–Hurwitz test for stability of scalar systems. Let $R \in \mathbb{R}[\xi]$ be a Hurwitz polynomial. Hence $R(\frac{d}{dt})w = 0$ defines an asymptotically stable scalar system. Take, for the derivative of the Lyapunov function,

$$(4.11) \quad \Phi(\zeta, \eta) = -\frac{1}{2}R(-\zeta)R(-\eta) = -\frac{1}{2}R^*(\zeta)R^*(\eta).$$

Then obviously, since R has no imaginary axis roots, (R, R^*) is an observable (i.e., a coprime) pair. The polynomial matrix Lyapunov equation (4.3) yields $X(\xi) = -\frac{1}{4}R(\xi)$ as a solution. Take $Y(\zeta, \eta) = X(\eta)$. Then (4.7) yields

$$(4.12) \quad B(\zeta, \eta) = \frac{1}{2} \frac{R(\zeta)R(\eta) - R(-\zeta)R(-\eta)}{\zeta + \eta}$$

as a Lyapunov function. Note that this Lyapunov function can be written directly from the system parameters, without having to solve linear equations! This fact is actually well known, even though it is not presented in the vein of providing a higher order Lyapunov function ([6], [13], [14]).

The two-variable polynomial B defined by (4.12) is called the *Bezoutian* of R . Note that $\dot{B}(\zeta, \eta) \stackrel{\mathfrak{B}}{=} -\frac{1}{2}R(-\zeta)R(-\eta)$; B is R -canonical, but \dot{B} is not. However, \dot{B} is \mathfrak{B} -equivalent to $-\frac{1}{2}R(-\zeta)R(-\eta) + \frac{1}{2}R(\zeta)R(\eta)$, which is. If we take this for the Φ in Theorem 4.8, then the Lyapunov equation yields $X = 0$. Taking $Y = 0$ then also yields the Bezoutian (4.12) as the corresponding (hence R -canonical) Lyapunov function B .

A close examination of the arguments involved yields the equivalence of the following three conditions on a polynomial R of degree n and the corresponding $B \in \mathbb{R}[\zeta, \eta]$ given by (4.12):

1. R is Hurwitz,
2. $B \geq 0$ and (R, R^*) is coprime,
3. \tilde{B} (the constant matrix associated with B) has rank n and is ≥ 0 .

The Lyapunov function (4.12), the Bezoutian, is a very useful one for deriving various stability tests. It is a classical concept in stability (see [11] for a recent reference). Let us illustrate its usefulness by deriving the Routh stability test from it.

Let $R \in \mathbb{R}[\xi]$ be a polynomial of degree n . Decompose R in its even and odd parts as

$$R(\xi) = E_0(\xi^2) + \xi E_1(\xi^2).$$

Form the *Routh table* by computing the polynomials E_2, E_3, \dots, E_n as

$$E_k(\xi) = \xi^{-1}(E_{k-1}(0)E_{k-2}(\xi) - E_{k-2}(0)E_{k-1}(\xi)).$$

Assume for simplicity that $R(0) = E_0(0) \geq 0$. Routh's stability criterion states that R is Hurwitz iff all elements of the *Routh array* $E_0(0), E_1(0), \dots, E_n(0)$ are positive. Define $R_k(\xi) = E_{k-1}(\xi^2) + \xi E_k(\xi^2)$ for $k = 1, \dots, n$, and let B_k be the Bezoutian associated with R_k . Examining expression (4.12) yields, after a simple calculation

$$(4.13) \quad E_k(0)B_k(\zeta, \eta) = \zeta \eta B_{k+1}(\zeta, \eta) + E_{k-1}(0)E_k(\zeta^2)E_k(\eta^2)$$

for $k = 1, \dots, n$ (define $B_{n+1} = 0$). Assume that $E_0(0), E_1(0), \dots, E_n(0)$ are all positive. Then we obtain (note that $B = B_1$)

$$B(\zeta, \eta) = \sum_{k=1}^n \alpha_k \zeta^{k-1} \eta^{k-1} E_k(\zeta^2) E_k(\eta^2),$$

where $\alpha_k = E_{k-1}(0)/E_1(0)E_2(0)\dots E_k(0)$. Obviously $\tilde{B} \geq 0$ and has rank n . Therefore R is Hurwitz. To show the converse, assume that R is Hurwitz. Then $E_0(0) > 0$. Also, $\tilde{B} \geq 0$ and has rank n . Therefore, by (4.13), $\tilde{B}_2 \geq 0$ and has rank $n - 1$. Hence R_2 is Hurwitz, and $E_1(0) > 0$. Now proceed by induction.

The key point thus is that

$$\sum_{k=1}^n \alpha_k \left(E_k \left(\frac{d^2}{dt^2} \right) \frac{d^{k-1}}{dt^{k-1}} w \right)^2$$

is a QDF which is well defined and nonnegative definite when the Routh conditions are satisfied. It has derivative

$$\frac{1}{2} \left(\left(R \left(-\frac{d}{dt} \right) w \right)^2 - \left(R \left(\frac{d}{dt} \right) w \right)^2 \right),$$

which is obviously nonnegative definite along solutions of $R(\frac{d}{dt})w = 0$.

Example 4.14. Let E_1, E_2, \dots, E_N and $O_1, O_2, \dots, O_{N'}$ be two sets of real polynomials, and assume that $R_{k,\ell}(\xi) := E_k(\xi^2) + \xi O_\ell(\xi^2)$ is Hurwitz for all $k = 1, 2, \dots, N$ and $\ell = 1, 2, \dots, N'$. Then any combination

$$R(\xi) = \sum_{k=1}^N \alpha_k E_k(\xi^2) + \sum_{\ell=1}^{N'} \beta_\ell \xi O_\ell(\xi^2)$$

is also Hurwitz whenever all the α_k 's and β_ℓ 's are positive. In order to see this, simply observe that, in the obvious notation, (4.12) yields

$$B(\zeta, \eta) = \sum_{k=1}^N \sum_{\ell=1}^{N'} \alpha_k \beta_\ell B_{k,\ell}(\zeta, \eta)$$

and the conclusion follows.

This may be applied to interval polynomials. Assume that $R \in R[\xi]$ is given by $R(\xi) = R_0 + R_1\xi + \dots + R_n\xi^n$, with $R_k \in [a_k, A_k]$. The question arises under what conditions all these polynomials are Hurwitz. The *weak Kharitonov* test states that this is the case iff the 2^n extreme polynomials, that is, those obtained by replacing each R_k by a_k or A_k , are all Hurwitz. This result is an immediate consequence of the above. With a little bit of extra work, we can also obtain the *strong Kharitonov* test [15] which states that the interval polynomials are Hurwitz iff the four *Kharitonov* polynomials obtained by taking the initial sequences a_0, a_1, A_2, \dots , or a_0, A_1, A_2, \dots , or A_0, A_1, a_2, \dots , or A_0, a_1, a_2, \dots , and continuing by alternating between two consecutive maxima and minima, are all Hurwitz. Indeed (see [20]), observe that for all $\omega, R(i\omega)$ lies in the rectangle in the complex plane spanned by the four points obtained by taking for R the Kharitonov polynomials. This rectangle does not contain the origin since, by the above, the convex hull of the Kharitonov polynomials contains only Hurwitz polynomials if the Kharitonov polynomials are themselves Hurwitz.

5. Average positivity. Up to now, we have considered positivity of QDFs and its use in establishing stability through Lyapunov functions. However, in many applications, especially in control theory, we are interested in an average type of positivity. In section 3, we already discussed when $\int Q_\Phi$ is zero. We now study when it is positive. With an eye towards applications in LQ and H_∞ control we have to distinguish several (unfortunately not less than three) types of average positivity. All of them have quite logical definitions.

DEFINITION 5.1. Let $\Phi \in \mathbb{R}_s^{q \times q}[\zeta, \eta]$. The QDF Q_Φ (or simply Φ) is said to be

1. average nonnegative, denoted $\int Q_\Phi \geq 0$, if $\int_{-\infty}^{+\infty} Q_\Phi(w)dt \geq 0$ for all $w \in \mathfrak{D}(\mathbb{R}, \mathbb{R}^q)$,
2. average positive, denoted by $\int Q_\Phi > 0$, if $\int Q_\Phi \geq 0$ and if $\int_{-\infty}^{+\infty} Q_\Phi(w)dt = 0$ implies $w = 0$,
3. strongly average positive, denoted $\int Q_\Phi \overset{\text{per}}{>} 0$, if for all nonzero periodic $w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^q)$ there holds $\frac{1}{T} \int_0^T Q_\Phi(w)dt > 0$, where T denotes the period of w .

Note that (3) looks somewhat different from the other definitions in this paper since, for the first time, periodic functions are involved. Actually, wherever in the paper definitions refer to compact support functions, they could have been written just as well in terms of periodic functions. However, strong average positivity is the only instance where the converse is not true.

PROPOSITION 5.2. Let $\Phi \in \mathbb{R}_s^{q \times q}[\zeta, \eta]$. Then

- (i) $(\int Q_\Phi \geq 0) \iff (\partial\Phi(i\omega) \geq 0 \ \forall \omega \in \mathbb{R})$.
- (ii) $(\int Q_\Phi > 0) \iff (\partial\Phi(i\omega) \geq 0 \ \forall \omega \in \mathbb{R} \text{ and } \det(\partial\Phi) \neq 0)$.
- (iii) $(\int Q_\Phi \overset{\text{per}}{>} 0) \iff (\partial\Phi(i\omega) > 0 \ \forall \omega \in \mathbb{R})$.

Proof. For the proof, see the appendix.

Concerning the equivalence (ii), note that $\partial\Phi(i\omega) \geq 0 \ \forall \omega \in \mathbb{R}$ and $\det(\partial\Phi) \neq 0$ is equivalent to: $\partial\Phi(i\omega) > 0$ for all but finitely many $\omega \in \mathbb{R}$.

Intuitively, we think of $Q_\Phi(w)$ as the power going into a physical system. In many applications, the power is indeed a quadratic differential form of some system variables. For example, in mechanical systems, it is $\sum_k F_k \frac{dq_k}{dt}$ with F_k the external force acting on the system, and q_k the position of the k th pointmass; in electrical circuits it is $\sum_k V_k I_k$, with V_k the potential and I_k the current going into the circuit at the k th terminal. Note that in these examples the variables are themselves also related. When this relation is expressed as an image representation, then we obtain a general QDF in terms of latent variables for the power delivered to a system.

Average nonnegativity states that the net flow of energy going into the system is nonnegative: the system dissipates energy. Of course, sometimes energy flows into the system, while at other times it flows out of it. This outflow is due to the fact that energy is stored. However, because of dissipation, the rate of increase of storage cannot exceed the supply. This interaction between supply, storage, and dissipation is now formalized.

DEFINITION 5.3. Let $\Phi \in \mathbb{R}_s^{q \times q}[\zeta, \eta]$ induce the QDF Q_Φ . The QDF Q_Ψ induced by $\Psi \in \mathbb{R}_s^{q \times q}[\zeta, \eta]$ is said to be a storage function for Φ if

$$(5.1) \quad \frac{d}{dt} Q_\Psi \leq Q_\Phi.$$

A QDF Q_Δ induced by $\Delta \in \mathbb{R}_s^{q \times q}[\zeta, \eta]$ is said to be a dissipation function for Φ if

$$(5.2) \quad \Delta \geq 0 \text{ and } \int Q_\Phi = \int Q_\Delta.$$

The next proposition shows that one can always interpret average positivity by an instantaneous positivity condition involving the difference between the rate of change of storage function and the supply rate.

PROPOSITION 5.4. *The following conditions are equivalent:*

1. $\int Q_\Phi \geq 0$,
2. Φ admits a storage function,
3. Φ admits a dissipation function.

Moreover, there is a one-one relation between storage and dissipation functions, Ψ and Δ , respectively, defined by

$$\frac{d}{dt} Q_\Psi(w) = Q_\Phi(w) - Q_\Delta(w)$$

equivalently, $\dot{\Psi} = \Phi - \Delta$, i.e.,

$$(5.3) \quad \Psi(\zeta, \eta) = \frac{\Phi(\zeta, \eta) - \Delta(\zeta, \eta)}{\zeta + \eta}.$$

Proof. For the proof, see the appendix.

Of course, we should expect that a storage function is related to memory, to state. The question, however, is: *the state of which system?* After all, we are considering a QDF, not a dynamical system. However, the factorization of Φ as

$$(5.4) \quad \Phi(\zeta, \eta) = M^T(\zeta) \Sigma_M M(\eta)$$

discussed earlier in section 3 allows us to introduce a state for the QDF Q_Φ . Indeed, (5.4) induces the dynamical system in image representation

$$(5.5) \quad v = M \left(\frac{d}{dt} \right) w.$$

Note that in (5.5) we are considering w as the latent variable and v as the manifest one. This is in keeping with the idea that $v^T \Sigma_M v$, the supply rate, is the variable of interest and that w is a latent variable that explains it. We are hence considering the behavior of the possible trajectories v . Assume that M has r rows, i.e., that

$M \in \mathbb{R}^{r \times q}[\xi]$. Thus, (5.5) defines a system $\mathfrak{B} \in \mathfrak{L}^r$ with $\mathfrak{B} = \text{im}(M(\frac{d}{dt}))$ and $M(\frac{d}{dt})$ viewed as a map from $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^q)$ to $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^r)$. Hence this system has a state representation. Assume that

$$(5.6) \quad x = X \left(\frac{d}{dt} \right) w$$

induces such a state representation. Thus $X \in \mathbb{R}^{\bullet \times q}[\xi]$ is a polynomial matrix defining a state map for $\mathfrak{B} = \text{im}(M(\frac{d}{dt}))$. Let $\Psi \in \mathbb{R}_s^{q \times q}[\zeta, \eta]$. Then the QDF Q_Ψ is said to be a *state function* (relative to the state of Φ) if there exists a real (symmetric) matrix P such that

$$(5.7) \quad Q_\Psi(w) = \|X \left(\frac{d}{dt} \right) w\|_P^2.$$

It is said to be a *state/supply function* if there exists a real (symmetric) matrix E such that

$$(5.8) \quad Q_\Psi(w) = \left\| \begin{bmatrix} M(\frac{d}{dt}) \\ X(\frac{d}{dt}) \end{bmatrix} w \right\|_E^2$$

where, as always, $\|a\|_A^2$ denotes $a^T A a$. Note that the factorization (5.4) is not unique. However, any such factorization is related in a simple way to a canonical one, say, to

$$(5.9) \quad \Phi(\zeta, \eta) = \hat{M}^T(\zeta) \Sigma_\Phi \hat{M}(\eta)$$

by the existence of a matrix $F \in \mathbb{R}^{\bullet \times \bullet}$ such that $\hat{M}(\xi) = F M(\xi)$. This relation has as a consequence that any (possibly nonminimal) state map X for the system in image representation (5.5) is related in a static way to a minimal state map \hat{X} associated with the system in image representation

$$(5.10) \quad \hat{v} = \hat{M} \left(\frac{d}{dt} \right) w$$

based on a canonical factorization. Indeed, there exists a matrix $L \in \mathbb{R}^{\bullet \times \bullet}$ such that

$$(5.11) \quad \hat{X}(\xi) = L X(\xi).$$

Thus, considering arbitrary (i.e., not necessarily canonical) factorizations and arbitrary (i.e., not necessarily minimal) state representations yields a (rather than *the*) state of Q_Φ . Thus, the situation with the state is similar to the situation with the state of a system $\mathfrak{B} \in \mathfrak{L}^q$.

We have the following important result.

THEOREM 5.5. *Let $\int Q_\Phi \geq 0$, and let $\Psi \in \mathbb{R}_s^{q \times q}[\zeta, \eta]$ be a storage function for Φ , i.e., $\dot{\Psi} \leq \Phi$. Then Ψ is a state function. Let $\Delta \in \mathbb{R}_s^{q \times q}[\zeta, \eta]$ be a dissipation function for Φ . Then Δ is a state/supply function. In fact, if $X \in \mathbb{R}^{\bullet \times q}[\xi]$ is a state map for Φ , then there exist real symmetric matrices P and E such that*

$$(5.12) \quad \Psi(\zeta, \eta) = X^T(\zeta) P X(\eta),$$

$$(5.13) \quad \Delta(\zeta, \eta) = \begin{bmatrix} M(\zeta) \\ X(\zeta) \end{bmatrix}^T E \begin{bmatrix} M(\eta) \\ X(\eta) \end{bmatrix}.$$

Equivalently

$$L_{\Psi}(w_1, w_2) = \left(X \left(\frac{d}{dt} \right) w_1 \right)^T P X \left(\frac{d}{dt} \right) w_2,$$

$$L_{\Delta}(w_1, w_2) = \begin{bmatrix} M(\frac{d}{dt})w_1 \\ X(\frac{d}{dt})w_1 \end{bmatrix}^T E \begin{bmatrix} M(\frac{d}{dt})w_2 \\ X(\frac{d}{dt})w_2 \end{bmatrix}$$

for all $w_1, w_2 \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^q)$.

Proof. For the proof, see the appendix.

Let $\Gamma \in \mathbb{R}^{q \times q}[\xi]$ be para-Hermitian: $\Gamma^* = \Gamma$. An $F \in \mathbb{R}^{q \times q}[\xi]$ is said to induce a *symmetric factorization* of Γ if $\Gamma(\xi) = F^T(-\xi)F(\xi)$. It is said to be a *symmetric Hurwitz factorization* if F is square and Hurwitz and a *symmetric anti-Hurwitz factorization* if F^* is square and Hurwitz. It is easy to see that for a symmetric factorization to exist we need to have $\Gamma(i\omega) \geq 0 \forall \omega \in \mathbb{R}$ and for an (anti-)Hurwitz one to exist we must have $\Gamma(i\omega) > 0 \forall \omega \in \mathbb{R}$. The converses are also true but not at all trivial in the matrix case. This result is well known (see, e.g., [9], [10], [18], [21]), and we state it for easy reference.

PROPOSITION 5.6. *Let $\Gamma \in \mathbb{R}^{q \times q}[\xi]$ be para-Hermitian. Then*

- (i) Γ allows a symmetric factorization iff $\Gamma(i\omega) \geq 0$ for all $\omega \in \mathbb{R}$.
- (ii) Γ allows a symmetric Hurwitz factorization iff $\Gamma(i\omega) > 0$ for all $\omega \in \mathbb{R}$. Such a factorization $\Gamma(\xi) = F^T(-\xi)F(\xi)$ is unique up to premultiplication of $F(\xi)$ by an orthogonal matrix.
- (iii) Γ allows a symmetric anti-Hurwitz factorization iff $\Gamma(i\omega) > 0$ for all $\omega \in \mathbb{R}$. Such a factorization $\Gamma(\xi) = F^T(-\xi)F(\xi)$ is unique up to premultiplication of $F(\xi)$ by an orthogonal matrix.

An important issue of concern is the uniqueness of the storage function, and therefore of the dissipation function, because of the one-to-one relation between the two. When $\int Q_\Phi = 0$, then the associated storage function is unique ($\Psi(\zeta, \eta) = \frac{\Phi(\zeta, \eta)}{\zeta + \eta}$) and the dissipation function is zero. However, in general there are many possibilities.

THEOREM 5.7. *Let $\int Q_\Phi \geq 0$. Then there exist storage functions Ψ_- and Ψ_+ for Φ such that any other storage function Ψ for Φ satisfies*

$$(5.14) \quad \Psi_- \leq \Psi \leq \Psi_+.$$

If $\int Q_\Phi \stackrel{per}{>} 0$ then Ψ^- and Ψ^+ may be constructed as follows. Let $\partial\Phi(\xi) = H^T(-\xi)H(\xi)$ and $\partial\Phi(\xi) = A^T(-\xi)A(\xi)$ be, respectively, Hurwitz and anti-Hurwitz factorizations of $\partial\Phi$. Then

$$(5.15) \quad \Psi_+(\zeta, \eta) = \frac{\Phi(\zeta, \eta) - A^T(\zeta)A(\eta)}{\zeta + \eta}$$

and

$$(5.16) \quad \Psi_-(\zeta, \eta) = \frac{\Phi(\zeta, \eta) - H^T(\zeta)H(\eta)}{\zeta + \eta}.$$

Proof. For the proof, see the appendix.

We close this section with a few remarks.

Remark 5.8. In this section we have studied average positivity with, in $Q_\Phi(w)$, $w \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^q)$ or $\mathcal{D}(\mathbb{R}, \mathbb{R}^q)$, but otherwise free. It is of interest to generalize these

concepts to the case that $w \in \mathfrak{B}$ with \mathfrak{B} a given element of \mathfrak{L}^q . Of course, in section 4 we have considered precisely such situations for \mathfrak{B} 's that are *autonomous*. Actually, it turns out that the theory of section 5 is immediately applicable to systems $\mathfrak{B} \in \mathfrak{L}^q$ that are *controllable*. Indeed, let $\mathfrak{B} \in \mathfrak{L}^q$ be controllable and assume that we want to study when

$$(5.17) \quad \int_{-\infty}^{+\infty} Q_{\Phi}(w)dt \geq 0 \quad \text{or} \quad \int_{-\infty}^{+\infty} Q_{\Phi}(w)dt = 0$$

holds for all $w \in \mathfrak{B} \cap \mathfrak{D}(\mathbb{R}, \mathbb{R}^q)$. Simply construct an image representation for \mathfrak{B} , say,

$$(5.18) \quad w = M \left(\frac{d}{dt} \right) \ell.$$

Upon substituting (5.18) in (5.17), we see that the issue then becomes one of studying when

$$\int_{-\infty}^{+\infty} Q_{\Phi} \left(M \left(\frac{d}{dt} \right) \ell \right) dt \geq 0 \quad \text{or} \quad \int_{-\infty}^{+\infty} Q_{\Phi} \left(M \left(\frac{d}{dt} \right) \ell \right) dt = 0$$

for all $\ell \in \mathfrak{D}(\mathbb{R}, \mathbb{R}^{\bullet})$. Since obviously $Q_{\Phi}(M(\frac{d}{dt})\ell) = Q_{\Phi'}(\ell)$ with

$$\Phi'(\zeta, \eta) := M^T(\zeta)\Phi(\zeta, \eta)M(\eta),$$

the problem reduces to studying Φ' . For example, the existence of a storage function is established as follows. Without loss of generality, take (5.18) to be an observable image representation. Then M has a polynomial left inverse M^\dagger . By Proposition 5.4, there exists Ψ' such that $\frac{d}{dt}Q_{\Psi'}(\ell) \leq Q_{\Phi'}(\ell)$ for all $\ell \in \mathfrak{D}(\mathbb{R}, \mathbb{R}^{\bullet})$. Now define $\Psi(\zeta, \eta) := M^{\dagger T}(\zeta)\Psi'(\zeta, \eta)M^\dagger(\eta)$. Then for $w = M(\frac{d}{dt})\ell$ we have $Q_{\Psi}(w) = Q_{\Psi'}(M^\dagger(\frac{d}{dt})w) = Q_{\Psi'}(\ell)$ and $Q_{\Phi}(w) = Q_{\Phi'}(\ell)$, so we obtain $\frac{d}{dt}Q_{\Psi}(w) \leq Q_{\Phi}(w)$.

The case that $\mathfrak{B} \in \mathfrak{L}^q$ is neither controllable nor autonomous will be studied in a later publication. The next comment is relevant to the question of what the appropriate definition of dissipativity is in that case.

Remark 5.9. Finding an appropriate definition of a dissipative system is an issue that has attracted considerable attention (see [29], [12], [24], [27]). Of course, this is at the root of the issues discussed in the present article. Let $\mathfrak{B} \in \mathfrak{L}^q$. There are many examples where the instantaneous rate of supply (say, of energy) into the system is given, not by a static function of the external variables, but by a QDF, $Q_{\Phi}(w)$. The study of supply rates that are themselves dynamic is one of the novel aspects of the present paper. When would one want to call \mathfrak{B} dissipative with respect to $Q_{\Phi}(w)$? Lossless? Conservative? When would one want to say that \mathfrak{B} absorbs some of the supply? The definitions of average nonnegativity for dissipativeness, and $\int Q_{\Phi} = 0$ for losslessness (= conservativeness), are fully adequate *provided that \mathfrak{B} is controllable* (see Remark 5.8). However, Proposition 5.4 points to another definition which does not need controllability and which, in the controllable case, reduces to it. Thus, we arrive at the following definition as the most general: $\mathfrak{B} \in \mathfrak{L}^q$ is said to be *dissipative* with respect to the supply rate Q_{Φ} if there exists a Q_{Ψ} such that $\frac{d}{dt}Q_{\Psi}(w) \leq Q_{\Phi}(w)$ for all $w \in \mathfrak{B}$, and *lossless, or conservative*, if this holds with equality. The unfortunate aspect of this definition is its *existential* nature — it shares this notorious feature with the first and second law of thermodynamics. It does not seem an easy matter in the noncontrollable case to reduce this to a statement involving only Q_{Φ} , and without

invoking a to-be-constructed Q_Ψ . In Theorem 5.5 we have unraveled this existence question a bit by proving that this Q_Ψ will be a state function.

Note that the proposed definition of dissipativity and losslessness is an interesting generalization of the notion of a Lyapunov function since, for autonomous systems, it is natural to take the external supply $\Phi = 0$. Also note that this definition holds for any \mathfrak{B} and Φ and does not require the introduction of the notion of state. In other words, dynamical systems with free variables that allow interaction with the environment relate to flows on manifolds, just as dissipative systems relate to Lyapunov functions.

Remark 5.10. Let \mathfrak{B} be controllable and assume that it is dissipative with respect to the supply rate $Q_\Phi(w)$ (see Remark 5.9). Also in this general case every storage function is a state function, and every dissipation function is a state/supply function. However, this time, not simply the state of Φ is involved, but the state of a system obtained by combining the dynamics of Φ and \mathfrak{B} . This is elaborated in [26].

Remark 5.11. Let $\Phi(\zeta, \eta) = M^T(\zeta)\Sigma_M M(\eta)$. Consider the system in image representation (5.5). Then it can be shown that for Φ to be average nonnegative, there must be an input/output partition for this system so that all the input components correspond to +1's in Σ_Φ . In other words, the supply rate $v^T \Sigma_M v$ is always of the form $\|u\|^2 + \|y_1\|^2 - \|y_2\|^2$, with u an input, and y_1, y_2 outputs.

Remark 5.12. It follows from Theorem 5.5 that a factorization of the polynomial matrix $\Phi(-\xi, \xi) = M^T(-\xi)\Sigma_M M(\xi)$ into $F^T(-\xi)F(\xi)$ always leads to a situation in which the McMillan degree of M is equal to that of $\text{col}(M, F)$. This means that the factorization is a *regular* factorization (as this property is called). In the H_∞ -problem factorization, questions are encountered in which the existence of a regular factorization poses a serious problem.

Remark 5.13. It is easy to see that the set of storage functions corresponding to a given supply rate is convex. Moreover, in the case of average positivity, $\Psi_- \neq \Psi_+$ and hence, in this case, there are an infinite number of possible storage functions. Actually, in this respect it is worth mentioning the following refinement of Theorem 5.7, which follows immediately from our proof of this theorem. If $\Phi(-\xi, \xi)$ satisfies $\Phi(-i\omega, i\omega) \geq 0$ (but not $\Phi(-i\omega, i\omega) > 0$) for all $\omega \in \mathbb{R}$, then a symmetric Hurwitz factorization does not exist. In this case, there are two possibilities: either $\det(\partial\Phi) \neq 0$ or $\det(\partial\Phi) = 0$. In the former case, $\Phi(-\xi, \xi)$ allows a factorization $\Phi(-\xi, \xi) = H^T(-\xi)H(\xi)$ with H “almost Hurwitz” (i.e., H has all its singularities in $\Re(\lambda) \leq 0$). In the latter case, there exists a unimodular matrix U such that

$$\Phi(\zeta, \eta) = U^T(\zeta)\Phi'(\zeta, \eta)U(\eta),$$

with Φ' of the form

$$\Phi' = \begin{bmatrix} \Phi_1 & \Phi_2 \\ \Phi_2^* & \Phi_3 \end{bmatrix},$$

with $\det(\partial\Phi_1) \neq 0$ and $\partial\Phi_2 = 0, \partial\Phi_3 = 0$. Factor $\Phi_1(-\xi, \xi)$ as before as $H_1^T(-\xi)H_1(\xi)$. Then

$$H = \begin{bmatrix} H_1 & 0 \\ 0 & 0 \end{bmatrix} U_1$$

yields an almost Hurwitz-like factorization of $\partial\Phi$. Similarly, we can define an almost anti-Hurwitz-like factorization $\partial\Phi = A^*A$ of any Φ satisfying $\Phi(-i\omega, i\omega) \geq 0$ for all $\omega \in \mathbb{R}$. The computation of Ψ_+ and Ψ_- given in Theorem 5.7 holds unaltered with

this A and H . Note that in the lossless case ($\partial\Phi = 0$) this yields $\Psi_+ = \Psi_-$, whence the uniqueness of Ψ .

Remark 5.14. It is easy to deduce from the proof of Theorem 5.7 that Ψ_- and Ψ_+ have the following interpretations. Let $x = X(\frac{d}{dt})w$ be the state (see 5.6). Let $a \in \mathbb{R}^n$. Consider all $w \in \mathfrak{D}(\mathbb{R}, \mathbb{R}^q)$ such that $(X(\frac{d}{dt})w)(0) = a$. Denote this set by \mathfrak{B}_a . By Theorem 5.5 we know that Q_{Ψ_-} is a state function, say, $Q_{\Psi_-}(w) = \|X(\frac{d}{dt})w\|_{K_-}$, for some symmetric matrix $K_- \in \mathbb{R}^{n \times n}$. Hence for $w \in \mathfrak{B}_a$ we have $Q_{\Psi_-}(w)(0) = a^T K_- a$. Similarly $Q_{\Psi_+}(w)(0) = a^T K_+ a$ for some symmetric matrix K_+ . Then it can be shown that

$$(5.19) \quad a^T K_- a = \sup_{w \in \mathfrak{B}_a} \left(- \int_0^{+\infty} Q_{\Phi}(w) dt \right)$$

and

$$(5.20) \quad a^T K_+ a = \inf_{w \in \mathfrak{B}_a} \left(\int_{-\infty}^0 Q_{\Phi}(w) dt \right).$$

For this reason, $Q_{\Psi_-}(w)(0)$ is called the *available storage* and $Q_{\Psi_+}(w)(0)$ the *required supply* at $t = 0$ due to w . In this inf and sup, one keeps the past, respectively, the future of w fixed.

6. Half-line positivity. In section 5, we studied QDFs for which $\int_{-\infty}^{+\infty} Q_{\Phi}(w) dt \geq 0$. The intuitive idea was that this expresses that the net supply (of “energy”) is directed into the system: energy is being absorbed and dissipated in the system. There are, however, situations where at any moment in time the system has absorbed energy, i.e., $\int_{-\infty}^t Q_{\Phi}(w)(\tau) d\tau \geq 0$ for all $t \in \mathbb{R}$. For example, electrical circuits and mechanical devices at rest are in a state of minimum energy, and therefore the energy delivered *up to any time* is nonnegative. This type of positivity is studied in this section. It plays a crucial role in H_{∞} problems.

DEFINITION 6.1. Let $\Phi \in \mathbb{R}_s^{q \times q}[\zeta, \eta]$. The QDF Q_{Φ} (or simply Φ) is said to be half-line nonnegative, denoted by $\int^t Q_{\Phi} \geq 0$, if $\int_{-\infty}^0 Q_{\Phi}(w) dt \geq 0$ for all $w \in \mathfrak{D}(\mathbb{R}, \mathbb{R}^q)$, and half-line positive, denoted $\int^t Q_{\Phi} > 0$, if in addition $\int_{-\infty}^0 Q_{\Phi}(w) dt = 0$ implies $w(t) = 0$ for $t \leq 0$.

Note that half-line nonnegativity implies average nonnegativity, and that half-line positivity implies average positivity.

Write $\Phi(\zeta, \eta) = M^T(\zeta) \Sigma_M M(\eta)$ and partition M conform Σ_M as

$$(6.1) \quad M = \begin{bmatrix} P \\ N \end{bmatrix}$$

so that $\Phi(\zeta, \eta) = P^T(\zeta)P(\eta) - N^T(\zeta)N(\eta)$ and hence $Q_{\Phi}(w) = \|P(\frac{d}{dt})w\|^2 - \|N(\frac{d}{dt})w\|^2$. In the following, for $\lambda \in \mathbb{C}$, let $\bar{\lambda}$ denote its complex conjugate.

PROPOSITION 6.2. Let $\Phi \in \mathbb{R}_s^{q \times q}[\zeta, \eta]$. Then

- (i) $(\int^t Q_{\Phi} \geq 0) \Rightarrow (\Phi(\bar{\lambda}, \lambda) \geq 0 \ \forall \lambda \in \mathbb{C}, \Re(\lambda) \geq 0)$
- (ii) $(\int^t Q_{\Phi} > 0) \Rightarrow (\Phi(\bar{\lambda}, \lambda) \geq 0 \ \forall \lambda \in \mathbb{C}, \Re(\lambda) \geq 0 \text{ and } \det(\partial\Phi) \neq 0)$.

Proof. For the proof, see the appendix.

As noted before, it immediately follows from the definitions that half-line nonnegativity implies average nonnegativity, etc. Thus, Proposition 5.4 implies the existence of a storage function. It is the nonnegativity of the storage function that allows us to conclude the *half-line* positivity.

THEOREM 6.3. Let $\Phi \in \mathbb{R}_s^{q \times q}[\zeta, \eta]$. Then the following statements are equivalent.

1. $\int^t Q_\Phi \geq 0$,
2. *there exists a storage function $\Psi \geq 0$ for Φ ,*
3. *Φ admits a storage function, and the storage function Ψ_+ defined in Theorem 5.7 satisfies $\Psi_+ \geq 0$.*

Proof. For the proof, see the appendix.

In order to check half-line nonnegativity, one could thus in principle proceed as follows. Verify that $\Phi(-i\omega, i\omega) \geq 0$ for all $\omega \in \mathbb{R}$, compute Ψ_+ , and check whether $\Psi_+ \geq 0$. In some situations, it is actually possible to verify this condition in a more immediate fashion; for example, when $\Phi(\zeta, \eta) = \Phi_0$, a constant matrix, with $\Phi_0 \geq 0$ (trivial, but that is the case that occurs in standard LQ theory!), or when in (6.1) P is square and $\det(P) \neq 0$. Then, under the assumption that a storage function exists (equivalently: $N^T(-i\omega)N(i\omega) \leq P^T(-i\omega)P(i\omega)$ for all $\omega \in \mathbb{R}$), all storage functions are actually nonnegative if one of them is nonnegative. In fact, in this case the following theorem holds.

THEOREM 6.4. *Let $\Phi \in \mathbb{R}_s^{q \times q}[\zeta, \eta]$. Assume it is factored as $\Phi(\zeta, \eta) = P^T(\zeta)P(\eta) - N^T(\zeta)N(\eta)$ with P square and $\det(P) \neq 0$. Let $X \in \mathbb{R}^{\bullet \times q}[\xi]$ be a minimal state map for the \mathfrak{B} given in image representation by (6.1). The following statements are equivalent:*

1. $\int^t Q_\Phi \geq 0$,
2. $\Phi(\bar{\lambda}, \lambda) \geq 0$ for all $\lambda \in \mathbb{C}$, $\Re(\lambda) \geq 0$,
3. NP^{-1} has no poles in $\Re(\lambda) \geq 0$ and $\Phi(-i\omega, i\omega) \geq 0$ for all $\omega \in \mathbb{R}$,
4. *there exists a storage function $\Psi \geq 0$ for Φ ,*
5. *there exists a storage function for Φ and every storage function Ψ for Φ satisfies $\Psi \geq 0$,*
6. *there exists a real symmetric matrix $K > 0$ such that $Q_K(w) := \|X(\frac{d}{dt})w\|_K^2$ is a storage function for Φ ,*
7. *there exists a storage function for Φ and every real symmetric matrix K such that $Q_K(w) := \|X(\frac{d}{dt})w\|_K^2$ is a storage function for Φ satisfies $K > 0$.*

Furthermore, if $\begin{bmatrix} P \\ N \end{bmatrix}$ is observable, then any of the above statements is equivalent with 3'. P is Hurwitz and $\Phi(-i\omega, i\omega) \geq 0$ for all $\omega \in \mathbb{R}$.

Proof. For the proof, see the appendix.

7. Observability. One of the noticeable features of QDFs is that a number of interesting systems theory concepts generalize very nicely to QDFs. We have already seen that the state of a symmetric canonical factorization of Φ functions as the state of the QDF Q_Φ . In this section we introduce observability of a QDF. In a later section we will discuss duality of QDFs.

For $\Phi \in \mathbb{R}^{q_1 \times q_2}[\zeta, \eta]$ and $w_1 \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^{q_1})$ fixed, the linear map $w_2 \mapsto L_\Phi(w_1, w_2)$ is denoted by $L_\Phi(w_1, \bullet)$. For $w_2 \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^{q_2})$ fixed, the linear map $w_1 \mapsto L_\Phi(w_1, w_2)$ is denoted by $L_\Phi(\bullet, w_2)$. The BLDF Φ is called observable if $L_\Phi(w_1, \bullet)$ and $L_\Phi(\bullet, w_2)$ determine w_1 and w_2 uniquely. Equivalently we have the following.

DEFINITION 7.1. *Let $\Phi \in \mathbb{R}^{q_1 \times q_2}[\zeta, \eta]$. We call Φ observable if, for all $w_1 \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^{q_1})$ and for all $w_2 \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^{q_2})$, we have*

$$L_\Phi(w_1, \bullet) = 0 \Leftrightarrow w_1 = 0$$

and

$$L_\Phi(\bullet, w_2) = 0 \Leftrightarrow w_2 = 0.$$

The following theorem gives necessary and sufficient conditions for observability purely in terms of the two-variable polynomial matrix Φ and in terms of the (one-

variable) polynomial matrices N and M occurring in any canonical factorization of Φ .

THEOREM 7.2. *Let $\Phi(\zeta, \eta) = N^T(\zeta)M(\eta)$ be a canonical factorization. The following statements are equivalent:*

1. Φ is observable,
2. for every $\lambda \in \mathbb{C}$, the rows of $\Phi(\lambda, \xi) \in \mathbb{R}^{q_1 \times q_2}[\xi]$, and the columns of $\Phi(\xi, \lambda) \in \mathbb{R}^{q_1 \times q_2}[\xi]$ are linearly independent over \mathbb{C} ,
3. $N(\lambda)$ and $M(\lambda)$ have full column rank for all $\lambda \in \mathbb{C}$; equivalently, the image representations $v_1 = N(\frac{d}{dt})w_1$ and $v_2 = M(\frac{d}{dt})w_2$ are observable,

Proof. For the proof, see the appendix.

If Φ is symmetric, then we have $\Phi^T(\lambda, \xi) = \Phi(\xi, \lambda)$, so condition (ii) above can be replaced by a single statement on the independence of the rows of $\Phi(\lambda, \xi)$. Also, in this case the maps $L_\Phi(w, \bullet)$ and $L_\Phi(\bullet, w)$ coincide. Furthermore, for the particular symmetric canonical factorization $\Phi(\zeta, \eta) = M_c^T(\zeta)\Sigma_\Phi M_c(\eta)$ obtained from (3.15), we have $|\Phi|(\zeta, \eta) = M_c^T(\zeta)M_c(\eta)$. Hence observability of Φ is also equivalent with $|\Phi| > 0$ and with the condition $|\Phi|(\bar{\lambda}, \lambda) > 0$ for all $\lambda \in \mathbb{C}$. This immediately yields the following.

COROLLARY 7.3. *Let $\Phi \in \mathbb{R}_s^{q \times q}[\zeta, \eta]$ and let $\Phi(\zeta, \eta) = M^T(\zeta)\Sigma_\Phi M(\eta)$ be a symmetric canonical factorization. Then the following statements are equivalent:*

1. Φ is observable,
2. $L_\Phi(w, \bullet) = 0 \Leftrightarrow w = 0$,
3. for every $\lambda \in \mathbb{C}$, the rows of $\Phi(\lambda, \xi) \in \mathbb{R}^{q_1 \times q_2}[\xi]$ are linearly independent over \mathbb{C} ,
4. $M(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$, equivalently, the image representation $v = M(\frac{d}{dt})w$ is observable,
5. $|\Phi| > 0$,
6. $|\Phi|(\bar{\lambda}, \lambda) > 0$ for all $\lambda \in \mathbb{C}$.

8. Strict positivity. Throughout this section we assume that $\Phi \in \mathbb{R}_s^{q \times q}[\zeta, \eta]$ is observable. We now introduce and develop the notion of strict positivity. The concept of strict half-line positivity given here is very analogous to that used by Meinsma [19].

DEFINITION 8.1. *Let $\Phi \in \mathbb{R}_s^{q \times q}[\zeta, \eta]$ be observable. We call the QDF Q_Φ strictly positive, denoted $\Phi \gg 0$, if there exists $\epsilon > 0$ such that $\Phi - \epsilon|\Phi| \geq 0$. We call it strictly average positive, denoted by $\int Q_\Phi \gg 0$, if there exists $\epsilon > 0$ such that*

$$(8.1) \quad \int_{-\infty}^{+\infty} Q_\Phi(w)dt \geq \epsilon \int_{-\infty}^{+\infty} Q_{|\Phi|}(w)dt$$

for all $w \in \mathfrak{D}(\mathbb{R}, \mathbb{R}^q)$. We call it strictly half-line positive, denoted $\int^t Q_\Phi \gg 0$, if there exists an $\epsilon > 0$ such that

$$(8.2) \quad \int_{-\infty}^0 Q_\Phi(w)dt \geq \epsilon \int_{-\infty}^0 Q_{|\Phi|}(w)dt$$

for all $w \in \mathfrak{D}(\mathbb{R}, \mathbb{R}^q)$. Note that (because of observability) strict positivity implies positivity, and similarly for the other cases.

These notions of strict positivity involve $|\Phi|$ which may be difficult to evaluate. However, it is possible to relate it to any canonical factorization of Φ . This is stated in the next proposition. For simplicity we state only the case of strict average positivity. However, completely analogous statements hold for simple strict positivity or for strict half-line positivity.

PROPOSITION 8.2. Let $\Phi \in \mathbb{R}_s^{q \times q}[\zeta, \eta]$ be observable and let $\Phi(\zeta, \eta) = P^T(\zeta)P(\eta) - N^T(\zeta)N(\eta)$ be a symmetric canonical factorization of Φ (see (3.12)). Denote $M = \begin{bmatrix} P \\ N \end{bmatrix}$. The following are equivalent:

1. Φ is strictly average positive.
2. There exists an $\epsilon > 0$ such that

$$(8.3) \quad \int_{-\infty}^{+\infty} \left\| M \left(\frac{d}{dt} \right) w \right\|_{\Sigma_\Phi}^2 dt \geq \epsilon \int_{-\infty}^{+\infty} \left\| M \left(\frac{d}{dt} \right) w \right\|^2 dt$$

for all $w \in \mathfrak{D}(\mathbb{R}, \mathbb{R}^q)$. Here $\|a\|_{\Sigma_\Phi}^2$ denotes $a^T \Sigma_\Phi a$.

3. There exists an $\alpha < 1$ such that

$$(8.4) \quad \int_{-\infty}^{+\infty} \left\| N \left(\frac{d}{dt} \right) w \right\|^2 dt \leq \alpha \int_{-\infty}^{+\infty} \left\| P \left(\frac{d}{dt} \right) w \right\|^2 dt$$

for all $w \in \mathfrak{D}(\mathbb{R}, \mathbb{R}^q)$.

Moreover, also for a noncanonical factorization (3.11), (2) and (3) (with Σ_Φ replaced by Σ_M) are equivalent and imply (1).

Proof. For the proof, see the appendix.

9. A Pick matrix condition for half-line positivity. It is surprisingly difficult to establish some type of analogue of Proposition 5.2 for half-line positivity, and earlier attempts [28], [30], [1]) turned out to be flawed. In Proposition 6.4 such an analogue of Proposition 5.2 was given but only in the special case where $\Phi(\zeta, \eta) = P^T(\zeta)P(\eta) - N^T(\zeta)N(\eta)$ with $\det(P) \neq 0$. In this section we give a necessary and sufficient condition for strict half-line positivity in terms of Φ .

As is well known, the Pick matrix plays an important role in system and circuit theory, in particular in connection with passivity properties of linear dynamical systems; see [34], [4], [5]. We derive a Pick-matrix-type test for nonnegativity of Ψ_+ . This test is perhaps the most original specific result of this paper. For simplicity we consider only the case of strict half-line positivity. First, however, we need to define the Pick-type matrix which may be computed effectively from a $\Phi \in \mathbb{R}_s^{q \times q}[\zeta, \eta]$. Let $F \in \mathbb{R}^{q \times q}[\zeta]$, and assume that $\det(F) \neq 0$. We call F *semisimple* if for all $\lambda \in \mathbb{C}$ the dimension of the kernel of $F(\lambda)$ is equal to the multiplicity of λ as a root of $\det(F)$. Note that F is certainly semisimple if $\det(F)$ has distinct roots. We now define the matrix T_Φ . Since the expression is much simpler in the semisimple case, we explain that case first.

DEFINITION 9.1 (semisimple case). Let $\Phi \in \mathbb{R}_s^{q \times q}[\zeta, \eta]$ be observable, and assume that $\det(\partial\Phi)$ has no roots on the imaginary axis. Let $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{C}$ be the roots of $\det(\partial\Phi)$ with positive real part and let $a_1, a_2, \dots, a_n \in \mathbb{C}^q$ be such that $\partial\Phi(\lambda_i)a_i = 0$, and such that the a_k 's associated with the same λ_i form a basis of $\ker(\partial\Phi(\lambda_i))$. Then the Pick matrix of Φ is defined as

$$(9.1) \quad T_\Phi := \left[\frac{\bar{a}_i^T \Phi(\bar{\lambda}_i, \lambda_j) a_j}{\bar{\lambda}_i + \lambda_j} \right]_{i,j=1,\dots,n}.$$

In order to define the matrix T_Φ in the general case, we need to take into account the algebraic multiplicities of the roots λ_i .

DEFINITION 9.2 (general case). Let $\Phi \in \mathbb{R}_s^{q \times q}[\zeta, \eta]$ be observable, $\det(\partial\Phi) \neq 0$, and assume that $\det(\partial\Phi)$ has no roots on the imaginary axis. Let $\lambda_1, \lambda_2, \dots, \lambda_k \in \mathbb{C}$ be the distinct roots of $\det(\partial\Phi)$ with positive real part, and denote by n_i the multiplicity

of λ_i as a root of $\det(\partial\Phi)$. For $i = 1, 2, \dots, k$, there are n_i linearly independent vectors $a_{i,0}, a_{i,1}, \dots, a_{i,n_i-1}$ determined by the (n_i) linear equations

$$\sum_{j=\ell}^{n_i-1} \binom{j}{\ell} (\partial\Phi)^{(j-\ell)}(\lambda_i) a_{i,j} = 0, \quad (\ell = 0, 1, \dots, n_i - 1).$$

Here, $(\partial\Phi)^{(k)}(\xi)$ denotes the k th derivative of the polynomial matrix $\partial\Phi$.

For $i = 1, 2, \dots, k$, define

$$A_i := \begin{bmatrix} a_{i,0} & 0 & \cdots & 0 \\ a_{i,0} & a_{i,1} & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ a_{i,0} & a_{i,1} & \cdots & a_{i,n_i-1} \end{bmatrix} \in \mathbb{C}^{n_i q \times n_i}$$

Also, define $\Phi_{i,j} \in \mathbb{C}^{n_i q \times n_j q}$ by defining its (r, s) th block to be the $q \times q$ matrix

$$(\Phi_{i,j})_{r,s} := \Phi^{(r,s)}(\bar{\lambda}_i, \lambda_j), \quad r = 1, 2, \dots, n_i; \quad s = 1, 2, \dots, n_j,$$

where $\Phi^{(k,\ell)}$ means taking the k th partial derivative with respect to ζ and the ℓ th with respect to η .

Then we define the Pick matrix of Φ as the matrix T_Φ whose (i, j) th block is given by $T_{i,j} \in \mathbb{C}^{n_i \times n_j}$, with

$$T_{i,j} := \frac{1}{\lambda_i + \lambda_j} \bar{A}_i^T \Phi_{i,j} A_j.$$

Note that the sum $\sum_{i=1}^k n_i$ of the multiplicities is equal to $n := \frac{1}{2} \deg \det(\partial\Phi)$, and that T_Φ is a complex Hermitian matrix of size $n \times n$.

The next theorem is the most refined result of this paper. It shows, on the one hand, the relation between strict half-line positivity and positivity of a storage function, and, on the other hand, the relation with the positivity of the Pick matrix T_Φ .

We have seen in Theorem 5.5 that a storage function is a quadratic state function, i.e., $Q_\Phi(w)$ is of the form $x^T K x$, $K = K^T$, with $x = X(\frac{d}{dt})w$ a minimal state map for Φ . We call this state function *positive definite* if $K > 0$.

THEOREM 9.3. *Let $\Phi \in \mathbb{R}_s^{q \times q}[\zeta, \eta]$ be observable. The following are equivalent:*

1. $\int^t Q_\Phi \gg 0$,
2. (a) $\int Q_\Phi \gg 0$,
(b) *there exists a storage function that is a positive definite state function,*
3. (a) $\exists \epsilon > 0$ such that $\Phi(-i\omega, i\omega) \geq \epsilon |\Phi|(-i\omega, i\omega)$ for all $\omega \in \mathbb{R}$,
(b) $T_\Phi > 0$.

Proof. For the proof, see the appendix.

Remark 9.4. It follows from the proof of Theorem 9.3 that half-line nonnegativity implies that the Pick-type matrix T_Φ (see (9.1)) is ≥ 0 whenever any set of λ_i 's in the right half of the complex plane and any set of a_i 's are chosen. It is possible to prove that if T_Φ is nonnegative definite for any such choice for the λ_i 's and a_i 's, then we have half-line nonnegativity. The remarkable thing about Theorem 9.3 is that it suffices to evaluate T_Φ at the set of special λ_i 's and a_i 's obtained from the singularities of $\partial\Phi$.

Remark 9.5. It is well known that solvability of a certain Nevanlinna–Pick interpolation problem is equivalent to positive definiteness of a given Pick matrix. In fact, in [34] the necessity of the positive definite Pick matrix is shown using a half-line positivity argument. Theorem 9.3 states that positive definiteness of a given Pick matrix is also *sufficient* for half-line positivity.

Remark 9.6. It can be shown that if Φ is observable, then $\int Q_\Phi \gg 0$ implies that $\Psi_+ - \Psi_-$ is a positive definite state function, with Ψ_+ and Ψ_- as defined in Theorem 5.7.

Remark 9.7. It is possible to generalize the T_Φ -test of Theorem 9.3 to half-line positive (instead of strictly half-line positive QDFs) by including “infinite zeros” of $\det(\partial\Phi)$. However, the notation gets very involved, and therefore we will not do this here.

10. Duality. In the present section we discuss some remarkable relations between positivity of QDFs and their duals. These relations are of interest in their own right and will be of crucial importance in our treatment of the H_∞ -problem [25].

Let \mathfrak{B}_1 and $\mathfrak{B}_2 \in \mathfrak{L}^q$. We call \mathfrak{B}_1 and \mathfrak{B}_2 *complementary* if $\mathfrak{B}_1 \oplus \mathfrak{B}_2 = \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^q)$. It is easy to see that this implies that both \mathfrak{B}_1 and \mathfrak{B}_2 must be controllable: uncontrollable \mathfrak{B} ’s in \mathfrak{L}^q have no complement in \mathfrak{L}^q . We call them *dual* if they are complementary and if $\langle w_1, w_2 \rangle = 0$ for all $w_1 \in \mathfrak{B}_1 \cap \mathfrak{D}(\mathbb{R}, \mathbb{R}^q)$ and $w_2 \in \mathfrak{B}_2 \cap \mathfrak{D}(\mathbb{R}, \mathbb{R}^q)$, where $\langle w_1, w_2 \rangle$ denotes the usual inner product $\int_{-\infty}^{+\infty} w_1^T w_2 dt$. If this is the case, then we denote \mathfrak{B}_2 as \mathfrak{B}_1^\perp , since \mathfrak{B}_1 defines \mathfrak{B}_1^\perp uniquely. Obviously we also have $(\mathfrak{B}_1^\perp)^\perp = \mathfrak{B}_1$.

It is easy to see that $R(\frac{d}{dt})w = 0$ is a minimal kernel representation of the controllable \mathfrak{B} iff $v = R^T(-\frac{d}{dt})\ell$ is an observable image representation of \mathfrak{B}^\perp (a kernel representation $R(\frac{d}{dt})w = 0$ of \mathfrak{B} is called *minimal* if R has full row rank). Consequently, $w = M(\frac{d}{dt})\ell$ is an observable image representation of \mathfrak{B} iff $M^T(-\frac{d}{dt})v = 0$ is a minimal kernel representation of \mathfrak{B}^\perp . This duality can also be extended to state representations, in the following sense. If

$$(10.1) \quad E \frac{dx}{dt} + Fx + Gw = 0,$$

is an n -dimensional minimal state representation of \mathfrak{B} , then \mathfrak{B}^\perp admits an also n -dimensional minimal state representation (thus the dimensions of the minimal state representations are the same), say,

$$(10.2) \quad E' \frac{dz}{dt} + F'z + G'v = 0,$$

having the property that for all $(w, x) \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{q+n})$ satisfying (10.1), and for all $(v, z) \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{q+n})$ satisfying (10.2) there holds the following kind of duality involving the state

$$(10.3) \quad \frac{d}{dt} z^T x = v^T w.$$

In fact, this is an immediate consequence of the following.

PROPOSITION 10.1. *Let $R(\frac{d}{dt})w = 0$ and $w = M(\frac{d}{dt})\ell$ be a minimal kernel representation and an observable image representation, respectively, of the controllable system $\mathfrak{B} \in \mathfrak{L}^q$. Assume that $X \in \mathbb{R}^{n \times \bullet}[\xi]$ defines a minimal state map for \mathfrak{B} , i.e.,*

$x = X(\frac{d}{dt})\ell$ defines a minimal state of \mathfrak{B} . Then there exists a $Z \in \mathbb{R}^{n \times \bullet}[\xi]$ defining a minimal state map $Z(\frac{d}{dt})$ for \mathfrak{B}^\perp , such that for all $\ell, \ell' \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^\bullet)$, we have

$$(10.4) \quad \frac{d}{dt} \left(Z \left(\frac{d}{dt} \right) \ell' \right)^T X \left(\frac{d}{dt} \right) \ell = \left(R^T \left(-\frac{d}{dt} \right) \ell' \right)^T M \left(\frac{d}{dt} \right) \ell.$$

If we define $\Psi(\zeta, \eta) := Z^T(\zeta)X(\eta)$ and $\Phi(\zeta, \eta) := R(-\zeta)M(\eta)$, then (10.4) is equivalent to $\dot{\Psi} = \Phi$.

Proof. For the proof, see the appendix.

If a pair of minimal state maps (X, Z) of \mathfrak{B} and \mathfrak{B}^\perp satisfies (10.4), then we call it a *matched pair* of state maps.

We now associate with a QDF a dual one and relate their average nonnegativity and average positivity. Let $\Phi \in \mathbb{R}_s^{q \times q}[\zeta, \eta]$ and let $\Phi(\zeta, \eta) = M^T(\zeta)\Sigma_\Phi M(\eta)$ be a symmetric canonical factorization, with

$$\Sigma_\Phi = \begin{bmatrix} I_{r_+} & 0 \\ 0 & -I_{r_-} \end{bmatrix}.$$

Let us assume that $M \in \mathbb{R}^{r \times q}[\xi]$. Partition M conformably to Σ_Φ as

$$M = \begin{bmatrix} P \\ N \end{bmatrix}$$

so that Φ is written as

$$(10.5) \quad \Phi(\zeta, \eta) = P^T(\zeta)P(\eta) - N^T(\zeta)N(\eta).$$

Consider the dynamical system $\mathfrak{B} \in \mathcal{L}^r$ with image representation

$$(10.6) \quad w = M \left(\frac{d}{dt} \right) \ell.$$

There are a number of integers associated with M that are of interest to us:

$$(10.7) \quad r_+ = \text{rowdim}(P),$$

$$(10.8) \quad r_- = \text{rowdim}(N),$$

$$(10.9) \quad m = \text{rank}(M).$$

The number r_+ corresponds to the number of positive squares in Q_Φ , r_- to the number of negative squares, while m equals the number of inputs in any input/output or input/state/output representation of \mathfrak{B} . Since it is defined by an image representation, \mathfrak{B} is a controllable system and, as such, it admits a dual, $\mathfrak{B}^\perp \in \mathcal{L}^r$. Let $R(\frac{d}{dt})w = 0$ be a minimal kernel representation of \mathfrak{B} . Then

$$(10.10) \quad v = R^T \left(-\frac{d}{dt} \right) \ell'$$

is an observable image representation for \mathfrak{B}^\perp . Let $\Phi'(\zeta, \eta) := R(-\zeta)\Sigma_\Phi R^T(-\eta)$. Note that the QDFs $Q_\Phi(\ell) = (M(\frac{d}{dt})\ell)^T \Sigma_\Phi M(\frac{d}{dt})\ell$ and

$$Q_{\Phi'}(\ell') = \left(R^T \left(-\frac{d}{dt} \right) \ell' \right)^T \Sigma_\Phi R^T \left(-\frac{d}{dt} \right) \ell'$$

are in a sense also dual. Their positivity properties are very much related, as shown in the following theorem.

THEOREM 10.2. *Assume that $r_+ = m$. Then*

- (i) $\int Q_\Phi \geq 0 \Leftrightarrow \int Q_{\Phi'} \leq 0$,
- (ii) $\int Q_\Phi > 0 \Leftrightarrow \int Q_{\Phi'} < 0$,
- (iii) (assume Φ is observable) $\int Q_\Phi \gg 0 \Leftrightarrow \int Q_{\Phi'} \ll 0$,
- (iv) Let (X, Z) be a matched pair of minimal state maps for \mathfrak{B} and \mathfrak{B}^\perp . Assume that $\int Q_\Phi \geq 0$, and let Ψ define a storage function for Φ . By Theorem 5.5, Ψ is a state function, i.e., there exists a real symmetric matrix K such that

$$(10.11) \quad Q_\Psi(\ell) = \left(X \left(\frac{d}{dt} \right) \ell \right)^T K X \left(\frac{d}{dt} \right) \ell.$$

Assume that K is nonsingular. Then

$$(10.12) \quad Q_{\Psi'}(\ell') = - \left(Z \left(\frac{d}{dt} \right) \ell' \right)^T K^{-1} Z \left(\frac{d}{dt} \right) \ell'$$

is a storage function for $-Q_{\Phi'}$.

- (v) (assume Φ is observable) $\int_t Q_\Phi \gg 0 \Leftrightarrow \int_t Q_{\Phi'} \ll 0$. Here $\int_t Q_{\Phi'} \ll 0$ is defined as the property that there exists $\epsilon > 0$ such that $\int_0^\infty Q_{\Phi'}(w)dt \leq -\epsilon \int_0^\infty Q_{\Phi'}(w)dt$ for all $w \in \mathfrak{D}(\mathbb{R}, \mathbb{R}^q)$ (i.e., half-line positivity over the positive half-line).

Proof. For the proof, see the appendix.

We close this section by pointing out that it is of interest to generalize the notion of duality by using, instead of the usual inner product, an inner product that is itself induced by a QDF. These ramifications are a matter of future research.

11. Conclusions. In this paper we studied two-variable polynomial matrices and their role in a number of problems in linear system theory. The basic premise set forward is the following. Dynamic models lead naturally to the study of one-variable polynomial matrices. By substituting the time derivative for the indeterminate, and by letting the resulting differential operator act on a variable, one arrives at a dynamical system, which may then be in kernel or in image representation. The study of quadratic functionals in a variable and its derivative, on the other hand, leads to two-variable polynomial matrices. Important instances where dynamical systems occur in conjunction with functionals are, for example, Lyapunov theory, the theory of dissipative systems, and LQ and H_∞ control. We developed the former two applications in the present paper. The latter two will be discussed elsewhere.

Appendix.

Proof of Theorem 3.1. We prove the equivalence of the two statements in (1) at the end of the proof and proceed with the first statement by running the circle (1) \Rightarrow (3) \Rightarrow (2) \Rightarrow (1). Assume that $\int L_\Phi = 0$. Then obviously $\int_{-\infty}^\infty L_\Phi(v, w)dt = 0$ for all $v \in \mathfrak{D}(\mathbb{R}, \mathbb{C}^{q_1})$ and $w \in \mathfrak{D}(\mathbb{R}, \mathbb{C}^{q_2})$, with $L_\Phi(v, w)$ in this case (for complex functions) defined by $\sum_{k, \ell} \left(\frac{d^k v}{dt^k} \right)^T \Phi_{k, \ell} \left(\frac{d^\ell w}{dt^\ell} \right)$. Then

$$\int_{-\infty}^\infty \hat{v}^T(-i\omega) \Phi(-i\omega, i\omega) \hat{w}(i\omega) d\omega = 0$$

for all $\hat{v} \in L_2(\mathbb{C}, \mathbb{C}^{q_1})$, $\hat{w} \in L_2(\mathbb{C}, \mathbb{C}^{q_2})$ that are Fourier transforms of $v \in \mathfrak{D}(\mathbb{R}, \mathbb{C}^{q_1})$, and $w \in \mathfrak{D}(\mathbb{R}, \mathbb{C}^{q_2})$. This implies that $\partial \Phi = 0$. Assume to the contrary that there exist $\omega_0 \in \mathbb{R}$, $a \in \mathbb{C}^{q_1}$, $b \in \mathbb{C}^{q_2}$ such that $\bar{a}^T \Phi(-i\omega_0, i\omega_0) b \neq 0$. Define $v_N \in \mathfrak{D}(\mathbb{R}, \mathbb{C}^{q_1})$

for $N = 1, 2, \dots$, by

$$(A.1) \quad v_N(t) = \begin{cases} e^{i\omega_0 t} a & |t| \leq \frac{2\pi N}{\omega_0}, \\ \tilde{v}(t + \frac{2\pi N}{\omega_0}) & t < -\frac{2\pi N}{\omega_0}, \\ \tilde{v}(t - \frac{2\pi N}{\omega_0}) & t > \frac{2\pi N}{\omega_0}. \end{cases}$$

Define $w_N \in \mathfrak{D}(\mathbb{R}, \mathbb{C}^{q_2})$ analogously by replacing a by b . Note that \tilde{v} and \tilde{w} can be chosen independent of N , and obtain smoothness for all N : indeed, if v_1 is smooth, then by the periodic nature of v_N for $|t| \leq \frac{2\pi N}{\omega_0}$, v_N will also be smooth.

Next evaluate $\int_{-\infty}^{\infty} L_{\Phi}(v_N, w_N) dt$ and observe that this integral equals

$$\frac{4\pi N}{\omega_0} \bar{a}^T \Phi(-i\omega_0, i\omega_0) b + E$$

with E independent of N . It follows that $\int_{-\infty}^{\infty} L_{\Phi}(v_N, w_N) dt \neq 0$ for N sufficiently large. In order to obtain this for real-valued functions, consider the real and imaginary parts of v_N, w_N and the integrals. This establishes the contradiction. Hence (1) implies (3).

To prove (3) \Rightarrow (2), view $\Phi(\zeta, \eta)$ as a one-variable polynomial in ζ and carry out the division by $\zeta + \eta$. This yields $\Phi(\zeta, \eta) = (\zeta + \eta)d(\zeta, \eta) + r(\zeta, \eta)$. Hence $\partial\Phi = 0$ implies $r = 0$. This yields (2).

To show that (2) \Rightarrow (1), observe that $\int_{-\infty}^{\infty} L_{\Phi}(v, w) dt = \int_{-\infty}^{\infty} \frac{d}{dt} L_{\Psi}(v, w) dt$. The last term obviously vanishes since v and w have compact support.

To show the equivalence of the two statements in (1), observe that it follows trivially that $\int L_{\Phi} = 0$ implies path independence. Conversely, if $\partial\Phi = 0$ then, according to (3), there exists $\Psi \in \mathbb{R}^{q_1 \times q_2}[\zeta, \eta]$ such that $L_{\Phi} = \frac{d}{dt} L_{\Psi}$. Thus, for any pair of functions $v \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{q_1})$ and $w \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{q_2})$, and for any t_1 and t_2 , we have

$$\int_{t_1}^{t_2} L_{\Phi}(v, w) dt = \int_{t_1}^{t_2} \frac{d}{dt} L_{\Psi}(v, w) dt = L_{\Psi}(v, w)(t_2) - L_{\Psi}(v, w)(t_1).$$

Hence the integral depends only on the values taken on by v and w and their derivatives at the endpoints t_1 and t_2 . \square

Proof of Proposition 3.2. This proposition is proven following the standard proofs used in behavioral theory: reduce the problem to the scalar case using the Smith form. Let $R = U\Delta V$ with U, V unimodular and Δ diagonal. Define $\mathfrak{B}' = V(\frac{d}{dt})\mathfrak{B}$. Then \mathfrak{B}' has $\Delta(\frac{d}{dt})w = 0$ as kernel representation. To prove the proposition, note that the “if” parts are immediate.

To show the first “only if” part, we show that $D(\frac{d}{dt})\mathfrak{B} = 0$ implies that there exists F such that $D = FR$ or equivalently, with $D = D'V$, that $D'(\frac{d}{dt})\mathfrak{B}' = 0$ implies that there exists F' such that $D' = F'\Delta$. Let $\Delta = \text{diag}(d, \Delta')$, let d' be the first column of D' , and let w_1 be a solution of $d(\frac{d}{dt})w_1 = 0$. Since $\text{col}[w_1, 0, \dots, 0] \in \mathfrak{B}'$ and $D'(\frac{d}{dt})\mathfrak{B}' = 0$, it follows that $d'(\frac{d}{dt})w_1 = 0$. It is easily seen that $d(\frac{d}{dt})w_1 = 0$ implies $d'(\frac{d}{dt})w_1 = 0$ iff each element of the polynomial vector d' is a factor of d . Proceeding this way column by column yields $D' = F'\Delta$.

To show the second “only if” part, we prove first the analogous result for BLDFs. This states that with $\Phi \in \mathbb{R}^{q_1 \times q_2}[\zeta, \eta]$, the BLDF $L_{\Phi}(w_1, w_2) = 0$ for all $w_1 \in \mathfrak{B}_1$ and $w_2 \in \mathfrak{B}_2$ iff there exists F_1, F_2 such that

$$(A.2) \quad \Phi(\zeta, \eta) = R_1^T(\zeta)F_2(\zeta, \eta) + F_1(\zeta, \eta)R_2(\eta),$$

where R_1 and R_2 induce kernel representations of \mathfrak{B}_1 and \mathfrak{B}_2 . The “if” part is once again obvious. To prove the “only if” part, consider first the following lemma, which proves the scalar case $q_1 = q_2 = 1$.

LEMMA A.1. *Let $r_1, r_2 \in \mathbb{R}[\xi]$ and $\Phi \in \mathbb{R}[\zeta, \eta]$. Let $\mathfrak{B}_m \in \mathfrak{L}^1$, $m = 1, 2$ be given in kernel representation by $r_m(\frac{d}{dt})w_m = 0$. Then $Q_\Phi(w_1, w_2) = 0$ for all $w_m \in \mathfrak{B}_m$ iff there exists $f_m \in \mathbb{R}[\zeta, \eta]$ such that*

$$(A.3) \quad \Phi(\zeta, \eta) = r_1(\zeta)f_2(\zeta, \eta) + f_1(\zeta, \eta)r_2(\eta).$$

Proof. The “if” part is obvious. To show the “only if” part, let r_1 have degree n_1 and r_2 have degree n_2 , and assume that they are monic. Consider the term $\Phi_{k,\ell}\zeta^k\eta^\ell$ of $\Phi(\zeta, \eta)$. In the quadratic form $Q_\Phi(w_1, w_2)$ this term contributes $\Phi_{k,\ell}\frac{d^k w_1}{dt^k}\frac{d^\ell w_2}{dt^\ell}$. If w_1 satisfies $r_1(\frac{d}{dt})w_1 = 0$, and if $k \geq n_1$, then the contribution in $Q_\Phi(w_1, w_2)$ of $\Phi_{k,\ell}\zeta^k\eta^\ell$ is equivalent to that of $\Phi_{k,\ell}(\zeta^k - \zeta^{k-n_1}r_1(\zeta))\eta^\ell$. Proceeding analogously with the ℓ 's and the other terms shows that there exists $\Phi'(\zeta, \eta) = \sum_{k,\ell} \Phi'_{k,\ell}\zeta^k\eta^\ell$ with $\Phi'_{k,\ell} = 0$ for $k \geq n_1$ or $\ell \geq n_2$ such that

$$(A.4) \quad \Phi(\zeta, \eta) = r_1(\zeta)f_2(\zeta, \eta) + f_1(\zeta, \eta)r_2(\eta) + \Phi'(\zeta, \eta).$$

Obviously $Q_{\Phi'}(w_1, w_2) = Q_\Phi(w_1, w_2)$ for $w_m \in \mathfrak{B}_m$. Therefore $Q_{\Phi'}(w_1, w_2) = 0$ for $w_m \in \mathfrak{B}_m$. Consider $Q_{\Phi'}(w_1, w_2)(0)$ and observe that this is a quadratic form in $w_1(0), \frac{dw_1}{dt}(0), \dots, \frac{d^{n_1-1}w_1}{dt^{n_1-1}}(0)$ and $w_2(0), \frac{dw_2}{dt}(0), \dots, \frac{d^{n_2-1}w_2}{dt^{n_2-1}}(0)$. These initial conditions can be chosen arbitrarily in the sense that for any values of $w_m(0), \frac{dw_m}{dt}(0), \dots, \frac{d^{n_m-1}w_m}{dt^{n_m-1}}(0)$ there exist $w_m \in \mathfrak{B}_m$ having these initial values. It follows that $\Phi' = 0$. \square

Now return to the proof of the case for general q_1, q_2 . Bring R_1 and R_2 in Smith form, showing that it suffices to prove (A.2) for $R = \Delta_1$ and $R_2 = \Delta_2$ with Δ_1 and Δ_2 in Smith form. Let d_1 be the (k_1, k_1) th element of Δ_1 and d_2 the (k_2, k_2) th element of Δ_2 . Examine (A.2) and observe that we need to show that the (k_1, k_2) th element of Φ , $\Phi_{k_1 k_2}$, can be written as

$$(A.5) \quad \Phi_{k_1, k_2}(\zeta, \eta) = d_1(\zeta)f_2(\zeta, \eta) + f_1(\zeta, \eta)d_2(\eta)$$

whenever it holds that $d_1(\frac{d}{dt})v_1 = 0$ and $d_2(\frac{d}{dt})v_2 = 0$ implies that $L_{\Phi_{k_1 k_2}}(v_1, v_2) = 0$. Now use the previous lemma.

In order to prove Proposition 3.2 for $\Phi \in \mathbb{R}_s^{n_1 \times n_2}[\zeta, \eta]$, use the *-operator on (A.2), and add.

The image representation part of Proposition 3.2 is proven analogously. \square

Proof of Proposition 3.5. The proof follows exactly along the same lines as the proof of Proposition 3.2, and we can therefore be very brief. The Smith form once again implies that it suffices to prove the case $q_1 = q_2 = 1$. Denote a kernel representation of \mathfrak{B} by $r(\frac{d}{dt})w = 0$. Using (A.4) with $r_1 = r_2 = r$ shows that $Q_\Phi \geq 0$ on \mathfrak{B} iff $Q_{\Phi'} \geq 0$ on \mathfrak{B} . However, again by the arbitrariness of the initial conditions, $(Q_\Phi \geq 0$ on $\mathfrak{B})$ iff the matrix $\tilde{\Phi}'$ associated with Φ' is nonnegative definite. Part (i) of the proposition follows.

To show part (ii), factor Φ' (using $\tilde{\Phi}'$) as $\Phi'(\zeta, \eta) = D^T(\zeta)D(\eta)$ with $D \in \mathbb{R}^{\bullet \times 1}[\xi]$ having elements whose degree is less than that of r . It thus suffices to find conditions for $r(\frac{d}{dt})w = 0$ and $D(\frac{d}{dt})w = 0$ to imply $w = 0$. That, however, is exactly equivalent to the observability of the pair (r, D) . \square

Proof of Theorem 4.3. The “if” part is shown as follows. By Proposition 3.5 we know that $\dot{\Psi} \stackrel{\mathfrak{B}}{<} 0$ implies that $\dot{\Psi}(\zeta, \eta) \stackrel{\mathfrak{B}}{=} -D^T(\zeta)D(\eta)$ with $D \in \mathbb{R}^{\bullet \times q}[\xi]$ such that (R, D) is observable, with $R \in \mathbb{R}^{\bullet \times q}[\xi]$ a kernel representation of \mathfrak{B} . It also holds that

$$(A.6) \quad \frac{d}{dt}Q_{\Psi}(w) = Q_{\dot{\Psi}}(w).$$

Integrate this from 0 to T along a $w \in \mathfrak{B}$ and obtain

$$(A.7) \quad Q_{\Psi}(w)(T) - Q_{\Psi}(w)(0) = \int_0^T Q_{\dot{\Psi}}(w)dt = - \int_0^T \|D\left(\frac{d}{dt}\right)(w)\|^2 dt.$$

Using $\Psi \stackrel{\mathfrak{B}}{\geq} 0$, this yields

$$(A.8) \quad \int_0^T \|D\left(\frac{d}{dt}\right)(w)\|^2 dt \leq Q_{\Psi}(w)(0).$$

Therefore

$$(A.9) \quad \int_0^{\infty} \|D\left(\frac{d}{dt}\right)(w)\|^2 dt < \infty.$$

This implies the asymptotic stability of \mathfrak{B} . Assume that $ae^{\lambda t} \in \mathfrak{B}$, $a \neq 0$. Then $R(\lambda)a = 0$ and by (A.8) there must hold that either $D(\lambda)a = 0$ or $\Re(\lambda) < 0$. (Note that we silently use the obvious fact that (A.8) also holds for the complexification of \mathfrak{B} .) However, by observability of (R, D) , $R(\lambda)a = 0$ and $D(\lambda)a = 0$ imply $a = 0$. Hence all exponential solutions $ae^{\lambda t}$ of $R\left(\frac{d}{dt}\right)w = 0$ must have $\Re(\lambda) < 0$. It is well known from the theory of differential equations that this implies that all solutions approach zero as $t \rightarrow \infty$. The “only if” follows from the stronger Theorem 4.8 and will be proven then. \square

Proof of Corollary 4.6. $\dot{\Psi}(\zeta, \eta) = (\zeta + \eta)\Psi_0$. In the case at hand, $R(\xi) = A - \xi I$. Using Proposition 3.2, $(\zeta + \eta)\Psi \stackrel{\mathfrak{B}}{=} A\Psi_0 + \Psi_0 A^T$. Finally, observe that observability of $(A - \xi I, \sqrt{\Delta_0})$ (as a pair of polynomial matrices) is equivalent to that of $(A, \sqrt{\Delta_0})$ (as a pair of matrices) which is equivalent to that of (A, Δ_0) . \square

Proof of Proposition 4.7. Examine formula (A.8) in the proof of Theorem 4.3. It implies $Q_{\Psi}(w)(0) \geq \int_0^{\infty} \|D\left(\frac{d}{dt}\right)w\|^2 dt$. Therefore $Q_{\Psi}(w)(0) = 0$ implies $D\left(\frac{d}{dt}\right)w = 0$. However, by observability of D , $D\left(\frac{d}{dt}\right)w = 0$ in turn implies $w = 0$. \square

Proof of Theorem 4.8. The proof is organized as follows. First, we prove that (4.3) is solvable; second, that if R is square (4.4), (4.5) gives all Sits solutions; third, that (4.7) yields $\dot{\Psi} \stackrel{\mathfrak{B}}{=} \Phi$; fourth, that $\dot{\Psi}_1 \stackrel{\mathfrak{B}}{=} \dot{\Psi}_2$ implies $\Psi_1 \stackrel{\mathfrak{B}}{=} \Psi_2$; fifth, that $\Phi \stackrel{\mathfrak{B}}{\leq} 0$ yields $\Psi \stackrel{\mathfrak{B}}{\geq} 0$; and sixth, that $\Phi \stackrel{\mathfrak{B}}{<} 0$ yields $\Psi \stackrel{\mathfrak{B}}{\gg} 0$.

(i) First put R in Smith form: let

$$R = U \begin{bmatrix} D \\ 0 \end{bmatrix} V,$$

with D diagonal and U, V unimodular. Observe that it suffices to prove (4.3) with $R = D$. The (k, ℓ) th component of the matrix equation (4.3) in the obvious notation takes the form

$$(A.10) \quad x_{\ell k}(-\xi)d_{\ell}(\xi) + d_k(-\xi)x_{k\ell}(\xi) = \Phi_{k\ell}(-\xi, \xi).$$

Since d_k and d_ℓ are Hurwitz, $d_\ell(\xi)$ and $d_k(-\xi)$ are coprime and hence, by Bezout, (A.10) has a solution. This then yields a solution of the matrix version.

(ii) Again use the Smith form. Obtain that the difference of two solutions must satisfy

$$(A.11) \quad x_{\ell k}(-\xi)d_\ell(\xi) + d_k(-\xi)x_{k\ell}(\xi) = 0.$$

Hence again using coprimeness of $d_\ell(\xi)$ and $d_k(-\xi)$, there exists a polynomial $f_{k\ell}$ such that $x_{k\ell}(\xi) = f_{k\ell}(\xi)d_\ell(\xi)$. This yields (4.4). To show (4.5), obtain

$$(A.12) \quad R^T(-\xi)(F(\xi) + F^T(-\xi))R(\xi) = 0.$$

If R is square and $\det(R) \neq 0$, (4.5) follows by pre- and postmultiplying by $(R^T(-\xi))^{-1}$ and $(R(\xi))^{-1}$.

(iii) This proof is obvious.

(iv) Let $w \in \mathfrak{B}$ and assume that $\dot{\Psi}_1 \stackrel{\mathfrak{B}}{=} \dot{\Psi}_2$, i.e., $\dot{\Delta} \stackrel{\mathfrak{B}}{=} 0$, where $\Delta = \Psi_1 - \Psi_2$. Then

$$(A.13) \quad \int_0^t Q_{\dot{\Delta}}(w)dt = \int_0^t \frac{d}{dt}Q_{\Delta}(w)dt = Q_{\Delta}(w)(t) - Q_{\Delta}(w)(0).$$

Since $Q_{\dot{\Delta}}(w) = 0$, asymptotic stability of \mathfrak{B} implies $Q_{\Delta}(w)(t) \rightarrow 0$ as $t \rightarrow \infty$ and hence that $Q_{\Delta}(w)(0) = 0$. Therefore $\Delta \stackrel{\mathfrak{B}}{=} 0$.

(v) This follows immediately from (A.7), and Proposition 4.7 yields (vi). \square

Proof of Proposition 4.9. Existence of both D' and Ψ' follows from the algorithm given in the statement of the proposition. To show uniqueness of D' observe that $D' \stackrel{\mathfrak{B}}{=} D''$, i.e., $D'' - D' = FR$, and $D'R^{-1}, D''R^{-1}$ strictly proper, implies $F = 0$, i.e., $D' = D''$. In the two-variable case assume $\Psi' \stackrel{\mathfrak{B}}{=} \Psi''$, i.e.,

$$\Psi'(\zeta, \eta) = \Psi''(\zeta, \eta) + F^T(\eta, \zeta)R(\eta) + R^T(\zeta)F(\zeta, \eta).$$

Thus,

$$(R^T(-\zeta))^{-1}(\Psi' - \Psi'')(\zeta, \eta)(R(\eta))^{-1} = (R^T(\zeta))^{-1}F^T(\eta, \zeta) + F(\zeta, \eta)(R(\eta))^{-1}$$

Strict properness again implies $F = 0$. \square

Proof of Proposition 4.10. Let $\Psi \stackrel{\mathfrak{B}}{=} 0$. Then $\Psi(\zeta, \eta) = F^T(\eta, \zeta)R(\eta) + R^T(\zeta)F(\zeta, \eta)$. Pre- and postmultiply by $(R^T(\zeta))^{-1}$ and $(R(\eta))^{-1}$, respectively, and conclude that $F = 0$. The result follows. If $\Psi \stackrel{\mathfrak{B}}{\geq} 0$, use the same reasoning and Proposition 3.5 on $\Psi(\zeta, \eta) \stackrel{\mathfrak{B}}{=} D^T(\zeta)D(\eta)$ with D R -canonical. The case $\Psi \stackrel{\mathfrak{B}}{>} 0$ is similar. \square

Proof of Theorem 4.12. We first show that (4.4) has an R -canonical solution. Let X be any solution. Factor XR^{-1} as $XR^{-1} = P + S$ with P polynomial and S strictly proper. First observe that it follows from (4.3) that $P(\xi) + P^T(-\xi) = 0$. Next, show that $X - PR$ is a canonical solution. Uniqueness of this R -canonical solution follows from (4.4).

Next, we show that (4.10) yields an R -canonical Ψ . Simply pre- and postmultiply by $(R^T(\zeta))^{-1}$ and $R^{-1}(\eta)$ and observe properness. Uniqueness follows from $(\dot{\Psi}_1 \stackrel{\mathfrak{B}}{=} \Phi$ and $\dot{\Psi}_2 \stackrel{\mathfrak{B}}{=} \Phi) \implies (\dot{\Psi}_1 \stackrel{\mathfrak{B}}{=} \dot{\Psi}_2)$. Now apply Proposition 4.9.

The remaining statements follow from Proposition 4.10. \square

Proof of Proposition 5.2. The proof of all three statements is analogous. Therefore we only give the proof of (i). To prove (\Leftarrow) , let $w \in \mathfrak{D}(\mathbb{R}, \mathbb{R}^q)$ and let \hat{w} be its Fourier transform. Observe, using Parseval's Theorem, that

$$(A.14) \quad \int_{-\infty}^{+\infty} Q_{\Phi}(w)dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{w}(-i\omega)^T \Phi(-i\omega, i\omega) \hat{w}(i\omega) d\omega,$$

whence (\Leftarrow) . To show the converse, as in the proof of Theorem 3.1, we silently switch from \mathbb{R}^q as signal space to \mathbb{C}^q . Assume that there exists $a \in \mathbb{C}^q$ and $\omega_0 \in \mathbb{R}$ such that $\bar{a}^T \Phi(-i\omega_0, i\omega_0) a < 0$. Consider the function $w_N \in \mathfrak{D}(\mathbb{R}, \mathbb{C}^q)$ for $N = 1, 2, \dots$, defined exactly as v_N was in the proof of Theorem 3.1. Next evaluate $\int_{-\infty}^{+\infty} Q_{\Phi}(w_N)dt$ and observe (using the idea in the proof of Theorem 3.1) that this integral can be made negative by taking N sufficiently large. \square

Proof of Proposition 5.4. We will run the circle $(3) \Rightarrow (2) \Rightarrow (1) \Rightarrow (3)$. To see that $(3) \Rightarrow (2)$, assume that Δ is a dissipation function. Then $\Phi(-\xi, \xi) = \Delta(-\xi, \xi)$, by Theorem 3.1. Define

$$(A.15) \quad \Psi(\zeta, \eta) = \frac{\Phi(\zeta, \eta) - \Delta(\zeta, \eta)}{\zeta + \eta}.$$

Hence $\dot{\Psi} = \Phi - \Delta$. Use $\Delta \geq 0$ to conclude that Ψ is a storage function. To see that $(2) \Rightarrow (1)$, use $\dot{\Psi} \leq \Phi$ and Theorem 3.1 to conclude (1). To see that $(1) \Rightarrow (3)$, use Propositions 5.2 and 5.6 to construct a D such that $\Phi(-\xi, \xi) = D^T(-\xi)D(\xi)$. Observe that $\Delta(\zeta, \eta) := D^T(\zeta)D(\eta)$ defines a dissipation function. The one-one relation between Ψ and Δ is given by (A.15). \square

Proof of Theorem 5.5. By (5.11) it suffices to consider minimal state representations obtained from a canonical factorization of Φ . Let

$$v = M = \left(\frac{d}{dt} \right) w$$

be obtained from such a factorization, and let $x = X(\frac{d}{dt})w$ be a minimal state. There exists a permutation matrix P such that

$$PM = \begin{bmatrix} U \\ Y \end{bmatrix},$$

with $\det(U) \neq 0$ and such that YU^{-1} is a matrix of proper rational functions. Denote $u = U(\frac{d}{dt})w$. Consider $f = F(\frac{d}{dt})w$, where F is an arbitrary polynomial matrix. Then (see section 2) f is a state function, (i.e., there exists a matrix K such that $f = Kx$) iff FU^{-1} is strictly proper and a state/input function (i.e., there exists matrices L, J such that $f = Kx + Ju$) iff FU^{-1} is proper.

We first prove the second part of the theorem, i.e., that every dissipation function is a state/supply function. Let $\Delta(\zeta, \eta) = D^T(\zeta)D(\eta)$ be a dissipation function. Then

$$(A.16) \quad M^T(-\xi)\Sigma_{\Phi}M(\xi) = D^T(-\xi)D(\xi).$$

Pre- and postmultiply by U^{-1} , to obtain

$$(A.17) \quad (M(-\xi)U^{-1}(-\xi))^T \Sigma_{\Phi} M(\xi)U^{-1}(\xi) = (D(-\xi)U^{-1}(-\xi))^T D(\xi)U^{-1}(\xi).$$

Since the left-hand side is proper, so is the right-hand side. This obviously implies that $D(\xi)U^{-1}(\xi)$ is proper. Hence $D(\frac{d}{dt})w$ is a state/input function and equivalently,

$D(\xi) = KX(\xi) + JU(\xi)$ for suitable constant matrices K, J . From this it is readily seen that there exists a matrix E such that (5.13) holds.

We now prove the first part of the theorem, i.e., that every storage function is a state function. Let $\Psi(\zeta, \eta)$ define a storage function for Φ . Let $D(\xi)$ be such that

$$(\zeta + \eta)\Psi(\zeta, \eta) = M^T(\zeta)\Sigma_\Phi M(\eta) - D^T(\zeta)D(\eta).$$

By redefining

$$\tilde{M}^T(\zeta)\tilde{\Sigma}_\Phi\tilde{M}(\eta) = \begin{bmatrix} M(\zeta) \\ D(\zeta) \end{bmatrix}^T \begin{bmatrix} \Sigma_\Phi & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} M(\eta) \\ D(\eta) \end{bmatrix},$$

and observing that a minimal state for the system with image representation $v = M(\frac{d}{dt})w$ is also a minimal state for the system with image representation

$$v = \begin{bmatrix} M(\frac{d}{dt}) \\ D(\frac{d}{dt}) \end{bmatrix} w,$$

it suffices to prove the claim in the lossless case, i.e., when $\dot{\Psi} = \Phi$.

Assume that $\Phi(\zeta, \eta) = M^T(\zeta)\Sigma_\Phi M(\eta)$ is a symmetric canonical factorization of Φ and that $\Psi(\zeta, \eta) = N^T(\zeta)\Sigma_\Psi N(\eta)$ is a symmetric canonical factorization of Ψ .

Postmultiply the identity $\dot{\Psi} = \Phi$ by $U^{-1}(\eta)$ to obtain

$$(A.18) \quad (\zeta + \eta)N^T(\zeta)\Sigma_\Psi N(\eta)U^{-1}(\eta) = M^T(\zeta)\Sigma_\Phi M(\eta)U^{-1}(\eta).$$

Assume that $L_k\eta^k$ is the term of degree k in the polynomial part of the matrix of rational functions $N(\eta)U^{-1}(\eta)$. Using that the right-hand side of (A.18) is proper in η , by equating powers of η yields $N^T(\zeta)\Sigma_\Psi L_k = 0$. Express $N(\zeta)$ as

$$N(\zeta) = \begin{bmatrix} N_0 & N_1 & \dots & N_L \end{bmatrix} \begin{bmatrix} I \\ I\zeta \\ \vdots \\ I\zeta^L \end{bmatrix}$$

and use (A.18) to obtain $\begin{bmatrix} N_0 & N_1 & \dots & N_L \end{bmatrix}^T \Sigma_\Psi L_k = 0$. Since the factorization $\Psi(\zeta, \eta) = N^T(\zeta)\Sigma_\Psi N(\eta)$ is canonical, $\begin{bmatrix} N_0 & N_1 & \dots & N_L \end{bmatrix}$ is surjective. Hence (A.18) yields $L_k = 0$. This shows that $N(\xi)U^{-1}(\xi)$ is strictly proper and hence that $N(\frac{d}{dt})w$ is a state function as desired. Thus there exists a constant matrix K such that $N(\xi) = KX(\xi)$. This shows that there exists a matrix P such that (5.12) holds. This completes the proof of the theorem. \square

Proof of Theorem 5.7. We first prove the second part, the part regarding strong average positivity. In this case it follows from Proposition 5.3 that $\Phi(-i\omega, i\omega) > 0$ for all ω . Hence by Proposition 5.6, $\Phi(-\xi, \xi)$ has a Hurwitz and an anti-Hurwitz factorization. The associated storage functions, Ψ_+ and Ψ_- , satisfy

$$(A.19) \quad \frac{d}{dt}(Q_{\Psi_+}(w) - Q_{\Psi_-}(w)) = \|H\left(\frac{d}{dt}\right)w\|^2 - \|A\left(\frac{d}{dt}\right)w\|^2.$$

Let $x = X(\frac{d}{dt})w$ be a minimal state associated with a canonical factorization of Φ . By Theorem 5.5, there exist real symmetric matrices, say K_+ and K_- , such that for

all $w \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^q)$, we have

$$Q_{\Psi_+}(w) = X \left(\frac{d}{dt} w \right)^T K_+ X \left(\frac{d}{dt} w \right) w,$$

$$Q_{\Psi_-}(w) = X \left(\frac{d}{dt} w \right)^T K_- X \left(\frac{d}{dt} w \right) w.$$

Then if for all a there exists a solution w of $A(\frac{d}{dt})w = 0$ such that $(X(\frac{d}{dt})w)(0) = a$, we obtain, by integrating along this solution,

$$(A.20) \quad a^T K_+ a - a^T K_- a = Q_{\Psi_+}(w)(0) - Q_{\Psi_-}(w)(0) = \int_{-\infty}^0 \|H \left(\frac{d}{dt} w \right)\|^2 dt,$$

whence $K_+ \geq K_-$, so $\Psi_+ \geq \Psi_-$.

The problem is that there may not be a solution of $A(\frac{d}{dt})w = 0$ for all a such that $(X(\frac{d}{dt})w)(0) = a$. In order to circumvent this difficulty we first prove the statements of the second part under the additional assumption that $\Phi \geq \epsilon|\Phi|$ for some $\epsilon > 0$, in addition to the assumption that $\Phi(-i\omega, i\omega) > 0$ for all ω . Next, we modify Φ to Φ_ϵ such that these conditions hold for $\epsilon > 0$, and, finally, take the limit for $\epsilon \downarrow 0$.

Assume that $\Phi(-i\omega, i\omega) > 0$ for all $\omega \in \mathbb{R}$ and $\Phi \geq \epsilon|\Phi|$ for some $\epsilon > 0$. The system (5.5) allows an I/O representation, in the sense that there exists a permutation matrix P such that

$$(A.21) \quad Pv = \begin{bmatrix} U(\frac{d}{dt}) \\ Y(\frac{d}{dt}) \end{bmatrix} w,$$

with $\det(U) \neq 0$ and $G := YU^{-1}$ proper. Let $u = U(\frac{d}{dt})w$, $y = Y(\frac{d}{dt})w$. There exist constant matrices A, B, C , and D such that u, x , and y are related by $\frac{dx}{dt} = Ax + Bu, y = Cx + Du$. Since AU^{-1} is biproper, $A(\frac{d}{dt})w$ is of the form $Fx + Lu$ with L nonsingular. Using $u = -L^{-1}Fx$ and $x(0) = a$ in these equations then results in a solution of $A(\frac{d}{dt})w = 0$. To show that AU^{-1} is indeed biproper, use Proposition 5.2 to obtain

$$\begin{bmatrix} U(-i\omega) \\ Y(-i\omega) \end{bmatrix}^T P \Sigma_\Phi P^T \begin{bmatrix} U(i\omega) \\ Y(i\omega) \end{bmatrix} =$$

$$A^T(-i\omega)A(i\omega) \geq \epsilon \begin{bmatrix} U(-i\omega) \\ Y(-i\omega) \end{bmatrix}^T \begin{bmatrix} U(i\omega) \\ Y(i\omega) \end{bmatrix}.$$

After pre- and postmultiplying by $(U^{-1}(-i\omega))^T$ and $U^{-1}(i\omega)$, respectively, we obtain that

$$(A.22) \quad \begin{bmatrix} I \\ G(-i\omega) \end{bmatrix}^T P \Sigma_\Phi P^T \begin{bmatrix} I \\ G(i\omega) \end{bmatrix} = ((AU^{-1})(-i\omega))^T (AU^{-1})(i\omega) \geq \epsilon I.$$

Since G is proper, AU^{-1} is proper, by the equality on the left. The inequality on the right gives biproperness.

Consider a general Φ and define Φ_ϵ by $\Phi_\epsilon = \Phi + \epsilon|\Phi| + \epsilon I$. Then Φ_ϵ satisfies the above conditions and hence there exists (in the obvious notation) Ψ_ϵ^- and Ψ_ϵ^+ such that $\Psi_\epsilon^- \leq \Psi_\epsilon \leq \Psi_\epsilon^+$. Observe that for $0 < \epsilon_1 \leq \epsilon_2$ there holds $\Phi_{\epsilon_1} \leq \Phi_{\epsilon_2}$ and deduce

from $\Psi_{\epsilon_1}^+ \leq \Phi_{\epsilon_1} \leq \Phi_{\epsilon_2}$ that $\Psi_{\epsilon_2}^+ \geq \Psi_{\epsilon_1}^+$. Similarly, $\Psi_{\epsilon_2}^- \leq \Psi_{\epsilon_1}^-$. Consequently $\Psi_{\epsilon_2}^- \leq \Psi_{\epsilon_1}^- \leq \Psi_{\epsilon_1}^+ \leq \Psi_{\epsilon_2}^+$. Prove (using for example the associated matrix representations) that this monotonicity implies the existence of $\lim_{\epsilon \downarrow 0} \Psi_{\epsilon}^- =: \Psi_0^-$ and $\lim_{\epsilon \downarrow 0} \Psi_{\epsilon}^+ =: \Psi_0^+$.

We now prove that Ψ_0^- and Ψ_0^+ satisfy $\Psi_0^- \leq \Phi$ and $\Psi_0^+ \leq \Phi$, and subsequently that any storage function Ψ of Φ satisfies $\Psi_0^- \leq \Psi \leq \Psi_0^+$. To prove the first part, observe that $\Psi_{\epsilon}^- \leq \Phi_{\epsilon}$ and $\Psi_{\epsilon}^+ \leq \Phi_{\epsilon}$ for $\epsilon > 0$ and take the limit for $\epsilon \downarrow 0$. To prove the second part, assume that $\Psi \leq \Phi$. Then $\Psi \leq \Phi \leq \Phi_{\epsilon}$. Therefore $\Psi_{\epsilon}^- \leq \Psi \leq \Psi_{\epsilon}^+$. Now take the limit for $\epsilon \downarrow 0$.

We still have to prove the formulas (5.15) and (5.16) for the computation of Ψ_- and Ψ^+ for the case that we only have $\Phi(i\omega, -i\omega) > 0$ for all $\omega \in \mathbb{R}$ and not necessarily $\Phi \geq \epsilon|\Phi|$ for some $\epsilon > 0$. Let H_{ϵ} be a symmetric Hurwitz factor of $\Phi_{\epsilon}(-\xi, \xi)$: $\Phi_{\epsilon}(-\xi, \xi) = H_{\epsilon}^T(-\xi)H_{\epsilon}(\xi)$, as discussed in Proposition 5.6. In order to make it unique, normalize H_{ϵ} to $\sqrt{\Phi_{\epsilon}(0)}$. It holds that

$$H_{\epsilon}^T(\zeta)H_{\epsilon}(\eta) = \Phi_{\epsilon}(\zeta, \eta) - (\zeta + \eta)\Psi_{\epsilon}^-(\zeta, \eta).$$

Since $\Phi_{\epsilon} \rightarrow \Phi$ as $\epsilon \downarrow 0$ and $\Psi_{\epsilon}^- \rightarrow \Psi_0^-$ as $\epsilon \downarrow 0$, we also have that H_{ϵ} converges. Clearly the limit H_0 satisfies $\Phi(-\xi, \xi) = H_0^T(-\xi)H_0(\xi)$ and must be Hurwitz. The formula for $\Psi_0^- (= \Psi^-)$ follows. The situation for Ψ^+ is treated analogously. \square

Proof of Proposition 6.2: (i) Compute $\int_{-\infty}^0 Q_{\Phi}(w)$ for $w(t) = e^{\lambda t}a$ with $\Re(\lambda) > 0$ and $a \in \mathbb{C}^m$. This integral equals $\frac{\bar{a}^T \Phi(\bar{\lambda}, \lambda)a}{\lambda + \bar{\lambda}}$. This w is not of compact support, but an approximation argument can be used to complete the proof of (i). For $\Re(\lambda) = 0$ the result follows from Proposition 5.2. (ii) is proven similarly. \square

Proof of Theorem 6.3 : We prove that (3) \Rightarrow (2) \Rightarrow (1) \Rightarrow (3). That (3) \Rightarrow (2) is trivial. In order to see that (2) \Rightarrow (1), integrate $\frac{d}{dt}Q_{\Psi}(w) \leq Q_{\Phi}(w)$ from $-\infty$ to 0. We now prove that (1) \Rightarrow (3). Assume first that Φ satisfies the assumptions $\Phi(-i\omega, i\omega) > 0$ for all ω and $\Phi \geq \epsilon|\Phi|$ for some $\epsilon > 0$. By Theorem 5.7 we then have

$$(A.23) \quad \Psi_+(\zeta, \eta) = \frac{\Phi(\zeta, \eta) - A^T(\zeta)A(\eta)}{\zeta + \eta}.$$

This yields $\frac{d}{dt}Q_{\Psi_+}(w) = Q_{\Phi}(w) - \|A(\frac{d}{dt})w\|^2$ for all w . Let $x = X(\frac{d}{dt})w$ be a minimal state map of Φ . By Theorem 5.5, $Q_{\Psi_+}(w) = \|X(\frac{d}{dt})w\|_{K_+}^2$ for some real symmetric matrix K_+ . Using this expression in (A.23) and integrating from $-\infty$ to 0 yields that, for all a such that $X(\frac{d}{dt})w(0) = a$ and $A(\frac{d}{dt})w = 0$, we have $a^T K_+ a = \int_{-\infty}^0 Q_{\Phi}(w)dt$. This integral is ≥ 0 , so we must have $a^T K_+ a \geq 0$ (actually, such w does not have compact support but, by an approximation argument, the integral cannot be < 0). As in the proof of Theorem 5.7, it can be shown that for any initial condition a such w exists. This proves that $\Psi_+ \geq 0$. Take a general Φ . As in the proof of Theorem 5.7, first replace Φ by Φ_{ϵ} . By applying the previous to Φ_{ϵ} , we can conclude that (in the obvious notation) $\Psi_{\epsilon}^+ \geq 0$. Then take the limit for $\epsilon \downarrow 0$. \square

Proof of Theorem 6.4. We will first run the circle (1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (7) \Rightarrow (4) \Rightarrow (1).

(1) \Rightarrow (2). This was proven in Proposition 6.2.

(2) \Rightarrow (3). We have $P^T(\bar{\lambda})P(\lambda) \geq N^T(\bar{\lambda})N(\lambda)$ for $\lambda \in \mathbb{C}$, $\Re(\lambda) \geq 0$. Assume that NP^{-1} has a pole λ such that $\Re(\lambda) \geq 0$. Then there exists a vector $v \neq 0$ such that $P(\lambda)v = 0$ while $N(\lambda)v \neq 0$. This, however, contradicts the above inequality.

(3) \Rightarrow (7). Let $Q_K(w) = \|X(\frac{d}{dt})w\|_K^2$ be a storage function. We want to show that $K > 0$. First we show that NP^{-1} is a proper rational matrix. We have $P^T(-i\omega)P(i\omega) - N^T(-i\omega)N(i\omega) \geq 0$. Define $G := NP^{-1}$. Then for all ω such that $P(i\omega)$ is nonsingular, we have $G^T(-i\omega)G(i\omega) \leq I$, which shows that G is proper.

There exist matrices A, B, C , and D such that all smooth x, w_1 and w_2 satisfying $\dot{x} = Ax + Bw_2, w_1 = Cx + Dw_2$ can be written as $x = X(\frac{d}{dt})w, w_1 = N(\frac{d}{dt})w$, and $w_2 = P(\frac{d}{dt})w$ for some $w \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^\bullet)$. Since NP^{-1} has no poles in $\Re(\lambda) \geq 0$, the matrix A can be chosen such that its eigenvalues are in the open left half of the complex plane. Moreover, we may assume that the pair (C, A) is observable. Let $a \in \mathbb{R}^n$. Choose $w_2 = 0$, let x satisfy $\frac{d}{dt}x = Ax, x(0) = a$, and let $w_1 = Cx$. This shows that there exists $w \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^\bullet)$ such that $(X(\frac{d}{dt})w)(0) = a$ and $w_2 = P(\frac{d}{dt})w = 0$. Also, $X(\frac{d}{dt})w \in L_2[0, \infty)$ since A is a Hurwitz matrix. Since $w_1 = N(\frac{d}{dt})w = CX(\frac{d}{dt})w$, we also have that $N(\frac{d}{dt})w \in L_2[0, \infty)$. Thus we can integrate the dissipation inequality from 0 to ∞ to obtain

$$(A.24) \quad - \left\| \left(X \left(\frac{d}{dt} \right) w \right) (0) \right\|_K^2 \leq - \int_0^\infty \|N(\frac{d}{dt})w\|^2 dt.$$

This shows that $a^T K a \geq 0$. Assume that $a^T K a = 0$. Then we must have $w_1 = N(\frac{d}{dt})w = 0$. By observability of the pair (C, A) this implies that $a = 0$.

(7) \Rightarrow (4). Let Q_Ψ be any storage function. Since X is a state map, by Theorem 5.5 there exists a real symmetric matrix K such that $Q_\Psi(w) = \|X(\frac{d}{dt})w\|_K^2$ for all w . By assumption, K is positive definite.

(4) \Rightarrow (1). This was proven in Theorem 6.3.

The implications (7) \Rightarrow (5), (5) \Rightarrow (4), (7) \Rightarrow (6), and (6) \Rightarrow (4) are obvious.

Finally, if we assume observability, then the poles of NP^{-1} coincide with the singularities of the polynomial matrix P . This shows that, under this assumption, (3) and (3') are equivalent. This completes the proof. \square

Proof of Theorem 7.2. Consider the representation (3.10) of L_Φ . From the fact that the factorization is canonical, it is easily seen that the mappings $w_1 \mapsto (N(\frac{d}{dt})w_1)(0)$ and $w_2 \mapsto (M(\frac{d}{dt})w_2)(0)$ are surjective. Thus we have

$$L_\Phi(w_1, \bullet) = 0 \Leftrightarrow \left(N \left(\frac{d}{dt} \right) w_1 \right)^T M \left(\frac{d}{dt} \right) w_2 = 0 \text{ for all } w_2 \Leftrightarrow N \left(\frac{d}{dt} \right) w_1 = 0.$$

Similarly, $L_\Phi(\bullet, w_2) = 0$ iff $M(\frac{d}{dt})w_2 = 0$. From this, the equivalence of (1) and (3) is immediate.

To prove (1) \Rightarrow (2), assume that, for some $\lambda \in \mathbb{C}, a^T \Phi(\lambda, \xi) = 0$, where a is a complex vector. Define $w_1(t) := e^{\lambda t} \bar{a}$. For any w_2 and for all t , we then have

$$L_\Phi(w_1, w_2)(t) = e^{\lambda t} \left(a^T \Phi \left(\lambda, \frac{d}{dt} \right) w_2 \right) (t) = 0.$$

This implies $w_1 = 0$, so $a = 0$, which proves that the rows of $\Phi(\lambda, \xi)$ are linearly independent over \mathbb{C} . Similarly, we can prove that the columns of $\Phi(\xi, \lambda)$ are linearly independent.

Finally, we prove that (2) implies (3). Let $\lambda \in \mathbb{C}$ and put $M(\lambda)a = 0$ for some complex vector a . We want to prove that $a = 0$. We clearly get $N^T(\xi)M(\lambda)a = 0$ so $\Phi(\xi, \lambda)a = 0$. Since the columns of $\Phi(\xi, \lambda)$ are linearly independent over \mathbb{C} , this yields $a = 0$. Likewise we can prove that $N(\lambda)$ has full column rank for all λ . \square

Proof of Proposition 8.2. Write (8.3) as

$$(A.25) \quad (1 + \epsilon) \int_{-\infty}^{+\infty} \|N \left(\frac{d}{dt} \right) w\|^2 dt \leq (1 - \epsilon) \int_{-\infty}^{+\infty} \|P \left(\frac{d}{dt} \right) w\|^2 dt.$$

Put $\alpha = \frac{1-\epsilon}{1+\epsilon}$ and conclude that (2) and (3) are equivalent (also in the noncanonical case). To show that (1) \iff (2), observe that $|\Phi|(\zeta, \eta) = M_c^T(\zeta)M_c(\eta)$ for the special symmetric canonical factorization of $\Phi(\zeta, \eta)$ corresponding to (3.15). Hence statement (1) of the theorem is actually statement (2) for this special canonical factorization of Φ . It thus suffices to prove that if (2) holds for one canonical factorization, then it holds for any. From matrix theory, it follows that two canonical factorizations

$$(A.26) \quad M_1^T(\zeta)\Sigma_\Phi M_1(\eta) = M_2^T(\zeta)\Sigma_\Phi M_2(\eta)$$

are related by $M_1(\xi) = SM_2(\xi)$, with S a nonsingular matrix. Hence (2) for M_2 implies

$$\begin{aligned} \int_{-\infty}^{+\infty} \|M_1 \left(\frac{d}{dt} \right) w\|_{\Sigma_\Phi}^2 dt &= \int_{-\infty}^{+\infty} \|M_2 \left(\frac{d}{dt} \right) w\|_{\Sigma_\Phi}^2 dt \\ &\geq \epsilon_2 \int_{-\infty}^{+\infty} \|M_2 \left(\frac{d}{dt} \right) w\|^2 dt \\ &\geq \frac{\epsilon_2}{\|S\|^2} \int_{-\infty}^{+\infty} \|M_1 \left(\frac{d}{dt} \right) w\|^2 dt \end{aligned}$$

and (2) for M_1 follows. Obviously this proof can be reversed with M_1 playing the role of M_2 . If M_2 comes from a noncanonical factorization, then S may not be nonsingular and the proof goes through (but cannot be reversed). \square

Proof of Theorem 9.3. The proof is structured as follows. We first prove that (1) \iff (2). Subsequently, we show that (1) \implies (3) and finally that (3) \implies (1).

(1) \implies (2). That $\int^t Q_\Phi \gg 0$ implies (2a) is obvious. In order to prove (2b) we need the following lemma. Recall that a quadratic state function Q_Ψ (or simply Ψ), $Q_\Psi(w) = \|X(\frac{d}{dt})w\|_K^2$, is called *positive definite* if $K > 0$.

LEMMA A.2. *Let $M \in \mathbb{R}^{\bullet \times q}[\xi]$ be observable. Then there exists a positive definite state function Ψ such that*

$$(A.27) \quad \frac{d}{dt}Q_\Psi(w) \leq \|M \left(\frac{d}{dt} \right) w\|^2.$$

Proof. We show that Ψ_+ , the supremal storage function associated with $M^T(\zeta)M(\eta)$, fits the bill. By Theorem 5.5, Ψ_+ is a state function, say, $Q_{\Psi_+} = \|X(\frac{d}{dt})w\|_{K_+}^2$. Here we take X to be any minimal state map of M . Obviously $\Psi_+ \geq 0$, since $\Psi = 0$ satisfies (A.27) and $\Psi_+ \geq \Psi = 0$. In order to show that $K_+ > 0$, let $a \neq 0$ be arbitrary. We show that $a^T K_+ a > 0$. Factor $M^T(-\xi)M(\xi) = A^T(-\xi)A(\xi)$, with $A(\xi)$ anti-Hurwitz. Then

$$(A.28) \quad \frac{d}{dt}Q_{\Psi_+}(w) = \|M \left(\frac{d}{dt} \right) w\|^2 - \|A \left(\frac{d}{dt} \right) w\|^2$$

for all $w \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^q)$. As in the proof of Theorem 5.7, it is easily seen that there exists $w \neq 0$ such that $A(\frac{d}{dt})w = 0$ and $X(\frac{d}{dt})w(0) = a$ (show that AU^{-1} is biproper,

with $M = \text{col}(U, Y)$ an I/O partitioning). For this w in (A.28) we obtain $a^T K_+ a = \int_{-\infty}^0 \|M(\frac{d}{dt})w\|^2 dt > 0$, where the strict inequality follows from the observability of M . \square

We now return to the proof of (1) \Rightarrow (2b) of Theorem 9.3. Let $\Phi(\zeta, \eta) = M^T(\zeta)\Sigma_\Phi M(\eta)$ be the symmetric canonical factorization such that $|\Phi|(\zeta, \eta) = M^T(\zeta)M(\eta)$ (i.e., the one obtained by factoring $\tilde{\Phi} = \tilde{U}^T \Lambda \tilde{U}$, with Λ the diagonal matrix consisting of the nonzero eigenvalues of $\tilde{\Phi}$, and putting $\tilde{M} := \sqrt{|\Lambda|} \tilde{U}$). By the above lemma there exists a positive definite state function $\tilde{\Psi}$ such that $\frac{d}{dt} Q_{\tilde{\Psi}}(w) \leq \|M(\frac{d}{dt})w\|^2$. Now, $\int^t Q_{\tilde{\Psi}} \gg 0$ implies that there exists $\epsilon > 0$ such that $\int^t Q_{\Phi_\epsilon} \geq 0$, where $\Phi_\epsilon := \Phi - \epsilon|\Phi|$. Let Ψ_+^ϵ be the supremal storage function associated with Φ_ϵ . Clearly $\Psi_+^\epsilon \leq \Phi_\epsilon \leq \Phi$, so Ψ_+^ϵ is also a storage function for Φ . This immediately yields $\Psi_+^\epsilon \leq \Psi_+$. Consider the two-variable polynomial matrix $\Psi_+^\epsilon + \epsilon\tilde{\Psi}$. Clearly this defines a storage function for Φ as well, so $\Psi_+^\epsilon \leq \Psi_+^\epsilon + \epsilon\tilde{\Psi} \leq \Psi_+$. According to Theorem 6.3, $\Psi_+^\epsilon \geq 0$. Since $\tilde{\Psi}$ is a positive definite state function, this implies that Ψ_+ is a positive definite state function.

We show that (2) \Rightarrow (1). Let $X \in \mathbb{R}^{n \times q}[\xi]$ define a minimal state map for the system $v = M(\frac{d}{dt})w$. By (2b) Ψ_+ is a positive definite state function. Hence there exists $K_+ = K_+^T > 0$ such that $Q_{\Psi_+}(w) = \|X(\frac{d}{dt})w\|_{K_+}^2$. Factor $M^T(-\xi)\Sigma_M M(\xi) = A^T(-\xi)A(\xi)$ with A anti-Hurwitz. Then we have

$$(A.29) \quad \int_{-\infty}^0 \|M\left(\frac{d}{dt}\right)\|_{\Sigma_\Phi}^2 dt = \|X\left(\frac{d}{dt}\right)w(0)\|_{K_+}^2 + \int_{-\infty}^0 \|A\left(\frac{d}{dt}\right)w\|^2 dt$$

for all $w \in \mathfrak{D}(\mathbb{R}, \mathbb{R}^q)$. There exists a permutation matrix P such that $PM = \text{col}(U, Y)$, with $\det(U) \neq 0$ and YU^{-1} proper. Write $u = U(\frac{d}{dt})w$, $y = Y(\frac{d}{dt})w$, and $x = X(\frac{d}{dt})w$, with w ranging over $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^q)$. Write the associated input/state/output representation. Hence there are constant matrices A_1, B_1, C_1 , and D_1 such that these u, y , and x are exactly those that are related by the equations

$$(A.30) \quad \dot{x} = A_1x + B_1u, \quad y = C_1x + D_1u.$$

As in the proof of Theorem 5.7, by strict positivity we have that AU^{-1} is biproper. Thus there exist constant matrices F and L , $\det(L) \neq 0$ such that $A(\frac{d}{dt})w = Fx + Lu$. Solving this equation for u and substituting the result in (A.30) yields that the relation between $a := A(\frac{d}{dt})w$, $v = P\text{col}(u, y)$, and $x = X(\frac{d}{dt})w$ is given by linear equations of the form

$$(A.31) \quad \dot{x} = A_2x + B_2a \quad v = C_2x + D_2a$$

with (C_2, A_2) observable and where the eigenvalues of A_2 coincide with the singularities of the spectral factor $A(\xi)$. This shows that for given $a \in L_2((-\infty, 0], \mathbb{R}^\bullet)$ and final condition $x(0) = x_0$, the corresponding v is in $L_2((-\infty, 0], \mathbb{R}^\bullet)$. In other words (A.31) defines a bounded operator from $L_2((-\infty, 0], \mathbb{R}^\bullet) \times \mathbb{R}^n$ to $L_2((-\infty, 0], \mathbb{R}^\bullet)$, mapping (a, x_0) to v . Hence there exists constants C_1 and C_2 such that

$$\int_{-\infty}^0 \|v\|^2 dt \leq C_1 \int_{-\infty}^0 \|a\|^2 dt + C_2 \|x_0\|^2.$$

Since $K_+ > 0$, there exists $\epsilon > 0$ such that $\frac{1}{\epsilon}K_+ > C_2I$ and $\frac{1}{\epsilon} > C_1$. For this ϵ we have

$$\int_{-\infty}^0 \|v\|^2 dt \leq \frac{1}{\epsilon} \left(\int_{-\infty}^0 \|a\|^2 dt + x_0^T K_+ x_0 \right)$$

which, by (A.29), is equivalent to

$$\int_{-\infty}^0 \|M\left(\frac{d}{dt}\right)\|^2 dt \leq \frac{1}{\epsilon} \int_{-\infty}^0 \|M\left(\frac{d}{dt}\right)\|_{\Sigma_\Phi}^2 dt.$$

This shows that Φ is strictly half-line positive. Whence, (2) \Rightarrow (1).

Next we show that (1) \Rightarrow (3). We will only consider the semisimple case. That (1) \Rightarrow (3a) follows from Proposition 5.2. To prove (3b), calculate $\int_{-\infty}^0 Q_\Phi(a)dt$ for

$$a(t) = \sum_{k=1}^n \alpha_k e^{\lambda_k t} a_k$$

and obtain the result

$$(A.32) \quad \int_{-\infty}^0 Q_\Phi(a)dt = \begin{bmatrix} \bar{\alpha}_1 \\ \bar{\alpha}_2 \\ \vdots \\ \bar{\alpha}_n \end{bmatrix}^T T_\Phi \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}.$$

We know that, for some $\epsilon > 0$, $\int_{-\infty}^0 Q_\Phi(w)dt \geq \epsilon \int_{-\infty}^0 Q_{|\Phi|}(w)dt$ for all w of compact support. An approximation argument yields that this implies $\int_{-\infty}^0 Q_\Phi(a)dt > 0$ for $a \neq 0$, equivalently for $\text{col}(\alpha_1, \alpha_2, \dots, \alpha_n) \neq 0$. Hence, $T_\Phi > 0$.

Finally, we turn to (3) \Rightarrow (2). The implication (3) \Rightarrow (2a) follows from Proposition 5.2. To show that (3) \Rightarrow (2b), we show that $T_\Phi > 0$ implies that the supremal storage function Ψ_+ defines a positive definite state function. Let $\partial\Phi(\xi) = A^T(-\xi)A(\xi)$ be an anti-Hurwitz factorization. We claim that $\lambda_1, \lambda_2, \dots, \lambda_n$ are exactly the singularities of $A(\xi)$, with associated vectors a_1, a_2, \dots, a_n in the kernel of $A(\lambda_k)$, $k = 1, 2, \dots, n$. Indeed, if λ has $\Re e(\lambda) > 0$, then $A^T(-\lambda)$ is nonsingular. Hence $\Phi(-\lambda_k, \lambda_k)a_k = 0$ and $\Re e(\lambda_k) > 0$ implies $A(\lambda_k)a_k = 0$.

According to Theorem 5.7, it holds that

$$(A.33) \quad \frac{d}{dt}Q_{\Psi_+}(w) = Q_\Phi(w) - \|A\left(\frac{d}{dt}\right)w\|^2.$$

For any solution $w = \sum_{k=1}^n \alpha_k e^{\lambda_k t} a_k$ of $A\left(\frac{d}{dt}\right)w = 0$, we thus have

$$(A.34) \quad Q_{\Psi_+}(w)(0) = \int_{-\infty}^0 Q_\Phi(w)dt = \begin{bmatrix} \bar{\alpha}_1 \\ \bar{\alpha}_2 \\ \vdots \\ \bar{\alpha}_n \end{bmatrix}^T T_\Phi \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}.$$

Also, there exists a real symmetric matrix K_+ such that

$$Q_{\Psi_+}(w) = \|X\left(\frac{d}{dt}\right)w\|_{K_+}^2,$$

where $X\left(\frac{d}{dt}\right)$ is a minimal state map. Since $T_\Phi > 0$, (A.34) implies that $K_+ > 0$. Indeed, let $a \neq 0$ be arbitrary. Let $w \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^q)$ be such that $A\left(\frac{d}{dt}\right)w = 0$, say,

$w = \sum_{k=1}^r a_k e^{\lambda_k t} \alpha_k$, and $X(\frac{d}{dt})w(0) = a$. Such w exists by strict positivity (see the proof of Theorem 5.7). Thus we have

$$a^T K_+ a = \left\| X \left(\frac{d}{dt} \right) w(0) \right\|_{K_+}^2 = \begin{bmatrix} \bar{\alpha}_1 \\ \bar{\alpha}_2 \\ \vdots \\ \bar{\alpha}_n \end{bmatrix}^T T_{\Phi} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} > 0.$$

This completes the proof of Theorem 9.3. □

Proof of Proposition 10.1. Let $R(\frac{d}{dt})w = 0$ and $w = M(\frac{d}{dt})\ell$ be, respectively, a kernel and an observable image representation of \mathfrak{B} . Then we have $R(\xi)M(\xi) = 0$. Furthermore, $v = R^T(-\frac{d}{dt})\ell'$ is an image representation of \mathfrak{B}^\perp . Consider the BLDF $(R^T(-\frac{d}{dt})\ell')^T M(\frac{d}{dt})\ell$. Note that this is the BLDF associated with the two-variable polynomial matrix $R(-\zeta)M(\eta)$. By Theorem 3.1, there exists $\Psi(\zeta, \eta)$ such that $\frac{d}{dt}Q_\Psi(\ell', \ell) = (R^T(-\frac{d}{dt})\ell')^T M(\frac{d}{dt})\ell$, and by Theorem 5.5 $Q_\Psi(\ell', \ell)$ is a state function; in other words, if $X(\frac{d}{dt})$ and $\tilde{Z}(\frac{d}{dt})$ are minimal state maps of \mathfrak{B} and \mathfrak{B}^\perp , then $Q_\Psi(\ell, \ell') = (\tilde{Z}(\frac{d}{dt})\ell')^T K X(\frac{d}{dt})\ell$ for some matrix K . The proposition follows by taking $Z := K^T \tilde{Z}$ if we can show that K is nonsingular. To show this, assume to the contrary that $Ka = 0$. Let $\tilde{w} \in \mathfrak{B} \cap \mathfrak{D}(\mathbb{R}, \mathbb{R}^q)$ be a trajectory emanating at $t = 0$ from $x(0) = a$. It follows that

$$\int_0^\infty v^T \tilde{w} dt = 0$$

for all $v \in \mathfrak{B}^\perp \cap \mathfrak{D}(\mathbb{R}, \mathbb{R}^q)$. Consider the function $\hat{w} : \mathbb{R} \rightarrow \mathbb{R}^q$ such that $w(0) = 0$ for $t \leq 0$ and $\hat{w}(t) = \tilde{w}(t)$ for $t \geq 0$. Obviously it holds that

$$\int_{-\infty}^{+\infty} v^T \hat{w} dt = 0$$

for all $v \in \mathfrak{B}^\perp \cap \mathfrak{D}(\mathbb{R}, \mathbb{R}^q)$. Therefore \hat{w} belongs to the $L_2(\mathbb{R}, \mathbb{R}^q)$ closure of \mathfrak{B} (this is the one point in this paper where \mathcal{C}^∞ solutions are inadequate). Since $\hat{w}(t) = 0$ for $t \leq 0$, it must hold that $x(0) = 0$. Hence $Ka = 0$ implies $a = 0$, yielding the result. □

In order to prove Theorem 10.2 we use the following lemma. Recall that the inertia of an $n \times n$ complex Hermitian matrix H is the triple (π_-, π_0, π_+) , with π_- the number of negative eigenvalues, π_+ the number of positive eigenvalues, and $\pi_0 (= n - \pi_- - \pi_+)$ the multiplicity of the zero eigenvalue.

LEMMA A.3. *Let \mathcal{L} be a linear subspace of \mathbb{R}^n . Consider the quadratic form $x^T Q x$ on \mathbb{R}^n with $Q = Q^T$ nonsingular. Let the inertia of Q be $(\pi_-, 0, \pi_+)$ and assume that $\pi_+ = \dim(\mathcal{L})$. Then $a^T Q a > 0$ for all $0 \neq a \in \mathcal{L}$ iff $a^T Q^{-1} a < 0$ for $0 \neq a \in \mathcal{L}^\perp$, and $a^T Q a \geq 0$ for all $a \in \mathcal{L}$ iff $a^T Q^{-1} a \leq 0$ for $a \in \mathcal{L}^\perp$.*

Proof. Let $\mathcal{L} = \ker(R) = \text{im}(M)$ with R surjective and M injective. Then $\mathcal{L}^\perp = \text{im}(R^T) = \ker(M^T)$. Furthermore, $a^T Q a > 0$ for $0 \neq a \in \mathcal{L}$ means $M^T Q M > 0$. Consider the relations

$$\begin{bmatrix} M^T \\ RQ^{-1} \end{bmatrix} Q \begin{bmatrix} M & Q^{-1}R^T \end{bmatrix} = \begin{bmatrix} M^T Q M & 0 \\ 0 & RQ^{-1}R^T \end{bmatrix},$$

$$\begin{bmatrix} M^T \\ R \end{bmatrix} Q \begin{bmatrix} M & Q^{-1}R^T \end{bmatrix} = \begin{bmatrix} M^T Q M & 0 \\ RQ M & RR^T \end{bmatrix}.$$

The second relation shows that $[M \ Q^{-1}R^T]$ is nonsingular. The first shows that

$$\text{in}(M^TQM) + \text{in}(RQ^{-1}R^T) = \text{in}(Q).$$

Hence $M^TQM > 0$ implies $RQ^{-1}R^T < 0$. To get the \geq case, replace Q by $Q + \epsilon I$ and let $\epsilon \downarrow 0$. \square

Proof of Theorem 10.2. To prove (i) and (ii), combine Proposition 5.2 and Lemma A.3 in the following way. For $\omega \in \mathbb{R}$ fixed, define $\mathcal{L} := \text{im}(M(i\omega)) = \ker(R(i\omega))$. Define $Q := \Sigma_\Phi$. Note that $Q^{-1} = \Sigma_\Phi$ as well. Using that $M(i\omega)$ and $R^T(-i\omega)$ are injective, we get the equivalence

$$M^T(i\omega)\Sigma_\Phi M(i\omega) > 0 \Leftrightarrow R(i\omega)\Sigma_\Phi R^T(-i\omega) < 0$$

which yields statement (ii). Statement (i) follows from the second assertion of Lemma A.3, which yields the same equivalence with nonstrict inequalities.

We now prove (iii). Again this can be proven using Lemma A.3, this time with $Q = \Sigma_\Phi - \epsilon I$. We then get

$$M^T(i\omega)(\Sigma_\Phi - \epsilon I)M(i\omega) \geq 0 \Leftrightarrow R(i\omega)(\Sigma_\Phi - \epsilon I)^{-1}R^T(-i\omega) \leq 0.$$

Using the formula $\Sigma_\Phi + \epsilon I = (1 - \epsilon^2)(\Sigma_\Phi - \epsilon I)^{-1}$, the latter is equivalent with

$$R(i\omega)\Sigma_\Phi R^T(-i\omega) \leq -\epsilon R(i\omega)R^T(-i\omega).$$

This shows (iii).

In order to prove (iv), we need the following lemma.

LEMMA A.4. *Let (X, Z) be a matched pair of minimal state maps for \mathfrak{B} and \mathfrak{B}^\perp . Define subspaces $\mathcal{L} \subset \mathbb{R}^{r+2n}$, $\mathcal{M} \subset \mathbb{R}^{r+2n}$ by*

$$(A.35) \mathcal{L} := \left\{ \begin{bmatrix} w \\ x \\ a \end{bmatrix} \in \mathbb{R}^{r+2n} \mid \exists \ell \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^\bullet) \begin{bmatrix} w \\ x \\ a \end{bmatrix} = \begin{bmatrix} M(\frac{d}{dt})\ell \\ X(\frac{d}{dt})\ell \\ \frac{d}{dt}X(\frac{d}{dt})\ell \end{bmatrix} (0) \right\},$$

$$(A.36) \mathcal{M} := \left\{ \begin{bmatrix} v \\ b \\ z \end{bmatrix} \in \mathbb{R}^{r+2n} \mid \exists \ell' \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^\bullet) \begin{bmatrix} v \\ b \\ z \end{bmatrix} = \begin{bmatrix} R^T(-\frac{d}{dt})\ell' \\ -\frac{d}{dt}Z(\frac{d}{dt})\ell' \\ -Z(\frac{d}{dt})\ell' \end{bmatrix} (0) \right\}.$$

Then $\dim(\mathcal{L}) = n + m$ and $\mathcal{L}^\perp = \mathcal{M}$.

Proof. There exists a permutation matrix P such that

$$PM = \begin{bmatrix} U \\ Y \end{bmatrix}$$

with $U \in \mathbb{R}^{m \times m}[\xi]$ and YU^{-1} a proper rational matrix. If we define $u = U(\frac{d}{dt})\ell$ and $y = Y(\frac{d}{dt})\ell$, then u has the usual properties of input and y has the usual properties of output of \mathfrak{B} . There exist matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$, and $D \in \mathbb{R}^{p \times m}$ (with $p = r - m$, the number of outputs) such that $x = X(\frac{d}{dt})\ell$, $u = U(\frac{d}{dt})\ell$, and $y = Y(\frac{d}{dt})\ell$ are exactly related by $\frac{dx}{dt} = Ax + Bu$, $y = Cx + Du$. Thus for all $\ell \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^\bullet)$ we have:

$$\begin{bmatrix} P & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} M(\frac{d}{dt})\ell \\ X(\frac{d}{dt})\ell \\ \frac{d}{dt}X(\frac{d}{dt})\ell \end{bmatrix} (0) = \begin{bmatrix} 0 & I_m \\ C & D \\ I_n & 0 \\ A & B \end{bmatrix} \cdot \begin{bmatrix} X(\frac{d}{dt})\ell \\ U(\frac{d}{dt})\ell \end{bmatrix} (0).$$

This implies

$$\begin{bmatrix} P & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \mathcal{L} \subset \text{im} \begin{bmatrix} 0 & I_m \\ C & D \\ I_n & 0 \\ A & B \end{bmatrix}.$$

Here, in fact, equality holds. Indeed, given $\text{col}(x_0, u_0)$, take any $x \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^n)$ and $u \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^m)$ such that $x(0) = x_0$ and $u(0) = u_0$, and such that $\frac{dx}{dt} = Ax + Bu$, $y = Cx + Du$. There exists $\ell \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^\bullet)$ such that $x = X(\frac{d}{dt})\ell$, $u = U(\frac{d}{dt})\ell$. This shows that equality holds and that $\dim(\mathcal{L}) = n + m$.

We now prove that $\mathcal{L}^\perp = \mathcal{M}$. For all $\ell \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^\bullet)$ and $\ell' \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^\bullet)$ we have that (10.4) holds. By evaluating this for $t = 0$, we immediately obtain that $\mathcal{L} \perp \mathcal{M}$. Thus it suffices to show that $\dim \mathcal{L} = n + (r - m)$. This is, however, an immediate consequence of the fact that the number of inputs of \mathfrak{B}^\perp , $m(\mathfrak{B}^\perp)$ is equal to $r - m$. \square

We now return to the proof of (iv) of Theorem 10.2. Assume that $\int Q_\Phi \geq 0$ and let $\Psi(\zeta, \eta) = X^T(\zeta)KX(\eta)$, with $K = K^T$ a storage function for Φ , i.e., $\dot{\Psi} \leq \Phi$. In terms of $w = M(\frac{d}{dt})\ell$, $x = X(\frac{d}{dt})\ell$, $\frac{dx}{dt} = \frac{d}{dt}X(\frac{d}{dt})\ell$ this inequality yields, in particular,

$$(A.37) \quad \begin{bmatrix} w(0) \\ x(0) \\ \frac{dx}{dt}(0) \end{bmatrix}^T \begin{bmatrix} \Sigma_\Phi & 0 & 0 \\ 0 & 0 & -K \\ 0 & -K & 0 \end{bmatrix} \begin{bmatrix} w(0) \\ x(0) \\ \frac{dx}{dt}(0) \end{bmatrix} \geq 0.$$

Denote the symmetric matrix in (3.16) by Q . Note that (A.37) says that $a^T Q a \geq 0$ for all $a \in \mathcal{L}$, with \mathcal{L} defined by (A.35). Since $\dim(\mathcal{L}) = n + m = n + r_+$, which is exactly the number of positive eigenvalues of Q , it follows from Lemma A.3 that $a^T Q^{-1} a \leq 0$ for all $a \in \mathcal{L}^\perp = \mathcal{M}$. More explicitly,

$$(A.38) \quad a^T \begin{bmatrix} \Sigma_\Phi & 0 & 0 \\ 0 & 0 & -K^{-1} \\ 0 & -K^{-1} & 0 \end{bmatrix} a \leq 0$$

for $a \in \mathcal{L}^\perp$. A typical element of \mathcal{M} has the form

$$a = \begin{bmatrix} R^T(-\frac{d}{dt})\ell' \\ -\frac{d}{dt}Z(\frac{d}{dt})\ell' \\ -Z(\frac{d}{dt})\ell' \end{bmatrix} (t),$$

where $\ell' \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^\bullet)$. By letting $t \in \mathbb{R}$ be arbitrary, the inequality (A.38) yields exactly the dissipation inequality

$$\frac{d}{dt} \|Z\left(\frac{d}{dt}\right)\ell'\|_{-K^{-1}}^2 \leq \|R^T\left(-\frac{d}{dt}\right)\ell'\|_{-\Sigma_\Phi}^2$$

which is the content of (4). To show (v), use (iv) and Theorem 9.3. \square

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